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# THÈSE

pour obtenir le diplôme de Doctorat en Mathématiques

Option:

Analyse et contrôle des systèmes

Présentée par:

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THÈME:

# Comportement de la solution de certains systèmes d'évolution semi linéaire

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# Contents

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# Abstract

In the first part of this work, we consider a nonlinear hyperbolic equation with variable damping and source terms. Our aim is to prove that the solution with negative initial energy blows up in finite time. After that, we consider a coupled system of nonlinear wave equations with variable exponents in the damping terms. By using the multiplier method, we prove the decay estimates for the solution under appropriate assumptions on these exponents. In the third part, we study the wave equation with damping, source and nonlinear first order perturbation terms. Our aim is to prove that if the damping terms dominated the first order perturbation term then the energy is decreasing and the solutions with sufficiently negative initial energy blow up in finite time.

**Key words**. Wave equation; Coupled system, Viscoelastic term; Damping term; Source term; First order perturbation term; Variable exponents ; Blow up; General decay.

# Résumé

Dans la première partie de ce travail, on considère une équation hyperbolique non linéaire avec des termes dissipatif et source variables. Notre objectif est de prouver que la solution avec une énergie initiale négative explose en un temps fini. Ensuite, on considère un système couplé d'équations des ondes non linéaires avec des exposants variables. En utilisant la méthode du multiplicateur, on montre les estimations de décroissance pour la solution sous des hypothèses sur ces exposants. Dans la troisième partie, on étudie l'équation des ondes avec des termes dissipatif, source et une perturbation du premier ordre non linéaires. Notre but est de montrer que si les termes dissipatifs dominent le terme de perturbation alors l'énergie diminue et les solutions avec une énergie initiale suffisamment négative explosent en un temps fini.

***Mots clés :*** Equation des ondes, Système couplé, Terme viscoélastique, Terme dissipatif, Terme source, Perturbation du premier ordre, Exposant variable, Explosion

# الملخص

في الجزء الأول من هذا العمل، نعتبر المعادلة غير الخطية الزائدية مع معامل الكبح ومعامل المنبع المتغيرة. هدفنا هو إثبات أن الحل ذو طاقة ابتدائية سالبة ينفجر في وقت منته. بعد ذلك، نعتبر جملة مزدوجة من المعادلات الزائدية غير الخطية وبأسس متغيرة. باستخدام طريقة الضرب، نثبت تناقص طاقة الحل ضمن فرضيات لهذه الأسس المتغيرة. في الجزء الثالث، ندرس معادلة الموجة بمعامل الكبح،المنبع والاضطراب من الدرجة الأولى غير الخطية. هدفنا هو إثبات أنه إذا كان معامل الكبح يهيمن على معامل الاضطراب من الدرجة الأولى فإن الطاقة تتناقص والحل ذو طاقة ابتدائية سالبة بما فيه الكفاية ينفجر في وقت منته.

***الكلمات المفتاحية:*** معادلة الأمواج، جملة معادلتين، معامل لزج، معامل الكبح، معامل المنبع، اضطراب من الدرجة الأولى، الأس المتغير، الانفجار.

# General Introduction

## 1. Literature Review

A considerable and great effort has been devoted to the study of linear and nonlinear wave equations in the case of constant and variable exponent nonlinearities. Our aim, here, is to give an overview of the existing results and introduce some other ones. In the chapter two of this thesis, we consider the following system

$$\begin{cases} u_{tt} - div\,(A\nabla u) + u_t\,|u_t|^{m(.)-2} = u\,|u|^{p(.)-2} \quad \text{in } \Omega\times(0,T)\,, \\ \\ u = 0 \quad \text{on } \partial\Omega\times(0,T)\,, \\ \\ u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 \quad \text{in } \Omega. \end{cases} \tag{$P1$}$$

Where $T>0$, $\Omega$ is a bounded domain of $\mathbb{R}^n$ $(n\in\mathbb{N}^*)$ with a smooth boundary $\partial\Omega$. $A = A(x,t)$ is an $n\times n$ symmetric matrix with real coefficients. The exponents $m(.)$ and $p(.)$ are given measurable functions on $\Omega$.

When $A =$ Identity in $(P1)$, the bibliography of works concerning problems of existence and nonexistence of global solution is truly long. In the case of constant damping and source terms, Ball [**10**] in 1977 considered the wave equation with source term and proved the blow up of solution when the energy of the initial data is negative. Haraux and Zuazua [**27**] in 1988 proved that the damping term of polynomial or arbitrary growth assured the global estimates of the wave equation for arbitrary initial data. The interaction between the damping and the source term was considered by Levine [**45**] in 1974, in the linear damping case $m = 2$. He showed that the solutions with negative initial energy blow up in finite time. Georgiev and Todorova [**24**] in 1994 extended Levine's result to the nonlinear damping case $m>2$. They showed that solutions with any initial data is global if the damping term dominated the source term then blow up in finite time if the source term dominated the damping term and the initial energy is sufficiently negative. Without imposing the condition that the initial energy is sufficiently negative. Messaoudi [53] in 2001 proved that any negative initial energy solution blows up in finite time. In the case of variable damping and source term, these problems have been considered by many authors using the Lebesgue spaces with variable exponents [17]. For instance, Antontsev [8] in 2011, considered the wave equation with $p(x,t)$-Laplacian and variable source term. In his work, he proved existence and blow up results under some assumptions on the initial energy data. Recently, Messaoudi and Talahmeh [56] considered in 2017 the quasilinear wave equation with variable exponents nonlinearities and proved that the solution with negative or positive initial energy blows up in finite time. In the same year, Messaoudi et al. [57] considered the nonlinear wave equation with variable source and damping terms and proved the blow up of solution with negative energy of initial data. In 2018, Ghegal et al. [23] considered the same system. They used the stable set method to prove the global existence result. Then, by some integral inequality they showed the stability of this solution. Noting that, there has been a lot of interest in Mathematical models of parabolic, elliptic and hyperbolic equations with variable exponents. Variable exponents Lebesgue spaces appeared in the literature for the first time already in a paper by Orlicz [62] in 1931. In 2001, Rajagopal and Ruzicka [65] presented the Mathematical theory for the application of variable exponents spaces in electro-rheological fluids. Problems with variable exponents growth conditions, also, appear in the Mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids [7] in 2005 and nonlinear elastics [75] in 2008 and the references therein.

When $A(x,t)=a(x,t)$ where $a$ is a given function, Sun et al. [69] showed in 2016 a result of blow up of solution when the energy of initial data is positive.

When $A = A(x,t)$, Boukhatem and Benyatou [13], in 2012, considered the hyperbolic equation with constants damping and source terms. They obtained a result of blows up of solution when the initial energy is positive.

After that, we consider in chapter three, the following initial boundary value system

$$\begin{cases} u_{tt} - \Delta u + a\left|u_t\right|^{m(.)-2} u_t + \alpha(u-v) = 0 & \text{in } \Omega\times(0,T), \\ v_{tt} - \Delta v + b\left|v_t\right|^{r(.)-2} v_t + \alpha(v-u) = 0 & \text{in } \Omega\times(0,T), \\ u = v = 0 & \text{on } \partial\Omega\times(0,T), \quad (P2) \\ u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 & \text{in } \Omega, \\ v(0) = v_0 \quad \text{and} \quad v_t(0) = v_1 & \text{in } \Omega, \end{cases}$$

where $a, b, \alpha$ are positive constants and the exponents $m(.)$ and $r(.)$ are given measurable functions on $\Omega$.

Coupled systems of two nonlinear wave equations has been treated by several authors. Indeed, in the case of constant exponents, we consider the initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds + h_1(u_t) = f_1(x,u) & \text{in } \Omega\times(0,+\infty), \\ v_{tt} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds + h_2(v_t) = f_2(x,u) & \text{in } \Omega\times(0,+\infty), \\ u = v = 0 & \text{on } \partial\Omega\times(0,T), \\ u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 & \text{in } \Omega, \\ v(0) = v_0 \quad \text{and} \quad v_t(0) = v_1 & \text{in } \Omega, \end{cases} \tag{1.1}$$

with the presence of the memory term ($g_i \neq 0, i = 1, 2$), there are numerous results related to the asymptotic behavior and blow up of solutions of viscoelastic systems. For example, Liang and Gao [47] in 2011 studied problem (1.1) with $h_1(u_t) = -\Delta u_t$ and $h_2(v_t) = -\Delta v_t$. They obtained, under suitable conditions on the functions $g_i, f_i, i = 1, 2$ and certain initial data in the stable set, that the decay rate of the energy functions is exponential. On the contrary, for certain initial data in the unstable set, there are solutions with positive initial energy that blow up in finite time. For $h_1(u_t) = |u_t|^{m-1} u_t$ and $h_2(v_t) = |v_t|^{r-1} v_t$, Hun and Wang [26] in 2009 established several results related to local existence, global existence and finite time blow up when the initial energy is negative. This has been later improved by Messaoudi and Said Houari [52] in 2010 by using the same method as in [66] and some estimates obtained in [51]. They proved a global nonexistence result of certain solutions with positive initial energy.

Conversely, in the absence of the viscoelastic terms in (1.1). Agre and Rammaha [5] in 2006 proved several results concerning local and global existence of a weak solution and showed that any weak solution with negative initial energy blow up in finite time, using the same techniques as in [24]. Alves et al.[6] in 2009 investigated the existence, uniform decay rates and blow up of solutions. After that, the blow up result was improved by Said Houari [66] in 2010. Also, He [67] in 2012 showed that the local solution obtained in [5] is global and this solution has a decay property. When the functions $g_i, h_i$ and $f_i$ ($i = 1, 2$) are not taken into account in (1.1). Aassila [1] in 1999 obtained the decay estimates of the energy of solutions to compactly coupled wave equations with a nonlinear boundary dissipation given by

$$\begin{cases} u_{1tt} - \Delta u_1 + \alpha (u_1 - u_2) = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \\ u_{2tt} - \Delta u_2 + \alpha (u_2 - u_1) = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u_i = 0 & \text{on } \Gamma_0 \times \mathbb{R}_+, \quad i = 1, 2, \\ \frac{\partial u_i}{\partial \nu} + a_i u_i + g_i (u_{it}) = 0 & \text{on } \Gamma_1 \times \mathbb{R}_+, \quad i = 1, 2, \\ u_i(0) = u_{i0} \text{ and } u_{it}(0) = u_{i1} & \text{in } \Omega, \quad i = 1, 2, \end{cases}$$

where $\{\Gamma_0, \Gamma_1\}$ is a partition of its boundary and $\alpha : \Omega \to \mathbb{R},\ a_1, a_2 : \Gamma_1 \to \mathbb{R},\ g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are some given functions.

Erhan et al. [19] in 2017 considered the following coupled nonlinear wave equations

$$\begin{cases} u_{tt} + u_t + |u_t|^{p-1} u_t = \operatorname{div}\left(\rho\left(|\nabla u|^2\right) \nabla u\right) + f_1(u, v) & (x,t) \in \Omega \times (0, T), \\ \\ v_{tt} + v_t + |v_t|^{q-1} v_t = \operatorname{div}\left(\rho\left(|\nabla v|^2\right) \nabla v\right) + f_2(u, v) & (x,t) \in \Omega \times (0, T), \\ u(x,t) = v(x,t) = 0 & (x,t) \in \partial\Omega \times (0, T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x) & x \in \Omega \end{cases} \tag{1.2}$$

and proved the exponential growth for sufficiently large initial data. In the absence of the linear weak damping $u_t$ and $v_t$ terms. Wu and Li [72] in 2011 obtained the blow up of the solution of problem (1.2), for negative initial energy. Wu et al. [73] in 2010 studied the last system and proved a global existence and blow up of the solution under some suitable conditions. This blow up result has been improved by Fei and Hongjun [20] in 2011 for a large class of initial data in positive initial energy, using some techniques as in Payne and Sattinger [64] and some estimates used firstly by Vitillaro [71]. Recently, Erhan and Polat [18] in 2013 studied the local and global existence, energy decay and blow up of the solution.

To our best of knowledge, very little is known for the coupled systems with variable exponents, global existence and stability. The first tentative was given by Bouhoufani and

Hamchi [12] in 2020. They considered the following coupled system of two nonlinear hyperbolic equations

$$\begin{cases} u_{tt} - \operatorname{div}(A\nabla u) + |u_t|^{m(x)-2} u_t = f_1(x,u,v) & \text{in } \Omega \times (0,T), \\ v_{tt} - \operatorname{div}(B\nabla v) + |v_t|^{r(x)-2} v_t = f_2(x,u,v) & \text{in } \Omega \times (0,T), \\ u = v = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \text{ and } u_t(0) = u_1 & \text{in } \Omega, \\ v(0) = v_0 \text{ and } v_t(0) = v_1 & \text{in } \Omega \end{cases}$$

and proved, under suitable assumptions on the initial data and the variable exponents, the global existence theorem by using the Stable set method and established a decay estimate of the solution energy by Komornik's integral inequalities.

Finally, in chapter four, we consider the following system

$$\begin{cases} u_{tt} - \Delta u + g * \Delta u + a u_t + F(t, \nabla u) = |u|^{p-2} u \quad \text{in } \Omega \times (0,T), \\ u = 0 \qquad \text{on } \partial\Omega \times (0,T), \\ u(.,0) = u_0(.) \quad \text{and} \quad u_t(.,0) = u_1(.) \quad \text{in } \Omega, \end{cases} \qquad (P3)$$

where $p > 2$, $a > 0$ are constants, $g$ and $F$ are functions satisfying some conditions to be specified later. Noting that $(g * v)(t) = \int_0^t g(t-\tau)v(\tau)d\tau$ for all $t \geq 0$.

When $F = 0$, the problem of existence and nonexistence of global solution has been extensively studied by many researches. In the absence of the polynomial source term $|u|^{p-2} u$, Messaoudi [55] in 2005 considered system $(P3)$ with $a = 0$. He obtained an exponential decay result of the global solution under some conditions on the relaxation function $g$. In 1988 and 1989, Haraux, Zuazua and Kopackova [27, 41] proved that if $g = 0$, then a nonlinear damping term of polynomial or arbitrary growth assured the global estimates for arbitrary initial data. Cavalcanti et al [15] in 2002 proved that the global solution of the semilinear viscoelastic wave equation with localized damping term decays exponentially to zero. In the presence of the polynomial source term $|u|^{p-2}u$, Ball [10] in 1977 proved that if the damping terms are absent, that is for $a = 0$ and $g = 0$, then the solutions blow up when the energy of the initial data is negative. Berrimi and Messaoudi [11] in 2006 considered the case of $a = 0$ and $g \neq 0$, they proved that the solutions decay exponentially or polynomially depending on the relaxation function $g$. The case of $a \neq 0$ and $g = 0$ was considered by Levine [44] in 1974, he showed that the solutions blow up in finite time under some assumptions on the initial energy. Messaoudi [53] in 2001 proved that if $g = 0$ and the source term dominated the polynomial damping term, then the solutions with negative initial energy blow up in finite time. In 2003, the same author [54] considered, the wave equation with damping terms (polynomial and viscoelastic). Under some assumptions on $g$, he proved that if the source term dominated the polynomial damping term then the solutions with negative initial energy blow up in finite time and if the polynomial damping term dominated the source term then for any initial data the global solution exists. In 2006, he considered the same system and proved that under some conditions on the relaxation function $g$, damping and sources terms the solutions with positive initial energy blow up too [51].

When $F \neq 0$ and the polynomial source term is absent, the systems of the second order hyperbolic equation with linear or nonlinear first order perturbation term have been considered in [14, 16, 21, 22]. Noting that, the inclusion of this term produce serious additional difficulties since we do not have any information about their influence on the energy of the solution, specially, about the signal of the derivative of the energy. In 2008, Hamchi [25] considered the case of linear first order perturbation term, she introduced a new multiplier to remove the condition of smallness imposed in the literature on the linear perturbation term.

## 2. Objectives

In this thesis, we consider the system $(P1)$ where we show that the solution with negative initial energy blows up in finite time.

Then, we give the global existence and stability of the hyperbolic coupled system with nonlinearties of variable exponents type $(P2)$ under suitable assumptions.

Finally, we prove that if the damping terms (linear and viscoelastic) dominated the nonlinear first order perturbation term in $(P3)$ then the energy is decreasing. So, we can define the auxiliary functional $L$. After that, we show that the solutions with sufficiently negative initial energy blow up in finite time.

## 3. Organization of the thesis

The main body of this work consists of four chapters in addition to the general introduction and conclution.

In the **first Chapter**, we gather the tools used throughout this thesis. In section 1, we recall some useful preliminaries on the constant exponent Lebesgue and Sobolev spaces also some definitions and results needed in our proofs later. Section 2 is about the variable exponent spaces which include the history of the Lebesgue and Sobolev spaces, also, we mention some definitions and properties of those spaces.

In **Chapter two**, we study the blow up of solutions of the nonlinear hyperbolic equation with variable damping and source terms (*system* $(P1)$). In section 1, we give the assumption and preliminary results needed to obtain our result. We, also, give the energy identity associated to the solution. In section 2, we state and prove the blow up result for the solution.

In **Chapter three**, we study the coupled system of nonlinear wave equations with a variable exponents $m(.)$ and $r(.)$ in the damping terms (*system* $(P2)$). In Section 1, we state and prove the theorem of existence and uniqueness of a weak solution to this problem. In section 2, we prove the decay estimates for the solution under appropriate assumptions on these variable exponents.

In **Chapter four**, we investigate the blow up of solutions of the nonlinear wave equation with damping, source and nonlinear first order perturbation terms (*system* $(P3)$). In Section 1, we show that the energy of the solution is a decreasing function. In section 2, we prove that if the damping terms (linear and viscoelastic) dominated the first order perturbation term then we obtain the blow up result for the solution with sufficiently negative initial energy in finite time.

# Notations

Let $T > 0$ and $\Omega$ be a bounded domain of $\mathbb{R}^n$ $(n \in \mathbb{N}^*)$ with a smooth boundary $\partial\Omega$. The following standard notations are used in the dissertation

- $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + ... + \partial_{x_n}^2$
- $\nabla = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$
- $u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}$
- $C^1(\Omega)$ denotes the space of all continuously differentiable functions on $\Omega$,
- $C_0^1(\Omega)$ denotes the space of all continuously differentiable functions with compact support in $\Omega$. The support of a continuous function $f$ defined on $\Omega$ is the closure of the set of point where $f(x)$ is nonzero. That is

$$\text{supp}(f) := \overline{\{x \in \Omega \mid f(x) \neq 0\}}.$$

- $C_0^\infty(\Omega)$ denotes the space of all continuously functions with compact support in $\Omega$, having continuous derivatives of all orders.

Let $Z$ be a real Banach space with a norm $\|\cdot\|$. We have

- The space $L^p(0,T;Z)$ consists of all measurable functions $u : [0,T] \to Z$ with

$$\|u\|_{L^p(0,T';Z)} := \left( \int_0^T \|u(t)\|^p dt \right)^{1/p} < +\infty \quad \text{for} \quad 1 \leq p < \infty$$

  and

$$\|u\|_{L^\infty(0,T;Z)} := \text{esssup}_{0 \leq t \leq T} \|u(t)\| < +\infty \quad \text{for} \quad p = \infty.$$

- The space $L^p_{loc}(0,T;Z)$ consists of all measurable functions $u : (0,T) \to Z$ with $u \in L^p([a,b];Z)$ for every closed interval $[a,b] \subset (0,T)$.

- The space $C([0,T],Z)$ consists of all continuous functions $u:[0,T]\to Z$ with

$$\|u\|_{C([0,T],Z)} := \max_{0\le t\le T} \|u\| < +\infty$$

- The space $C^1([0,T],Z)$ consists of all continuously differentiable functions $u:[0,T]\to Z$ with

$$\|u\|_{C^1([0,T],Z)} := \max_{0\le t\le T} \|u\| + \max_{0\le t\le T} \left\|\frac{du}{dt}\right\| < +\infty$$

CHAPTER 1

# Preliminaries

The objective of this chapter is to provide the basic tools necessary to understand the notions and results that will be handled throughout this work.

## 1. The constant exponent Spaces

In this section, we present the definitions of the Lebesgue and Sobolev spaces with constant exponent. Then, we present some useful inequalities that are related to this spaces in which we will need them later in our proofs.

**1.1. Constant Exponent Lebesgue space.** The Lebesgue space is presented in the following.

Definition 1.1. *Let $p \in \mathbb{R}^*$. The Lebesgue space is defined as:*

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\; \textit{measurable and} \;\; \int_\Omega |u|^p \, dx < +\infty \right\} \quad \textit{if} \;\; 1 \leq p < \infty.$$

*$L^p(\Omega)$ is equiped with the norm*

$$\|u\|_{L^p(\Omega)} = \|u\|_p = \left( \int_\Omega |u|^p dx \right)^{\frac{1}{p}} < +\infty.$$

*and*

$$L^\infty(\Omega) = \{ u : \Omega \to \mathbb{R} \; \textit{measurable and} \; \exists M > 0 : |u| \leq M \;\; a.e \; on \; \Omega \} \quad \textit{if} \;\; p = +\infty,$$

*$L^\infty(\Omega)$ is equiped with the norm*

$$\|u\|_{L^\infty(\Omega)} = \|u\|_{(\Omega)} = inf \left\{ M > 0 : |u| \leq M, a.e. \;\; on \; \Omega \right\}.$$

**1.2. Constant Exponent Sobolev Space.** The Sobolev space is presented in the following.

Definition 1.2. *Let $m \in \mathbb{N}^*$. The Sobolev space $W^{m,p}(\Omega)$ is defined as*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega), \partial^\alpha u \in L^p(\Omega); \alpha \in \mathbb{N} : |\alpha| \leq m\} \quad \textit{if} \ \ 1 \leq p < \infty,.$$

*$W^{m,p}(\Omega)$ is endowed with the norm bellow*

$$\|u\|_{W^{m,p}(\Omega)} = \left( \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_p^p \right)^{\frac{1}{p}}.$$

*and*

$$W^{m,\infty}(\Omega) = \{u \in L^\infty(\Omega), \partial^\alpha u \in L^\infty(\Omega); \alpha \in \mathbb{N} : |\alpha| \leq m\} \quad \textit{if} \ \ p = +\infty$$

*$W^{m,\infty}(\Omega)$ is endowed with the norm bellow*

$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_\infty .$$

Remark 1.3.

- *For $p = 2$ and $m = 1$, we note $W^{1,2}(\Omega) = H^1(\Omega)$. So*

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) \ / \ \frac{\partial u}{\partial x_i} \in L^2(\Omega) \ \textit{for all} \ \ i = 1, n \right\}.$$

- *We note by $H_0^1(\Omega)$ the spaces given by*

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega) \ / \ u \mid_{\partial\Omega} = 0 \right\}.$$

- *We note by $H^2(\Omega)$ the spaces given by*

$$H^2(\Omega) = \left\{ u \in L^2(\Omega) \ / \ \frac{\partial u}{\partial x_i}, \ \frac{\partial^2 u}{\partial x_i^2}, \ \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2(\Omega) \ \textit{for all} \ \ i, j = \overline{1, n} \right\}.$$

Lemma 1.4. *The Sobolev space $H_0^1(\Omega)$ is a Hilbert space with the scalar product defined by*

$$\langle u,v\rangle_{H_0^1(\Omega)} = \int_\Omega \nabla u.\nabla v dx \quad \text{for all} \quad u,v \in H_0^1(\Omega)$$

*and with the norm*

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 dx\right)^{\frac{1}{2}} \quad \text{for all} \quad u \in H_0^1(\Omega)$$

## 1.3. Important Lemmas, formulas and inequalities with constant exponent.

Lemma 1.5. ***(Poincare's inequality)*** *There exists a constant $C > 0$, depending on $\Omega$, such that*

$$\|u\|_{L^2(\Omega)} \leq C\|u\|_{H_0^1(\Omega)} \quad \text{for all} \quad u \in H_0^1(\Omega). \tag{1.1}$$

Lemma 1.6. ***(Holder's inequality)*** *Let $0 < p,q,r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ then $fg \in L^r(\Omega)$, we have*

$$\|fg\|_r \leq \|f\|_p \|g\|_q. \tag{1.2}$$

Lemma 1.7. ***(Schwarz inequality)*** *For all $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega)$, we have*

$$\left|\langle u,v\rangle_{L^2(\Omega)}\right| \leq \|u\|_{L^2(\Omega)} . \|v\|_{L^2(\Omega)} . \tag{1.3}$$

Lemma 1.8. ***(Green formula)*** *For all $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, we have*

$$\int_\Omega \Delta u v dx = -\int_\Omega \nabla u \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\nu. \tag{1.4}$$

Lemma 1.9. ***(Young's inequality)*** *Let $X$ and $Y$ be two positive reals. Let $\mu$ and $\theta$ be two strictly positive reals satisfying $\frac{1}{\mu} + \frac{1}{\theta} = 1$*
*For all $\delta > 0$, we have*

$$XY \leq \frac{\delta^\mu}{\mu} X^\mu + \frac{\delta^{-\theta}}{\theta} Y^\theta. \tag{1.5}$$

LEMMA 1.10. ***(Algebraic inequality)*** *Let $m \geq 1$. For all $\mu, \theta > 0$, we have*

$$(\mu+\theta)^m \leq 2^m(\mu^m+\theta^m). \tag{1.6}$$

## 2. The variable exponent spaces

**2.1. History of Variable Exponent Spaces.** Variable Lebesgue spaces were first introduced by Orlicz in 1931 in his article [63]. He started by looking for necessary and sufficient conditions on a sequence $(y_i)$ in $\mathbb{R}$ under which $\sum x_i y_i$ converges, for any sequence $(x_i)$ in $\mathbb{R}$ such that $\sum x_i^{p_i}$ converges, where $(p_i)$ is a sequence of real numbers with $p_i > 1$. He, also, considered the variable exponent function space $L^{p(\cdot)}$ on the real line. Orlicz later concentrated much on the theory of the function spaces that were named after him ( see [59]). In the theory of Orlicz spaces, the space $L^{\varphi}$ is defined as follows:

$$L^{\varphi} := \left\{ u : \Omega \to \mathbb{R} \text{ such that } \varrho(\lambda u) = \int_{\Omega} \varphi(\lambda |u(x)|) dx < +\infty \right\},$$

for some $\lambda > 0$, where $\varphi$ is a real valued function that may depend on $x$ and satisfies some additional conditions. Putting certain properties of $\varrho$ in an abstract setting, a more general class of function spaces, called modular spaces, was first studied by Nakano [60, 61]. Following the work of Nakano, modular spaces were investigated be several people, most importantly by groups at Sapporo (Japan), Voronezh (U. S. S. R) and Leiden (the Netherlands). An Explicit version of modular function spaces was investigated by Polish Mathematicians, like Hudzik [28, 38] and Kaminska [35, 39].

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as the Orlics space $L^{\varphi_{p(\cdot)}}(\Omega)$ where

$$\varphi_{p(.)}(t) = t^{p(.)} \quad \text{or} \quad \varphi_{p(.)}(t) = \frac{t^{p(.)}}{p(.)},$$

where

$$p(.) : \Omega \longrightarrow [1, \infty) \text{ is a measurable function.}$$

So,

$$L^{\varphi_{p(.)}}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \varrho(\lambda u) = \int_\Omega \varphi_{p(x)}(\lambda |u(x)|)dx < +\infty \right\},$$

for some $\lambda > 0$ equipped with the Luxemburg norm

$$\|u\|_{p(.)} = \inf \left\{ \lambda > 0 \text{ such that } \int_\Omega \varphi_{p(x)} \left( \left| \frac{u(x)}{\lambda} \right| \right) dx \leq 1 \right\}.$$

Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. Their result originated in 1961 in a paper by Tsenov [70]. The Luxemburg norm was introduced by Sharapudinov [68] for the Lebesgue space. He showed that this Space is Banach if the exponent satisfies $1 < essinf\ p \leq esssup\, p < +\infty$. In the mid-80s, Zhikov [75] started a new line of investigation of variable exponent spaces, by considering variational integrals with non standard growth conditions. The next major step in the study of variable exponent spaces was by Kovacik and Rakosnk [42] in the early 90's. In their paper, they established many of the basic properties of Lebesgue and Sobolev spaces in $\mathbb{R}^n$.

In the beginning of the new millennium, a great development has been made for the rigorous study of variable exponent spaces. In particular,

- A connection was made between the variable exponent spaces and the variational integrals with non standard growth and coercivity conditions.
- Modelling of some physical phenomena such as flows of electro rheological fluids or fluids with temperature dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing which give rise to equations with nonstandard growth conditions. That is, equations with variable exponents of nonlinearities. These models include hyperbolic, parabolic or elliptic equations that are nonlinear in gradient of the unknown solution and with variable exponents of nonlinearity.

**2.2. Variable Exponent Lebesgue Space.** In this subsection, we present some definition about Lebesgue spaces with variable exponents.

DEFINITION 2.1. *Let $(\Omega, \sum, \mu)$ be a $\sigma$–finite, complete measure space. We define $P(\Omega, \mu)$ to be the set of all $\mu$–measurable functions $p : \Omega \to [1, \infty)$. The functions $p \in P(\Omega, \mu)$ are called variable exponents on $\Omega$. We define*

$$p^- := essinf_{y \in A} p(y) \quad and \quad p^+ := esssup_{y \in A} p(y).$$

*If $p^+ < \infty$ then we call $p$ a bounded variable exponent. If $p \in P(\Omega, \mu)$ then we define $p' \in P(\Omega, \mu)$ by*

$$\frac{1}{p(y)} + \frac{1}{p'(y)} = 1, \quad where \quad \frac{1}{\infty} := 0.$$

*The function $p'$ is called the dual variable exponent of $p$. In the special case that $\mu$ is the $n$–dimentional Lebesgue measure and $\Omega$ is an open subset of $\mathbb{R}^n$, we abbreviate $P(\Omega) := P(\Omega, \mu)$.*

DEFINITION 2.2. *We define the Lebesgue space with a variable exponent $p(.)$ by*

$$L^{p(.)}(\Omega) := \left\{ u : \Omega \to \mathbb{R};\ measurable\ in\ \Omega : \lim_{\lambda \to 0} \varrho_{p(.)}(\lambda u) = 0 \right\}$$

*or equivalently*

$$L^{p(.)}(\Omega) := \left\{ u : \Omega \to \mathbb{R};\ measurable\ in\ \Omega : \varrho_{p(.)}(\lambda u) < +\infty,\ for\ some\ \lambda > 0 \right\},$$

*where*

$$\varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx$$

*$L^{p(.)}(\Omega)$ is endowed with the following Luxembourg-type norm*

$$\|u\|_{p(.)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

LEMMA 2.3. [9] *If $p(x) \equiv p$, where $p$ is a constant. Then,*

$$\|u\|_{p(.)} = \lambda_0 = \left( \int_\Omega |u(x)|^p\, dx \right)^{\frac{1}{p}}. \tag{2.1}$$

Now, we introduce the most important condition on the variable exponent, called the log-Hölder continuity condition, which is necessary to obtain the Poincaré inequality in the variable case.

Definition 2.4. *We say that a function $q:\Omega \to \mathbb{R}$ is log-Hölder continuous on $\Omega$, if there exist constant $C > 0$ such that for all $0 < \delta < 1$, we have*

$$|q(x) - q(y)| \leq -\frac{C}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \delta. \tag{2.2}$$

Lemma 2.5. [43] *Let $\Omega$ be a domain of $\mathbb{R}^n$. If $p:\Omega \to \mathbb{R}$ is a Lipchitz function, then it is log-Hölder continuous on $\Omega$.*

Remark 2.6. *The log-Hölder continuity condition on $p$ can be replaced by $p \in C(\overline{\Omega})$ , if $\Omega$ is bounded.*

*The following results are very important and useful in the sequal.*

Theorem 2.7. [43] *If $p \in P(\Omega, \mu)$, then $L^{p(\cdot)}(\Omega, \mu)$ is a Banach space.*

Lemma 2.8. [43] *If $p:\Omega \to [1, \infty)$ is a measurable function with $p^+ < +\infty$ then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.*

In the following lemma, we present the relation between the function $\varrho_{p(.)}(u)$, called the modular function, and the norm $\|u\|_{p(.)}$.

Lemma 2.9. [43] *If $1 < p^- \leq p(x) \leq p^+ < +\infty$ then*

$$\min\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\} \leq \varrho_{p(.)}(u) \leq \max\left\{\|u\|_{p(.)}^{p^-}, \|u\|_{p(.)}^{p^+}\right\},$$

*for any $u \in L^{p(\cdot)}(\Omega)$.*

Remark 2.10. *If the exponent $p$ is constant then $p^- = p^+$ and hence $\varrho_{p(.)}(u) = \|u\|_p^p$.*

LEMMA 2.11. ***(Young's Inequality).*** [43] *Let $\delta, \delta', s \geq 1$ be a measurable functions defined on $\Omega$ such that*

$$\frac{1}{s(y)} = \frac{1}{\delta(y)} + \frac{1}{\delta'(y)} \text{ for a.e } y \in \Omega.$$

*Then, for all $a, b \geq 0$, we have*

$$\frac{(ab)^{s(.)}}{s(.)} \leq \frac{a^{\delta(.)}}{\delta(.)} + \frac{b^{\delta'(.)}}{\delta'(.)}. \tag{2.3}$$

*By taking $s = 1$ and $1 < \delta, \delta' < +\infty$, it follows that, for any $\epsilon > 0$, we have*

$$ab \leq \epsilon a^{\delta} + C_{\epsilon} b^{\delta'},$$

*where $C_{\epsilon} = 1/\delta'(\epsilon\delta)^{\frac{\delta'}{\delta}}$.*
*For $p = q = 2$, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}.$$

LEMMA 2.12. ***(Hölder's Inequality)****. Let $p, q, s \geq 1$ be a measurable functions defined on $\Omega$ satisfying*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} \text{ for a.e } y \in \Omega.$$

*If $f \in L^{p(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$, then $fg \in L^{s(.)}(\Omega)$ and*

$$\|fg\|_{s(.)} \leq 2 \|f\|_{p(.)} \|g\|_{q(.)}. \tag{2.4}$$

*By taking $p = q = 2$, we have the* ***Cauchy Schwarz inequality****.*

**2.3. Variable exponent Sobolev spaces.** In this subsection, we study some functional analysis type properties of Sobolev spaces with variable exponents. We start by recalling the definition of the weak derivative.

DEFINITION 2.13. ***(Weak derivative)*** *Let $\Omega \subset \mathbb{R}^n$ be a domain. Assume that $u \in L^1_{loc}(\Omega)$. Let $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be a multi-index and let $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If there exists $g \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega u \frac{\partial^{|\alpha|}\psi}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n} \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega \psi g \, \mathrm{d}x \textit{ for all } \psi \in C_0^\infty(\Omega),$$

*then $g$ is called a weak partial derivative of $u$ of order $\alpha$. The function $g$ is denoted by $\partial_\alpha u$ or by $\frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n}$.*

DEFINITION 2.14. *Let $k \in \mathbb{N}$. We define the variable exponent Sobolev space $W^{k,p(\cdot)}(\Omega)$ by*

$$W^{k,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega) \textit{ such that } \partial^{|\alpha|}u \in L^{p(\cdot)}(\Omega),\ \forall |\alpha| \leq k \right\}$$

*equipped with the following norm*

$$\|u\|_{W^{k,p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \varrho_{W^{k,p(\cdot)}(\Omega)}\left(\frac{u}{\lambda}\right) \leq 1 \right\} := \sum_{0 \leq |\alpha| \leq k} \|\partial_\alpha u\|_{p(\cdot)},$$

*with*

$$\varrho_{W^{k,p(\cdot)}(\Omega)}(u) = \sum_{0 \leq |\alpha| \leq k} \varrho_{L^{p(\cdot)}(\Omega)}(\partial_\alpha u).$$

*Clearly*

$$W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$$

*and*

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \textit{ such that } \nabla u \textit{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

*equipped with the norm*

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

THEOREM 2.15. [9] *Let $p \in \mathcal{P}(\Omega, \mu)$. The space $W^{k,p(\cdot)}(\Omega)$ is a Banach space, which is separable if $p$ is bounded and reflexive if $1 < p^- \leq p^+ < +\infty$.*

DEFINITION 2.16. *The closure of the set of $W^{k,p(\cdot)}(\Omega)$-functions with compact support in $W^{k,p(\cdot)}(\Omega)$ is the Sobolev space $W_0^{k,p(\cdot)}(\Omega)$ "with zero boundary trace",* *i.e.,*

$$W_0^{k,p(\cdot)}(\Omega) = \overline{\left\{u \in W^{k,p(\cdot)}(\Omega) : u = u\chi_K \text{ for a compact } K \subset \Omega\right\}}.$$

*Furthermore, we denote by* $H_0^{k,p(\cdot)}(\Omega)$ *the closure of* $C_0^\infty(\Omega)$ *in* $W^{k,p(\cdot)}(\Omega)$ *and by* $W^{-1,p'(\cdot)}(\Omega)$ *the dual space of* $W_0^{1,p(\cdot)}(\Omega)$*, in the same way as the usual Sobolev spaces, where* $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

REMARK 2.17. (1) $H_0^{k,p(\cdot)}(\Omega) \subset W_0^{k,p(\cdot)}(\Omega)$.

(2) *If* $p$ *is log-Hölder continuous on* $\Omega$ *then* $W_0^{k,p(\cdot)}(\Omega) = H_0^{k,p(\cdot)}(\Omega)$.

(3) *If* $p = 2$ *and* $k = 1$ *then we set* $W_0^{1,2(\cdot)}(\Omega) = H_0^1(\Omega)$.

THEOREM 2.18. [9] *Let* $p \in \mathcal{P}(\Omega, \mu)$*. The space* $W_0^{k,p(\cdot)}(\Omega)$ *is a Banach space, which is separable if* $p$ *is bounded and reflexive if* $1 < p^- \leq p^+ < +\infty$.

THEOREM 2.19. [43] ***(Poincaré's inequality)*** *Let* $\Omega$ *be a bounded domain of* $\mathbb{R}^n$*. If* $p$ *satisfies the Log-Hölder inequality on* $\Omega$ *then*

$$\|u\|_{p(\cdot)} \leq C\|\nabla u\|_{p(\cdot)} \text{ for all } u \in W_0^{1,p(\cdot)}(\Omega), \tag{2.5}$$

*where* $C$ *is a positive constant depends on* $p(\cdot)$ *and* $\Omega$ *only. In particular, the space* $W_0^{1,p(\cdot)}(\Omega)$ *has an equivalent norm given by*

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)}.$$

We end this section with some essential embedding results.

LEMMA 2.20. [9, 43] ***(Embedding Property)*** *Let* $\Omega$ *be a bounded domain in* $\mathbb{R}^n$ *with a smooth boundary* $\partial\Omega$*. Assume that* $p, q \in C(\overline{\Omega})$ *such that*

$$1 < p^- \leq p(x) \leq p^+ < +\infty \text{ and } 1 < q^- \leq q(x) \leq q^+ < +\infty, \text{ for all } x \in \overline{\Omega}.$$

*and* $p(x) < q^*(x)$ *in* $\bar{\Omega}$ *with* $q^*(x) = \begin{cases} \frac{nq(x)}{n-q(x)}, & \text{if } q^+ < n \\ \infty, & \text{if } q^+ \geq n \end{cases}$.

*Then the embedding* $W^{1,q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ *is continuous and compact.*

COROLLARY 2.21. *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Assume that $p:\bar{\Omega}\to(1,\infty)$ is a continuous function such that*

$$2<p^{-}\leq p(x)\leq p^{+}<\frac{2n}{n-2},\quad n\geq 3.$$

*Then, the embedding $W_0^{1,P(\cdot)}(\Omega)\hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous and compact.*

CHAPTER 2

# Blow up of nonlinear hyperbolic equation with variable damping and source terms

This chapter is the subject of a submitted paper by

Soumia Abdelhadi and Ilhem Hamchi .

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In this chapter, we consider the following system

$$\begin{cases} u_{tt} - div\,(A\nabla u) + u_t\,|u_t|^{m(.)-2} = u\,|u|^{p(.)-2} \quad \text{in } \Omega \times (0,T)\,, \\ \\ u = 0 \quad \text{on } \partial\Omega \times (0,T)\,, \\ \\ u(0) = u_0 \quad \text{and} \quad u_t(0) = u_1 \quad \text{in } \Omega. \end{cases} \qquad (P1)$$

Where $A = A(x,t)$ is an $n \times n$ symmetric matrix with real coefficients. The exponents $m(.)$ and $p(.)$ are given measurable functions on $\Omega$.

In this chapter, we consider the case of variable coefficients ($A = A(x,t)$), variable damping and source terms where we show that the solution of $(P1)$ with negative initial energy blows up in finite time.

This chapter consists of two sections. In section 1, we give the assumptions and preliminary results needed to obtain our main result. In section 2, we prove the main result.

## 1. Assumptions and preliminary results

In this chapter, we study the blow up question of the system $(P1)$ under the following assumptions:

- (H1) *For the matrix A,* assume that

(1) $A$ is of class $C^1\left(\bar{\Omega}\times[0,+\infty[\right)$.

(2) There exists a constant $a_0>0$ such that for all $\xi\in\mathbb{R}^n$ we have

$$A\xi\cdot\xi\geq a_0\mid\xi\mid^2 \quad \text{and} \quad A'\xi\xi\leq 0.$$

- (H2) *For the exponents:* The exponents $m(.)$ and $p(.)$ are measurable functions on $\Omega$ such that:

(1) The following log-Holder continuity condition

$$|q(x)-q(y)|\leq -\frac{C}{\log|x-y|}, \qquad \text{for all } x,y\in\Omega, \quad \text{with} \quad |x-y|<\delta$$

$$\text{where} \quad C>0 \quad \text{and} \quad 0<\delta<1.$$

(2)

$$2\leq m_1\leq m(x)\leq m_2, \quad n=1,2.$$
$$2<m_1\leq m(x)\leq m_2\leq\frac{2n}{n-2}, \quad n\geq 3,$$

with $m_1:=ess\inf_{x\in\Omega}m(x)$ and $m_2:=ess\sup_{x\in\Omega}m(x)$.

(3)

$$2\leq p_1\leq p(x)\leq p_2, \quad n=1,2.$$
$$2<p_1\leq p(x)\leq p_2\leq 2\frac{n-1}{n-2}, \quad n\geq 3,$$

with $p_1:=ess\inf_{x\in\Omega}p(x)$ and $p_2:=ess\sup_{x\in\Omega}p(x)$.

(4)

$$m_2<p_1\leq p_2.$$

- (H3)

$$E(0) < 0$$

where

$$E(0) := \frac{1}{2} \parallel u_1 \parallel_2^2 + \frac{1}{2} \int_\Omega A(x,0) \nabla u_0 \nabla u_0 dx - \int_\Omega \frac{1}{p(x)} \mid u_0 \mid^{p(x)} dx.$$

Now, we introduce some preliminary results needed to prove our main result. The existence and uniqueness result for problem $(P1)$ is given in the following theorem.
In the beginning, let us introduce the definition of a weak solution for our system

DEFINITION 1.1. *Let $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. The functions $u$ such that*

$$u \in L^\infty\left([0,T), H_0^1(\Omega)\right),\ u_t \in L^\infty\left([0,T), L^2(\Omega)\right) \cap L^{m(\cdot)}(\Omega \times (0,T)),$$

*is called a weak solution of $(P1)$ on $[0,T)$, if*

$$\begin{cases} \frac{d}{dt} \int_\Omega u_t \phi\ dx + \int_\Omega A \nabla u \cdot \nabla \phi\ dx + \int_\Omega |u_t|^{m(x)-2} u_t \phi\ dx - \int_\Omega |u|^{p(x)-2} u \phi\ dx = 0, \\ u(0) = u_0, u_t(0) = u_1, \end{cases}$$

*for a.e. $t \in (0,T)$ and the test function $\phi \in H_0^1(\Omega)$.*

THEOREM 1.2. *Under the above assumptions and for $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the problem $(P1)$ has a unique local weak solution $u$ on $[0,T)$, in the sense of Definition 1.1.*

PROOF. As in [**57**]. □

We assume the following result.

THEOREM 1.3. *For $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, then the problem $(P1)$ has a strong solution*

$$u \in L^\infty\left((0,T), H^2(\Omega) \cap H_0^1(\Omega)\right),\ u_t \in L^\infty\left((0,T), H_0^1(\Omega)\right) \cap L^{m(\cdot)}(\Omega \times (0,T)),$$
$$u_{tt} \in L^\infty\left((0,T), L^2(\Omega)\right)$$

*Moreover for any weak solution $(u,u_t)$ of $(P1)$, there exists a sequence $(u_n,u_{nt})$, $n \geq 1$, of strong solutions of $(P1)$ such that*

$$(u_n,u_{nt}) \longrightarrow (u,u_t) \text{ in } L^\infty\Big((0,T);H_0^1(\Omega)\Big) \times L^\infty\Big((0,T),L^2(\Omega)\Big) \cap L^{m(\cdot)}(\Omega \times (0,T)).$$

We define the energy functional for the local solution $u$ of problem $(P1)$ by

$$E(t) = \frac{1}{2} \parallel u_t \parallel_2^2 + \frac{1}{2}\int_\Omega A\nabla u \nabla u dx - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\,dx, \qquad \forall t \in [0,T].$$

By the definition of the solution, we can prove the following lemma which shows that $E$ is a non increasing function of $t$.

Lemma 1.4. *For any strong solution of $(P1)$, we have*

$$E'(t) = \frac{1}{2}\int_\Omega A'\nabla u \nabla u dx - \int_\Omega |\, u_t \,|^{m(x)}\,dx \leq 0, \qquad \forall\, t \in [0,T].$$

Now, we set

$$H(t) = -E(t), \qquad \forall t \in [0,T].$$

Lemma 1.5. *We have*

$$0 < H(0) \leq H(t) \leq \frac{1}{p_1}\int_\Omega |\, u \,|^{p(x)}\,dx, \qquad \forall t \in [0,T]. \tag{1.1}$$

Proof. • Since $E(0) < 0$ then $H(0) = -E(0) > 0$.

• From the definition of $H$ and the non increasing of $E$, we have

$$H(0) \leq H(t), \qquad \forall t \in [0,T].$$

- We have

$$H(t) = -\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\int_\Omega A\nabla u \nabla u dx + \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx.$$

$(H1-2)$ implies that

$$H(t) \leq \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx.$$

By $(H2-3)$, we arrive at

$$H(t) \leq \frac{1}{p_1}\int_\Omega |u|^{p(x)} dx.$$

□

Let $C$ be a generic positive constant and it may changes from line to line. The following two lemmas are needed, too, in our work.

LEMMA 1.6. *There exists a constant $C > 0$ such that*

$$\int_\Omega \mid u \mid^{p(x)} dx \geq C\|u\|_{p_1}^{p_1} \tag{1.2}$$

*and*

$$\int_\Omega \mid u \mid^{m(x)} dx \leq C\left(\left(\int_\Omega \mid u \mid^{p(x)} dx\right)^{\frac{m_2}{p_1}} + \left(\int_\Omega \mid u \mid^{p(x)} dx\right)^{\frac{m_1}{p_1}}\right). \tag{1.3}$$

PROOF. .

**Proof of (1.2):** We have

$$\int_\Omega \mid u \mid^{p(x)} dx = \int_{\Omega_+} \mid u \mid^{p(x)} dx + \int_{\Omega_-} \mid u \mid^{p(x)} dx, \tag{1.4}$$

where

$$\Omega_+ = \{x \in \Omega / \mid u(x,t) \mid \geq 1\} \quad \text{and} \quad \Omega_- = \{x \in \Omega / \mid u(x,t) \mid < 1\}.$$

We have

$$\int_{\Omega_+} | u |^{p(x)} \, dx \geq \int_{\Omega_+} | u |^{p_1} \, dx \tag{1.5}$$

and

$$\int_{\Omega_-} | u |^{p(x)} \, dx \geq \int_{\Omega_-} | u |^{p_2} \, dx.$$

Since $p_1 \leq p_2$, then

$$\int_{\Omega_-} | u |^{p(x)} \, dx \geq C \left( \int_{\Omega_-} | u |^{p_1} \, dx \right)^{\frac{p_2}{p_1}}. \tag{1.6}$$

Replacing (1.5) and (1.6) in (1.4) to obtain

$$\int_{\Omega} | u |^{p(x)} \, dx \geq \int_{\Omega_+} | u |^{p_1} \, dx + C \left( \int_{\Omega_-} | u |^{p_1} \, dx \right)^{\frac{p_2}{p_1}}.$$

This implies that

$$\int_{\Omega} | u |^{p(x)} \, dx \geq \int_{\Omega_+} | u |^{p_1} \, dx \quad \text{and} \quad C \left( \int_{\Omega} | u |^{p(x)} \, dx \right)^{\frac{p_1}{p_2}} \geq \int_{\Omega_-} | u |^{p_1} \, dx.$$

By addition, we find

$$\int_{\Omega} | u |^{p(x)} \, dx + C \left( \int_{\Omega} | u |^{p(x)} \, dx \right)^{\frac{p_1}{p_2}} \geq \|u\|_{p_1}^{p_1}.$$

So

$$\left[ 1 + C \left( \int_{\Omega} | u |^{p(x)} \, dx \right)^{\frac{p_1}{p_2} - 1} \right] \int_{\Omega} | u |^{p(x)} \, dx \geq \|u\|_{p_1}^{p_1}.$$

But, by (1.1) and $(H2-3)$ we find

$$(p_1 H(0))^{\frac{p_1}{p_2} - 1} \geq \left( \int_{\Omega} | u |^{p(x)} \, dx \right)^{\frac{p_1}{p_2} - 1}.$$

Then

$$\left[1+C\left(p_1H(0)\right)^{\frac{p_1}{p_2}-1}\right]\int_\Omega \mid u\mid^{p(x)}dx \geq \|u\|_{p_1}^{p_1}.$$

Consequently, we obtained (1.2).

**Proof of (1.3):** We have

$$\begin{aligned}\int_\Omega \mid u\mid^{m(x)}dx &= \int_{\Omega_+}\mid u\mid^{m(x)}dx+\int_{\Omega_-}\mid u\mid^{m(x)}dx\\ &\leq \int_{\Omega_+}\mid u\mid^{m_2}dx+\int_{\Omega_-}\mid u\mid^{m_1}dx.\end{aligned}$$

Since $m_1 \leq m_2 < p_1$ then

$$\begin{aligned}\int_\Omega \mid u\mid^{m(x)}dx &\leq C\left[\left(\int_{\Omega_+}\mid u\mid^{p_1}dx\right)^{\frac{m_2}{p_1}}+\left(\int_{\Omega_-}\mid u\mid^{p_1}dx\right)^{\frac{m_1}{p_1}}\right]\\ &\leq C\left(\|u\|_{p_1}^{m_2}+\|u\|_{p_1}^{m_1}\right).\end{aligned}$$

By (1.2), we find the desired result. □

LEMMA 1.7. *For all*

$$0<\alpha\leq \min\left\{\frac{p_1-2}{2p_1},\frac{p_1-m_2}{p_1(m_2-1)}\right\}\quad and\quad k>1,$$

*we have*

$$\int_\Omega H^{\alpha(m(x)-1)}(t)\mid u\mid^{m(x)}dx \leq C\left(\int_\Omega A\nabla u\nabla u dx+\int_\Omega \mid u\mid^{p(x)}dx\right) \tag{1.7}$$

*and*

$$\begin{aligned}\int_\Omega \mid u\mid\mid u_t\mid^{m(x)-1}dx &\leq C\frac{k^{1-m_1}}{m_1}\left(\int_\Omega A\nabla u\nabla u dx+\int_\Omega \mid u\mid^{p(x)}dx\right)\\ &+ \frac{(m_2-1)k}{m_2}H^{-\alpha}(t)H'(t).\end{aligned} \tag{1.8}$$

Proof. .

**Proof of** (1.7): We have

$$\int_\Omega H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx = \int_\Omega \left[\frac{H(t)}{H(0)}\right]^{\alpha(m(x)-1)} [H(0)]^{\alpha(m(x)-1)} \mid u \mid^{m(x)} dx.$$

Since $\dfrac{H(t)}{H(0)} \geq 1$, then by $(H2-2)$ we find

$$\begin{aligned}\int_\Omega H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx &\leq \int_\Omega \left[\frac{H(t)}{H(0)}\right]^{\alpha(m_2-1)} [H(0)]^{\alpha(m(x)-1)} \mid u \mid^{m(x)} dx \\ \leq \quad [H(t)]^{\alpha(m_2-1)} &\int_\Omega [H(0)]^{\alpha(m(x)-m_2)} \mid u \mid^{m(x)} dx. \end{aligned} \tag{1.9}$$

But

$$[H(0)]^{\alpha(m(x)-m_2)} \leq C \quad \text{for all} \quad x \in \Omega.$$

Indeed

$$\text{If} \quad H(0) \leq 1 \quad \text{then} \quad [H(0)]^{\alpha(m(x)-m_2)} \leq [H(0)]^{\alpha(m_1-m_2)}.$$

$$\text{If} \quad H(0) > 1 \quad \text{then} \quad [H(0)]^{\alpha(m(x)-m_2)} \leq [H(0)]^{\alpha(m_2-m_2)} = 1.$$

Then, (1.9) becomes

$$\int_\Omega H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx \leq C\,[H(t)]^{\alpha(m_2-1)} \int_\Omega \mid u \mid^{m(x)} dx.$$

By (1.1) and (1.3), we find

$$\begin{aligned} &\int_\Omega H^{\alpha(m(x)-1)}(t) \mid u \mid^{m(x)} dx \\ \leq \quad & C\left(\left(\int_\Omega \mid u \mid^{p(x)} dx\right)^{\frac{m_2}{p_1}+\alpha(m_2-1)} + \left(\int_\Omega \mid u \mid^{p(x)} dx\right)^{\frac{m_1}{p_1}+\alpha(m_2-1)}\right). \end{aligned}$$

We apply Lemma 4.1 in [57] for

$$2 \leq s = m_1 + \alpha p_1(m_2 - 1) \leq p_1$$

then for

$$2 \leq s = m_2 + \alpha p_1(m_2 - 1) \leq p_1$$

and by $(H1-1)$ we obtain (1.7).

**Proof of** (1.8): By Young inequality (1.5), with

$$X = |u|, \quad Y = | u_t |^{m(x)-1}, \quad \mu = m(x) \quad \text{and} \quad \theta = \frac{m(x)}{m(x)-1}$$

we find

$$\begin{aligned}
\int_\Omega | u || u_t |^{m(x)-1} \, dx \quad &\leq \quad \int_\Omega \frac{\delta^{m(x)}}{m(x)} | u |^{m(x)} \, dx + \int_\Omega \frac{m(x)-1}{m(x)} \delta^{-\frac{m(x)}{m(x)-1}} | u_t |^{m(x)} \, dx \\
&\leq \quad \frac{1}{m_1} \int_\Omega \delta^{m(x)} | u |^{m(x)} \, dx \\
&+ \quad \frac{m_2 - 1}{m_1} \int_\Omega \delta^{-\frac{m(x)}{m(x)-1}} | u_t |^{m(x)} \, dx.
\end{aligned}$$

Let $k > 0$. If we take

$$\delta = \left( kH^{-\alpha}(t) \right)^{-\frac{m(x)-1}{m(x)}} > 0$$

then we find

$$\begin{aligned}
\int_\Omega | u || u_t |^{m(x)-1} \, dx \quad &\leq \quad \frac{1}{m_1} \int_\Omega k^{1-m(x)} H^{\alpha(m(x)-1)}(t) \, | u |^{m(x)} \, dx \\
&+ \quad \frac{(m_2-1)k}{m_1} H^{-\alpha}(t) \int_\Omega | u_t |^{m(x)} \, dx. \qquad (1.10)
\end{aligned}$$

But, from the definition of $H$, Lemma 1.4 and $(H1-2)$ we have

$$\int_\Omega |u_t|^{m(x)} \, dx = \frac{1}{2} \int_\Omega A' \nabla u \nabla u dx + H'(t) \leq H'(t).$$

Then, for $k>1$, (1.10) becomes

$$\int_\Omega |\, u \,||\, u_t \,|^{m(x)-1}\, dx \leq \frac{k^{1-m_1}}{m_1}\int_\Omega H^{\alpha(m(x)-1)}(t)\,|\, u\,|^{m(x)}\, dx + \frac{(m_2-1)k}{m_2} H^{-\alpha}(t)H'(t).$$

By (1.7), we obtain the result.

□

## 2. Main result

In this section, we state and prove our main result .

THEOREM 2.1. *The solution of problem $(P1)$ blows up in finite time.*

Because of the Theorem 1.3, it is sufficient to prove the result for strong solutions.

PROOF. We proceed in 4 steps:

**Step 1** For $\epsilon>0$, we consider the following functional

$$L(t) = H^{1-\alpha}(t) + \epsilon \int_\Omega u u_t dx, \qquad \forall t \in [0,T].$$

If we derive the function $L$ with respect to $t$ we obtain

$$L'(t) = (1-\alpha)\, H^{-\alpha}(t)H'(t) + \epsilon \, \|u_t\|_2^2 + \epsilon \int_\Omega u u_{tt} dx, \quad \forall t \in [0,T]. \tag{2.1}$$

But

$$\begin{aligned}\int_\Omega u u_{tt} dx &= \int_\Omega u div(A\nabla u) dx - \int_\Omega u u_t \,|\, u_t \,|^{m(x)-2}\, dx \\ &+ \int_\Omega |\, u\,|^{p(x)}\, dx.\end{aligned}$$

By the generalized Green formula (1.4), we obtain

$$\int_\Omega u u_{tt} dx \;=\; -\int_\Omega A\nabla u \nabla u dx - \int_\Omega u u_t \,|\, u_t\,|^{m(x)-2}\, dx + \int_\Omega |\, u\,|^{p(x)}\, dx. \tag{2.2}$$

Replacing (2.2) in (2.1), we find

$$
\begin{aligned}
L'(t) &\geq (1-\alpha) H^{-\alpha}(t) H'(t) + \epsilon \|u_t\|_2^2 - \epsilon \int_\Omega A\nabla u \nabla u dx \\
&- \epsilon \int_\Omega | u || u_t |^{m(x)-1} dx + \epsilon \int_\Omega | u |^{p(x)} dx.
\end{aligned}
$$

By (1.8), we obtain

$$
\begin{aligned}
L'(t) &\geq \left[1-\alpha-\epsilon\frac{(m_2-1)k}{m_2}\right] H^{-\alpha}(t) H'(t) + \epsilon \|u_t\|_2^2 \\
&- \epsilon\left(1+C\frac{k^{1-m_1}}{m_1}\right)\int_\Omega A\nabla u \nabla u dx \\
&+ \epsilon\left(1-C\frac{k^{1-m_1}}{m_1}\right)\int_\Omega | u |^{p(x)} dx. \qquad (2.3)
\end{aligned}
$$

Add and subtract $\epsilon(1-\eta)p_1 H(t)$ for $0 < \eta < 1 - \frac{2}{p1}$ in the right-hand side of (2.3) and use the definition of $H$ to obtain

$$
\begin{aligned}
L'(t) &\geq \left[1-\alpha-\epsilon\frac{(m_2-1)k}{m_2}\right] H^{-\alpha}(t) H'(t) + \epsilon(1-\eta)p_1 H(t) + \epsilon \|u_t\|_2^2 \\
&- \epsilon\left(1+C\frac{k^{1-m_1}}{m_1}\right)\int_\Omega A\nabla u \nabla u dx + \epsilon\left(1-C\frac{k^{1-m_1}}{m_1}\right)\int_\Omega | u |^{p(x)} dx \\
&- \epsilon(1-\eta)p_1\left(-\frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}\int_\Omega A\nabla u \nabla u dx + \frac{1}{p_1}\int_\Omega | u |^{p(x)} dx\right). \qquad (2.4)
\end{aligned}
$$

Then

$$
\begin{aligned}
L'(t) &\geq \left[1-\alpha-\epsilon\frac{(m_2-1)k}{m_2}\right] H^{-\alpha}(t) H'(t) + \epsilon(1-\eta)p_1 H(t) \\
&+ \epsilon\left(\eta-C\frac{k^{1-m_1}}{m_1}\right)\int_\Omega | u |^{p(x)} dx + \epsilon\left(\frac{(1-\eta)p_1}{2}+1\right)\epsilon \|u_t\|_2^2 \\
&+ \epsilon\left(\frac{p_1-2}{2} - \frac{\eta p_1}{2} - C\frac{k^{1-m_1}}{m_1}\right)\int_\Omega A\nabla u \nabla u dx.
\end{aligned}
$$

For $k$ sufficiently large, we arrive at

$$\begin{aligned} L'(t) &\geq \left[1-\alpha-\epsilon\frac{(m_2-1)k}{m_2}\right] H^{-\alpha}(t)H'(t)+\epsilon\gamma\left[H(t)+\int_\Omega |\, u\,|^{p(x)}\, dx+\epsilon\|u_t\|_2^2\right] \\ &+ \epsilon\beta\int_\Omega A\nabla u\nabla u dx, \end{aligned} \tag{2.5}$$

where

$$\gamma=\min\left\{(1-\eta)p_1, \eta-C\frac{k^{1-m_1}}{m_1}, \frac{(1-\eta)p_1}{2}+1\right\}>0.$$

and

$$\beta=\frac{p_1-2}{2}-\frac{\eta p_1}{2}-C\frac{k^{1-m_1}}{m_1}=\frac{p_1-2}{2}-\eta\left(1+\frac{p_1}{2}\right)+\eta-C\frac{k^{1-m_1}}{m_1}>0.$$

If we choose $\epsilon$ small enough such that

$$1-\alpha-\epsilon\frac{m_2-1}{m_2}k\geq 0.$$

Then, by (1.2) inequality (2.5) takes the form

$$L'(t)\geq \epsilon C\left[H(t)+\|\, u\,\|_{p_1}^{p_1}+\|\, u_t\,\|_2^2\right]. \tag{2.6}$$

**Step 2** Since

$$L(0)=H^{1-\alpha}(0)+\epsilon\int_\Omega u_0(x)u_1(x)dx>0.$$

then from the increase of $L$ (See (2.6)), we find

$$L(t)\geq 0, \quad \forall t\in[0,T].$$

**Step 3** By the definition of $L$, we find

$$L^{\frac{1}{1-\alpha}}(t) \;\;\leq\;\; \left[H^{1-\alpha}(t)+\epsilon\int_\Omega |\, u\,||\, u_t\,|\, dx\right]^{\frac{1}{1-\alpha}}.$$

By the Algebraic inequality (1.6) with

$$\mu = H^{1-\alpha}(t),\ \theta = \epsilon \int_{\Omega} |\, u \,||\, u_t \,|\, dx \text{ and } m = \frac{1}{1-\alpha}$$

we obtain

$$L^{\frac{1}{1-\alpha}}(t) \quad \leq \quad 2^{\frac{1}{1-\alpha}} \left[ H(t) + \left( \epsilon \int_{\Omega} |\, u \,||\, u_t \,|\, dx \right)^{\frac{1}{1-\alpha}} \right].$$

But, by Schwarz inequality (1.3), we have

$$\left( \int_{\Omega} |\, u \,||\, u_t \,|\, dx \right)^{\frac{1}{1-\alpha}} \quad \leq \quad \|\, u \,\|_2^{\frac{1}{1-\alpha}} \|\, u_t \,\|_2^{\frac{1}{1-\alpha}} .$$

From the embedding $L^{p_1}(\Omega) \hookrightarrow L^2(\Omega)$, we find

$$\left( \int_{\Omega} |\, u \,||\, u_t \,|\, dx \right)^{\frac{1}{1-\alpha}} \leq C \,\|\, u \,\|_{p_1}^{\frac{1}{1-\alpha}} \|\, u_t \,\|_2^{\frac{1}{1-\alpha}} .$$

Apply Young inequality (1.5) with

$$X = \|\, u \,\|_{p_1}^{\frac{1}{1-\alpha}}, \quad Y = \|\, u_t \,\|_2^{\frac{1}{1-\alpha}}, \quad \mu = \frac{2(1-\alpha)}{1-2\alpha} \quad \textit{and} \quad \theta = 2(1-\alpha)$$

we have

$$\left( \int_{\Omega} |\, u \,||\, u_t \,|\, dx \right)^{\frac{1}{1-\alpha}} \leq C \left( \|\, u \,\|_{p_1}^{\frac{2}{1-2\alpha}} + \|\, u_t \,\|_2^2 \right).$$

We apply Corollary 4.4 in [**57**] with $2 \leq s = \frac{2}{1-2\alpha} \leq p_1$ to find

$$L^{\frac{1}{1-\alpha}}(t) \leq C \left[ H(t) + \|\, u \,\|_{p_1}^{p_1} + \|\, u_t \,\|_2^2 \right], \quad \forall t \in [0,T]. \tag{2.7}$$

**Step 4** By combining (2.6) and (2.7), we arrive at

$$L'(t) \quad \geq C L^{\frac{1}{1-\alpha}}(t), \quad \text{for all } t \geq 0$$

A simple integration over $(0,t)$ gives

$$L(t) \geq \frac{1}{\left[L^{\frac{-\alpha}{1-\alpha}}(0) - \frac{\alpha C t}{(1-\alpha)}\right]^{\frac{1-\alpha}{\alpha}}}, \quad \text{for all } t \geq 0.$$

From which follows, via (2.7), the desired result. □

CHAPTER 3

# Global existence and stability for a coupled system of nonlinear wave equations with variable exponents in the damping terms

This chapter is the subject of a submitted paper by

Soumia Abdelhadi and Abderrahmane Youkana .

In this work, we consider the following initial boundary value problem

$$\begin{cases} u_{tt}-\Delta u+a\left|u_t\right|^{m(.)-2}u_t+\alpha(u-v)=0 & \text{in } \Omega\times(0,T), \\ v_{tt}-\Delta v+b\left|v_t\right|^{r(.)-2}v_t+\alpha(v-u)=0 & \text{in } \Omega\times(0,T), \\ u=v=0 & \text{on } \partial\Omega\times(0,T), \\ u(0)=u_0 \quad \text{and} \quad u_t(0)=u_1 & \text{in } \Omega, \\ v(0)=v_0 \quad \text{and} \quad v_t(0)=v_1 & \text{in } \Omega, \end{cases} \qquad (P2)$$

where $a,b,\alpha$ are positive constants and the exponents $m(.)$ and $r(.)$ are given measurable functions on $\Omega$ satisfying

$$2\leq m(x)\leq\infty, \quad \text{if } n=1,2,$$

$$2\leq m^{-}\leq m(x)\leq m^{+}<\frac{2n}{n-2}, \qquad \text{if } n\geq 3 \qquad (0.1)$$

and

$$2 \le r(x) \le \infty, \quad \text{if } n = 1, 2,$$
$$2 \le r^- \le r(x) \le r^+ < \frac{2n}{n-2}, \quad \text{if } n \ge 3, \tag{0.2}$$

where

$$m^- = \operatorname{ess}\inf_{x\in\Omega} m(x),\ m^+ = \operatorname{ess}\sup_{x\in\Omega} m(x) \tag{0.3}$$

and

$$r^- = \operatorname{ess}\inf_{x\in\Omega} r(x),\ r^+ = \operatorname{ess}\sup_{x\in\Omega} r(x). \tag{0.4}$$

Also, we assume that $m(.)$ and $r(.)$ satisfy the log-Holder continuity condition :

$$|q(x) - q(y)| \le -\frac{C}{\log|x-y|}, \quad \text{for all } x, y \in \Omega \text{ with } |x-y| < \delta, \quad 0 < \delta < 1,\ C > 0. \tag{0.5}$$

Our purpose in this chapter is to study the global existence and the stability of the solution of $(P2)$ under suitable assumptions on the parameters of this system. The content of this chapter is as follows. In Section 1, we state and prove the existence result. In Section 2, we prove that the decay estimates of the energy function are exponential or polynomial depending on the exponents $m(.)$ and $r(.)$.

## 1. Global Existence Result

Before investigating the decay of solutions, we give the following definition of a weak solution, then, we state and prove the theorem of existence and uniqueness of solution to problem $(P2)$ the proof can be established similarly [**12, 58**].

Definition 1.1. *Let* $(u_0,u_1)$, $(v_0,v_1)\in H_0^1(\Omega)\times L^2(\Omega)$. *Any pair of functions* $(u,v)$ *such that*

$$u,v\in L^\infty\left([0,T),H_0^1(\Omega)\right),\ u_t\in L^\infty\left([0,T),L^2(\Omega)\right)\cap L^{m(\cdot)}(\Omega\times(0,T)),$$
$$v_t\in L^\infty\left([0,T),L^2(\Omega)\right)\cap L^{r(\cdot)}(\Omega\times(0,T)),$$

*is called a weak solution of* $(P2)$ *on* $[0,T)$, *if*

$$\begin{cases} \frac{d}{dt}\int_\Omega u_t\phi\ dx+\int_\Omega\nabla u\cdot\nabla\phi\ dx+a\int_\Omega|u_t|^{m(x)-2}u_t\phi\ dx+\int_\Omega\alpha(u-v)\phi\ dx=0,\\ \frac{d}{dt}\int_\Omega v_t\psi\ dx+\int_\Omega\nabla v\cdot\nabla\psi\ dx+b\int_\Omega|v_t|^{r(x)-2}v_t\psi\ dx+\int_\Omega\alpha(v-u)\psi\ dx=0,\\ u(0)=u_0,u_t(0)=u_1,v(0)=v_0,v_t(0)=v_1,\end{cases}$$

*for a.e.* $t\in(0,T)$ *and all test functions* $\phi,\psi\in H_0^1(\Omega)$.

Theorem 1.2. *Under the above assumptions on* $m(.),r(.)$ *and for* $(u_0,u_1),(v_0,v_1)\in H_0^1(\Omega)\times L^2(\Omega)$, *the problem* $(P2)$ *has a unique local weak solution* $(u,v)$ *on* $[0,T)$, *in the sense of Definition 1.1.*

Proof. **Uniqueness :**

Let $(u_1,v_1)$ and $(u_2,v_2)$ be two solutions of problem $(P2)$ in the sense of Definition 1.1. Taking $\phi=u_{1t}-u_{2t}$ and $\psi=v_{1t}-v_{2t}$ in this definition. Then, $(u,v)=(u_1-u_2,v_1-v_2)$ satisfying the following identities

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|u_t|^2\ dx+\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla u|^2\ dx+\alpha\frac{1}{2}\frac{d}{dt}\int_\Omega|u_1-u_2|^2\ dx\\ +a\int_\Omega\left(|u_{1t}|^{m(x)-2}u_{1t}-|u_{2t}|^{m(x)-2}u_{2t}\right)(u_{1t}-u_{2t})\ dx\\ -\alpha\int_\Omega vu_t\ dx=0\end{aligned}\tag{1.1}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|v_t|^2dx+\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla v|^2dx+\alpha\frac{1}{2}\frac{d}{dt}\int_\Omega|v_1-v_2|^2dx\\ +b\int_\Omega\left(|v_{1t}|^{r(x)-2}v_{1t}-|v_{2t}|^{r(x)-2}v_{2t}\right)(v_{1t}-v_{2t})\ dx\\ -\alpha\int_\Omega uv_t\ dx=0.\end{aligned}\tag{1.2}$$

Using the fact that for all $Y, Z \in \mathbb{R}$ and $x \in \Omega$

$$\left(|Y|^{q(x)-2}Y - |Z|^{q(x)-2}Z\right)(Y-Z) \geq 0, \qquad q(x) \geq 2, \tag{1.3}$$

and summing up the inequalities (1.1) and (1.2), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(|u_t|^2 + |v_t|^2 + |\nabla u|^2 + |\nabla v|^2 + \alpha(u-v)^2\right)dx = 0.$$

Integrating over $(0,t)$, we get

$$\int_\Omega \left(|u_t|^2 + |v_t|^2 + |\nabla u|^2 + |\nabla v|^2 + \alpha(u-v)^2\right)dx = 0,$$

which gives us,

$$u_t(x,.) = v_t(x,.) = 0 \quad \text{ on } \quad \Omega, \qquad \nabla u(.,t) = \nabla v(.,t) = 0, \quad \text{for all t } \in (0,T).$$

As a result

$$u = v = c = 0 \text{ on } \Omega \times (0,T).$$

Since $u = v = 0$ on $\partial\Omega \times (0,T)$, this implies the uniqueness.

**Existence :**

To prove the existence of a local solution to problem $(P2)$ , we proceed in several steps.

**Step 1: Faedo-Galerkin approximation.**

Let $\{w_j\}_{j=1}^{\infty}$ an orthonormal basis of $H_0^1(\Omega)$. For all $k \geq 1$, let $(u^k, v^k)$ be a sequence in the finite-dimensional subspace $W_k = span\{w_1, \ldots, w_k\}$ defined by

$$u^k(x,t) = \sum_{j=1}^{k} a_j(t)w_j(x) \qquad \text{and} \qquad v^k(x,t) = \sum_{j=1}^{k} b_j(t)w_j(x), \quad \text{for all}\, x \in \Omega, \quad t \in (0,T),$$

which satisfy the following approximate problems, denoted by $(P_k)$:

$$\int_\Omega u^k_{tt}(x,t)w_j(x)\ dx + \int_\Omega \nabla u^k(x,t)\nabla w_j(x)\ dx + a\int_\Omega u^k_t(x,t)\left|u^k_t(x,t)\right|^{m(x)-2} w_j(x)\ dx$$
$$+\int_\Omega \alpha(u^k(x,t)-v^k(x,t))w_j(x)\ dx = 0, \quad \forall j=1,2,\ldots,k \tag{1.4}$$

and

$$\int_\Omega v^k_{tt}(x,t)w_j(x)\ dx + \int_\Omega \nabla v^k(x,t)\nabla w_j(x)\ dx + b\int_\Omega v^k_t(x,t)\left|v^k_t(x,t)\right|^{r(x)-2} w_j(x)\ dx$$
$$+\int_\Omega \alpha(v^k(x,t)-u^k(x,t))w_j(x)\ dx = 0, \quad \forall j=1,2,\ldots,k, \tag{1.5}$$

with initial condition

$$u^k(0)=u^k_0=\sum_{i=1}^k (u_0,w_i)w_i,\ u^k_t(0)=u^k_1=\sum_{i=1}^k (u_1,w_i)w_i,$$
$$v^k(0)=v^k_0=\sum_{i=1}^k (v_0,w_i)w_i,\ v^k_t(0)=v^k_1=\sum_{i=1}^k (v_1,w_i)w_i,$$

such that

$$u^k_0 \longrightarrow u_0 \quad \text{and} \quad v^k_0 \longrightarrow v_0 \quad \text{in} \quad H^1_0(\Omega),$$
$$u^k_1 \longrightarrow u_1 \quad \text{and} \quad v^k_1 \longrightarrow v_1 \quad \text{in} \quad L^2(\Omega).$$

This generates a system of $k$ nonlinear ordinary differential equations, which admits a unique local solution $(u^k, v^k)$ in $[0,T_k)$, where $T_k < T$, by standard ODE theory. In the following step, we will show, by a priory estimates, that $T_k = T, \forall k \geq 1$.

**Step 2. A priori Estimates.**

Multiplying (1.4) and (1.5) by $a'_j(t)$ and $b'_j(t)$ respectively. We sum each result over $j$, from

1 to $k$, to find

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega |\ u_t^k(x,t)\ |^2\ \ dx+\int_\Omega \left|\nabla u^k(x,t)\right|^2\ \ dx+\alpha\int_\Omega |\ u^k(x,t)\ |^2\ \ dx\right\} \\ + & \ \ a\int_\Omega |\ u_t^k(x,t)\ |^{m(x)}\ \ dx-\alpha\int_\Omega v^k(x,t)u_t^k(x,t)\ dx=0 \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega |\ v_t^k(x,t)\ |^2\ \ dx+\int_\Omega \left|\nabla v^k(x,t)\right|^2\ \ dx+\alpha\int_\Omega |\ v^k(x,t)\ |^2\ dx\right\} \\ + & \ \ b\int_\Omega |\ v_t^k(x,t)\ |^{r(x)}\ \ dx-\alpha\int_\Omega u^k(x,t)v_t^k(x,t)\ dx=0. \end{aligned} \tag{1.7}$$

Summing up the identities (1.6) and (1.7), we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_\Omega \left|u_t^k(x,t)\right|^2\ dx+\frac{1}{2}\frac{d}{dt}\int_\Omega \left|v_t^k(x,t)\right|^2\ dx+\frac{1}{2}\frac{d}{dt}\int_\Omega |\ \nabla u^k(x,t)\ |^2\ dx+\frac{1}{2}\frac{d}{dt}\int_\Omega |\ \nabla v^k(x,t)\ |^2\ dx \\ & +\alpha\frac{1}{2}\frac{d}{dt}\int_\Omega \left|u^k(x,t)\right|^2\ dx+\alpha\frac{1}{2}\frac{d}{dt}\int_\Omega \left|v^k(x,t)\right|^2\ dx-\alpha\int_\Omega v^k(x,t)u_t^k(x,t)\ dx-\alpha\int_\Omega u^k(x,t)v_t^k(x,t)\ dx \\ & \quad +a\int_\Omega \left|u_t^k(x,t)\right|^{m(x)}\ dx+b\int_\Omega \left|v_t^k(x,t)\right|^{r(x)}\ dx=0, \end{aligned} \tag{1.8}$$

which yields

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_\Omega \left(\left|u_t^k(x,t)\right|^2+\left|v_t^k(x,t)\right|^2+|\ \nabla u^k(x,t)\ |^2+|\ \nabla v^k(x,t)\ |^2+\alpha(u^k(x,t)-v^k(x,t))^2\right)dx \\ & \quad +a\int_\Omega \left|u_t^k(x,t)\right|^{m(x)}dx+b\int_\Omega \left|v_t^k(x,t)\right|^{r(x)}\ dx=0. \end{aligned} \tag{1.9}$$

Integrating (1.9) over $(0,t)$ to obtain

$$\begin{aligned} & \frac{1}{2}\int_\Omega \left(\left|u_t^k(x,t)\right|^2+\left|v_t^k(x,t)\right|^2+|\ \nabla u^k(x,t)\ |^2+|\ \nabla v^k(x,t)\ |^2+\alpha(u^k(x,t)-v^k(x,t))^2\right)dx \\ & \quad +a\int_0^t\int_\Omega \left|u_t^k(x,s)\right|^{m(x)}dxds+b\int_0^t\int_\Omega \left|v_t^k(x,s)\right|^{r(x)}dxds \\ & \quad \le \frac{1}{2}\int_\Omega \left(|u_1|^2+|v_1|^2+|\ \nabla u_0\ |^2+|\ \nabla v_0\ |^2+\alpha(u_0-v_0)^2\right)dx=c, \end{aligned}$$

where $c$ is a positive constant, for all $T_k<T$ and $k\ge 1$. Therefore, the local solution $(u^k,v^k)$ of system $(P_k)$ can be extended to $(0,T)$ for all $k\ge 1$.

Furthermore, we deduce that

$$(u^k)_k, (v^k)_k \quad \text{are bounded in} \quad L^\infty\left((0,T), H_0^1(\Omega)\right),$$

$$(u_t^k)_k \quad \text{is bounded in} \quad L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{m(.)}\left(\Omega \times (0,T)\right),$$

$$(v_t^k)_k \quad \text{is bounded in} \quad L^\infty\left((0,T), L^2(\Omega)\right) \cap L^{r(.)}\left(\Omega \times (0,T)\right).$$

Consequently, we can extract two subsequences $\{u^k\}_k$ and $\{v^k\}_k$ which we denote by $\{u^l\}_l$ and $\{v^l\}_l$ respectively, such that when $l \longrightarrow \infty$, we have

$$(u^l) \longrightarrow u \quad \text{and} \quad (v^l) \longrightarrow v \qquad \text{weakly* in} \quad L^\infty\left((0,T), H_0^1(\Omega)\right),$$

$$(u_t^l) \longrightarrow u_t \quad \text{weakly* in} \quad L^\infty\left((0,T), L^2(\Omega)\right) \quad \text{and weakly in} \quad L^{m(.)}\left(\Omega \times (0,T)\right),$$

$$(v_t^l) \longrightarrow v_t \quad \text{weakly* in} \quad L^\infty\left((0,T), L^2(\Omega)\right) \quad \text{and weakly in} \quad L^{r(.)}\left(\Omega \times (0,T)\right).$$

**Step 3. The Nonlinear terms.**

In this step, we show that

$$u_t^l \mid u_t^l \mid^{m(.)-2} \rightarrow u_t \mid u_t \mid^{m(.)-2} \quad \text{weakly in} \quad L^{\frac{m(.)}{m(.)-1}}\left(\Omega \times (0,T)\right)$$

and

$$v_t^l \mid v_t^l \mid^{r(.)-2} \rightarrow v_t \mid v_t \mid^{r(.)-2} \quad \text{weakly in} \quad L^{\frac{r(.)}{r(.)-1}}\left(\Omega \times (0,T)\right).$$

By Lions Lemma [48], we obtain $u \in C\left([0,T], L^2(\Omega)\right)$ and $v \in C\left([0,T], L^2(\Omega)\right)$, so that $u(x,0)$ and $v(x,0)$ have a meaning. Since $\left(u_t^l\right)$ is bounded in $L^{m(\cdot)}(\Omega \times (0,T))$, it follows that, there exists a subsequence of $(u_t^l \mid u_t^l \mid^{m(.)-2})_l$, still denoted by $(u_t^l \mid u_t^l \mid^{m(.)-2})_l$, for simplicity, such that

$$\left|u_t^l\right|^{m(\cdot)-2} u_t^l \rightarrow \chi_1 \text{ weakly in } L^{\frac{m(\cdot)}{m(\cdot)-1}}(\Omega \times (0,T)).$$

Similarly, we find

$$\left|v_t^l\right|^{r(\cdot)-2} u_t^l \to \chi_2 \text{ weakly in } L^{\frac{r(\cdot)}{r(\cdot)-1}}(\Omega \times (0,T)).$$

In what follows, we prove that $\chi_1 = | u_t |^{m(.)-2} u_t$ and $\chi_2 = | v_t |^{r(.)-2} v_t$. We have

$$\int_\Omega u_t^l w_j \, dx - \int_\Omega u_1^l w_j \, dx + \int_0^t \int_\Omega \nabla u^l \cdot \nabla w_j \, dxds + a \int_0^t \int_\Omega \left|u_t^l\right|^{m(x)-2} u_t^l w_j \, dxds$$
$$+\alpha \int_0^t \int_\Omega (u^l - v^l) w_j \, dxds = 0, \qquad \forall j < l$$

and

$$\int_\Omega v_t^l w_j \, dx - \int_\Omega v_1^l w_j \, dx + \int_0^t \int_\Omega \nabla v^l \cdot \nabla w_j \, dxds + b \int_0^t \int_\Omega \left|v_t^l\right|^{r(x)-2} v_t^l w_j \, dxds$$
$$+\alpha \int_0^t \int_\Omega (v^l - u^l) w_j \, dxds = 0, \qquad \forall j < l.$$

As $l$ goes to $+\infty$, we easily check that

$$\int_\Omega u_t w_j \, dx - \int_\Omega u_1 w_j \, dx + \int_0^t \int_\Omega \nabla u \cdot \nabla w_j \, dxds + a \int_0^t \int_\Omega \chi_1 w_j \, dxds$$
$$+\alpha \int_0^t \int_\Omega (u - v) w_j \, dxds = 0, \quad \forall j \geq 1$$

and

$$\int_\Omega v_t w_j \, dx - \int_\Omega v_1 w_j \, dx + \int_0^t \int_\Omega \nabla v \cdot \nabla w_j \, dxds + b \int_0^t \int_\Omega \chi_2 w_j \, dxds$$
$$+\alpha \int_0^t \int_\Omega (v - u) w_j \, dxds = 0, \quad \forall j \geq 1.$$

Consequently,

$$\int_\Omega u_t w \, dx - \int_\Omega u_1 w \, dx + \int_0^t \int_\Omega \nabla u \cdot \nabla w \, dxds + a \int_0^t \int_\Omega \chi_1 w \, dxds$$
$$+\alpha \int_0^t \int_\Omega (u - v) w \, dxds = 0, \quad \forall w \in H_0^1(\Omega) \tag{1.10}$$

and

$$\int_\Omega v_t w \ dx - \int_\Omega v_1 w \ dx + \int_0^t \int_\Omega \nabla v \cdot \nabla w \ dxds + b \int_0^t \int_\Omega \chi_2 w \ dxds$$
$$+\alpha \int_0^t \int_\Omega (v-u) w \ dxds = 0, \quad \forall w \in H_0^1(\Omega). \tag{1.11}$$

All terms define absolute continuous functions; so we get, for a.e $t \in [0,T]$,

$$\frac{d}{dt}\int_\Omega u_t w \ dx + \int_\Omega (\nabla u \cdot \nabla w \ + a\chi_1 w + \alpha(u-v)) \ dx = 0, \qquad \forall w \in H_0^1(\Omega) \tag{1.12}$$

and

$$\frac{d}{dt}\int_\Omega v_t w \ dx + \int_\Omega (\nabla v \cdot \nabla w + b\chi_2 w + \alpha(v-u)) \ dx = 0, \qquad \forall w \in H_0^1(\Omega). \tag{1.13}$$

This implies that

$$u_{tt} - \Delta u + a \ \chi_1 + \alpha(u-v) = 0, \qquad \text{in } D'(\Omega \times (0,T))$$

and

$$v_{tt} - \Delta v + b \ \chi_2 + \alpha(v-u) = 0, \qquad \text{in } D'(\Omega \times (0,T)).$$

Next, we set $\Psi(w) = | w |^{m(x)-2} w$ and define the following sequence, for all $l \geq 1$, see [48],

$$X^l \qquad = a \int_0^T \int_\Omega \Big(\Psi(u_t^l) - \Psi(w)\Big)(u_t^l - w) dxdt, \qquad \forall w \in L^{m(.)}\Big((0,T), H_0^1(\Omega)\Big).$$

By the inequality (1.3), we have Then

$$X^l \geq 0, \quad \forall l \geq 1.$$

Replacing $u^k$ by $u^l$ in (1.6) and integrating the result over $(0,T)$, we find

$$\begin{aligned} &\frac{1}{2}\int_\Omega |\ u_t^l(x,T)\ |^2\ dx + \frac{1}{2}\int_\Omega \left|\nabla u^l(x,T)\right|^2\ dx + \frac{\alpha}{2}\int_\Omega |\ u^l(x,T)\ |^2\ dx \\ + &\ a\int_0^T\int_\Omega |\ u_t^l(x,t)\ |^{m(x)}\ dxdt - \int_0^T \alpha\int_\Omega v^l(x,t)u_t^l(x,t)\ dxdt \\ &\le \frac{1}{2}\int_\Omega \left(\left|u_1^l\right|^2 + |\ \nabla u_0^l\ |^2 + \alpha(u_0^l)^2\right)\ dx. \end{aligned} \tag{1.14}$$

Using the fact that $v^l$ and $u_t^l$ are bounded in $L^\infty((0,T),L^2(\Omega))$ , we obtain

$$-\int_0^T \alpha\int_\Omega v^l(x,t)u_t^l(x,t)\ dxdt \le \alpha\int_0^T\int_\Omega |u_t^l(x,t)|^2\ dxdt + \alpha\int_0^T\int_\Omega |v^l(x,t)|^2\ dxdt \le M,$$

where $M$ is a constant independent of $l$. Hence (1.14) takes the form

$$\begin{aligned} \frac{1}{2}\int_\Omega |\ u_t^l(x,T)\ |^2\ dx + \frac{1}{2}\int_\Omega \left|\nabla u^l(x,T)\right|^2 dx + \frac{\alpha}{2}\int_\Omega |\ u^l(x,T)\ |^2\ dx + a\int_0^T\int_\Omega |\ u_t^l(x,t)\ |^{m(x)}\ dxdt \\ \le \frac{1}{2}\int_\Omega \left(\left|u_1^l\right|^2 + |\ \nabla u_0^l\ |^2 + \alpha(u_0^l)^2\right) dx + M. \end{aligned} \tag{1.15}$$

Therefore

$$\begin{aligned} X^l \ \le\ & \frac{1}{2}\int_\Omega \left(\left|u_1^l\right|^2 + |\ \nabla u_0^l\ |^2 + \alpha(u_0^l)^2\right) dx + M \\ -\ & \frac{1}{2}\int_\Omega |\ u_t^l(x,T)\ |^2\ dx - \frac{1}{2}\int_\Omega \left|\nabla u^l(x,T)\right|^2 dx - \frac{\alpha}{2}\int_\Omega |\ u^l(x,T)\ |^2\ dx \\ -\ & a\int_0^T\int_\Omega \Psi(u_t^l)w\ dxdt - a\int_0^T\int_\Omega \Psi(w)(u_t^l - w)\ dxdt, \quad \forall w \in L^{m(.)}\left((0,T),H_0^1(\Omega)\right). \end{aligned}$$

Taking $l \longrightarrow +\infty$, we obtain

$$\begin{aligned}
0 \leq \limsup_l X^l &\leq \frac{1}{2}\int_\Omega \left(|u_1|^2 + |\nabla u_0|^2 + \alpha(u_0)^2\right) dx + M \\
&- \frac{1}{2}\int_\Omega \left(|u_t(x,T)|^2 + |\nabla u(x,T)|^2 + \alpha(u(x,T)))^2\right) dx \\
&- a\int_0^T \int_\Omega \chi_1 w \ dxdt - a\int_0^T \int_\Omega \Psi(w)(u_t - w) \ dxdt.
\end{aligned} \tag{1.16}$$

Using the fact $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, replacing $w$ by $u_t$ in (1.10) , $v_t$ in (1.11) and integrating over $(0,T)$, to get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u_t(x,T)|^2 \ dx - \frac{1}{2}\int_\Omega |u_1|^2 \ dx + \frac{1}{2}\int_\Omega |\nabla u(x,T)|^2 \ dx - \frac{1}{2}\int_\Omega |\nabla u_0|^2 \ dx \\
&+a\int_0^T \int_\Omega \chi_1 u_t \ dxdt + \alpha\int_0^T \int_\Omega (u-v)u_t \ dxdt = 0
\end{aligned} \tag{1.17}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |v_t(x,T)|^2 \ dx - \frac{1}{2}\int_\Omega |v_1|^2 \ dx + \frac{1}{2}\int_\Omega |\nabla v(x,T)|^2 \ dx - \frac{1}{2}\int_\Omega |\nabla v_0|^2 \ dx \\
&+b\int_0^T \int_\Omega \chi_2 v_t \ dxdt + \alpha\int_0^T \int_\Omega (v-u)v_t \ dxdt = 0.
\end{aligned}$$

Addition of (1.16) and (1.17) yields

$$0 \leq \limsup_l X^l \leq -a\int_0^T \int_\Omega \chi_1 u_t \ dxdt - a\int_0^T \int_\Omega \chi_1 w \ dxdt - a\int_0^T \int_\Omega \Psi(w)(u_t - w) \ dxdt + 2M.$$

Which gives for all $\forall w \in L^{m(\cdot)}\left((0,T), H_0^1(\Omega)\right)$

$$a\int_0^T \int_\Omega (\chi_1 - \Psi(w))(u_t - w) \ dxdt \geq 0.$$

Hence, by the density of $H_0^1(\Omega)$ in $L^{m(\cdot)}(\Omega)$, we obtain

$$\int_0^T \int_\Omega (\chi_1 - \Psi(w))(u_t - w) \ dxdt \geq 0, \quad \forall w \in L^{m(\cdot)}(\Omega \times (0,T)).$$

Let $w=\lambda z+u_t$, for $z\in L^{m(\cdot)}(\Omega\times(0,T))$. So, we get

$$-\lambda\int_0^T\int_\Omega(\chi_1-\Psi(\lambda z+u_t))\,z\;dxdt\geq 0,\quad \forall\lambda\neq 0,\quad \forall z\in L^{m(\cdot)}(\Omega\times(0,T)).$$

For $\lambda>0$, we have

$$\int_0^T\int_\Omega(\chi_1-\Psi(\lambda z+u_t))\,z\;dxdt\leq 0,\quad \forall z\in L^{m(\cdot)}(\Omega\times(0,T)).$$

As $\lambda\to 0$ and using the continuity of $\Psi$ with respect to $\lambda$, we get

$$\int_0^T\int_\Omega(\chi_1-\Psi(u_t))\,z\;dxdt\leq 0,\quad \forall z\in L^{m(\cdot)}(\Omega\times(0,T)).$$

Similarly, for $\lambda<0$, we get

$$\int_0^T\int_\Omega(\chi_1-\Psi(u_t))\,z\;dxdt\geq 0,\quad \forall z\in L^{m(\cdot)}(\Omega\times(0,T)).$$

which gives $\chi_1=\Psi(u_t)$. By the same steps, we deduce $\chi_2=\Psi(v_t)$.

Therefore, (1.12) and (1.13) take the form

$$\int_\Omega\Big(u_{tt}w+\nabla u\cdot\nabla w+a\,|u_t|^{m(x)-2}\,u_tw+\alpha(u-v)w\Big)\,dx=0,\quad \forall w\in L^{m(\cdot)}\Big((0,T)\times H_0^1(\Omega)\Big)$$

and

$$\int_\Omega\Big(v_{tt}w+\nabla v\cdot\nabla w+b\,|v_t|^{r(x)-2}\,v_tw+\alpha(v-u)w\Big)\,dx=0,\quad \forall w\in L^{r(\cdot)}\Big((0,T)\times H_0^1(\Omega)\Big).$$

Hence, we obtain

$$u_{tt}-\Delta u+a\,|u_t|^{m(\cdot)-2}\,u_t+\alpha(u-v)=0,\qquad \text{in } D'(\Omega\times(0,T))$$

and

$$v_{tt}-\Delta v+b\,|v_t|^{r(\cdot)-2}\,v_t+\alpha(v-u)=0,\qquad \text{in } D'(\Omega\times(0,T)).$$

**Step 4: The Initial conditions**

To handle the initial conditions, we note that

$$u^l \to u \text{ weakly}^* \text{ in } L^\infty\Big((0,T),H_0^1(\Omega)\Big)$$

and

$$u_t^l \to u_t \text{ weakly}^* \text{ in } L^\infty\Big((0,T),L^2(\Omega)\Big).$$

Similarly

$$v^l \to v \text{ weakly}^* \text{ in } L^\infty\Big((0,T),H_0^1(\Omega)\Big)$$

and

$$v_t^l \to v_t \text{ weakly}^* \text{ in } L^\infty\Big((0,T),L^2(\Omega)\Big).$$

Thus, using Lions' Lemma [48] , we obtain, up to a subsequence,

$$u^l \to u \text{ in } C\Big([0,T],L^2(\Omega)\Big) \quad \text{and} \quad v^l \to v \text{ in } C\Big([0,T],L^2(\Omega)\Big).$$

Therefore, $(u^l(x,0),v^l(x,0))$ make sense and $u^l(x,0) \to u(x,0)$ in $L^2(\Omega)$ and $v^l(x,0) \to v(x,0)$ in $L^2(\Omega)$. Also we have that

$$u^l(x,0) = u_0^l(x) \to u_0(x) \text{ in } H_0^1(\Omega) \quad \text{and} \quad v^l(x,0) = v_0^l(x) \to v_0(x) \text{ in } H_0^1(\Omega).$$

Hence,

$$u(x,0) = u_0(x) \qquad \text{and} \qquad v(x,0) = v_0(x).$$

As in [49], let $\phi \in C_0^\infty(0,T)$ and replacing $\left(u^k,v^k\right)$ by $\left(u^l,v^l\right)$ respectively, we obtain, from (1.4) and (1.5) and for any $j \le l$,

$$\begin{aligned}
&-\int_0^T \int_\Omega u_t^l(x,t)w_j(x)\phi'(t)\, dxdt \\
&\quad = -\int_0^T \int_\Omega \nabla u^l(x,t)\nabla w_j(x)\phi(t)\, dxdt - a\int_0^T \int_\Omega \left|u_t^l(x,t)\right|^{m(x)-2} u_t^l(x,t)w_j(x)\phi(t)\, dxdt \\
&\qquad + \alpha\int_0^T \int_\Omega (u^l(x,t) - v^l(x,t))w_j(x)\phi(t)\, dxdt
\end{aligned}$$

and

$$-\int_0^T\int_\Omega v_t^l(x,t)w_j(x)\phi'(t)\;dxdt$$
$$=-\int_0^T\int_\Omega \nabla v^l(x,t)\nabla w_j(x)\phi(t)\;dxdt - b\int_0^T\int_\Omega \left|v_t^l(x,t)\right|^{r(x)-2} v_t^l(x,t)w_j(x)\phi(t)\;dxdt$$
$$+\alpha\int_0^T\int_\Omega (v^l(x,t)-u^l(x,t))w_j(x)\phi(t)\;dxdt.$$

As $l\to+\infty$, we obtain that

$$-\int_0^T\int_\Omega u_t(x,t)w_j(x)\phi'(t)\;dxdt$$
$$=-\int_0^T\int_\Omega \nabla u(x,t)\nabla w_j(x)\phi(t)\;dxdt - a\int_0^T\int_\Omega |u_t(x,t)|^{m(x)-2} u_t(x,t)w_j(x)\phi(t)\;dxdt$$
$$+\alpha\int_0^T\int_\Omega (u(x,t)-v(x,t))w_j(x)\phi(t)\;dxdt,$$

for all $j\geq 1$. This implies

$$-\int_0^T\int_\Omega u_t(x,t)w(x)\phi'(t)\;dxdt$$
$$=\int_0^T\int_\Omega \left[\Delta u(x,t) - a|u_t(x,t)|^{m(x)-2}u_t(x,t)+\alpha(u(x,t)-v(x,t))\right]w(x)\phi(t)\;dxdt, \qquad \forall w\in H_0^1(\Omega).$$

As a result $u_{tt}\in L^{\frac{m(\cdot)}{m(\cdot)-1}}\left([0,T),H^{-1}(\Omega)\right)$ and $u$ solves the equation

$$u_{tt}-\Delta u + a|u_t|^{m(\cdot)-2}u_t+\alpha(u-v)=0, \quad \text{in } D'((0,T)\times\Omega).$$

Since $u_t\in L^\infty\left([0,T),L^2(\Omega)\right)$ and, $u_{tt}\in L^{\frac{m(\cdot)}{m(\cdot)-1}}\left([0,T),H^{-1}(\Omega)\right)$, we deduce

$$u_t\in C\left([0,T),H^{-1}(\Omega)\right).$$

Therefore, $u_t^l(x,0)$ makes sense. Then we have

$$u_t^l(x,0)\to u_t(x,0) \text{ in } H^{-1}(\Omega).$$

By the definition of $u_1^\ell$, we have

$$u_t^l(x,0)=u_1^l(x)\to u_1(x) \text{ in } L^2(\Omega).$$

Consequently,

$$u_t(x,0) = u_1(x).$$

Similarly, we can prove that $v_t(x,0) = v_1(x)$.

Finally, we deduce that $(u,v)$ is a unique local solution of $(P2)$ . This ends the proof of Theorem 1.2. □

We assume the following result.

THEOREM 1.3. *For $u_0, v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1, v_1 \in H_0^1(\Omega)$, then the problem $(P2)$ has a strong solution*

$$u,v \in L^\infty\Big((0,T), H^2(\Omega)\cap H_0^1(\Omega)\Big),\ u_t \in L^\infty\Big((0,T), H_0^1(\Omega)\Big) \cap L^{m(\cdot)}(\Omega\times(0,T)),$$
$$v_t \in L^\infty\Big((0,T), H_0^1(\Omega)\Big) \cap L^{r(\cdot)}(\Omega\times(0,T)),\ u_{tt}, v_{tt} \in L^\infty\Big((0,T), L^2(\Omega)\Big)$$

*Moreover for any weak solution $(u,u_t)$ and $(v,v_t)$ of $(P1)$, there exists a sequence $(u_n,u_{nt})$ and $(v_n,v_{nt})$, $n\geq 1$, of strong solutions of $(P2)$ such that*

$$(u_n,u_{nt}) \longrightarrow (u,u_t) \ \textit{in}\ L^\infty\Big((0,T); H_0^1(\Omega)\Big) \times L^\infty\Big((0,T), L^2(\Omega)\Big) \cap L^{m(\cdot)}(\Omega\times(0,T))$$

*and*

$$(v_n,v_{nt}) \longrightarrow (v,v_t) \ \textit{in}\ L^\infty\Big((0,T); H_0^1(\Omega)\Big) \times L^\infty\Big((0,T), L^2(\Omega)\Big) \cap L^{r(\cdot)}(\Omega\times(0,T)).$$

## 2. Decay result

In this section, we state and prove our main decay results. For this purpose we define the energy of the solution by

$$E(t) = \frac{1}{2}\int_\Omega \Big[|u_t|_2^2 + |v_t|_2^2 + |\nabla u|_2^2 + |\nabla v|_2^2 + \alpha(u-v)^2\Big]\,dx. \tag{2.1}$$

Formal calculations show

$$E'(t) = -a\int_\Omega |u_t|^{m(x)}\ dx - b\int_\Omega |v_t|^{r(x)}\ dx \leq 0. \tag{2.2}$$

Now, we state and prove our main result.

THEOREM 2.1. *Suppose conditions* (0.1)- (0.5) *hold. Then there exist two positive constants* $c,\omega > 0$ *such that the energy satisfies, for all* $t \geq 0$, *we have*

$$E(t) \leq \begin{cases} \frac{c}{(1+t)^{2/(\rho^{+}-2)}}, & \text{if } \rho^{+} > 2, \\ ce^{-\omega t}, & \text{if } \rho^{+} = 2, \end{cases}$$

*where*

$$\rho^{+} = max\left\{m^{+}, r^{+}\right\}.$$

PROOF. Because of the Theorem 1.3, it is sufficient to prove the result for strong solutions.

Let $T > S > 0$ and $q \geq 0$ to be specified later. By definition 1.1, we obtain

$$\int_{S}^{T}\int_{\Omega} E^{q}(t)\left[(u(t)u_{t}(t))_{t} - |u_{t}(t)|^{2} + |\nabla u(t)|^{2} + au(t)u_{t}(t)|u_{t}(t)|^{m(x)-2} + \alpha u(t)(u(t)-v(t))\right]dxdt$$
$$= 0 \tag{2.3}$$

and

$$\int_{S}^{T}\int_{\Omega} E^{q}(t)\left[(v(t)v_{t}(t))_{t} - |v_{t}(t)|^{2} + |\nabla v(t)|^{2} + bv(t)v_{t}(t)|v_{t}(t)|^{r(x)-2} + \alpha v(t)(v(t)-u(t))\right]dxdt$$
$$= 0. \tag{2.4}$$

Adding and subtracting the following two terms

$$\int_{S}^{T}\int_{\Omega} E^{q}(t)\left[|u_{t}(t)|^{2}\right]dxdt$$

and

$$\int_{S}^{T}\int_{\Omega} E^{q}(t)\left[|v_{t}(t)|^{2}\right]dxdt$$

to (2.3) and (2.4) respectively, then we obtain

$$\int_S^T \int_\Omega E^q(t)\left[|u_t(t)|^2 + |\nabla u(t)|^2 + \alpha u(t)(u(t) - v(t))\right] dxdt$$
$$= -\int_S^T \int_\Omega E^q(t)(u(t)u_t(t))_t\, dxdt + 2\int_S^T \int_\Omega E^q(t)|u_t(t)|^2\, dxdt$$
$$-\int_S^T \int_\Omega E^q(t)\, au(t)u_t(t)|u_t(t)|^{m(x)-2}\, dxdt$$

and

$$\int_S^T \int_\Omega E^q(t)\left[|v_t(t)|^2 + |\nabla v(t)|^2 + \alpha v(t)(v(t) - u(t))\right] dxdt$$
$$= -\int_S^T \int_\Omega E^q(t)(v(t)v_t(t))_t\, dxdt + 2\int_S^T \int_\Omega E^q(t)|v_t(t)|^2\, dxdt$$
$$-\int_S^T \int_\Omega E^q(t)\, bv(t)v_t(t)|v_t(t)|^{r(x)-2}\, dxdt.$$

The addition of the two results yields

$$\int_S^T \int_\Omega E^q(t)\left[|u_t(t)|^2 + |\nabla u(t)|^2 + \alpha u(t)(u(t) - v(t)) + |v_t(t)|^2 + |\nabla v(t)|^2\right] dxdt$$
$$+\int_S^T \int_\Omega E^q(t)\left[\alpha v(t)(v(t) - u(t))\right] dxdt = -\int_S^T \int_\Omega E^q(t)(u(t)u_t(t) + v(t)v_t(t))_t\, dxdt$$
$$+2\int_S^T \int_\Omega E^q(t)\left(|u_t(t)|^2 + |v_t(t)|^2\right) dxdt$$
$$-\int_S^T \int_\Omega E^q(t)\left(au(t)u_t(t)|u_t(t)|^{m(x)-2} + bv(t)v_t(t)|v_t(t)|^{r(x)-2}\right) dxdt. \tag{2.5}$$

Recalling the expression of $E$, (2.5) yields

$$2\int_S^T E^{q+1}(t)\, dt = -\int_s^T E^q(t)\int_\Omega (u(t)u_t(t) + v(t)v_t(t))_t\, dxdt$$
$$+2\int_S^T E^q(t)\int_\Omega \left(|u_t(t)|^2 + |v_t(t)|^2\right) dxdt$$
$$-\int_S^T E^q(t)\int_\Omega \left(au(t)u_t(t)|u_t(t)|^{m(x)-2} + bv(t)v_t(t)|v_t(t)|^{r(x)-2}\right) dxdt. \tag{2.6}$$

On the other hand, we have for a.e. t $\in [S,T]$

$$\begin{aligned}\frac{d}{dt}\left(E^{q}(t)\int_{\Omega}(u(t)u_{t}(t)+v(t)v_{t}(t))\,dx\right) &= (E^{q}(t))'\int_{\Omega}(u(t)u_{t}(t)+v(t)v_{t}(t))\,dx \\ &+ E^{q}(t)\int_{\Omega}(u(t)u_{t}(t)+v(t)v_{t}(t))_{t}\,dx,\end{aligned}$$

which gives,

$$\begin{aligned}E^{q}(t)\int_{\Omega}(u(t)u_{t}(t)+v(t)v_{t}(t))_{t}\,dx &= \frac{d}{dt}\left(E^{q}(t)\int_{\Omega}(u(t)u_{t}(t)+v(t)v_{t}(t))\,dx\right) \\ &- (qE^{q-1}(t)E'(t))\int_{\Omega}(u(t)u_{t}(t)+v(t)v_{t}(t))\,dx.\end{aligned} \tag{2.7}$$

Substituting (2.7) in (2.6), we obtain

$$\begin{aligned}2\int_{S}^{T}E^{q+1}(t)\mathrm{d}t = &-\int_{S}^{T}\frac{\mathrm{d}}{\mathrm{d}t}\left(E^{q}(t)\int_{\Omega}(u(t)u_{t}(t)+v(t)v_{t}(t))\,\mathrm{d}x\right)dt \\ &+q\int_{S}^{T}E^{q-1}(t)E'(t)\int_{\Omega}(u(t)u_{t}(t)+v(t)v_{t}(t))\,\mathrm{d}x dt \\ &+2\int_{S}^{T}E^{q}(t)\int_{\Omega}(|u_{t}(t)|^{2}+|v_{t}(t)|^{2})\,dxdt - a\int_{S}^{T}E^{q}(t)\int_{\Omega}|u_{t}(t)|^{m(x)-2}u_{t}(t)u(t)\,dxdt \\ &-b\int_{S}^{T}E^{q}(t)\int_{\Omega}|v_{t}(t)|^{r(x)-2}v_{t}(t)v(t)\,dxdt = I_{1}+I_{2}+I_{3}+I_{4}+I_{5}.\end{aligned}$$

Using Young's (2.3) and Poincare's inequalities (2.5) and the definition of $E$, we obtain

$$\begin{aligned}|I_{1}| &= \left|-\int_{S}^{T}\frac{d}{dt}\left(E^{q}(t)\int_{\Omega}(u(t)u_{t}(t)+v(t)v_{t}(t))dx\right)dt\right| \\ &= \left|E^{q}(S)\int_{\Omega}(uu_{t}+vv_{t})(x,S)dx - E^{q}(T)\int_{\Omega}(uu_{t}+vv_{t})(x,T)dx\right| \\ &\leq E^{q}(S)\left[\frac{1}{2}\int_{\Omega}(u^{2}+v^{2})(x,S)dx+\frac{1}{2}\int_{\Omega}(u_{t}^{2}+v_{t}^{2})(x,S)dx\right] \\ &+E^{q}(T)\left[\frac{1}{2}\int_{\Omega}(u^{2}+v^{2})(x,T)dx+\frac{1}{2}\int_{\Omega}(u_{t}^{2}+v_{t}^{2})(x,T)dx\right] \\ &\leq E^{q}(S)\left[\frac{1}{2}c_{*}\int_{\Omega}(|\nabla u(x,S)|^{2}+|\nabla v(x,S)|^{2})dx+E(S)\right] \\ &+E^{q}(T)\left[\frac{1}{2}c_{*}\int_{\Omega}(|\nabla u(x,T)|^{2}+|\nabla v(x,T)|^{2})dx+E(T)\right],\end{aligned}$$

where $c_*$ is the embedding constant.

Using the fact that $E$ is nonincreasing function, then we get

$$|I_1| \leq cE^{q+1}(S) + cE^{q+1}(T) \leq cE^{q+1}(S) \leq cE^{q}(0)E(S) = cE(S). \tag{2.8}$$

and

$$\begin{aligned}
|I_2| = \left| q \int_S^T E^{q-1}(t)E'(t) \int_\Omega (uu_t + vv_t)\, \mathrm{d}x \right| &\leq c \int_S^T E^{q-1}(t) \left(-E'(t)\right) E(t)\mathrm{d}t \\
&= cE^{q+1}(S) - cE^{q+1}(T) \\
&\leq cE^{q+1}(S) \leq cE(S).
\end{aligned} \tag{2.9}$$

Next, we estimate the terms $I_4$ and $I_5$, we have

$$\begin{aligned}
I_4 + I_5 &\leq a \int_S^T E^q(t) \int_\Omega |u(t)|\, |u_t(t)|^{m(x)-1}\, \mathrm{d}x dt + b \int_S^T E^q(t) \int_\Omega |v(t)|\, |v_t(t)|^{r(x)-1}\, \mathrm{d}x dt \\
&= aJ_1 + bJ_2.
\end{aligned} \tag{2.10}$$

By applying Young's inequality (2.3) with

$$\delta(x) = \frac{m(x)}{m(x)-1} \qquad \text{and} \qquad \delta'(x) = m(x),$$

we obtain, for all $\epsilon > 0$

$$J_1 \leq \int_S^T E^q(t) \left[ \epsilon \int_\Omega |u(t)|^{m(x)} dx + \int_\Omega c_\epsilon(x)\, |u_t(t)|^{m(x)} dx \right] dt,$$

where

$$c_\epsilon(x) = \frac{[m(x)-1]^{m(x)-1}}{[m(x)]^{m(x)} \epsilon^{m(x)-1}}.$$

Likewise

$$J_2 \leq \int_S^T E^q(t) \left[ \epsilon \int_\Omega |v(t)|^{r(x)} dx + \int_\Omega c'_\epsilon(x)\, |v_t(t)|^{r(x)} dx \right] dt,$$

where

$$c'_\epsilon(x) = \frac{[r(x)-1]^{r(x)-1}}{[r(x)]^{r(x)} \epsilon^{r(x)-1}}.$$

Therefore

$$\begin{aligned}
aJ_1 \leq & a\epsilon \int_S^T E^q(t) \int_\Omega |u(t)|^{m(x)} \mathrm{d}x dt + a \int_S^T E^q(t) \int_\Omega c_\epsilon(x) \, |u_t(t)|^{m(x)} \, \mathrm{d}x dt \\
& \leq a\epsilon \int_S^T E^q(t) \left[ \int_\Omega |u(t)|^{m^-} \, \mathrm{d}x + \int_\Omega |u(t)|^{m^+} \, \mathrm{d}x \right] dt \\
& + a \int_S^T E^q(t) \int_\Omega c_\epsilon(x) \, |u_t(t)|^{m(x)} \, \mathrm{d}x dt \\
\leq & a\epsilon \int_S^T E^q(t) \left[ c_1^* \|\nabla u(t)\|_2^{m^-} + c_2^* \|\nabla u(t)\|_2^{m^+} \right] dt \\
& + a \int_S^T E^q(t) \int_\Omega c_\epsilon(x) \, |u_t(t)|^{m(x)} \, \mathrm{d}x dt,
\end{aligned}$$

where $c_1^*$, $c_2^*$ are two positive constants independent of $\epsilon$.

By using the definition of (2.1), we get

$$\begin{aligned}
aJ_1 \leq & a\epsilon c \int_S^T E^{q+1}(t)(E(t))^{\frac{m^-}{2}-1} dt + a\epsilon c \int_S^T E^{q+1}(t)(E(t))^{\frac{m^+}{2}-1} dt \\
& + a \int_S^T E^q(t) \int_\Omega c_\epsilon(x) \, |u_t(t)|^{m(x)} \, \mathrm{d}x dt \\
\leq & a \; c\epsilon \left( (E(0))^{\frac{m^-}{2}-1} + (E(0))^{\frac{m^+}{2}-1} \right) \int_S^T E^{q+1}(t) \mathrm{d}t \\
& + a \int_S^T E^q(t) \int_\Omega c_\epsilon(x) \, |u_t(t)|^{m(x)} \, \mathrm{d}x dt.
\end{aligned} \tag{2.11}$$

Similarly

$$\begin{aligned}
bJ_2 \leq & \, b \; c\epsilon \left( (E(0))^{\frac{r^-}{2}-1} + (E(0))^{\frac{r^+}{2}-1} \right) \int_S^T E^{q+1}(t) \mathrm{d}t \\
& + b \int_S^T E^q(t) \int_\Omega c'_\epsilon(x) \, |v_t(t)|^{r(x)} \, \mathrm{d}x dt.
\end{aligned} \tag{2.12}$$

Thus, using (2.11) and (2.12), we get

$$\begin{aligned}
I_4 + I_5 \leq & \epsilon c \left[ a \left( (E(0))^{\frac{m^-}{2}-1} + (E(0))^{\frac{m^+}{2}-1} \right) + b \left( (E(0))^{\frac{r^-}{2}-1} + (E(0))^{\frac{r^+}{2}-1} \right) \right] \int_S^T E^{q+1}(t) \mathrm{d}t \\
& + \int_S^T E^q(t) \int_\Omega \left( a \; c_\epsilon(x) \, |u_t(t)|^{m(x)} + b c'_\epsilon(x) \; |v_t(t)|^{r(x)} \right) \mathrm{d}x dt.
\end{aligned}$$

For the third term $I_3$, we set as in [**50**] and exploit Holder (2.4) and Young's inequalities (2.3) and as follows

$$\Omega_+ = \{ x \in \Omega / \, |u_t(x,t)| \geq 1 \} \quad \text{and} \quad \Omega_- = \{ x \in \Omega / \, |u_t(x,t)| < 1 \}.$$

Therefore,

$$\begin{aligned}\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt &= \int_S^T E^q(t)\left[\int_{\Omega_-}|u_t(t)|^2\,dx+\int_{\Omega_+}|u_t(t)|^2\,dx\right]dt\\ &\le c\int_S^T E^q(t)\left[\left(\int_{\Omega_-}|u_t(t)|^{\rho^+}\,dx\right)^{2/\rho^+}+\left(\int_{\Omega_+}|u_t(t)|^{\rho^-}\,dx\right)^{2/\rho^-}\right]dt\\ &\le c\int_S^T E^q(t)\left[\left(\int_{\Omega_-}|u_t(t)|^{m(x)}\,dx\right)^{2/\rho^+}+\left(\int_{\Omega_+}|u_t(t)|^{m(x)}\,dx\right)^{2/\rho^-}\right]dt,\end{aligned}$$

where

$$\rho^- = \min\left\{m^-,r^-\right\},\rho^+ = \max\left\{m^+,r^+\right\}.$$

Which gives

$$\begin{aligned}\int_S^T E^q(t)\int_\Omega |u_t(t)|^2\,dxdt &\le c\int_S^T E^q(t)\left[\left(\int_\Omega |u_t(t)|^{m(x)}\,dx\right)^{2/\rho^+}+\left(\int_\Omega |u_t(t)|^{m(x)}\,dx\right)^{2/\rho^-}\right]dt\\ &\le c\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^+}dt+c\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^-}dt.\end{aligned}$$

Similarly, we obtain

$$\int_S^T E^q(t)\int_\Omega |v_t(t)|^2\,dxdt \le c\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^+}dt+c\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^-}dt.$$

By addition, this yields

$$|I_3| \quad \le c\textstyle\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^+}dt+c\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^-}dt. \tag{2.13}$$

Now, we distinguish two cases:

**Case 1:** if $\rho^+ = 2$, then $\rho^- = 2$. Hence, (2.13) takes the form

$$|I_3| \le c\int_S^T E^q(t)\left(-E'(t)\right)dt \le \epsilon c\int_S^T E^{q+1}(t)dt + c_\epsilon E(S).$$

**Case 2 :** if $\rho^+ > 2$, we use Young's inequality (2.3) with

$$\delta = (q+1)/q \text{ and } \delta' = q+1$$

to obtain

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^+} dt \le \epsilon c \int_S^T E^{q+1}(t)dt + c_\epsilon \int_S^T \left(-E'(t)\right)^{2(q+1)/\rho^+} dt.$$

By taking $q = \rho^+/2 - 1$, it comes

$$\begin{aligned}\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^+} dt &\le \epsilon c \int_S^T E^{q+1}(t)dt + c_\epsilon \int_S^T \left(-E'(t)\right) dt \\ &\le \epsilon c \int_S^T E^{q+1}(t)dt + c_\epsilon E(S).\end{aligned} \tag{2.14}$$

For the second term in the right-hand side of (2.13), we use Young's inequality (2.3) with

$$\delta = \rho^-/\left(\rho^- - 2\right) \text{ and } \delta' = \rho^-/2,$$

to obtain

$$\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^-} \mathrm{d}t \le \epsilon C \int_S^T E(t)^{q\rho^-/\left(\rho^- - 2\right)} \mathrm{d}t + C_\epsilon E(S).$$

Since, $q\rho^-/\left(\rho^- - 2\right) = q + 1 + \left(\rho^+ - \rho^-\right)/\left(\rho^- - 2\right)$, then

$$\begin{aligned}\int_S^T E^q(t)\left(-E'(t)\right)^{2/\rho^-} \mathrm{d}t &\le \epsilon c (E(S))^{\left(\rho^+ - \rho^-\right)/\left(\rho^- - 2\right)} \int_S^T E^{q+1}(t)\mathrm{d}t + c_\epsilon E(S) \\ &\le \epsilon c \int_S^T E^{q+1}(t)\mathrm{d}t + c_\epsilon E(S).\end{aligned} \tag{2.15}$$

By (2.14) and (2.15), estimate (2.13) takes the form

$$|I_3| \le \epsilon c \int_S^T E^{q+1}(t)\mathrm{d}t + c_\epsilon E(S). \tag{2.16}$$

Combining (2.8)-(2.16), we see that

$$\begin{aligned}2\int_S^T E^{q+1}(t)\mathrm{d}t &\le \int_S^T E^q(t)\int_\Omega \left(a\ c_\epsilon(x)\,|u_t|^{m(x)} + c'_\epsilon(x)\ b\,|v_t|^{r(x)}\right)\mathrm{d}x dt \\ &\quad + \epsilon c\left[1 + a\left((E(0))^{\frac{m^-}{2}-1} + (E(0))^{\frac{m^+}{2}-1}\right) + b\left((E(0))^{\frac{r^-}{2}-1} + (E(0))^{\frac{r^+}{2}-1}\right)\right]\int_S^T E^{q+1}(t)\mathrm{d}t \\ &\quad + c_\epsilon E(S).\end{aligned}$$

Choose $\epsilon > 0$ so small that

$$\epsilon c\left[1+a\left((E(0))^{\frac{m^-}{2}-1}+(E(0))^{\frac{m^+}{2}-1}\right)+b\left((E(0))^{\frac{r^-}{2}-1}+(E(0))^{\frac{r^+}{2}-1}\right)\right]<1.$$

Once $\epsilon$ is fixed, then $m(x), r(x)$ are bounded. Hence, there exist two positive constants $M_1, M_2$ such that $c_\epsilon(x) \leq M_1$ and $c_\epsilon^{'}(x) \leq M_2$. Since $q = \frac{\rho^+}{2} - 1$, then we obtain

$$\begin{aligned}
\int_S^T E^{\frac{\rho^+}{2}}(t)\mathrm{d}t &\leq cE(S) + M\int_S^T E^q(t)\int_\Omega (a|u_t|^{m(x)} + b|v_t|^{r(x)})\mathrm{d}x \\
&\leq cE(S) - M\int_S^T E^q(t)E'(t)\mathrm{d}t \\
&\leq cE(S) + c\left(E^{\frac{\rho^+}{2}}(S) - E^{\frac{\rho^+}{2}}(T)\right) \\
&\leq c\left(E(S) + E^{\frac{\rho^+}{2}}(S)\right) \\
&\leq c\left(1 + E^{\frac{\rho^+}{2}-1}(0)\right)E(S) \leq cE(S), \quad \forall T > S > 0,
\end{aligned}$$

where $M = \max\{M_1, M_2\}$. Thus, we arrive at

$$\int_S^T E^{\frac{\rho^+}{2}}(t)\ dt \leq cE(S).$$

By letting $T \to +\infty$, we get

$$\int_S^{+\infty} E^{\frac{\rho^+}{2}}(t)\ dt \leq cE(S).$$

By Komornik's integral inequality [40], we obtain the desired result. $\square$

CHAPTER 4

# Blow up of the wave equation with nonlinear first order perturbation term

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In this chapter, we consider the following system

$$\begin{cases} u_{tt} - \Delta u + g * \Delta u + a u_t + F(t, \nabla u) = |u|^{p-2} u \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(.,0) = u_0(.) \quad \text{and} \quad u_t(.,0) = u_1(.) \quad \text{in } \Omega, \end{cases} \tag{P3}$$

where $p > 2$, $a > 0$ are constants, $g$ and $F$ are a functions satisfying some conditions to be specified later. Noting that $(g * v)(t) = \int_0^t g(t-\tau)v(\tau)d\tau$ for all $t \geq 0$.

Our aim, in this chapter, is to prove that if the damping terms (linear and viscoelastic) dominated the nonlinear first order perturbation term then the energy is decreasing. So, we can define the auxiliary functional $L$. After that, we show that the solutions with sufficiently negative initial energy blow up in finite time.

This chapter is organized as follows. In section 1, we present some preliminary results needed for our work. In section 2, we give the proof of main result of our problem.

## 1. Preliminary results

In this section, we shall give some preliminary results which will be used throughout this work.

The existence and uniqueness result for system $(P3)$ is given in the following theorem

Theorem 1.1. *Suppose that*

$$p \leq \frac{2(n-1)}{n-2} \quad \text{if} \quad n \geq 3.$$

*$g$ is a $C^1(\mathbb{R}^+)$ positive decreasing function satisfying*

$$1 - \int_0^\infty g(s)ds > 0$$

*and $F$ is a $C^1(\mathbb{R}^+ \times \mathbb{R}^n)$ function.*

*If $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$ then there exists a unique maximal strong solution of system $(P3)$*

$$u \in L^\infty\left((0,T), H^2(\Omega) \cap H_0^1(\Omega)\right),$$
$$u_t \in L^\infty\left((0,T), H_0^1(\Omega)\right),$$
$$u_{tt} \in L^\infty\left((0,T), L^2(\Omega)\right).$$

Proof. As in [**74**]. □

*Now, we consider the energy functional for the local solution $u$ of $(P3)$ defined for all $t \in [0,T)$ by*

$$E(t) = \frac{1}{2} \parallel u_t(t) \parallel_2^2 + \frac{1}{2}\left(1 - \int_0^t g(\tau)d\tau\right) \parallel \nabla u(t) \parallel_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p} \parallel u(t) \parallel_p^p,$$

*where*

$$(g \circ v)(t) = \int_0^t g(t-\tau) \parallel v(\tau) - v(t) \parallel_2^2 d\tau.$$

*We have*

LEMMA 1.2. *Assume that the hypotheses of the Theorem 1.1 are verified and suppose that*

$$| F(t,U) |^2 \leq 2ag(t) | U |^2, \quad \forall\ t \geq 0, \quad \forall\ U \in \mathbb{R}^n. \tag{1.1}$$

*Then $E$ is a decreasing function.*

PROOF. We multiply the first equation in $(P3)$ by $u_t(t)$, integrate it over $\Omega$ and use Green formula (1.4) to obtain for all $t \in [0,T)$

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - a \parallel u_t(t) \parallel_2^2 - \frac{1}{2}g(t) \parallel \nabla u(t) \parallel_2^2 - \int_\Omega F(t,\nabla u)u_t(t)dx.$$

Since $g' \leq 0$ then

$$E'(t) \leq -a \parallel u_t(t) \parallel_2^2 - \frac{1}{2}g(t) \parallel \nabla u(t) \parallel_2^2 + \int_\Omega | F(t,\nabla u) || u_t(t) | dx.$$

If we use the Young inequality (1.5), with

$$X = | F(t,\nabla u) |,\ Y = | u_t(t) | \quad \text{and} \quad \mu = \theta = 2$$

we find

$$E'(t) \quad \leq \quad \frac{1}{2}\int_\Omega \left[\delta \mid F(t,\nabla u) \mid^2 - g(t) \mid \nabla u(t) \mid^2\right] dx + \left(\frac{1}{2\delta} - a\right) \parallel u_t(t) \parallel_2^2 .$$

If we take $\delta = \dfrac{1}{2a}$ we obtain

$$E'(t) \quad \leq \quad \frac{1}{2}\int_\Omega [\frac{1}{2a} \mid F(t,\nabla u) \mid^2 - g(t) \mid \nabla u(t) \mid^2]dx.$$

By (1.1), we find

$$E'(t) \leq 0, \quad \forall t \in [0,T).$$

□

*Consider the following functional*

$$H(t) = -E(t), \qquad \forall\, t \in [0,T).$$

Lemma 1.3. *Suppose the conditions of Lemma 1.2 hold. Assume further that $E(0) < 0$. Then*

$$0 < H(0) \leq H(t) \leq \frac{1}{p} \parallel u(t) \parallel_p^p, \qquad \forall\, t \in [0, T).$$

Proof. From the definition of $E$, $H$ and the decreasing of $E$. □

## 2. Main result

*In this section, we shall discuss the blow up question of system $(P3)$.*

Theorem 2.1. *Under the assumptions of Lemma 1.3 and assume that $a$ and $g$ verify*

$$\alpha := \frac{p-2}{2} - \frac{3C_* \sqrt{ag(0)}}{2} - (\frac{p-2}{2} + \frac{1}{p}) \int_0^\infty g(\tau) d\tau > 0, \tag{2.1}$$

*where $C_*$ is the best constant of the Poincare inequality then the solution of problem $(P3)$ blows up in finite time.*

Proof. We proceed in 4 steps:

**Step 1** Since $H$ is positive then we can define for all $\epsilon > 0$ the auxiliary functional $L$ as follow

$$L(t) = \mathrm{e}^{at} H^{\frac{p+2}{2p}}(t) + \epsilon \mathrm{e}^{at} \int_\Omega u(t) u_t(t) dx, \quad \forall t \in [0, T).$$

We derive the functional $L$ with respect to $t$ we obtain

$$\begin{aligned} L'(t) &= a\mathrm{e}^{at} H^{\frac{p+2}{2p}}(t) + \frac{p+2}{2p} \mathrm{e}^{at} H^{\frac{2-p}{2p}}(t) H'(t) + \epsilon a \mathrm{e}^{at} \int_\Omega u(t) u_t(t) dx \\ &+ \epsilon \mathrm{e}^{at} \parallel u_t(t) \parallel_2^2 + \epsilon \mathrm{e}^{at} \int_\Omega u(t) u_{tt}(t) dx. \end{aligned}$$

Since $H$ and $H'$ are positive then

$$L'(t) \geq \epsilon a \mathrm{e}^{at} \int_\Omega u(t) u_t(t) dx + \epsilon \mathrm{e}^{at} \parallel u_t(t) \parallel_2^2 + \epsilon \mathrm{e}^{at} \int_\Omega u(t) u_{tt}(t) dx. \tag{2.2}$$

We multiply the first equation of $(P3)$ by $u(t)$ and integrate it over $\Omega$ we obtain

$$\begin{aligned}\int_\Omega u(t)u_{tt}(t)dx &= \int_\Omega \Delta u(t)u(t)dx - \int_\Omega\int_0^t g(t-\tau)\Delta u(\tau)u(t)d\tau dx \\ &- a\int_\Omega u_t(t)u(t)dx - \int_\Omega F(t,\nabla u)u(t)dx + \| u(t) \|_p^p .\end{aligned}$$

We use Green formula (1.4) and boundary conditions we find

$$\begin{aligned}\int_\Omega u(t)u_{tt}(t)dx &= -\| \nabla u(t) \|_2^2 + \int_0^t g(t-\tau)\int_\Omega \nabla u(t).\nabla u(\tau)dxd\tau \\ &- a\int_\Omega u_t(t)u(t)dx - \int_\Omega F(t,\nabla u)u(t)dx + \| u(t) \|_p^p .\end{aligned} \tag{2.3}$$

First, we have

$$\begin{aligned}\int_0^t g(t-\tau)\int_\Omega \nabla u(t).\nabla u(\tau)dxd\tau &= \int_0^t g(t-\tau)\int_\Omega \nabla u(t).[\nabla u(\tau)-\nabla u(t)]dxd\tau \\ &+ \int_0^t g(\tau)d\tau \| \nabla u(t) \|_2^2 .\end{aligned}$$

By Schwarz inequality (1.3), we find

$$\begin{aligned}\int_0^t g(t-\tau)\int_\Omega \nabla u(t).\nabla u(\tau)dxd\tau &\geq -\int_0^t g(t-\tau) \| \nabla u(t) \|_2 \| \nabla u(\tau)-\nabla u(t) \|_2 \, d\tau \\ &+ \int_0^t g(\tau)d\tau \| \nabla u(t) \|_2^2 .\end{aligned}$$

If we exploit Young inequality (1.5) with

$$X = \sqrt{g(t-\tau)} \| \nabla u(\tau)-\nabla u(t) \|_2, \; Y = \sqrt{g(t-\tau)} \| \nabla u(t) \|_2 \quad \text{and} \quad \mu=\theta=2$$

we obtain for all $\beta_1 > 0$

$$\int_0^t g(t-\tau)\int_\Omega \nabla u(t).\nabla u(\tau)dxd\tau \geq -\beta_1(g\circ\nabla u)(t) + \left(1-\frac{1}{4\beta_1}\right)\int_0^t g(\tau)d\tau \| \nabla u(t) \|_2^2. \tag{2.4}$$

On the other hand, if we use Young and Poincare inequalities (See inequalities (1.1) and (1.5) in chapter1) we find for all $\beta_2 > 0$

$$-\int_\Omega F(t,\nabla u)u(t)dx \geq -\frac{1}{2\beta_2}\int_\Omega | F(t,\nabla u) |^2 dx - \frac{\beta_2}{2}C_*^2 \parallel \nabla u(t) \parallel_2^2, \quad \forall t \in [0,T).$$

By (1.1), we find

$$-\int_\Omega F(t,\nabla u)u(t)dx \geq -\left(\frac{ag(t)}{\beta_2} + \frac{\beta_2}{2}C_*^2\right) \parallel \nabla u(t) \parallel_2^2, \quad \forall t \in [0,T).$$

Since $g' \leq 0$ then

$$-\int_\Omega F(t,\nabla u)u(t)dx \geq -\left(\frac{ag(0)}{\beta_2} + \frac{\beta_2}{2}C_*^2\right) \parallel \nabla u(t) \parallel_2^2, \quad \forall t \in [0,T). \tag{2.5}$$

Now, by the definition of $H$ we have

$$\parallel u(t) \parallel_p^p = \frac{p}{2} \parallel u_t(t) \parallel_2^2 + \frac{p}{2}\left(1 - \int_0^t g(\tau)d\tau\right) \parallel \nabla u(t) \parallel_2^2 + \frac{p}{2}(g \circ \nabla u)(t) + pH(t) \tag{2.6}$$

Replacing (2.4), (2.5) and (2.6) in (2.3) to obtain

$$\begin{aligned}
\int_\Omega u(t)u_{tt}(t)dx &\geq -\parallel \nabla u(t) \parallel_2^2 - \beta_1(g \circ \nabla u)(t) + \left(1 - \frac{1}{4\beta_1}\right)\int_0^t g(\tau)d\tau \parallel \nabla u(t) \parallel_2^2 \\
&- a\int_\Omega u_t(t)u(t)dx - \left(\frac{ag(0)}{\beta_2} + \frac{\beta_2}{2}C_*^2\right) \parallel \nabla u(t) \parallel_2^2 + \frac{p}{2} \parallel u_t(t) \parallel_2^2 \\
&+ \frac{p}{2}\left(1 - \int_0^t g(\tau)d\tau\right) \parallel \nabla u(t) \parallel_2^2 + \frac{p}{2}(g \circ \nabla u)(t) + pH(t) \\
&= (\frac{p}{2} - \beta_1)(g \circ \nabla u)(t) - a\int_\Omega u_t(t)u(t)dx + \frac{p}{2} \parallel u_t(t) \parallel_2^2 + pH(t) \\
&+ \left[\frac{p-2}{2} - \frac{ag(0)}{\beta_2} - \frac{\beta_2 C_*^2}{2} - (\frac{p-2}{2} + \frac{1}{4\beta_1})\int_0^t g(\tau)d\tau\right] \parallel \nabla u(t) \parallel_2^2 .
\end{aligned}$$

If we take $\beta_1 = \frac{p}{4}$ and $\beta_2 = \frac{\sqrt{ag(0)}}{C_*}$ we find

$$
\begin{aligned}
\int_\Omega u(t)u_{tt}(t)dx &\geq \frac{p}{4}(g\circ\nabla u)(t) - a\int_\Omega u_t(t)u(t)dx + \frac{p}{2}\parallel u_t(t)\parallel_2^2 + pH(t) \\
&+ \left[\frac{p-2}{2} - \frac{3C_*\sqrt{ag(0)}}{2} - (\frac{p-2}{2}+\frac{1}{p})\int_0^t g(\tau)d\tau\right]\parallel \nabla u(t)\parallel_2^2
\end{aligned}
$$

Since $g \geq 0$ then

$$
\begin{aligned}
\int_\Omega u(t)u_{tt}(t)dx &\geq \frac{p}{4}(g\circ\nabla u)(t) - a\int_\Omega u_t(t)u(t)dx + \frac{p}{2}\parallel u_t(t)\parallel_2^2 + pH(t) \\
&+ \alpha\parallel \nabla u(t)\parallel_2^2 . \qquad (2.7)
\end{aligned}
$$

Replacing (2.7) in (2.2) to find

$$
\begin{aligned}
L'(t) &\geq \epsilon e^{at}(1+\frac{p}{2})\parallel u_t(t)\parallel_2^2 + \epsilon e^{at}\frac{p}{4}(g\circ\nabla u)(t) + \epsilon e^{at}pH(t) \\
&+ \epsilon e^{at}\alpha\parallel \nabla u(t)\parallel_2^2 .
\end{aligned}
$$

Let $\beta > 0$. By writing $p = 2\beta + (p-2\beta)$ and since

$$
H(t) \geq \frac{1}{p}\parallel u(t)\parallel_p^p - \frac{1}{2}\parallel u_t(t)\parallel_2^2 - \frac{1}{2}\parallel \nabla u(t)\parallel_2^2 - \frac{1}{2}(g\circ\nabla u)(t)
$$

we obtain

$$
\begin{aligned}
L'(t) &\geq \epsilon e^{at}(1+\frac{p}{2}-\beta)\parallel u_t(t)\parallel_2^2 + \epsilon e^{at}(\frac{p}{4}-\beta)(g\circ\nabla u)(t) + \epsilon e^{at}(p-2\beta)H(t) \\
&+ \epsilon e^{at}(\alpha-\beta)\parallel \nabla u(t)\parallel_2^2 + \epsilon e^{at}\frac{2\beta}{p}\parallel u(t)\parallel_p^p . \qquad (2.8)
\end{aligned}
$$

If we take $\beta < \min\left(\frac{p}{4}, \alpha\right)$, inequality (2.8) takes the form

$$
L'(t) \geq Ce^{at}[H(t) + \parallel u_t(t)\parallel_2^2 + (g\circ\nabla u)(t) + \parallel u(t)\parallel_p^p], \quad \forall t\in[0,T]. \qquad (2.9)
$$

**Step 2** We have

$$
L(0) = H^{\frac{p+2}{2p}}(0) + \epsilon\int_\Omega u_0u_1dx.
$$

If

$$\int_\Omega u_0 u_1 dx \geq 0,$$

then

$$L(0) \geq 0.$$

If

$$\int_\Omega u_0 u_1 dx < 0,$$

then, if we take

$$\epsilon < \frac{-H^{\frac{p+2}{2p}}(0)}{\int_\Omega u_0 u_1 dx}$$

we obtain

$$L(0) \geq 0.$$

then from the increase of $L$, we find that

$$L(t) \geq 0, \quad \forall t \in [0,T).$$

**Step 3** By the definition of $L$ and the algebraic inequality (1.6), with

$$\mu = H(t), \quad \theta = \epsilon \int_\Omega |\, u(t) \,||\, u_t(t) \,|\, dx \quad \text{and} \quad m = \frac{2p}{p+2}$$

we obtain

$$\begin{aligned} L^{\frac{2p}{p+2}}(t) &\leq 2^{\frac{2p}{p+2}} \mathrm{e}^{\frac{2pat}{p+2}} \left[ H(t) + \epsilon^{\frac{2p}{p+2}} (\int_\Omega |\, u(t) \,||\, u_t(t) \,|\, dx)^{\frac{2p}{p+2}} \right] \\ &\leq C \mathrm{e}^{\frac{2pat}{p+2}} \left[ H(t) + \left( \int_\Omega |\, u(t) \,||\, u_t(t) \,|\, dx \right)^{\frac{2p}{p+2}} \right], \end{aligned}$$

where $C$ is a generic positive constant.

If we use Schwarz inequality (1.3) and the embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$ we find

$$L^{\frac{2p}{p+2}}(t) \leq C\mathrm{e}^{\frac{2pat}{p+2}}\left[H(t)+ \| u(t) \|_p^{\frac{2p}{p+2}} \| u_t(t) \|_2^{\frac{2p}{p+2}}\right].$$

If we use Young inequality (1.5) with

$$X = \| u(t) \|_p^{\frac{2p}{p+2}}, Y = \| u_t(t) \|_2^{\frac{2p}{p+2}}, \mu = \frac{p+2}{2} \quad \text{and} \quad \theta = \frac{p+2}{p}$$

we find

$$L^{\frac{2p}{p+2}}(t) \quad \leq \quad C\mathrm{e}^{\frac{2pat}{p+2}}\left[H(t)+ \| u_t(t) \|_2^2 + \| u(t) \|_p^p\right]. \tag{2.10}$$

**Step 4** By combining (2.9) and (2.10), we arrive at

$$L'(t)L^{-\frac{2p}{p+2}}(t) \geq C\mathrm{e}^{\frac{(2-p)at}{p+2}}.$$

A simple integration over $(0,t)$ gives

$$L(t) \geq \frac{1}{\left[L^{\frac{2-p}{p+2}}(0) - C\left(1-\mathrm{e}^{\frac{(2-p)at}{p+2}}\right)\right]^{\frac{p-2}{p+2}}}$$

From which follows, via (2.10), the desired result. □

# *Conclusion*

*In this thesis, we have studied some nonlinear hyperbolic problems involving nonlinearities.*

*The first study focuses on the nonlinear wave equation with damping and source terms of variable exponent types. By some assumptions on the initial data, we showed that the energy is decreasing and we obtained a blow up result for the solution in finite time.*

*The second study considered a coupled system of nonlinear wave equations with variable exponents in the damping terms. By using the Faedo Galerkin method, the existence and the uniqueness of a weak solution is established. We, also, proved decay estimates for the solutions under appropriate assumptions on the variable exponents using the multipliers method.*

*Finally, we considered the nonlinear wave equation with damping, source and nonlinear first order perturbation terms. We showed that when the damping terms (linear and viscoelastic) dominated the first order perturbation term, the usual energy is decreasing and the solutions with sufficiently negative initial energy blow up in finite time.*

## *Open Problems*

*The following open questions can be made regarding the material presented in this thesis.*

- *Study of the properties for the solutions of the system considered in chapter 2 in the case when the initial energy is positive.*
- *Establish sufficient conditions so that the system considered in chapter 4 is stable (exponentially or polynomially ).*

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