République Algérienne Démocratique et Populaire
Ministaire de l'Enseignement Supérieur et de la Recherche Scientifique
Université de Batna 2
Faculté de Mathématiques et Informatique
Département de Mathématiques

Thèse
Présentée en vue de l'obtention du Diplôme de
Doctorat en Sciences
Spécialité: Mathématiques Appliquées

Ptrésentée Par
Boussaïd Samira

Thème

Existence Globale et Comportement Asymptotique des Solutions d'Equation de Réaction Diffusion

Soutenue le: 26/09/2017

Devant le Jury Composé de:

S. REBIAI	Pr	Université de Batna 2	Président
A. YOUKANA	Pr	Université de Batna 2	Rapporteur
M. GUEDDA	Pr	Université de Picardie Jules Verne France	Examinateur
R. KERSNER	Pr	Université de Pécs Hongrie	Examinateur
A.Z. MOKRANE	M.C.A	Université de Batna 2	Examinateur
M.S. MOULAY	Pr	U.S.T.H-B Alger	Examinateur
F.BENTELHA	Pr	Université de Batna 2	Invitée

Remerciements

Je tiens tout d'abord à remercier le Professeur Amar Youkana pour avoir diriger cette thèse, de m'avoir fait confiance malgré qu'au début je ne connaissais pas grand chose aux problèmes de Réaction-Diffusion. Pour m'avoir laisser une large liberté dans le choix et les méthodes de traitements des problèmes considérés.

Je remercie également le Professeur Salah Eddine Rebiai pour avoir accepter de présider ce jury, il me fait ainsi un grand honneur vu qu'il avait déjà présidé le jury de soutenance de mon Magister à l'Université de Sétif 1 (il y a fort longtemps!!!). Comme je tiens à remercier Messieurs: le Professeur Robert Kersner, le Docteur Ahmed Zerrouk Mokrane, et le Professeur Mohamed Said Moulay d'avoir accepter de faire partie des jury de cette thèse.

Mes remerciements vont également à Madame Danielle Hilhorst, directrice de recherches au CNRS et au laboratoire LMO de l'université de Paris Sud, pour m'avoir accueilli dans son groupe de recherche, pour son hospitalité durant mes stages de courtes durées. Merci de m'avoir proposé les sujets des chapitres 1 et 2, pour ton extrême gentillesse et pour tous tes conseils et le temps que tu m'as consacré.

Je tiens à exprimer ma profonde gratitude au Professeur Mohamed Guedda, qui m'a aidé à décrocher la bourse PNE en acceptant d'être mon co-encadreur à l'étranger et en me manifestant son entière disponibilité pour effectuer toutes les lourdes et pénibles tâches administratives en France et en Algérie, de m'avoir intégrer au sein du laboratoire LAMFA à l'université de Jules Vernes Picardie (Amiens-France), de m'avoir soutenue durant les moments les plus difficiles de ma vie. Je le remecie également pour m'avoir proposé et aider à traiter le chapitre 3. Et pour avoir accepter de faire partie des membres du jury de cette thèse.

Je remercie toute l'équipe du laboratoire LAMFA pour m'avoir accueilli dans le groupe A3. Comme je remercie tous mes amis et collègues (enseignants et administrateurs sans exception) de l'université de Batna 1 et Batna 2, ainsi que mes étudiants.

Je rends hommage à Monsieur Berkane Ahmed, qui m'a soutenue et encouragé le jour où j'ai touché le fond et qui avait cru en moi.

Merci à tous mes enseignants durant mon cursus scolaire et universitaire.

Finalement je remercie profondément mes très chers parents Malika et Abdallah pour tous leurs sacrifices et devouements surtout ces derniers temps où ils m'ont offert la vie une deuxième fois. Merci à mes très chers frères et soeurs, mes neveux et ma nièce.

Sans oublier de remercier ma magnifique petite famille: mon mari Fayçal, Lyna, Zakarya et Hatem et vous demande pardon d'avoir était mes souffres douleurs, je vous promets de vous consacrer beaucoup plus de temps et surtout de ne pas gacher vos vacances.

Milles excuses si j'ai oublié de citer qui que se soit.

Contents

Introduction

Reaction-Diffusion equations have enjoyed a considerable amount of scientific interest, because of their practical relevance and the fact that they model several natural phenomena found in chemistry, biology, geology, ecology and physics. From a qualitative point of view, a reaction-diffusion problem describes how the concentration of one or more substances vary over time and space under the influence of two terms: Reaction term or source term, in which concentration is generated by local interaction. Diffusion term which causes the substances to spread out in space. A reaction diffusion problem have the form

$$\frac{\partial u}{\partial t} - d.\Delta u = f(u), \text{ in } \Omega \cdots (E)$$

$\Omega \subset \mathbb{R}^n$, $d > 0, \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $f : \mathbb{R} \to \mathbb{R}$, together with some appropriate boundary and initial conditions $(t = 0)$ imposed on u. Where the unknown is a function $u : \Omega \times \mathbb{R} \to \mathbb{R}$.
This thesis is devoted to the study of the existence and uniqueness of a global solutions to a nonlocal reaction diffusion problems and to the asymptotic behavior of this solutions using the notion of attractor. We assume in the beginning that we can find a phase space H (usually a Hilbert or a Banach space), such that for $u_0(x) = u(x, 0) \in H$, the equation has a unique solution $u(u_0, t)$ for all positive times. In this case we can define a C^0-semigroup of solution operators $S(t) : H \to H$ by $S(t)u_0 = u(u_0, t)$, enjoying the following properties

$$S(0) = I, (I\text{: Identity in } H)$$
$$S(t + s) = S(t).S(s) = S(s).S(t), \forall s, t \geq 0.$$

And

$$u(t) = S(t)u_0$$
$$u(t + s) = S(t).u(s) = S(s).u(t), \forall s, t \geq 0$$

where we say that the pair $(H, S(t))$ is the dynamical system associated with our problem.
A global attractor is a compact maximal bounded invariant set which attracts the trajectories as time goes to infinity. This set, if it exists, is unique and is essentially thinner than the initial phase space H. It is also not difficult to prove that is the smallest (for the inclusion) closed set enjoying the attraction property. But it may present several defaults, it may attract the trajectories at a slow rate. And in general, it is very difficult, to express the convergence rate in terms of the physical parameters of the problem. It may also change drastically under small perturbations. Furthermore, in many situations, the global attractor may not be observable in experiments or in numerical simulations. This can be due to the fact that it has a very complicated geometric structure. Finally, in some situations, the global attractor may fail to capture important transient behaviors.
In order, to overcome these difficulties, the notion of inertial manifold was proposed, which is a smooth

finite dimensional hyperbolic positively invariant manifold that contains the global attractor and attracts exponentially the trajectories. Unfortunately, all known constructions of inertial manifolds are based on a restrictive condition, the so-called spectral gap condition. Consequently, the existence of inertial manifolds is not known for many physically important equations. Thus, as an intermediate object between the two ideal objects that the global attractor and an inertial manifold are, the notion of exponential attractor (inertial set) was introduced, which is a compact positively invariant set that contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories. So, compared with the global attractor, an exponential attractor is more robust under perturbations and numerical approximations.

The Allen Cahn equation has the form

$$\frac{\partial u}{\partial t} = \varepsilon^2 \Delta u - F'(u), x \in \Omega, t \geq 0,$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $\varepsilon > 0$ one (small) parameter and F' the derivative of a double well potential, u is an order parameter which represents for example the arrangement by unity cell in a crystal lattice and the well of F corresponds to the two phases of the material.

For $\varepsilon = 0$, our equation is reduced to an ordinary differential equation and $u(x,t)$ evolves towards $+1$ or -1 as $u_0 > 0$ or $u_0 < 0$. The term $\varepsilon^2 \Delta u$ (the diffusion term) occurs in a time scale slower than the reaction $F'(u)$. A typical choice of the potential is $F(s) = \frac{1}{4}(s^2-1)^2, s \in \mathbb{R}$.

Our equation is usually obtained as a gradient flow (in the sense that the evolution generated by the equation possesses a Lyapounov functional which is the energy functional decreasing in time) of

$$E(u) = \int_\Omega \left[\frac{\varepsilon^2}{2} |\nabla u|^2 + F(u) \right] dx,$$

for the scalar product in Ω.

The term $F(u)$ represents the energy for a uniform parameter u and the term $\dfrac{\varepsilon^2}{2} |\nabla u|^2$ represents the interface introduced by Cahn and Hilliard (1958). In such a model the discontinuity of u is not allowed, and the interface is represented by a thin layer of a transition from a phase to an other owing a little thickness.

In the first chapter we took an Allen Cahn problem proposed by J. Rubinstein and P. Sternberg [33], which is a model of a binary mixture undergoing phase separation. More precisely, we proposed to study the problem

$$(P) \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) - \dfrac{1}{|\Omega|} \displaystyle\int_\Omega f(u) dx, & x \in \Omega, t > 0, \quad (0.1) \\ \partial_\nu u = 0 & x \in \partial\Omega, t \geq 0 \quad (0.2) \\ u(x,0) = u_0(x) & x \in \Omega, \quad (0.3) \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with outer unit normal ν and total volume $|\Omega|$. This model is mass conserved, namely

$$\frac{1}{|\Omega|} \int_\Omega u(x,t) dx = \frac{1}{|\Omega|} \int_\Omega u_0(x) dx = M.$$

Where the nonlinearity f is a polynomial function of odd degree

$$f(s) = \sum_{i=0}^{2p-1} a_i s^i, \text{ with } a_{2p-1} < 0, p \geq 2$$

and $f = -F'$; F is a smooth double well potential.
We proved the existence of a unique global solution, and of a global and exponential attractor were determined for this problem. This study was proposed by Madame Danielle Hilhorst a Reaserch Director in the University of Paris Sud in France, and also a part of our results was published in a paper in a collaboration with D. Hilhorst and T.N. Nguyen [5].
In the second chapter we took the Allen Cahn problem proposed in the first chapter with a singular potential, where for $0 < \theta < \theta_c$ a critical temperature

$$\begin{aligned} &F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\Phi(s), \\ &\Phi(s) = (1+s)\ln(1+s) + (1-s)\ln(1-s), \text{ for } s \in (-1,1), \end{aligned}$$

so that

$$\begin{aligned} &f(s) = \theta_c s - \frac{\theta}{2}\varphi(s), \\ &\varphi(s) = \ln\left(\frac{1+s}{1-s}\right), \text{ for } s \in (-1,1), \end{aligned}$$

and then problem (P) will have the form

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \theta_c(u-M) - \dfrac{\theta}{2}\left[\varphi(u) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \varphi(u)dx\right], & x \in \Omega, t > 0 \quad (0.4) \\ \partial_\nu u = 0 & x \in \partial\Omega, t \geq 0. \quad (0.5) \\ u(x,0) = u_0(x) & x \in \Omega \quad (0.6) \end{cases}$$

A global and unique solution to this problem was found, where the set $\{x \in \Omega, |u(x,t)| = 1\}$ has measure zero. And also a complete study of the existence of a global attractor to this problem was given. This chapter was done in collaboration with D. Hilhorst a research director in the university of Paris Sud in France.
The third chapter was devoted to the study of the existence of a unique global solution and its asymptotic behavior of a nonlocal reaction-diffusion system

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = av - bu\displaystyle\int_\Omega vdx, & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} - \Delta v = h - \alpha u - \beta v\displaystyle\int_\Omega vdx, & x \in \Omega, t > 0, \\ \partial_\nu u = 0, \partial_\nu v = 0 & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0 & x \in \Omega, \end{cases}$$

where the coefficients a, b, h, α and β are supposed to be positive, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with outer unit normal ν and total volume $|\Omega|$. The difficulty for this system is that the reaction terms do not have a constant sign, and this means that none of the equations are good in the sense that neither u nor v is a priori bounded in order to apply the well known regularizing effect to deduce the global existence in time for problem (P).
Problem (P) is a nonlocal reaction-diffusion system that could arise in physics. It models the effects of an external field on the rheological properties of a dilute suspension of rigid spherical particles containing embedded dipoles [6]. These permanent dipoles may be gravitational, magnetic or electric in nature.

Rotary Brownian motion is assumed negligible. Free rotation of the suspended particles resulting from the shear is hindered by the action of the field. This gives rise to a system of body couples and, hence, to a state of antisymmetric stress.

This system was proposed and done in collaboration with Professor Mohamed Guedda from Jules Verne University of Picardie in France, where we used the framework of (positively) invariant region $\Sigma \subset \mathbb{R}^2$; which means that if $(u_0(x), v_0(x)) \in \Sigma, \forall x \in \Omega$, then $(u(x,t), v(x,t)) \in \Sigma, \forall t > 0$. Due to the problem form this invariant region is a rectangle (see Smoller [34]). The technique used here to determinate Σ is inspired by Pao [29].

The region Σ can likewise be thought as an attracting region for the problem (P), which provides a compactness argument, leading to the proof of the existence of a global solution, and to establish a global attractor.

Chapter I

The mass conserved Allen Cahn problem with a polynomial potential

Let Ω be an open bounded domain from $\mathbb{R}^N (N \geq 1)$.
We are dealing with the following nonlocal reaction-diffusion problem

$$(P) \begin{cases} u_t = \Delta u + f(u(x,t)) - \frac{1}{|\Omega|}\int_\Omega f(u(x,t))dx & x \in \Omega, t > 0 \\ \partial_\nu u = 0 & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

where u_0 for simplicity, is taken to be a function satisfying the Neumann condition $\partial_\nu u = 0$.
This problem form was originally presented by Rubinstein and Sternberg [33] to model a binary mixture undergoing phase separation. It is **mass preserving**, that means that, viewing u as an order parameter in the mixture or simply as the concentration of one of the species, one can readily check that

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx = m, \forall t > 0$$

which is crucial to the model.
We will consider the function f as a polynomial of odd degree, more precisely

$$f(s) = \sum_{i=0}^{2p-1} a_i s^i, \text{ with } a_{2p-1} < 0, p \geq 2$$

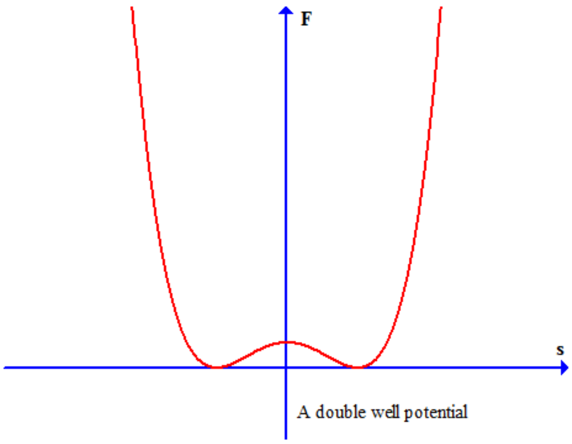
where $f = -F'$; F is a smooth double well potential. The nonlinearity f, have the following properties $\exists C_i > 0,$ for $i = 1, \cdots, 7$, with $M = \frac{m}{|\Omega|}$, such that

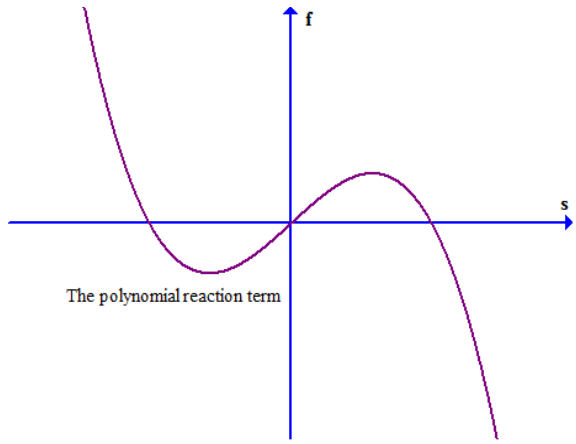
(P_1) $-C_2.(s-M)^{2p} - C_3 \leq f(s)(s-M) \leq -C_1.(s-M)^{2p} + C_3,$ for $p \geq 2$

(P_2) $f'(s) \leq C_4$,

(P_3) $-C_5((s-M)^{2p}+1) \leq -F(s) \leq -C_6((s-M)^{2p}-1)$, with $F(s) = -\int_0^s f(\tau)d\tau$.

(P_4) $|f(s)| \leq C_7(|s-M|^{2p-1}+1)$.





1 Existence of a global attractor

1.1 Existence of the semigroup

Let's give in the beginning the variational formulation of (P);
multiplying our equation with a test function $v \in H^1(\Omega)$ integrating it in Ω and using the Green's formula with the Neumann boundary condition, the resulting equation will be

$$\int_\Omega \frac{du}{dt} v dx + \int_\Omega \nabla u \nabla v dx = \int_\Omega v f(u) dx - \frac{1}{|\Omega|}\int_\Omega v dx \int_\Omega f(u) dx, \forall v \in H^1(\Omega),$$

but v is time independent so the result will be

$$\frac{d}{dt}\int_\Omega u v dx + \int_\Omega \nabla u \nabla v dx = \int_\Omega v f(u) dx - \frac{1}{|\Omega|}\int_\Omega v dx \int_\Omega f(u) dx, \forall v \in H^1(\Omega),$$

The following result asserts the existence of a semigroup $\{S(t)\}_{t\geq 0}$ such that $u(x,t) = S(t)u_0(x)$.

Theorem 1.1. *For $u_0 - M$ given in $L^2(\Omega)$, there exists a unique solution $u - M$ to problem (P) satisfying $u - M \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^{2p}(0,T;L^{2p}(\Omega))$ with $u_t \in L^2(0,T;(H^1(\Omega))')$ for all $T > 0$, and $u - M \in C([0,+\infty);L^2(\Omega))$.*
The mapping $u_0 \to u(t)$ is Lipschitz continuous on $L^2(\Omega)$.
If furthermore $u_0 - M \in H^1(\Omega) \cap L^{2p}(\Omega)$, then $u - M \in L^\infty(0,T;H^1(\Omega) \cap L^{2p}(\Omega)) \cap L^2(0,T;H^2(\Omega))$, and $u - M \in C([0,T];H^1(\Omega))$.

Proof. The proof relies on a Galerkin method, where we denote by $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots$ the eigenvalues of the operator $A = -\Delta : H^1(\Omega) \to (H^1(\Omega))'$ associated to the bilinear form

$$a(u,\tilde{u}) = \int_\Omega \nabla u . \nabla \tilde{u} dx,$$

with a homogenous Neumann boundary condition, and denote by $\omega_i \in H^1(\Omega) \cap L^{2p}(\Omega), i = 1, \cdots$ their corresponding unit eigenfunctions.
For each integer m we look for an approximate solution $u_m - M$ of the form

$$u_m(t) - M = \sum_{i=1}^{m} d_{im}(t)\omega_i,$$

satisfying

$$(\frac{\partial}{\partial t}(u_m - M), \omega_j) + a(u_m - M, \omega_j) = (\left[f(u_m) - \frac{1}{|\Omega|}(f(u_m), 1)\right], \omega_j), \tag{1.1}$$

for $j = 1, \cdots, m$

and $u_m(0) = u_{m0} \longrightarrow u_0$ in $L^2(\Omega)$ as $m \longrightarrow +\infty$. (1.2)

Problem (1.1) is an initial value problem of m ordinary differential equations, so by standard existence of solution argument we can state the existence of a unique solution on $(0, T_m), T_m > 0$. And if our sequence is bounded uniformly we will have then $T_m = +\infty$.
We multiply (1.1) by $d_{jm}(t)$ and sum on $j = 1, \cdots, m$ to obtain

$$(\frac{\partial}{\partial t}(u_m - M), u_m - M) + a(u_m - M, u_m - M) = (f(u_m), u_m - M), \text{ for } j = 1, \cdots, m$$

where

$$(\frac{1}{|\Omega|}, u_m - M) = \frac{1}{|\Omega|}\int_\Omega u_m dx - M = 0,$$

or

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_m - M)^2 dx\right) + \int_\Omega |\nabla(u_m - M)|^2 \, dx = \int_\Omega (u_m - M) f(u_m) dx, \tag{1.3}$$

and thanks to property (P_1), (1.3) will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_m - M)^2 dx\right) + \int_\Omega |\nabla(u_m - M)|^2 \, dx + C_1 \int_\Omega (u_m - M)^{2p} dx \le C_3 |\Omega|,$$

integrating it from 0 to T, gives

$$\frac{1}{2}\int_\Omega (u_m - M)^2(T) dx + \int_0^T \int_\Omega |\nabla(u_m - M)|^2 \, dx + C_1 \int_0^T \int_\Omega (u_m - M)^{2p} dx \le \frac{1}{2}\int_\Omega (u_0 - M)^2(x) dx + C_3 |\Omega| T.$$

So

$$\exists K = \frac{1}{2}\int_\Omega (u_0 - M)^2(x) dx + C_3 |\Omega| T,$$

such that

$$\sup_{t\in[0,T]} \left(\int_\Omega (u_m - M)^2(x,t) dx\right) \le 2K, \int_0^T \int_\Omega |\nabla(u_m - M)|^2 \, dxds \le K, \int_0^T \int_\Omega (u_m - M)^{2p} dxds \le K/C_1,$$

so $u_m - M$ is bounded independently of m in $L^\infty(0, T; L^2(\Omega)), L^2(0, T; H^1(\Omega))$ and $L^{2p}(0, T; L^{2p}(\Omega))$.
Hence there exists a subsequence of u_m still denoted u_m such that

$$u_m - M \rightharpoonup u - M \text{ in } L^2(0, T; H^1(\Omega)) \text{ and } L^{2p}(0, T; L^{2p}(\Omega)) \text{ weakly}. \tag{1.4}$$

$$u_m - M \rightharpoonup u - M \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weak star}. \tag{1.5}$$

$$f(u_m) \rightharpoonup \chi \text{ in } L^{(2p)'}(0, T; L^{(2p)'}(\Omega)) \text{ weakly } \left((2p)' = \frac{1}{1 - \frac{1}{2p}}, p \ge 2\right). \tag{1.6}$$

The last result is given by (P_1) and (P_4), where for

$$\frac{1}{2p}+\frac{1}{(2p)'}=1,$$

$$||f(u_m)||^{(2p)'}_{L^{(2p)'}(0,T;L^{(2p)'}(\Omega))}=\int_0^T\int_\Omega |f(u_m)|^{(2p)'}dxds\leq$$
$$\leq\int_0^T\left(C_7\int_\Omega(|u_m-M|^{2p-1}+1)dx\right)^{(2p)'}ds\leq C\int_0^T\int_\Omega|u_m-M|^{(2p-1)(2p)'}dxds,$$

which means that the bound on u_m-M in $L^{2p}(0,t;L^{2p}(\Omega))$ gives a bound on $f(u_m)$ in $L^{(2p)'}(0,t;L^{(2p)'}(\Omega))$.

Thus passing to the limits in (1.1) and (1.2) we find

$$((u-M)_t,v)+a((u-M),v)=(\chi,v),\forall v\in H^1(\Omega)\cap L^{2p}(\Omega),$$
$$u(0)=u_0.$$

This shows that

$$\frac{\partial(u-M)}{\partial t}=-A(u-M)+\chi \text{ is in } L^2(0,T;(H^1(\Omega))') \text{ and } L^{(2p)'}(0,t;L^{(2p)'}(\Omega)),$$

where $L^2(0,T;(H^1(\Omega))')$ and $L^{(2p)'}(0,t;L^{(2p)'}(\Omega))$ are in duality with $L^2(0,T;H^1(\Omega))$ and $L^{2p}(0,T;L^{2p}(\Omega))$.
Now we can apply the following theorem

Theorem (Compactness theorem)**.** *Let $X\subset\subset H\subset Y$ be Banach spaces, with X reflexive. Suppose that u_n is a sequence that is uniformly bounded in $L^2(0,T;X)$, and $\frac{du_n}{dt}$ is uniformly bounded in $L^p(0,T;Y)$, for some $p>1$. Then there is a subsequence that converges strongly in $L^2(0,T;H)$.*

So $u-M\in C([0,T];L^2(\Omega))$.
It remains to check that $\chi=f(u)$.
Besides the results (1.4), (1.5) and (1.6), we can state by the theorem 8.1 p214 in [32] that the subsequence u_m-M is relatively compact in $L^2([0,T];L^2(\Omega))$, so there exists a subsequence of u_m-M still denoted u_m-M such that

$$u_m-M\longrightarrow u-M \text{ in } L^2([0,T];L^2(\Omega)),$$

so by corollary 1.2 p27 in [32] we can say that

$$u_m-M\longrightarrow u-M \text{ a.e. in } \Omega\times(0,+\infty).$$

But f is continuous, thus

$$f(u_m)\longrightarrow f(u) \text{ in } \Omega\times(0,+\infty),$$

where $\{f(u_m)\}_{m\in\mathbb{N}}$ is bounded in $L^{(2p)'}(0,T;L^{(2p)'}(\Omega))$, with $(2p)'=\frac{1}{1-\frac{1}{2p}}, p\geq 2$, so applying lemma 8.3 p 218 in [32] we obtain that

$$f(u_m)\longrightarrow f(u) \text{ in } L^{(2p)'}(0,T;L^{(2p)'}(\Omega)),$$

but in (1.6) we had

$$f(u_m)\longrightarrow\chi \text{ in } L^{(2p)'}(0,T;L^{(2p)'}(\Omega)),$$

and by the uniqueness of the weak limit $\chi = f(u)$ a.e. in $\Omega \times (0, +\infty)$.
To check that $u(0) = u_0$, let's choose $\phi \in C^1(0,T; H^1(\Omega) \cap L^{2p}(\Omega))$ with $\phi(T) = 0$ and so $\phi \in L^{2p}(0,T; H^1(\Omega)) \cap L^{2p}(0,T; L^{2p}(\Omega))$, and using

$$(\frac{\partial (u-M)}{\partial t}, v) + a(u-M, v) = (f(u), v), \forall v \in H^1(\Omega) \cap L^{2p}(\Omega),$$

which we integrate by parts in the t variable in $[0,T]$, and for $v = \phi$, we know that

$$\begin{aligned}\int_0^T ((u-M)_t, \phi) ds &= (u-M, \phi)|_0^T - \int_0^T (u-M, \phi') ds \\ &= -(u(0)-M, \phi(0)) - \int_0^T (u-M, \phi') ds,\end{aligned}$$

thus

$$\int_0^T -(u-M, \phi') ds + \int_0^T a(u-M, \phi) ds = \int_0^T (f(u), \phi) ds + (u(0)-M, \phi(0)).$$

Doing the same in (1.1) yields

$$\int_0^T -(u_m-M, \phi') ds + \int_0^T a(u_m-M, \phi) ds = \int_0^T (f(u_m), \phi) ds + (u_m(0)-M, \phi(0)),$$

passing to the limit for m gives

$$\int_0^T -(u-M, \phi') ds + \int_0^T a(u-M, \phi) ds = \int_0^T (f(u), \phi) ds + (u_0-M, \phi(0)).$$

and so $u(0) = u_0$.
To prove the uniqueness of the solution u of problem (P), suppose the existence of two solutions u and $\tilde{u}$ satisfying problem (P), with $u_0 - M \in L^2(\Omega)$ and $\tilde{u_0} - \tilde{M} \in L^2(\Omega)$, then take $\omega = u - \tilde{u}$ where $\int_\Omega \omega dx = \int_\Omega u_0 dx - \int_\Omega \tilde{u_0} dx$ and for $M' = M - \tilde{M}$, then $\int_\Omega \omega dx = M'|\Omega|$.
ω satisfies then

$$\frac{\partial \omega}{\partial t} - \Delta\omega = f(u) - f(\tilde{u}) - \frac{1}{|\Omega|}\left(\int_\Omega f(u) - \int_\Omega f(\tilde{u}) dx\right). \tag{1.7}$$

Multiplying (1.7) by ω and integrating over Ω, then applying the Green's formula with boundary condition, the result will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2 dx = \int_\Omega [f(u) - f(\tilde{u})]\omega dx - \frac{1}{|\Omega|}\int_\Omega \omega dx \int_\Omega f(u) - f(\tilde{u}) dx,$$

where by (P_2) it will have the form

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2 dx \le C_4 \int_\Omega \omega^2 dx + \frac{1}{|\Omega|}\int_\Omega |\omega| \times \int_\Omega |f(u) - f(\tilde{u})| dx.$$

Again by (P_2)

$$\int_\Omega |f(u) - f(\tilde{u})| dx \le C_4 \int_\Omega |\omega| dx,$$

it will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2 dx \le C_4 \int_\Omega \omega^2 dx + \frac{C_4}{|\Omega|}\left(\int_\Omega |\omega|\right)^2$$

and by Hölder's inequality we will have

$$\left(\int_\Omega |\omega| dx\right)^2 \le |\Omega| \int_\Omega \omega^2 dx,$$

and so

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2 dx \le 2C_4 \int_\Omega \omega^2 dx,$$

that is

$$\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) \le 4C_4 \int_\Omega \omega^2 dx,$$

thus applying the Gronwall's lemma gives

$$\int_\Omega \omega^2 dx \le e^{4C_4 t} \int_\Omega \omega_0^2 dx,$$

and so the uniqueness is given for $u_0 = \tilde{u_0}$
To prove the second part of the theorem let's multiply (1.1) with $\lambda_j d_{jm}$ and summing on $j = 1, \cdots, m$, the result will be

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla(u_m - M)|^2 dx + \int_\Omega |\Delta(u_m - M)|^2 dx = \int_\Omega f'(u_m)|\nabla(u_m - M)|^2 dx \le C_4 \int_\Omega |\nabla(u_m - M)|^2 dx, \tag{1.8}$$

and so

$$\frac{d}{dt}\int_\Omega |\nabla(u_m - M)|^2 dx \le 2C_4 \int_\Omega |\nabla(u_m - M)|^2 dx,$$

and by the Gronwall's lemma the result will be

$$\int_\Omega |\nabla(u_m - M)|^2 dx \le \left(\int_\Omega |\nabla(u_0 - M)|^2 dx\right) e^{2C_4 t}.$$

Again from (1.8) we can assert that

$$\int_0^T \int_\Omega |\Delta(u_m - M)|^2 dx ds \le C_4 \int_0^T \int_\Omega |\nabla(u_m - M)|^2 dx ds + \frac{1}{2}\int_\Omega |\nabla(u_0(x) - M)|^2 dx. \tag{1.9}$$

And if we multiply (1.1) by $(u_{jm} - M)_t$, sum on $j = 1, \cdots, m$

$$\int_\Omega [(u_m - M)_t]^2 dx + \int_\Omega |\nabla(u_m - M)|^2 dx = \int_\Omega (u_m - M)_t f(u_m) dx,$$

where

$$(u_m - M)_t f(u_m) = -F'(u_m),$$

thus
$$\int_\Omega [(u_m - M)_t]^2 dx + \frac{d}{dt}\int_\Omega \left[\frac{1}{2}\left|\nabla(u_m - M)\right|^2 dx + F(u_m)\right] dx = 0,$$
and obviously
$$\frac{d}{dt}\int_\Omega \left[\frac{1}{2}\left|\nabla(u_m - M)\right|^2 + F(u_m)\right] dx \leq 0$$
which we integrate in t and by using (P_3) the result will be
$$\frac{1}{2}\int_\Omega |\nabla(u_m(T) - M)|^2 dx - \frac{1}{2}\int_\Omega |\nabla(u_0 - M)|^2 dx \leq -\int_\Omega F(u_m) dx \leq -C_6\left(\int_\Omega ((u_m - M)^{2p} - 1) dx\right),$$
and so
$$\frac{1}{2}\int_\Omega |\nabla(u_m(T) - M)|^2 dx + C_6\int_\Omega (u_m - M)^{2p} dx \leq \frac{1}{2}\int_\Omega |\nabla(u_0 - M)|^2 dx + C_6|\Omega|.$$

□

1.2 Existence of absorbing sets and of the maximal attractor

1.2.1 Generalities

We assume in the beginning that we can find a phase space H (usually a Hilbert or a Banach space), such that for $u_0 \in H$ the equation has a unique solution $u(t; u_0)$ for all positive times. In this case, we can define a C^0- semigroup of solution operators $S(t) : H \to H$ by $S(t)u_0 = u(t; u_0)$. These enjoy the usual semigroup properties

$$S(0) = I, (I\text{: Identity in } H) \tag{1.10}$$
$$S(t+s) = S(t).S(s) = S(s).S(t), \forall s, t \geq 0 \tag{1.11}$$

And

$$u(t) = S(t)u_0 \tag{1.12}$$
$$u(t+s) = S(t).u(s) = S(s).u(t), \forall s, t \geq 0 \tag{1.13}$$

where we consider the semidynamical system $(H, S(t))_{t\geq 0}$.

We say that an equation is dissipative if all the solutions are bounded, provided that this bound is uniform over all the trajectories.
For u_0, the orbit or trajectory starting at u_0 is the set $\cup_{t\geq 0} S(t)u_0 = \cup_{t\geq 0}\{u(t)\}$. A complete orbit containing u_0 is the union of the positive and negative orbit through u_0.
We say that a set $B \subset H$ is invariant for the semigroup $S(t)$ if

$$S(t)B = B, \forall t > 0.$$

The following definition gives dissipativity for a semigroup by the existence of an absorbing set.

Definition 1.1 (Absorbing set). *Let $\mathcal{B}$ be a subset of H and $\mathcal{U}$ an open set containing $\mathcal{B}$. We say that $\mathcal{B}$ is absorbing in $\mathcal{U}$ if the orbit of any bounded set of $\mathcal{U}$ enters into $\mathcal{B}$ after a certain time (which may depends on the set)*
$$\begin{cases} \forall \mathcal{B}_0 \subset \mathcal{U}, \mathcal{B}_0 \text{ bounded}, \\ \exists t_1(\mathcal{B}_0) \text{ such that } S(t)\mathcal{B}_0 \subset \mathcal{B}, \forall t \geq t_1(\mathcal{B}_0). \end{cases}$$
we say that $\mathcal{B}$ absorbs the bounded sets of $\mathcal{U}$.

An other example of invariant sets is given by ω-limit sets; these sets are also essential in view of the construction of global attractors.

Definition 1.2 (Limit Sets)**.** *The ω-limit set of a set X consists of all limit points of the orbit of X,*

$$\omega(X) = \cap_{s\geq 0}\overline{\cup_{t\geq s} S(t)X}. \tag{1.14}$$

Remark 1.1. The ω-limit sets could be characterized by:
$x \in \omega(X)$ if and only if there exist sequences $(x_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$, with $x_k \in X, \forall k \in \mathbb{N}$, and $t_k \to +\infty$ as $k \to +\infty$, such that $S(t_k)x_k \to x$ as $k \to +\infty$

The construction of attractors will be based on the following result.

Proposition 1.1. *Let $X \subset H$. If, for some $t_0 > 0$, the set*

$$\overline{\cup_{t\geq t_0} S(t)X} \tag{1.15}$$

is compact, then $\omega(X)$ is nonempty, compact, and invariant.

Definition 1.3 (The Global Attractor)**.** *A global (universal, or maximal) attractor is a compact set $\mathcal{A} \subset H$ that enjoys the following properties*

- *i- $\mathcal{A}$ is an invariant set ($S(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$).*
- *ii- $\mathcal{A}$ possesses an open neighborhood $\mathcal{U}$ such that for every u_0 in $\mathcal{U}$, $S(t)u_0$ converges to $\mathcal{A}$ as $t \to \infty$*

$$d(S(t)u_0, \mathcal{A}) \to 0 \text{ as } t \to \infty. \tag{1.16}$$

Of course $d(x, \mathcal{A}) = \inf_{y\in\mathcal{A}} d(x, y)$.

The following result shows that $\omega(B)$ is the global attractor, provided that $S(t)$ is dissipative and B is an absorbing set. But $\omega(B)$ is already nonempty, compact, and invariant, so it will remain to show that it attracts trajectories as in (1.16).

Theorem 1.2. *If $S(t)$ is dissipative and B is a compact absorbing set then there exists a global attractor $\mathcal{A} = \omega(B)$. If H is connected then so is $\mathcal{A}$.*

1.2.2 The existence result

The following result ensures the existence of a maximal attractor, which is a suitable set for the study of the asymptotic behavior of the problem in hand.

Theorem 1.3. *With properties (P_1) and (P_2), the semigroup $S(t)$ associated to problem (P) is such that*

- *i- There exist absorbing sets in $L^2(\Omega)$ and $H^1(\Omega) \cap L^{2p}(\Omega)$.*
- *ii- There exists a maximal attractor $\mathcal{A}$ which is bounded in $H^1(\Omega) \cap L^{2p}(\Omega)$, compact and connected in $L^2(\Omega)$.*

Proof. i- First we will show that there exists an absorbing set in $L^2(\Omega)$ and in $H^1(\Omega) \cap L^{2p}(\Omega)$.

i1- Existence of an absorbing set in $L^2(\Omega)$

Recall that we had

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u-M)^2\right)+\int_\Omega |\nabla(u-M)|^2\,dx + C_1\int_\Omega (u-M)^{2p}dx \le C_3|\Omega|,$$

and so

$$\frac{d}{dt}\left(\int_\Omega (u-M)^2\right)+2\int_\Omega (|\nabla(u-M)|^2+(u-M)^2)dx+2C_1\int_\Omega (u-M)^{2p}dx \le 2\int_\Omega (u-M)^2dx+2C_3|\Omega|.$$

Remark that thanks to the Hölder's inequality for $\frac{1}{p}+\frac{1}{q}=1$

$$\int_\Omega (u-M)^2dx = \int_\Omega \left|((u-M)^{2p})^{\frac{1}{p}}\right| dx \le \left(\int_\Omega \left|(((u-M)^{2p})^{1/p})^p\right| dx\right)^{1/p}.\left(\int_\Omega dx\right)^{1/q} =$$

$$= \left(\int_\Omega (u-M)^{2p}dx\right)^{1/p}.\,|\Omega|^{1/q}$$

But

$$\left(\int_\Omega (u-M)^{2p}dx\right)^{1/p}.\,|\Omega|^{1/q} = \left(\frac{2}{C_1}\right)^{1/p}\left(\frac{C_1}{2}\int_\Omega (u-M)^{2p}dx\right)^{1/p}.\,|\Omega|^{1/q}$$

thus

$$\int_\Omega (u-M)^2dx \le \left(\frac{2}{C_1}\right)^{1/p}|\Omega|^{1/q}.\left(\frac{C_1}{2}\int_\Omega (u-M)^{2p}dx\right)^{1/p}.$$

And by the Young's inequality for $\frac{1}{p}+\frac{1}{q}=1$ we remark that

$$\left[\left(\frac{2}{C_1}\right)^{1/p}|\Omega|^{1/q}\right]\left[\frac{C_1}{2}\int_\Omega (u-M)^{2p}dx\right]^{1/p} \le \frac{1}{p}\left[\left(\frac{C_1}{2}\int_\Omega (u-M)^{2p}dx\right)^{1/p}\right]^p+\frac{1}{q}\left[\left(\frac{2}{C_1}\right)^{1/p}|\Omega|^{1/q}\right]^q =$$

$$= \frac{1}{q}\left(\frac{2}{C_1}\right)^{p/q}|\Omega|+\frac{1}{p}\frac{C_1}{2}\int_\Omega (u-M)^{2p}dx,$$

so for $1\le p\le +\infty$

$$\int_\Omega (u-M)^2dx \le \frac{1}{p}\frac{C_1}{2}\int_\Omega (u-M)^{2p}dx+\frac{1}{q}\left(\frac{2}{C_1}\right)^{p/q}|\Omega| \le \frac{C_1}{2}\int_\Omega (u-M)^{2p}dx+\frac{1}{q}\left(\frac{2}{C_1}\right)^{p/q}|\Omega|$$

thus for $C_8=\frac{1}{q}\left(\frac{2}{C_1}\right)^{p/q}$

$$\int_\Omega (u-M)^2dx \le C_8\,|\Omega|+\frac{C_1}{2}\int_\Omega (u-M)^{2p}dx,$$

so

$$\frac{d}{dt}\int_\Omega (u-M)^2dx+2\int_\Omega \left(|\nabla(u-M)|^2+(u-M)^2\right)dx+2C_1\int_\Omega (u-M)^{2p}dx \le 2(C_3+C_8)\,|\Omega|+$$

$$+\,C_1\int_\Omega (u-M)^{2p}dx,$$

or

$$\frac{d}{dt}\int_\Omega (u-M)^2dx+2\int_\Omega \left(|\nabla(u-M)|^2+(u-M)^2\right)dx+C_1\int_\Omega (u-M)^{2p}dx \le 2(C_3+C_8)\,|\Omega| = C_9, \tag{1.17}$$

and by applying the Gronwall's lemma to

$$\frac{d}{dt}\int_\Omega (u-M)^2dx \le -2\int_\Omega (u-M)^2dx + C_9$$

we can say that

$$\int_\Omega (u-M)^2dx \le \left(\int_\Omega (u_0-M)^2dx\right)e^{-2t} + \frac{1}{2}C_9(1-e^{-2t}). \tag{1.18}$$

Thus

$$\sup_t \int_\Omega (u-M)^2dx \le \int_\Omega (u_0-M)^2dx + \frac{1}{2}C_9$$

and

$$\lim_{t\to\infty}\sup_t \int_\Omega (u-M)^2dx \le \rho_1^2 \text{ where } \rho_1^2 = \frac{1}{2}C_9.$$

We deduce from (1.18) that any ball of $L^2(\Omega)$ centered at 0 and of radius $\rho_2 > \rho_1 = \sqrt{\frac{1}{2}C_9}$ is an absorbing set in $L^2(\Omega)$. Indeed if $\mathcal{B}_0$ is a bounded set of $L^2(\Omega)$, included in a ball $B(0,R)$ of $L^2(\Omega)$ centered at 0 of radius R, then $S(t)\mathcal{B}_0 \subset B(0,\rho_2)$ for $t \ge t_0 = t_0(\mathcal{B}), t_0 = \frac{1}{2}\ln\left(\frac{R^2}{\rho_2^2-\rho_1^2}\right)$, because
for $\mathcal{B}_0 \subset L^2(\Omega)$ a bounded set, we want to find $t_0 = t_0(\mathcal{B}_0)$ such that

$$\forall u_0 - M \in \mathcal{B}_0 : \int_\Omega (u-M)^2dx \le \rho_2^2, \forall t \ge t_0,$$

and the boundedness of $\mathcal{B}_0$ gives the existence of $R > 0$ such that $\mathcal{B}_0 \subset B(0,R)$, where for $u_0 - M \in \mathcal{B}_0$

$$\int_\Omega (u-M)^2dx \le R^2e^{-2t} + \rho_1^2,$$

and we want that

$$\int_\Omega (u-M)^2dx \le \rho_2^2,$$

thus we must have

$$R^2e^{-2t} + \rho_1^2 \le \rho_2^2 \text{ with } \rho_1 < \rho_2,$$

which gives $t_0 = \frac{1}{2}\ln\left(\frac{R^2}{\rho_2^2-\rho_1^2}\right)$.
We conclude that the set $\mathcal{B} = B(0,\rho_2) = \{u - M \in L^2(\Omega), \int_\Omega (u-M)^2dx \le \rho_2^2, \rho_2 > \rho_1\}$ is an absorbing set in $L^2(\Omega)$.
From (1.17) we had

$$\frac{d}{dt}\int_\Omega (u-M)^2dx + 2\int_\Omega |\nabla(u-M)|^2dx + C_1\int_\Omega (u-M)^{2p}dx \le C_9,$$

thus

$$\int_t^{t+r}\frac{d}{ds}\left(\int_\Omega (u-M)^2dx\right)ds+2\int_t^{t+r}\int_\Omega |\nabla(u-M)|^2\,dxds+C_1\int_t^{t+r}\int_\Omega (u-M)^{2p}dxds\leq rC_9,$$

where

$$\int_t^{t+r}\frac{d}{ds}\left(\int_\Omega (u-M)^2dx\right)ds=\int_\Omega (u-M)^2(x,t+r)dx-\int_\Omega (u-M)^2(x,t)dx$$

then

$$\int_\Omega (u-M)^2(x,t+r)dx+2\int_t^{t+r}\int_\Omega |\nabla(u-M)|^2\,dxds+C_1\int_t^{t+r}\int_\Omega (u-M)^{2p}dxds\leq$$
$$\leq rC_9+\int_\Omega (u-M)^2(x,t)dx$$

for $u_0-M\in\mathcal{B}'\subset B(0,R)$ and $t\geq t_0$

$$2\int_t^{t+r}\int_\Omega |\nabla(u-M)|^2\,dxds+C_1\int_t^{t+r}\int_\Omega (u-M)^{2p}dxds\leq rC_9+\rho_2^2. \tag{1.19}$$

i2- Existence of an absorbing set in $H^1(\Omega)\cap L^{2p}(\Omega)$.
We equip $H^1(\Omega)\cap L^{2p}(\Omega)$ with the supremum of the norm of $H^1(\Omega)$ and of $L^{2p}(\Omega)$.
Let's multiply the following equation by $\dfrac{\partial(u-M)}{\partial t}$

$$\frac{\partial u}{\partial t}-\Delta u=f(u)-\frac{1}{|\Omega|}\int_\Omega f(u)dx,$$

after integration in the space, using the mass conservation property and applying of the Green's formula, the resulting equation will be

$$\int_\Omega\left(\frac{d(u-M)}{dt}\right)^2dx+\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\nabla(u-M)|^2\,dx\right)=\int_\Omega\frac{d(u-M)}{dt}f(u)dx=-\frac{d(u-M)}{dt}\left(\int_\Omega F(u)dx\right)$$

or

$$\int_\Omega\left(\frac{d(u-M)}{dt}\right)^2dx+\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\nabla(u-M)|^2\,dx\right)+\frac{d(u-M)}{dt}\left(\int_\Omega F(u)dx\right)=0,$$

so

$$\frac{d}{dt}\int_\Omega\left[\frac{1}{2}|\nabla(u-M)|^2+F(u)\right]\leq 0,$$

and after integration in time it will be

$$\int_\Omega |\nabla(u-M)(x,t)|^2\,dx+2\int_\Omega F(u(x,t))dx\leq\int_\Omega |\nabla(u_0(x)-M)|^2\,dx+2\int_\Omega F(u_0)dx,$$

and thanks to property (P_3) we will have

$$\int_\Omega |\nabla(u(x,t)-M)|^2\,dx+2C_6\int_\Omega (u-M)^{2p}(x,t)dx\leq\int_\Omega |\nabla(u_0(x)-M)|^2\,dx+$$
$$+2C_6\int_\Omega (u_0-M)^{2p}(x)dx+2\,|\Omega|\,(C_6+C_5),$$

so

$$\sup_{t\in[0,T]}\left(\int_\Omega |\nabla(u(x,t)-M)|^2\,dx + 2C_6\int_\Omega (u-M)^{2p}(x,t)dx\right)\le K',$$

for

$$K' = \int_\Omega |\nabla(u_0(x)-M)|^2\,dx + 2C_6\int_\Omega (u_0-M)^{2p}(x)dx + 2\,|\Omega|\,(C_6+C_5)$$

which will give us an estimation of $u-M$ in $L^\infty(0,T;H^1(\Omega)\cap L^{2p}(\Omega))$, and with application of the uniform Gronwall's lemma we can give to it an $L^\infty(\mathbb{R}^+,H^1(\Omega)\cap L^{2p}(\Omega))$ estimation

Lemma (Uniform Growall lemma)**.** *Let g,h,y be three locally integrable functions on $]t_0,+\infty[$ satisfying*

$$\frac{dy}{dt}\in L^1_{loc}\left(]t_0,+\infty[\right)\ \ \textit{and}\ \frac{dy}{dt}\le gy+h\ \textit{for}\ t\ge t_0,$$

$$\int_t^{t+r} g(s)ds\le a_1,\int_t^{t+r} h(s)ds\le a_2,\int_t^{t+r} y(s)ds\le a_3,\ \ \textit{for}\ t\ge t_0,$$

where a_1,a_2,a_3 and r are positive constants. Then

$$y(t+r)\le\left(\frac{a_3}{r}+a_2\right)e^{a_1},\forall t\ge t_0.$$

Remark that from (1.19) we can state that there exists a constant $a_3>0$ and for $t\ge t_0, u_0-M\in\mathcal{B}\subset B(0,R)$

$$\int_t^{t+r}\int_\Omega |\nabla(u-M)|^2\,dxds + \int_t^{t+r}\int_\Omega (u-M)^{2p}dxds\le a_3,$$

and if we set $y(t)=\int_\Omega\left[|\nabla(u-M)|^2+(u-M)^{2p}\right]dx$ then

$$\frac{d}{ds}y(s)ds\le 0.y(t)+0$$

thus by applying the uniform Gronwall's lemma we will have

$$\int_\Omega\left[|\nabla(u-M)|^2+(u-M)^{2p}\right]dx\le\left(\frac{a_3}{r}+0\right)e^0=\frac{a_3}{r}.$$

Thus the bounded set $\mathcal{B}_1=B\left(0,\left(\frac{a_3}{r}\right)^{1/2}\right)$ is an absorbing set in $H^1(\Omega)\cap L^{2p}(\Omega))$, and is relatively compact in $L^2(\Omega)$, because
for $u_0-M\in L^2(\Omega)$ where $u_0-M\in B(0,R_0)\subset B(0,\rho_2)$. Let $t_1=t_1(R_0)>0$ be such that

$$S(t)B(0,R_0)\subset B(0,\rho_2)\text{ for } t\ge t_1(R_0),$$

it means that $t\ge t_1$ implies that $\left(\int_\Omega (u-M)^2dx\right)^{1/2}\le\rho_2$. And by the uniform Gronwall's lemma

$$\int_\Omega\left(|\nabla(u-M)|^2+(u-M)^{2p}\right)dx\le\frac{a_3}{r},\forall t\ge t_1(R_0)+r.$$

Thus if $u(t)-M \in H^1(\Omega)\cap L^{2p}(\Omega), \forall t \geq t_1+r$, while $u_0-M \in L^2(\Omega)$ and $u_0-M \in B(0,R_0)$ then

$$\int_\Omega \left(|\nabla(u-M)|^2 + (u-M)^{2p}\right) dx \leq \frac{a_3}{r}, \forall t \geq t_1+r.$$

So for $t \geq t_1+r$, $u(t)-M$ is in $B\left(0,\left(\frac{a_3}{r}\right)^{1/2}\right)$ a bounded set in $H^1(\Omega)\cap L^{2p}(\Omega)$. Then for $t \geq t_1+r$, $S(t)$ transforms the bounded sets on $L^2(\Omega)$ in a bounded sets in $H^1(\Omega)\cap L^{2p}(\Omega)$, and knowing that $H^1(\Omega)\cap L^{2p}(\Omega)$ is compactly imbedded in $L^2(\Omega)$, these bounded sets are relatively compact in $L^2(\Omega)$.

ii- We proved in (i2) the existence of an absorbing set in $H^1(\Omega)\cap L^{2p}(\Omega)$, relatively compact in $L^2(\Omega)$, which is connected thus the global attractor is $\mathcal{A} = \omega\left(B\left(0,\left(\frac{a_3}{r}\right)^{1/2}\right)\right)$.

□

1.3 Regularity of the attractor

We have shown that the global attractor $\mathcal{A}$ is a bounded subset of $L^2(\Omega)$ and $H^1(\Omega)\cap L^{2p}(\Omega)$. Here we will prove that it is bounded in both $L^\infty(\Omega)$ and $H^2(\Omega)$.
For $u \in L^2(\Omega)$, we define

$$u_+(x) = \begin{cases} u(x) & \text{if } u(x) > 0 \\ 0 & \text{otherwise} \end{cases}, \qquad u_-(x) = \begin{cases} u(x) & \text{if } u(x) < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Clearly if $u \in L^2(\Omega)$, then so are u_+ and u_-, with

$$\int_\Omega u_+^2 dx \leq \int_\Omega u^2 dx, \text{ and } \int_\Omega u_-^2 dx \leq \int_\Omega u^2 dx.$$

Furthermore, if $u \in H^1(\Omega)$ then so are u_+ and u_- as given in the following result

Lemma 1.1. *If $u \in H^1(\Omega)$ then so are u_+ and u_- with*

$$\int_\Omega |\nabla u_+|^2 dx \leq \int_\Omega |\nabla u|^2 dx, \text{ and } \int_\Omega |\nabla u_-|^2 dx \leq \int_\Omega |\nabla u|^2 dx.$$

In fact

$$\nabla u_+(x) = \begin{cases} \nabla u(x) & \text{if } u(x) > 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \nabla u_-(x) = \begin{cases} \nabla u(x) & \text{if } u(x) < 0 \\ 0 & \text{otherwise} \end{cases}.$$

It follows immediately that

$$(Au, u_+) = a(u, u_+) = (\nabla u, \nabla u_+) = \int_\Omega |\nabla u_+|^2 dx.$$

Theorem 1.4. *The global attractor $\mathcal{A}$ is uniformly bounded in $L^\infty(\Omega)$, with*

$$||u-M||_\infty \leq \left(\frac{C_3}{C_1}\right)^{1/(2p)} \quad \text{for all } u-M \in \mathcal{A}.$$

Proof. From property (P_1) we had

$$f(s)(s-M) \leq -C_1(s-M)^{2p} + C_3, p \geq 2.$$

It follows that

$$f(s) \leq 0 \text{ when } s - M \geq \left(\frac{C_3}{C_1}\right)^{1/(2p)} \geq 0, \tag{1.20}$$

put then $\Pi = \left(\frac{C_3}{C_1}\right)^{1/(2p)}$.
Let's multiply our problem

$$u_t - \Delta u = f(u(x,t)) - \frac{1}{|\Omega|}\int_\Omega f(u(x,t))dx$$

by $((u-M)-\Pi)_+$, integrate it over Ω then using the Green's formula with the boundary condition, the result will be

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int_\Omega ((u-M)-\Pi)_+^2 dx\right) + \int_\Omega |\nabla((u-M)-\Pi)_+|^2 dx &= \int_\Omega ((u-M)-\Pi)_+ f(u)dx - \\ &- \frac{1}{|\Omega|}\int_\Omega ((u-M)-\Pi)_+ dx \int_\Omega f(u)dx = \\ &= \int_\Omega ((u-M)-\Pi)_+ f(u)dx + \Pi\int_\Omega f(u)dx \leq 0\end{aligned}$$

and so we can say that it exists a positive constant C such that

$$\frac{d}{dt}\left(\int_\Omega ((u-M)-\Pi)_+^2 dx\right) \leq -C\int_\Omega ((u-M)-\Pi)_+^2 dx,$$

an application of the Gronwall's inequality will give then

$$\int_\Omega ((u-M)-\Pi)_+^2 dx \leq e^{-Ct}\int_\Omega ((u_0(x)-M)-\Pi)_+^2 dx,$$

since the attractor is bounded in $L^2(\Omega)$ for any $v \in \mathcal{A}$, there exists u_0 such that $v = S(t)(u_0 - M)$, we have

$$\int_\Omega ((u-M)-\Pi)_+^2 dx = 0, \text{ for all } u - M \in \mathcal{A}.$$

Thus

$$||u-M||_\infty \leq \Pi \text{ for all } u - M \in \mathcal{A}.$$

In a similar manner, we can prove that

$$||u-M||_\infty \geq -\Pi \text{ for all } u - M \in \mathcal{A},$$

using $((u-M)-\Pi)_-$.

□

We will use bellow the L^∞ bound, to deduce that the attractor is bounded in $H^2(\Omega)$. To do that we will use the equation

$$\frac{du}{dt} + Au = f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)dx. \tag{1.21}$$

Multiply (1.21) by u_t and then integrate the result in Ω yields

$$\int_\Omega u_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_\Omega a(u,u)dx = \int_\Omega u_t f(u)dx = -\frac{d}{dt}\int_\Omega F(u)dx,$$

which we integrate from 0 to t to give

$$\int_0^t\int_\Omega u_s^2 dxds + \frac{1}{2}\int_\Omega a(u(x,t),u(x,t))dx = \frac{1}{2}\int_\Omega a(u_0(x),u_0(x))dx - \int_\Omega F(u(x,t))dx + \int_\Omega F(u_0(x))dx,$$

but $\mathcal{A}$ is bounded in $H^1(\Omega)\cap L^{2p}(\Omega)$ and in $L^\infty(\Omega)$, this gives for some $\kappa = \frac{1}{2}\int_\Omega a(u_0(x),u_0(x))dx + C_5\int_\Omega (u_0(x)-M)^{2p}dx + C_5|\Omega|$

$$\int_0^t\int_\Omega u_s^2 dxds + \frac{1}{2}\int_\Omega a(u(x,t),u(x,t))dx \le \kappa. \tag{1.22}$$

Now we will obtain a bound on u_t in $L^2(\Omega)$. For that we differentiate (1.21) to give

$$\frac{d}{dt}u_t + Au_t = f'(u)u_t - \frac{1}{|\Omega|}\int_\Omega f'(u)u_t dx,$$

and take the inner product with t^2u_t to have

$$\begin{aligned} t^2(u_t,\partial_t u_t) + t^2(u_t, Au_t) &= t^2(u_t, f'(u)u_t) - t^2(u_t, \frac{1}{|\Omega|}\int_\Omega f'(u)u_t dx) = \\ &= t^2(u_t, f'(u)u_t) - t^2\frac{1}{|\Omega|}\int_\Omega f'(u)u_t dx(u_t,1) = \\ &= t^2(u_t, f'(u)u_t) \text{ (because the mass conservation yields } (u_t,1)=0) \end{aligned}$$

and by property (P_2), we know that it exists $C_4 > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (tu_t)^2 dx\right) - t\int_\Omega u_t^2 dx + t^2\int_\Omega |\nabla u_t|^2 dx \le t^2 C_4\int_\Omega u_t^2 dx,$$

integrating between 0 and t gives

$$\frac{1}{2}\int_0^t \frac{d}{ds}\left(\int_\Omega (su_s)^2 dx\right)ds - \int_0^t\left(s\int_\Omega u_s^2 dx\right)ds + \int_0^t\left(s^2\int_\Omega |\nabla u_s|^2 dxds\right) \le C_4\int_0^t\left(s^2\int_\Omega u_s^2 dx\right)ds$$

and so

$$\frac{1}{2}\int_\Omega (tu_t)^2 dx + \int_0^t\left(s^2\int_\Omega |\nabla u_s|^2 dx\right)ds \le \int_0^t (s+C_4s^2)\left(\int_\Omega u_s^2 dx\right)ds$$

remark that $s + C_4s^2$ is bounded on $[0,1]$, we obtain then setting $t=1$

$$\frac{1}{2}\int_\Omega (u_t(1))^2 dx \le (1+C_4)\int_0^1\int_\Omega u_s^2 dxds$$

and by (1.22) it will be

$$\frac{1}{2}\int_{\Omega}(u_t(1))^2dx \leq (1+C_4)\kappa, \tag{1.23}$$

which is an $L^2(\Omega)$ bound on $u_t(1)$, uniform over all the attractor.
Since any $u-M \in \mathcal{A}$ is given as $S(1)v$ for some $v \in \mathcal{A}$, it follows that $f(u) - \frac{1}{|\Omega|}\int_{\Omega} f(u)dx - \frac{du}{dt}$ is uniformly bounded in $L^2(\Omega)$ over all of $\mathcal{A}$. And since Au is uniformly bounded in $L^2(\Omega)$ it follows that u is uniformly bounded in $H^2(\Omega)$.
We can now give and prove the following result

Theorem 1.5. *The global attractor $\mathcal{A}$ is bounded in $H^2(\Omega)$.*

Proof. We know that

$$Au = -\frac{du}{dt} + f(u) - \frac{1}{|\Omega|}\int_{\Omega} f(u)dx \tag{1.24}$$

holds for a.e. t along a trajectory. And since $u-M \in L^{\infty}(\Omega)$, so is $f(u) - \frac{1}{|\Omega|}\int_{\Omega} f(u)dx$, and thus $f(u) - \frac{1}{|\Omega|}\int_{\Omega} f(u)dx \in L^{\infty}(0,T;L^2(\Omega))$. And by the bound for u_t obtained in (1.23) we can say by (1.24), that $u(t)-M \in L^{\infty}(0,T;H^1(\Omega))$.
Since the trajectory is continuous into $L^2(\Omega)$, the bound on $|Au(t)|_{L^2(\Omega)}$ is uniform for all $t \in (0,T)$, and so the attractor is uniformly bounded in $H^2(\Omega)$. □

2 Existence of exponential attractor

By its definition an exponential attractor is an exponentially attracting compact set, with finite fractal dimension, which in contrast to a global attractor enjoys a uniform exponential rate of convergence of the solutions to it once the solution is inside an invariant absorbing set. Because of this, exponential attractors possess a deeper and more practical property; they remain more robust under perturbations and numerical approximations than global attractors.

2.1 Elementary notions

Let X be a compact subset of H, the fractal dimension of X is the following number

$$\dim_f X = \limsup_{\xi \to 0^+} \frac{\ln N_{\xi}(X)}{\ln \frac{1}{\xi}},$$

where $N_{\xi}(X)$ is the minimal number of balls of radius ξ in H which are necessary to cover X. In particular, if

$$N_{\xi}(X) \leq c\left(\frac{1}{\xi}\right)^d,$$

where c is independent of ξ, then

$$\dim_f X \leq d.$$

Definition 2.1 (Exponential Attractor)**.** *A compact set $\mathcal{M} \subset H$ is called an exponential attractor or an inertial set for $S(t)$ if it* *i- has finite fractal dimension* $dim_F \mathcal{M}$,

ii- is positively invariant, that is

$$S(t)\mathcal{M} \subseteq \mathcal{M}, \forall t \geq 0,$$

iii- attracts exponentially the bounded subsets of H *in the following sense*

$$\forall B \subset H, bounded, \exists c_0(B) > 0, \exists c_1(B) > 0, \text{ such that } dist(S(t)B, \mathcal{M}) \leq c_0 e^{-c_1 t}, t \geq 0$$

Remark 2.1. It follows from the definition that an exponential attractor, if it exists, always contains the global attractor and the existence of an exponential attractor actually yields the existence of a finite-dimensional global attractor (this follows from the continuity of the semigroup and the fact that an exponential attractor is a compact attracting set).

The first construction of exponential attractors, is due to Eden, Foias, Nocolaenko, and Temam [18], it consists in a way in constructing a fractal expansion of the global attractor $\mathcal{A}$. Where we consider an iterative process in which one adds, at each step, a cloud of points around the global attractor. But at each step the control of the dimension of this new cloud of points around the global attractor is needed, and also the new set must remain positively invariant, without increasing its dimension. The key idea of such a process is the so-called *squeezing property* which says, that either the higher modes are dominated by the lower ones or that the flow is contracted exponentially.

Definition 2.2 (The squeezing property). *A mapping* $S : B \longrightarrow B$, *where* B *is a compact subset of* H, *enjoys the squeezing property on* B *if, for some* $\delta \in (0, \frac{1}{4})$, *there exists an orthogonal projection* $P = P(\delta)$ *with finite rank such that, for every* $u, v \in B$, *either*

$$\|(I - P)(Su - Sv)\|_H \leq \|P(Su - Sv)\|_H$$

or

$$\|Su - Sv\|_H \leq \delta \|u - v\|_H$$

Remark 2.2. We can note that this property uses essentially orthogonal projectors with finite rank, so that the corresponding construction is valid only in Hilbert spaces.

Theorem 2.1 (Existence of exponential attractor). *If* $(\{S(t)\}_{t\geq 0}B)$ *satisfies the squeezing property on* B *and if* $S_* = S(t_*)$ *is Lipschitz on* B *with a Lipschitz constant* L *then, there exists an inertial set* $\mathcal{M}$ *for* $(\{S(t)\}_{t\geq 0}B)$ *such that*

$$d_f(\mathcal{M}) \leq N_0 \max\{1, \ln(16L + 1)/\ln 2\},$$

and

$$dist(S(t)u_0, \mathcal{M}) \leq c_0 e^{-(c_1/t_*)t}.$$

2.2 Existence of an exponential attractor

We want to show the existence of an inertial set to problem (P), let's recall it

$$(P) \begin{cases} u_t = \Delta u + f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)dx & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

to do that it is sufficient to show that our semigroup $S(t)$ satisfies the **squeezing property**.
We will consider the eigenvalues $\lambda_i, \forall i \in \mathbb{N}$ and eigenfunctions ω_i of the operator $-\Delta$ early taken, and take $H_n = span\{\omega_1, \cdots, \omega_n\}$ and the operator $P_n : (H^1(\Omega))' \to H_n$ which is an orthogonal projection and $Q_n = I - P_n$, where I is the identity on $(H^1(\Omega))'$.
We know from the study of the global attractor, that there exists a time $t_0 = t_0(\mathcal{B}_1)$ such that the set

$$B = \overline{\cup_{t \geq t_0} S(t)\mathcal{B}_1},$$

is compact invariant, where $\mathcal{B}_1$ is an absorbing set in $L^2(\Omega)$ and in $H^1(\Omega) \cap L^{2p}(\Omega)$, thus we will take $S(t) : B \to B$.
We know that for $\omega = u - \bar{u}$ our problem will be

$$\begin{cases} \omega_t = \Delta\omega + f(u) - f(\bar{u}) - \frac{1}{|\Omega|}\int_\Omega (f(u) - f(\bar{u}))\,dx & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu \omega = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \omega(x, 0) = u_0(x) - \bar{u}_0(x) & x \in \Omega, \end{cases}$$

we will try to answer to the following question
What is the right projection P_{N_0} that guarantees the **squeezing property**?
Multiplying our first equation by ω and integrating it over Ω the result will be

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2\,dx &= \int_\Omega \omega\,(f(u) - f(\bar{u}))\,dx - \frac{1}{|\Omega|}\int_\Omega \omega dx \times \int_\Omega \omega\,(f(u) - f(\bar{u}))\,dx \leq \\ &\leq \int_\Omega \omega\,|f(u) - f(\bar{u})|\,dx + \frac{1}{|\Omega|}\int_\Omega |\omega|\,dx \times \int_\Omega |f(u) - f(\bar{u})|\,dx, \end{aligned}$$

and by (P_4) we know that $\exists C_4 > 0$ such that $f'(s) \leq C_4$ thus

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2\,dx \leq C_4\int_\Omega \omega^2 dx + \frac{C_4}{|\Omega|}\left(\int_\Omega |\omega|\,dx\right)^2.$$

But

$$\left(\int_\Omega |\omega|\,dx\right)^2 \leq |\Omega|\int_\Omega \omega^2 dx,$$

and so

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2\,dx \leq 2C_4\int_\Omega \omega^2 dx$$

that is

$$\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) \leq 4C_4\int_\Omega \omega^2 dx$$

where by applying the Gronwall's Lemma the result will be

$$\int_\Omega \omega^2 \leq \left(\int_\Omega \omega_0^2\right) e^{4C_4 t},$$

thus $Lip_X(S(t)) \leq e^{4C_4 t}$.
Assume t_* has already been given and prove the **squeezing property**, that is, $\forall\delta, \exists N_0 = N_0(\delta)$ such that for u and $\bar{u}$ in B with $S_* = S(t_*)$, if

$$|Q_{N_0}(S_*u - S_*\bar{u})\,| > |P_{N_0}(S_*u - S_*\bar{u})| \;\text{ this gives? }\; |S_*u - S_*\bar{u}| < \delta\,|u - \bar{u}|\,.$$

For that, consider $\omega_* = S_*u - S_*\bar{u}$ and put $\lambda_* = \frac{\|\omega_*\|^2}{|\omega_*|^2}$, where $||\omega_*||^2 = \int_\Omega |\nabla\omega_*|^2 dx$, $|\omega_*|^2 = \int_\Omega \omega_*^2 dx$, then

$$\lambda_* = \frac{\|\omega_*\|^2}{|\omega_*|^2} = \frac{\|P_{N_0}\omega_* + Q_{N_0}\omega_*\|^2}{|P_{N_0}\omega_* + Q_{N_0}\omega_*|^2} = \frac{\|P_{N_0}\omega_*\|^2 + \|Q_{N_0}\omega_*\|^2}{|P_{N_0}\omega_*|^2 + |Q_{N_0}\omega_*|^2},$$

but we assumed from the **squeezing property** that

$$|Q_{N_0}\omega_*|^2 > |P_{N_0}\omega_*|^2 \Rightarrow 2\,|Q_{N_0}\omega_*|^2 > |P_{N_0}\omega_*|^2 + |Q_{N_0}\omega_*|^2\,,$$

thus

$$\lambda_* > \frac{\|Q_{N_0}\omega_*\|^2}{2\,|Q_{N_0}\omega_*|^2},$$

and

$$\|Q_{N_0}\omega_*\|^2 = \left|A^{1/2}Q_{N_0}\omega_*\right|^2 = (A^{1/2}Q_{N_0}\omega_*, A^{1/2}Q_{N_0}\omega_*) = (AQ_{N_0}\omega_*, Q_{N_0}\omega_*),$$

for λ_{N_0+1} the smallest eigenvalue of A over $Q_{N_0+1}\left(H^1(\Omega) \cap L^{2p}(\Omega)\right)$ we will have

$$\|Q_{N_0}\omega_*\|^2 = (AQ_{N_0}\omega_*, Q_{N_0}\omega_*) \geq (\lambda_{N_0+1}Q_{N_0}\omega_*, Q_{N_0}\omega_*) = \lambda_{N_0+1}\,|Q_{N_0}\omega_*|^2\,,$$

so

$$\lambda_* > \frac{\|Q_{N_0}\omega_*\|^2}{2\,|Q_{N_0}\omega_*|^2} \geq \frac{1}{2}\lambda_{N_0+1},$$

then

$$\lambda_* > \frac{1}{2}\lambda_{N_0+1}.$$

And to prove the **squeezing property** it will be sufficient to prove that if $\lambda_* > \frac{1}{2}\lambda_{N_0+1}$ then $|\omega_*| < \delta\,|u - \bar{u}|$.
But

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \frac{1}{2}\int_\Omega |\nabla\omega|^2\,dx \leq \frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2\,dx \leq 2C_4\int_\Omega \omega^2 dx$$

so

$$\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2\,dx - 4C_4\int_\Omega \omega^2 dx \leq 0$$

or

$$\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \left[\frac{\int_\Omega |\nabla\omega|^2\,dx}{\int_\Omega \omega^2 dx} - 4C_4\right]\int_\Omega \omega^2 dx \leq 0,$$

thus if we put $\lambda(t) = \frac{\int_\Omega |\nabla\omega|^2 dx}{\int_\Omega \omega^2 dx}$ and $\xi(t) = \frac{\omega(t)}{\sqrt{\int_\Omega \omega^2 dx}}$ we can state that

$$\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) \leq -\left(\lambda(t) - 4C_4\right)\int_\Omega \omega^2 dx,$$

and if we apply the Gronwall's Lemma we will have

$$\int_\Omega \omega^2 dx \leq \left(\int_\Omega \omega_0^2 dx\right) e^{\left[4C_4 t - \int_0^t \lambda(s)ds\right]}$$

and so for $t = t_*$

$$\int_\Omega \omega_*^2 dx \leq \left(\int_\Omega \omega_0^2 dx\right) e^{\left[4C_4 t_* - \int_0^{t_*} \lambda(s)ds\right]}.$$

Thus the last result will have the form

$$|\omega_*| = |S_* u - S_* \bar{u}| \leq \delta(t_*) |\omega_0| = \delta(t_*) |u_0 - \bar{u}_0| ,$$

and we want to have $0 < \delta = \delta(t_*) = e^{\left[2C_4 t_* - \frac{1}{2}\int_0^{t_*} \lambda(s)ds\right]} < \frac{1}{8}$.
At this stage we only know that $\lambda_* = \lambda(t_*) > \frac{1}{2}\lambda_{N_0+1}$ and that $\lim_{N_0 \to +\infty} \lambda_{N_0+1} = +\infty$, but the past behavior of the quotient norm $\lambda(s)$ for $s < t_*$ is not known, which we will state by the following result

Proposition 2.1. *Let* $\lambda(t) = \frac{\int_\Omega |\nabla \omega|^2 dx}{\int_\Omega \omega^2 dx} = \frac{\|\omega\|^2}{|\omega|^2}$ *and* $\xi(t) = \frac{\omega(t)}{\sqrt{\int_\Omega \omega^2 dx}} = \frac{\omega(t)}{|\omega|}$, *then* $\lambda(t)$ *satisfies the differential inequality*

$$\frac{d}{dt}\lambda(t) \leq C_4^2.$$

Moreover, if $\lambda(t_*) > \lambda_0$ *then* $\int_0^{t_*} \lambda(t)dt \geq \lambda_0 t_* - \frac{C_4^2}{2} t_*^2$.

Proof. For $\lambda(t) = \frac{\int_\Omega |\nabla \omega|^2 dx}{\int_\Omega \omega^2 dx} = \frac{\|\omega\|^2}{|\omega|^2}$ and $\xi(t) = \frac{\omega(t)}{\sqrt{\int_\Omega \omega^2}} = \frac{\omega(t)}{|\omega|}$ we have

$$\frac{1}{2}\frac{d\lambda(t)}{dt} = \frac{1}{|\omega|^2}\left(\omega_t, (A - \lambda(t))\omega\right),$$

where for

$$\omega_t = -A\omega + R(u) - R(\bar{u}) \text{ with}$$
$$R(u) = f(u) - \frac{1}{|\Omega|}\int f(u)dx \text{ and } R(\bar{u}) = f(\bar{u}) - \frac{1}{|\Omega|}\int f(\bar{u})dx,$$

but if $\lambda(t) = \frac{<\omega,\omega>}{(\omega,\omega)}$, then

$$\begin{aligned}
\frac{d\lambda(t)}{dt} &= \frac{2 < \omega_t, \omega > (\omega,\omega) - 2(\omega_t,\omega) < \omega,\omega >}{(\omega,\omega)^2} \\
&= \frac{2}{(\omega,\omega)}\left[< \omega_t, \omega > -(\omega_t,\omega)\frac{< \omega,\omega >}{(\omega,\omega)}\right] \\
&= \frac{2}{(\omega,\omega)}\left[< \omega_t, \omega > -(\omega_t,\omega)\lambda\right] \\
&= \frac{2}{(\omega,\omega)}\left[(\omega_t, A\omega) - (\omega_t,\omega)\lambda\right] \\
&= \frac{2}{(\omega,\omega)}\left(\omega_t, (A-\lambda)\omega\right) = \frac{2}{|\omega|}\left(\omega_t, (A-\lambda)\frac{\omega}{|\omega|}\right) \\
&= \frac{2}{|\omega|}\left(\omega_t, (A-\lambda)\xi\right) = \frac{2}{|\omega|}\left(-A\omega + R(u) - R(\bar{u}), (A-\lambda)\xi\right),
\end{aligned}$$

thus

$$\begin{aligned}
\frac{1}{2}\frac{d\lambda(t)}{dt} &= -\frac{1}{|\omega|}\left(A\omega, (A-\lambda)\xi\right) + \frac{1}{|\omega|}\left(R(u) - R(\bar{u}), (A-\lambda)\xi\right) \\
&= -\left((A-\lambda)\xi, (A-\lambda)\xi\right) + \frac{1}{|\omega|}\left(R(u) - R(\bar{u}), (A-\lambda)\xi\right) - \left(\lambda\xi, (A-\lambda)\xi\right).
\end{aligned}$$

But

$$(\lambda\xi, (A-\lambda)\xi) = (\lambda\xi, A\xi) - (\lambda\xi, \lambda\xi)$$
$$= \lambda\,(\xi, A\xi) - \lambda^2|\xi|^2 = \lambda||\xi||^2 - \lambda^2 = 0,$$

thus

$$\frac{1}{2}\frac{d\lambda(t)}{dt} + |(A-\lambda(t))\xi|^2 = \frac{1}{|\omega|}\left(R(u) - R(\bar{u}), (A-\lambda)\xi\right).$$

So

$$\frac{1}{2}\frac{d\lambda(t)}{dt} + |(A-\lambda(t))\xi|^2 = \frac{1}{|\omega|}\left(R(\bar{u}) - R(u), (A-\lambda)\xi\right)$$
$$\leq \frac{1}{|\omega|}\left|R(u) - R(\bar{u})\right|\left|(A-\lambda)\xi\right|.$$

And by Young's inequality it will be

$$\frac{1}{2}\frac{d\lambda(t)}{dt} + |(A-\lambda(t))\xi|^2 \leq \frac{1}{2\,|\omega|^2}\left|R(u) - R(\bar{u})\right|^2 + \frac{1}{2}\left|(A-\lambda)\xi\right|^2.$$

Thus

$$\frac{d\lambda(t)}{dt} \leq \frac{1}{|\omega|^2}\left|R(u) - R(\bar{u})\right|^2$$

But for $A = \left|R(u) - R(\bar{u})\right|^2$

$$A = \int_\Omega \left\{ f(u) - f(\bar{u}) - \frac{1}{|\Omega|}\int_\Omega \left(f(u) - f(\bar{u})\right)dx \right\}^2 dx$$
$$= \int_\Omega \left\{ \left(f(u) - f(\bar{u})\right)^2 + \frac{1}{|\Omega|^2}\left[\int_\Omega f(u) - f(\bar{u})dx\right]^2 - \frac{2}{|\Omega|}\left(f(u) - f(\bar{u})\right)\int_\Omega \left(f(u) - f(\bar{u})\right)dx \right\} dx$$
$$= \int_\Omega \left(f(u) - f(\bar{u})\right)^2 dx + \int_\Omega \frac{1}{|\Omega|^2}\left[\int_\Omega f(u) - f(\bar{u})dx\right]^2 dx -$$
$$- \frac{2}{|\Omega|^2}\int_\Omega \left[\left(f(u) - f(\bar{u})\right)\int_\Omega \left(f(u) - f(\bar{u})\right)dx\right] dx$$
$$= \int_\Omega \left(f(u) - f(\bar{u})\right)^2 dx - \frac{1}{|\Omega|^2}\left(\int_\Omega (f(u) - f(\bar{u}))dx\right)^2,$$

then

$$\left|R(u) - R(\bar{u})\right|^2 \leq \int_\Omega \left(f(u) - f(\bar{u})\right)^2 dx,$$

thus

$$\frac{d\lambda(t)}{dt} \leq \frac{1}{|\omega|^2}\int \left(f(u) - f(\bar{u})\right)^2 dx,$$

and using the property (P_2) of the function f we will have the result

$$\frac{d\lambda(t)}{dt} \leq C_4^2,$$

and by the Gronwall's lemma we can say that

$$\lambda(t) \leq C_4^2(t - t_0) + \lambda(t_0),$$

so by reversing the inequality for $0 \leq t_0 < t_*$

$$\lambda(t_0) \geq \lambda(t_*) + C_4^2(t_0 - t_*) > \lambda_0 + C_4^2(t_0 - t_*),$$

setting $t = t_*$ and integrating from $t_0 = 0$ to $t_0 = t_*$, we obtain

$$\int_0^{t_*} \lambda(t_0)dt_0 \geq \lambda_0 t_* - \frac{C_4^2}{2} t_*^2.$$

□

A simple consequence of this result is that for

$$\delta_* = e^{\left[2C_4 t_* - \frac{1}{2}\int_0^{t_*} \lambda(s)ds\right]},$$

then

$$\delta_* \leq \exp\left[(2C_4 - \frac{1}{4}\lambda_{N_0+1})t_* + \frac{C_4^2}{4}t_*^2\right].$$

If we choose initially that $t_* = 1$ then

$$\delta_* \leq \exp\left[2C_4 - \frac{1}{4}\lambda_{N_0+1} + \frac{C_4^2}{4}\right],$$

and if N_0 is large enough so that

$$\lambda_{N_0+1} > 8C_4 + C_4^2 + 12\ln(2)$$

which finishes the proof of the following result

Proposition 2.2. *Under the conditions $(P_1) - (P_4)$, there exists a time t_*, such that $S_* = S(t_*)$ satisfies the* ***squeezing property*** *with $\delta < \frac{1}{8}$.*

3 Convergence to Steady State [5]

Here we will make more assumptions that we did in Theorem (1.1), that is
Let Ω be a connected open bounded domain from $\mathbb{R}^N (N \geq 1)$.
Constants s_1, s_2: We suppose that $s_1 < s_2$ are two constants such that

$$f(s_2) < f(s) < f(s_1) \text{ for all } s \in (s_1, s_2). \tag{3.1}$$

Note that we can choose s_1, s_2 such that s_1 is negative with large absolute value and s_2 is arbitrary large.
Assumption on u_0: We will make the following assumption on the initial data:
(H): $u_0 \in L^2(\Omega)$ and $s_1 \leq u_0(x) \leq s_2$ for a.e. $x \in \Omega$.
The solution u of problem (P) will be such that

Proposition 3.1 (Invariant set). *Let $T > 0$, and assume that $u \in C^{2,1}(\bar{\Omega} \times (0,T]) \cap C(\bar{\Omega} \times [0,T])$ is a solution of problem (P) and that*

$$s_1 < u_0(x) < s_2 \text{ for all } x \in \Omega.$$

Then

$$s_1 < u(x,t) < s_2 \text{ for all } x \in \bar{\Omega}, 0 < t \leq T.$$

Proof. For the purpose of contradiction, we suppose that there exists a first time t_0 such that $u(x_0,t_0) = s_1$ or $u(x_0,t_0) = s_2$ for some $x_0 \in \bar{\Omega}$. Without loss of generality, assume that $u(x_0,t_0) = s_2$. By the continuity of u and the definition of t_0, we have

$$s_1 \leq u(x,t_0) \leq s_2 \text{ for all } x \in \bar{\Omega}, \text{ and } u(x,t) < s_2 \text{ for all } x \in \bar{\Omega} \text{ and } 0 \leq t < t_0. \tag{3.2}$$

Since $\partial_\nu u = 0$, we deduce from Hopf's maximum principle that $x_0 \in \Omega$. Therefore the function $u(.,t_0)$ attains its maximum at $x_0 \in \Omega$, which implies that $\Delta u(t_0,x_0) \leq 0$. By (3.2), we have

$$u_t(x_0,t_0) = \lim_{\Delta t \to 0^+} \frac{u(x_0,t_0-\Delta t) - u(x_0,t_0)}{-\Delta t} \geq 0,$$

which we substitute in problem (P) to obtain $\dfrac{1}{|\Omega|}\displaystyle\int_\Omega (f(s_2) - f(u(x,t_0)))\,dx \geq 0.$
Since $s_1 \leq u(x,t_0) \leq s_2$ for all $x \in \Omega$, it follows from (3.1) that $f(s_2) \leq f(u(x,t_0))$ for all $x \in \Omega$ so that $f(s_2) = f(u(x,t_0))$. Using (3.1) again, we obtain $u(x,t_0) = s_2$ for all $x \in \Omega$. As a consequence we have

$$\int_\Omega u(x,t_0)dx = s_2|\Omega| > \int_\Omega u_0(x)dx,$$

which contradicts the property of the mass preserving.

□

Theorem 3.1. *Assume that hypothesis **(H)** holds. Then problem (P) possesses a unique solution u such that*

$$\begin{aligned} &u \in C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{\Omega} \times [\varepsilon,\infty)) \text{ for all } \alpha \in (0,1), \varepsilon > 0, \\ &s_1 \leq u(x,t) \leq s_2 \text{ for all } x \in \bar{\Omega}, t > 0, \end{aligned} \tag{3.3}$$

and

$$\{u(t) : t \geq 1\} \text{ is relatively compact in } C^1(\bar{\Omega}) \tag{3.4}$$

In order to prove Theorem (3.1), we need some technical lemmas.

Lemma 3.1. *Let $u_0 \in L^2(\Omega)$, $g \in L^p(\Omega \times (0,T))$ for some $p \in (1,\infty)$ and let u be the solution of the time evolution problem*

$$\begin{cases} u_t - \Delta u = g & x \in \Omega, t > 0 \\ \partial_\nu u = 0 & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x) & x \in \Omega \end{cases}$$

Then for each $0 < \varepsilon < T$, there exists a positive constant $C_0(\varepsilon,\Omega,T)$ such that

$$||u||_{W_p^{2,1}(\Omega\times(\varepsilon,T))} \leq C_0\left(||u_0||_{L^2(\Omega)} + ||g||_{L^p(\Omega\times(0,T))}\right).$$

Lemma 3.2. *One has the following embedding*

$$W_p^{2,1}(\Omega \times (0,T)) \subset C^{\lambda,\lambda/2}((\bar{\Omega} \times [0,T]) \text{ with } \lambda = 2 - \frac{N+2}{2} \text{ and } p \neq N+2$$

Lemma (3.1) and Lemma (3.2) follow from [9] page 206. Now we can prove Theorem (3.1)

Proof. Let $\alpha \in (0,1)$, $p = \dfrac{N+2}{1-\alpha}$. Since $s_1 \le u(t) \le s_2$ for all $t \ge 0$, it follows that

$$\begin{aligned}\left\| f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)dx \right\|_{L^p(\Omega\times(0,1))} &\le |\Omega|^{1/p} \left\| f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)dx \right\|_{L^\infty(\Omega\times(0,1))} \\ &\le 2|\Omega|^{1/p} \sup_{s_1\le s\le s_2} |f(s)|.\end{aligned}$$

We apply Lemma (3.1) and the embedding in Lemma (3.2) on the domain $\Omega \times (0,1)$ to obtain

$$\begin{aligned}||u||_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}\times[\varepsilon,1])} &\le C\left(||u_0||_{L^2(\Omega)} + \left\| f(u) - \frac{1}{|\Omega|}\int_\Omega f(u)dx \right\|_{L^p(\Omega\times(0,1))} \right) \\ &\le C\left(|\Omega|^{1/2}||u_0||_{L^\infty(\Omega)} + 2|\Omega|^{1/p} \sup_{s_1\le s\le s_2} |f(s)| \right) \\ &\le C\left(|\Omega|^{1/2}(|s_1|+|s_2|) + 2|\Omega|^{1/p} \sup_{s_1\le s\le s_2} |f(s)| \right).\end{aligned}$$

Similarly, we apply Lemma (3.1) and the embedding in Lemma (3.2) on the domain $\Omega \times (k, k+1)$ and $\Omega \times (k+1/2, k+3/2)$ to obtain

$$||u||_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}\times[k+\varepsilon,k+1])} \le C\left(|\Omega|^{1/2}(|s_1|+|s_2|) + 2|\Omega|^{1/p} \sup_{s_1\le s\le s_2} |f(s)| \right),$$

and a similar one on the domain $\Omega \times (k+1/2, k+3/2)$. Finally, we deduce from the fact that k can be chosen arbitrary large that

$$||u||_{C^{1+\alpha,(1+\alpha)/2}(\bar{\Omega}\times[\varepsilon,\infty))} \le C\left(|\Omega|^{1/2}(|s_1|+|s_2|) + 2|\Omega|^{1/p} \sup_{s_1\le s\le s_2} |f(s)| \right).$$

□

3.1 A version of a Lojasiewicz inequaity

We will prove a version of Lojasiewicz inequality for the function E which coincides with the functional $\mathcal{E}$ on the solution orbits. We set

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \bar{F}(u)dx,$$

where $\bar{F} \in C_c^\infty(\mathbb{R})$ is such that

$$\bar{F}(s) = \begin{cases} F(s) & \text{if } s \in [s_1-1, s_2+1] \\ 0 & \text{otherwise} \end{cases}.$$

Then $E(u(t)) = \mathcal{E}(u(t))$ for all $t > 0$, where $\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - F(u)dx$.
In what follows we will prove the differentiability of E and compute its derivative. And then we will give a definition and some equivalent conditions of a critical point to prove the Lojasiewicz inequality.

3.1.1 Some preparations

We define the spaces
$H = \left\{u \in L^2(\Omega) : \int_\Omega u(x)dx = 0\right\}$, equipped with the norm $||.||_H = ||.||_{L^2(\Omega)}$,
$V = \left\{u \in H^1(\Omega) : \int_\Omega u(x)dx = 0\right\}$, equipped with the norm $||.||_V = ||.||_{H^1(\Omega)}$.
Let V^* be the dual space of V. We identify H with its dual to obtain:

$$V \hookrightarrow H \hookrightarrow V^*,$$

where this embeddings are continuous and compact. We denote by $\mathcal{L}(X,Y)$ the space of bounded linear operators from a Banach space X to a second Banach space$Y$, and write $\mathcal{L}(X) = \mathcal{L}(X,X)$.
We also define the spaces

$$\mathcal{L}^p(\Omega) = \left\{u \in L^p(\Omega) : \int_\Omega u(x)dx = 0\right\},$$

equipped with the norm $||.||_{\mathcal{L}^p(\Omega)} = ||.||_{L^p(\Omega)}$ and

$$X_p = \left\{u \in W^{2,p}(\Omega) : \int_\Omega u(x)dx = 0\right\},$$

equipped with the norm $||.||_{X_p} = ||.||_{W^{2,p}(\Omega)}$. Throughout the sequel, we denote by $C \geq 0$ a generic constant which may vary from line to line. We start with the following result.

Lemma 3.3. *Let $u, h \in L^1(\Omega), p \in [1,\infty)$ be arbitrary and let g be a continuously differentiable function from $\mathbb{R}$ to $\mathbb{R}$ such that*

$$|g(s)|, |g'(s)| \leq C \text{ for all } s \in \mathbb{R}. \tag{3.5}$$

Then

$$\int_0^1 g(u+\tau h)d\tau \to g(u) \text{ in } L^p(\Omega) \text{ as } ||h||_{L^1(\Omega)} \to 0.$$

Proof. By Jensen's inequality and (3.5),

$$\begin{aligned}\left|\int_0^1 (g(u+\tau h) - g(u))d\tau\right|^p &\leq \int_0^1 |(g(u+\tau h) - g(u))|^p d\tau \\ &\leq C\int_0^1 |(g(u+\tau h) - g(u))| d\tau \\ &\leq C|h|.\end{aligned}$$

Thus

$$\left(\int_\Omega \left|\int_0^1 (g(u+\tau h) - g(u))d\tau\right|^p\right)^{1/p} \leq C\left(\int_\Omega |h|\right)^{1/p}.$$

□

Lemma 3.4. *The functional E is twice continuously Fréchet differentiable on V. We denote by E', L be the first and second derivative of E, respectively. Then*

(*i*) *The first derivative*

$$E' : V \longrightarrow V^* \text{ is given by}$$

$$\langle E'(u), h\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h - \int_\Omega \bar{f}(u)h \quad \text{for all } u, h \in V. \tag{3.6}$$

(*ii*) *The second derivative*

$$L : V \longrightarrow \mathcal{L}(V, V^*) \text{ is given by}$$

$$\langle L(u)h, k\rangle_{V^*,V} = \int_\Omega \nabla h \nabla k - \int_\Omega \bar{f}'(u)hk \quad \text{for all } u, h, k \in V. \tag{3.7}$$

As a consequence,

$$\langle L(u)h, k\rangle_{V^*,V} = \langle h, L(u)k\rangle_{V,V^*}. \tag{3.8}$$

Proof. We write E as the difference of E_1 and E_2, where

$$E_1(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx \text{ and } E_2(u) = \int_\Omega \bar{F}(u)\,dx. \tag{3.9}$$

Obviously, E_1 is twice continuously Fréchet differentiable. Its derivatives are easily identified in the formula (3.6) and (3.7). We now prove the differentiability of E_2.

By Taylor's formula, there exists $\theta(x) \in (0, 1)$ such that

$$\bar{F}(u + h) - \bar{F}(u) = \bar{f}(u + \theta h)h \text{ for all } u, h \in V.$$

Il follows that

$$\begin{aligned}\left|E_2(u + h) - E_2(u) - \int_\Omega \bar{f}(u)h\,dx\right| &\le \int_\Omega |\bar{f}(u + \theta h) - \bar{f}(u)|\,|h|\,dx \\ &\le C\|\bar{f}(u + \theta h) - \bar{f}(u)\|_{L^2(\Omega)}\|h\|_V.\end{aligned}$$

Note that $u+\theta h$ tends to u in $H^1(\Omega)$ as $h \to 0$ in V; it follows from Lemma 3.3 that $\|\bar{f}(u+\theta h)-\bar{f}(u)\|_{L^2(\Omega)}$ tends to 0 as $h \to 0$ in V. Thus

$$\left|E_2(u + h) - E_2(u) - \int_\Omega \bar{f}(u)h\,dx\right| = o(\|h\|_V) \text{ as } h \to 0.$$

This implies that the first derivative E_2' exists and

$$\langle E_2'(u), h\rangle_{V^*,V} = \int_\Omega \bar{f}(u)h\,dx.$$

The Fréchet differentiability of E_2' is shown in a similar way. Choose $p > 2$ such that V is continuously embedded in $L^p(\Omega)$. Let T be a linear mapping from V to V^* given by

$$\langle T\,h, k\rangle_{V^*,V} = \int_\Omega \bar{f}'(u)h\,k\,dx.$$

We will use below a generalized Hölder's inequality based on the identity

$$\frac{1}{p} + \frac{1}{p} + \frac{p-2}{p} = 1.$$

For every $u, h, k \in V$, there exist $\eta(x) \in (0, 1)$ such that

$$\begin{aligned}&\left|\langle E_2'(u + h) - E_2'(u) - Th, k\rangle_{V^*,V}\right| \\ &\le \int_\Omega |\bar{f}'(u + \eta h) - \bar{f}'(u)|\,|h|\,|k|\,dx \\ &\le \|\bar{f}'(u + \eta h) - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}\|h\|_{L^p(\Omega)}\|k\|_{L^p(\Omega)} \\ &\le C\|\bar{f}'(u + \eta h) - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}\|h\|_V\|k\|_V,\end{aligned} \tag{3.10}$$

It follows from (3.10) that

$$\|E_2'(u+h) - E_2'(u) - T\,h\|_{V^*} \le C\|\bar{f}'(u+\eta h) - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}\|h\|_V.$$

Since $p/(p-2) < +\infty$ and since $\bar{f}'$ is bounded, $\|\bar{f}'(u+\eta h) - \bar{f}'(u)\|_{L^{p/(p-2)}(\Omega)}$ tends to 0 as $h \to 0$. Thus

$$\|E_2'(u+h) - E_2'(u) - Th\|_{V^*} = o(\|h\|_V)$$

which implies that

$$\langle E_2''(u)h, k\rangle_{V^*,V} = \int_\Omega \bar{f}'(u)h\,k \text{ for all } u, h, k \in V.$$

On the other hand,

$$\begin{aligned} |\langle (E_2''(u) - E_2''(v))h, k\rangle_{V^*,V}| &\le \int_\Omega |\bar{f}'(u) - \bar{f}'(v)|\,|h|\,|k|\,dx \\ &\le C\|\bar{f}'(u) - \bar{f}'(v)\|_{L^{p/(p-2)}(\Omega)}\|h\|_V\|k\|_V, \end{aligned}$$

so that

$$\|E_2''(u) - E_2''(v)\|_{\mathcal{L}(V,V^*)} \le C\|\bar{f}'(u) - \bar{f}'(v)\|_{L^{p/p-2}(\Omega)}.$$

This estimate implies the continuity of E_2''.

□

We define a continuous bilinear form from $V \times V \to \mathbb{R}$ by

$$a(u,v) = \int_\Omega \nabla u \nabla v\,dx.$$

The following lemma is an immediate consequence of the Lax-Milgram theorem, we omit then its proof.

Lemma 3.5. *There exists an isomorphism A from V onto V^* such that*

$$a(u,v) = \langle Au, v\rangle_{V^*,V} \text{ for all } u, v \in V. \tag{3.11}$$

Corollary 3.1. *The first and second derivatives of E can be represented in V^* as:*

$$E'(u) = Au - \bar{f}(u) + \frac{1}{|\Omega|}\int_\Omega \bar{f}(u)dx, \tag{3.12}$$

$$L(u)h = Ah - \bar{f}'(u)h + \frac{1}{|\Omega|}\int_\Omega \bar{f}'(u)h, \tag{3.13}$$

for all $u, h \in V$.

Proof. Since $\bar{f}$ is bounded, $\bar{f}(u) - \frac{1}{|\Omega|}\int_\Omega \bar{f}(u) \in H \hookrightarrow V^*$. Therefore,

$$Au - \bar{f}(u) + \frac{1}{|\Omega|}\int_\Omega \bar{f}(u) \in V^*.$$

Since

$$\int_\Omega \left(\frac{1}{|\Omega|}\int_\Omega \bar{f}(u)\right)h = \frac{1}{|\Omega|}\int_\Omega \bar{f}(u)\int_\Omega h = 0 \text{ for all } h \in V,$$

it follows that

$$\langle Au - \bar{f}(u) + \frac{1}{|\Omega|}\int_\Omega \bar{f}(u), h\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h - \int_\Omega \bar{f}(u)h.$$

This together with (3.6) implies that

$$E'(u) = Au - \bar{f}(u) + \frac{1}{|\Omega|}\int_\Omega \bar{f}(u).$$

Identity (3.13) may be proved in a similar way. □

Lemma 3.6. *Let $p \geq 2$, then for any $g \in \mathcal{L}^p(\Omega)$, there exists a unique solution $u \in X_p$ of the equation*

$$Au = g \text{ in } V^*. \tag{3.14}$$

Moreover, $\langle Aw, v\rangle = \langle -\Delta w, v\rangle$ for all $w \in X_p, v \in V$.

Proof. It follows from Lemma 3.5 that Equation (3.14) has a unique solution $u \in V$. We now claim that $u \in X_p$. Consider the elliptic problem

$$\begin{cases} -\Delta\tilde{u} = g & \text{in } \Omega, \\ \partial_\nu \tilde{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

First, since $g \in H$, it follows from the Fredholm alternative that this problem possesses a unique solution $\tilde{u} \in V$. Next, since $g \in \mathcal{L}^p(\Omega)$, we deduce from [3] that $\tilde{u} \in W^{2,p}(\Omega)$ so that also $\tilde{u} \in X_p$. In fact, $\tilde{u}$ satisfies Equation (3.14) since

$$\langle g, v\rangle_{V^*,V} = \langle -\Delta\tilde{u}, v\rangle_{V^*,V} = \int_\Omega \nabla\tilde{u}\nabla v\, dx = a(\tilde{u}, v) = \langle A\tilde{u}, v\rangle_{V^*,V}$$

for all $v \in V$. By the uniqueness of the solution of Equation (3.14), $u = \tilde{u} \in X_p$.
On the other hand, for all $w \in X_p, v \in V$

$$\langle -\Delta w, v\rangle_{V^*,V} = \int_\Omega \nabla w \nabla v\, dx = \langle Aw, v\rangle_{V^*,V},$$

so that $A = -\Delta$ on X_p. □

Definition 3.1. *We say that $\varphi \in V$ is a critical point of E if $E'(\varphi) = 0$.*

Lemma 3.7. *For every $\varphi \in V$, the following assertions are equivalent:*

(i) φ is a critical point of E,

(ii) $\varphi \in X_2$ and φ satisfies the equations

$$-\Delta\varphi - \bar{f}(\varphi) + \frac{1}{|\Omega|}\int_\Omega \bar{f}(\varphi) = 0 \text{ in } \Omega, \tag{3.15}$$

$$\partial_\nu\varphi = 0 \qquad \text{on } \partial\Omega. \tag{3.16}$$

Moreover, φ is $C^\infty(\overline{\Omega})$.

Proof. $(ii) \Rightarrow (i)$. It follows directly from Lemma 3.6 and the formula (3.12).
$(i) \Rightarrow (ii)$. Assume that $\varphi \in V$ is a critical point of E. We deduce from (3.12) that

$$A(\varphi) = \bar{f}(\varphi) - \frac{1}{|\Omega|}\int_\Omega \bar{f}(\varphi) \quad \text{in } V^*.$$

Since $\bar{f}(\varphi) - \frac{1}{|\Omega|}\int_\Omega \bar{f}(\varphi) \in H$, then $A(\varphi) \in H$. It follows from Lemma 3.6 that $\varphi \in X_2$ and $A = -\Delta$. Therefore φ satisfies (3.15).

Finally, we deduce that $\varphi \in C^\infty(\overline{\Omega})$ from the boundedness of $\bar{f}(\varphi) - \frac{1}{|\Omega|}\int_\Omega \bar{f}(\varphi)$, Sobolev embedding theorem and a standard bootstrap argument. □

3.1.2 The Lojasiewicz inequality

Theorem 3.2. *(Lojasiewicz inequality). Let $\varphi \in V$ be a critical point of the functional E such that $s_1 \le \varphi \le s_2$. Then there exist constants $\theta \in (0, \frac{1}{2}]$ and $C, \sigma > 0$ such that*

$$|E(u) - E(\varphi)|^{1-\theta} \le C\|E'(u)\|_{V^*}$$

for all $\|u - \varphi\|_V \le \sigma$. In this case we say that E satisfies the Lojasiewicz inequality in φ. The number θ will be called the Lojasiewicz exponent.

We check below that all hypotheses in [14, Corollary 3.11] are satisfied so that the result of Theorem 3.2 will follow from [14, Corollary 3.11]. We need the following result.

Lemma 3.8. *Let φ be a critical point of E. Then $L(\varphi)$ is a Fredholm operator from V to V^*. Moreover,*

- *(i)* $\ker L(\varphi)$ *is finite-dimensional and contained in* $C^\infty(\overline{\Omega})$.
- *(ii)* $\langle u, v\rangle_{V,V^*} = 0$ *for all* $u \in \ker L(\varphi)$ *and* $v \in \operatorname{Rg} L(\varphi)$,
- *(iii)* V^* *is the topological direct sum of* $\ker L(\varphi) \subset V \hookrightarrow V^*$ *and* $\operatorname{Rg} L(\varphi)$,
- *(iv) if* $g \in \mathscr{L}^p(\Omega) \cap \operatorname{Rg} L(\varphi)$ *for* $p \ge 2$ *and* $u \in V$ *solves the equation*

$$L(\varphi)u = g \text{ in } V^*$$

then $u \in X_p$. *Consequently,*

$$\operatorname{Rg}(L(\varphi)|_{X_p}) = \operatorname{Rg} L(\varphi) \cap \mathscr{L}^p(\Omega).$$

Proof. We first prove that the linear operator

$$\begin{aligned} T : V &\longrightarrow V^* \\ h &\longmapsto -\bar{f}'(\varphi)h + \frac{1}{|\Omega|}\int_\Omega \bar{f}'(\varphi)h. \end{aligned}$$

is compact. Indeed, it follows from the compact embedding $H \hookrightarrow V^*$ and the following estimate

$$\begin{aligned} \|Th\|_H &\le \|\bar{f}'(\varphi)h\|_{L^2(\Omega)} + \left\|\frac{1}{|\Omega|}\int_\Omega \bar{f}'(\varphi)h\right\|_{L^2(\Omega)} \\ &\le C(\|h\|_{L^2(\Omega)} + \|h\|_{L^1(\Omega)}) \\ &\le C\|h\|_V. \end{aligned}$$

Recall that since A is an isomorphism from V onto V^*, it is also a Fredholm operator of index

$$\operatorname{ind} A := \dim\ker A - \operatorname{codim}\operatorname{Rg} A = 0.$$

It follows that $L(\varphi) = A + T$, as a sum of a Fredholm operator and a compact operator, is also a Fredholm operator with the same index [7, p. 168]. Therefore,

$$\operatorname{Rg} L(\varphi) \text{ is closed in } V^* \text{ and } \dim\ker L(\varphi) = \operatorname{codim}\operatorname{Rg} L(\varphi) < \infty. \tag{3.17}$$

(i) Using similar arguments as the proof in Lemma 3.7, we deduce that if $h \in \ker L(\varphi)$ then $h \in X_2$ and satisfies the equation:

$$\begin{cases} -\Delta h - \bar{f}'(\varphi)h + \frac{1}{|\Omega|}\int_\Omega \bar{f}'(\varphi)h = 0 \text{ in } \Omega, \\ \partial_\nu h = 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{cases}$$

Note that $\bar{f}'(\varphi) \in C^\infty(\overline{\Omega})$; we deduce that $h \in C^\infty(\overline{\Omega})$ from a Sobolev embedding theorem and a bootstrap argument.
(ii) We may identify the linear operator $L(\varphi)$ with a bilinear symmetric form on $V \times V$ (e.g see [36, Section 10.5.3 p. 82]). Thus, for every $u \in \ker L(\varphi), v = L(\varphi)w, w \in V$,

$$\langle u, v\rangle_{V,V^*} = \langle u, L(\varphi)w\rangle_{V,V^*} = \langle L(\varphi)u, w\rangle_{V^*,V} = 0,$$

which implies (ii).
(iii) Using part (ii), we deduce that for every $u \in \ker L(\varphi) \cap \operatorname{Rg} L(\varphi)$, $\langle u, u\rangle_{V^*,V} = 0$, hence $u = 0$. It follows that $\ker L(\varphi) \cap \operatorname{Rg} L(\varphi) = \{0\}$. On the other hand, $\dim\ker L(\varphi) = \operatorname{codim}\operatorname{Rg} L(\varphi)$ so that V^* is the algebraic direct sum of $\ker L(\varphi)$ and $\operatorname{Rg} L(\varphi)$.

Since $\ker L(\varphi)$ is finite-dimensional, it is closed in V^*. It follows from (3.17) that $\operatorname{Rg} L(\varphi)$ is closed in V^*, thus V^* is the topological direct sum of $\ker L(\varphi)$ and $\operatorname{Rg} L(\varphi)$.
(iv) Since $g \in \operatorname{Rg} L(\varphi)$, there exists $u \in V$ satisfying

$$Au = \bar{f}'(\varphi)u - \frac{1}{|\Omega|}\int_\Omega \bar{f}'(\varphi)u + g.$$

We write

$$Au = \bar{f}'(\varphi)u - \frac{1}{|\Omega|}\int_\Omega \bar{f}'(\varphi)u + g \in H,$$

thus $u \in X_2$ and $A = -\Delta$. We have

$$-\Delta u - \bar{f}'(\varphi)u = -\frac{1}{|\Omega|}\int_\Omega \bar{f}'(\varphi)u + g \in L^p(\Omega),$$

note that $\bar{f}'(\varphi) \in C^\infty(\overline{\Omega})$ and use elliptic regularity theory to deduce that $u \in X_p$. From this we obtain (iv). $\square$

Before proving Theorem 3.2, we recall the definition of an analytic map on a neighborhood of a point (see [37, Definition 8.8, p. 362]). A map T from a Banach space X into a Banach space Y is called analytic on a neighborhood of $z \in X$ if it may be represented as

$$T(z+h) - T(z) = \sum_{k\geq 1} T_k(z)[h,\dots,h] \text{ in } Y, \text{ for any } h \in X, \|h\|_X \leq \varepsilon,\ \varepsilon \text{ small enough,}$$

where $T_k(z)$ is a symmetric $k-$linear form on X with values in Y and

$$\sum_{k\geq 1} \|T_k(z)\|_{\mathcal{L}_k(X,Y)}\|h\|^k < \infty.$$

Here, $\mathcal{L}_k(X,Y)$ is the space of bounded k-linear operators $X^k \to Y$.

Proof. of Theorem 3.2
In order to prove Theorem 3.2, we apply [14, Corollary 3.11] for

$$X = X_p, Y = \mathcal{L}^p(\Omega),$$

where $p > N$. In this case, there holds the embedding $W^{2,p} \subset C^{1,\lambda}(\bar{\Omega})$ with $\lambda = 1 - \frac{n}{p}$. Note that

$$E'(u) = -\Delta u - \bar{f}(u) + \frac{1}{|\Omega|}\int_\Omega \bar{f}(u) \in \mathcal{L}^p(\Omega),$$

for all $u \in X_p$. In view of Lemma 3.8, it is sufficient to prove that E' is analytic in a neighborhood of φ. Indeed, let ε be small enough such that for all $h \in X_p$ with $||h||_{X_p} \leq \varepsilon$, we have

$$||h||_{C(\bar{\Omega})} \leq ||h||_{X_p} < 1.$$

Since

$$\bar{f}(s) = f(s) = \sum_{i=0}^{n} a_i s^i \text{ for all } s \in (s_1 - 1, s_2 + 1),$$

we perform a Taylor's expansion to deduce for all $h \in X_p$ with $||h||_{X_p} \leq \varepsilon$, that

$$\bar{f}(\varphi(x) + h(x)) - \bar{f}(\varphi(x)) = \sum_{i=1}^{n} \frac{\bar{f}^{(i)}\varphi(x)}{i!} h^i(x).$$

It follows that

$$\begin{aligned} E'(\varphi + h) - E'(\varphi) &= -\Delta h + \sum_{i=1}^{n} \frac{\bar{f}^{(i)}\varphi}{i!} h^i - \sum_{i=1}^{n} \frac{1}{|\Omega|}\int_\Omega \frac{\bar{f}^{(i)}\varphi}{i!} h^i dx \\ &= \sum_{i=1}^{n} T_i[h, ..., h], \end{aligned}$$

where

$$T_1[h] := -\Delta h + \bar{f}'(\varphi)h - \frac{1}{|\Omega|}\int_\Omega \bar{f}'(\varphi)h$$

and

$$T_i[h, ..., h] := \frac{\bar{f}'(\varphi)}{i!} h^i - \frac{1}{|\Omega|}\int_\Omega \frac{\bar{f}'(\varphi)}{i!} h^i \text{ for all } 1 < i \leq n.$$

We now prove that $T_i \in \mathcal{L}_i(X_p, \mathcal{L}^p(\Omega))$. For all $h_1, ..., h_i \in X_p$, and $1 < i \leq n$, we have

$$\begin{aligned} \|T_i[h_1, ..., h_i]\|_{\mathcal{L}^p(\Omega)} &\leq C \|T_i[h_1, ..., h_i]\|_{L^\infty(\Omega)} \\ &\leq C \left\| \frac{\bar{f}'(\varphi)}{i!} h_1 ... h_i \right\|_{L^\infty(\Omega)} + C \left\| \frac{1}{\Omega}\int_\Omega \frac{\bar{f}'(\varphi)}{i!} h_1 ... h_i \right\|_{L^\infty(\Omega)} \\ &\leq C \prod_{j=1}^{i} ||h_j||_{L^\infty(\Omega)} \leq \prod_{j=1}^{i} ||h_j||_{X_p} \end{aligned}$$

which implies that $T_i \in \mathcal{L}_i(X_p, \mathcal{L}^p(\Omega))$ for all $1 < i \leq n$. In the case $i = 1$, since $-\Delta$ is linear, continuous from X_p to $\mathcal{L}^p(\Omega)$, we easily deduce that $T_1 \in \mathcal{L}_i(X_p, \mathcal{L}^p(\Omega))$. Therefore E' is analytic on a neighborhood of φ.

□

3.2 Large time behavior

Theorem 3.3. *Let **(H)** hold and let u be the unique solution of problem (P). Then there exists a function φ such that*

$$\lim_{t\to\infty} ||u(t) - \varphi||_{C^1(\bar{\Omega})} = 0 \text{ as } t \to \infty. \tag{3.18}$$

Moreover, $s_1 \leq \varphi \leq s_2$,

$$\int_\Omega \varphi = \int_\Omega u_0,$$

and φ is a solution of the stationary problem

$$(S) \begin{cases} -\Delta\varphi = -f(\varphi) + \frac{1}{|\Omega|}\int_\Omega f(\varphi) & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

This section is devoted to the proof of Theorem 3.3 by applying the Lojasiewicz inequality.

Lemma 3.9. *Suppose that **(H)** is satisfied and let u be the solution of problem (P). Then*

(i) For all $0 < s \leq t < \infty$,

$$E(u(s)) = E(u(t)) + \int_s^t \int_\Omega |u_t|^2 \, dx. \tag{3.19}$$

(ii) Further, $e := \lim_{t\to\infty} E(u(t))$ exists.

We denote by $S(t)$ the semigroup on H corresponding with problem (P) and define the ω-limit set of u_0 by

$$\omega(u_0) := \{\varphi \in H^1(\Omega) : \exists t_n \to \infty, u(t_n) \to \varphi \text{ in } H^1(\Omega) \text{ as } n \to \infty\}.$$

We have the following result

Lemma 3.10. *Suppose that **(H)** is satisfied and let u be the solution of problem (P). Then*

(i) $\omega(u_0)$ is a non-empty, compact set of $H^1(\Omega)$.

(ii) The functional E is constant on $\omega(u_0)$. If $\varphi \in \omega(u_0)$ then $s_1 \leq \varphi \leq s_2$ and φ is a critical point of E, i.e., $E'(\varphi) = 0$.

(iii) $d(u(t), \omega(u_0)) \to 0$ as $t \to \infty$. Where $d(u(t), \omega(u_0)) = \inf_{\varphi\in\omega(u_0)} ||u(t) - \varphi||_{H^1(\Omega)}$

Proof. (i) Since $\{u(t), t \geq 1\}$ is compact in $H^1(\Omega)$, we can easily show that $\omega(u_0)$ is non-empty, compact of $H^1(\Omega)$.
Next, note that if $\psi \in \omega(u_0)$, then there exists a sequence $t_n \to \infty$ such that $\psi = \lim_{n\to\infty} S(t_n)u_0$. For all $t \geq 0$, we have

$$S(t)\psi = \lim_{n\to\infty} S(t + t_n)u_0 \in \omega(u_0).$$

This shows that $\omega(u_0)$ is positive invariant.

(ii) First, we prove that E is constant on $\omega(u_0)$. Let $e = \lim_{t\to\infty} E(S(t)u_0)$ as in Lemma 3.9. For any $\varphi \in \omega(u_0)$, we have $\varphi = \lim_{n\to\infty} S(t_n)u_0$ for some sequence $t_n \to \infty$. Since E is continuous in $H^1(\Omega)$,

$$E(\varphi) = \lim_{n\to\infty} E(S(t_n)u_0) = e.$$

i.e., E is constant on $\omega(u_0)$.

Note that since $S(t_n)u_0 \to \varphi$ in $L^2(\Omega)$ so that we can extract a subsequence of $S(t_n)u_0$ which converges almost everywhere on Ω. On the other hand $s_1 \le S(t_n)u_0 \le s_2$ for all $n \ge 0$, therefore

$$s_1 \le \varphi \le s_2.$$

We now prove that φ is a critical of E. Since $\omega(u_0)$ is positive invariance, $E(S(t)\varphi) = E(\varphi)$ for all $t \ge 0$. It follows from Lemma 3.9 that

$$\int_0^t \int_\Omega |\varphi_t|^2 \, dxdt = 0 \text{ for all } t \ge 0,$$

so that $\varphi_t = 0$ for all $t \ge 0$. Hence, φ satisfies the equation

$$\begin{aligned} & -\Delta\varphi - f(\varphi) + \frac{1}{|\Omega|}\int_\Omega f(\varphi) = 0 \text{ in } \Omega, \\ & \partial_\nu \varphi = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{aligned}$$

Therefore, φ is a critical point of the functional E by Lemma 3.7.

(iii) Assume by contradiction that there exists a sequence $t_n \to \infty$ and $\varepsilon > 0$ such that

$$d(S(t_n)u_0, \omega(u_0)) \ge \varepsilon.$$

By compactness, there exists a subsequence $t_{n_k} \to \infty$ such that $S(t_{n_k})u_0 \to w \in \omega(u_0)$. Therefore, $d(S(t_{n_k})u_0, \omega(u_0)) = 0$ as $k \to \infty$, which is absurd. □

Proof. of Theorem 3.3
We will first apply the Lojasiewicz inequality in the case $\int_\Omega u_0(x) = 0$. Recall that since E is constant on $\omega(u_0)$, as in Lemma 3.10 we can write

$$e = E(v) \text{ for all } v \in \omega(u_0). \tag{3.20}$$

It follows from Theorem 3.2 that E satisfies the Lojasiewicz inequality in the neighborhood of every $\varphi \in \omega(u_0)$; in other words, we have that for every $\varphi \in \omega(u_0)$ there exist constants

$$\theta \in (0, \frac{1}{2}], C \ge 0 \text{ and } \delta > 0$$

such that

$$|E(v) - E(\varphi)|^{1-\theta} \le C\|E'(v)\|_{V^*} \text{ whenever } \|v - \varphi\|_V \le \delta. \tag{3.21}$$

Since the functional E is continuous on V, we may choose δ small enough so that

$$|E(v) - E(\varphi)| < 1 \text{ whenever } \|v - \varphi\|_V \le \delta. \tag{3.22}$$

It follows from the compactness of $\omega(u_0)$ in V that there exists a neighborhood $\mathcal{U}$ of $\omega(u_0)$ composed of finitely many balls $B_j, j=1,...,J$, with center φ_j and radius δ_j. In each of the ball B_j, inequality (3.22) and the Lojasiewicz inequality (3.21) hold for some constants θ_j and C_j. We define $\bar{\theta}=\min\{\theta_j, j=1,...,J\}$ and $\bar{C}=\max\{C_j, j=1,...,J\}$ to deduce from (3.20), (3.21) and (3.22) that

$$|E(v)-e|^{1-\bar{\theta}} \leq \bar{C}\|E'(v)\|_{V^*} \text{ for } v\in\mathcal{U}.$$

Using Lemma 3.10(iii), there exists

$$t_0 \geq 0$$

such that

$$\bar{u}(t)\in\mathcal{U} \text{ for all } t\geq t_0.$$

Hence, for every $t\geq t_0$, there holds

$$\begin{aligned} -\frac{d}{dt}|E(\bar{u}(t))-e|^{\bar{\theta}} &= \bar{\theta}|E(\bar{u}(t))-e|^{\bar{\theta}-1}\Big(-\frac{dE}{dt}(\bar{u}(t))\Big) \\ &\geq \frac{\bar{\theta}}{\bar{C}}\frac{\|\bar{u}_t\|^2_{L^2(\Omega)}}{\|E'(\bar{u}(t))\|_{V^*}}. \end{aligned} \tag{3.23}$$

Note that for all $t\geq t_0, \bar{u}(t)\in C^\infty(\overline{\Omega})$, so that $E'(\bar{u}(t))\in H$ and it can be written of the form

$$E'(\bar{u}(t)) = -\Delta\bar{u}-\bar{f}(\bar{u})+\frac{1}{|\Omega|}\bar{f}(\bar{u}) = -\bar{u}_t.$$

Applying continuous embedding $H\hookrightarrow V^*$, we have

$$\|E'(\bar{u}(t))\|_{V^*} \leq \bar{\bar{C}}\|E'(\bar{u}(t))\|_{L^2(\Omega)} = \bar{\bar{C}}\|\bar{u}_t\|_{L^2(\Omega)} \text{ for all } t\geq t_0, \tag{3.24}$$

where $\bar{\bar{C}}$ is a positive constant. Combining (3.23) and (3.24) we obtain

$$-\frac{d}{dt}|E(\bar{u}(t))-e|^{\bar{\theta}} \geq C_0\|\bar{u}_t\|_{L^2(\Omega)}.$$

Here, $C_0=\dfrac{\bar{\theta}}{\bar{C}\bar{\bar{C}}}$. Thus

$$\|\bar{u}(t_1)-\bar{u}(t_2)\|_{L^2} \leq \int_{t_1}^{t_2}\|\bar{u}_t\|_{L^2} \leq \frac{1}{C_0}(|E(\bar{u}(t_1))-e|^{\bar{\theta}}-|E(\bar{u}(t_2))-e|^{\bar{\theta}})$$

for all

$$t_0\leq t_1\leq t_2.$$

Therefore, $\|\bar{u}(t_1)-\bar{u}(t_2)\|_{L^2(\Omega)}$ tends to zero as $t_1\to\infty$ so that $\{\bar{u}(t)\}$ is a Cauchy sequence in H. Consequently, there exists $\varphi\in H$ such that $\lim_{t\to\infty}\bar{u}(t)=\varphi$ exists in H. □

3.3 Rate of the convergence

In this section, we estimate the rate of the convergence of $u(t)$ to φ. The proof is based on Lojasiewicz's inequality. We consider two cases, when Lojasiewicz exponent $\theta=\frac{1}{2}$ and $\theta\in(0,\frac{1}{2})$.

3.3.1 When Lojasiewicz exponent $\theta \in (0, \frac{1}{2})$

We need the following lemma.

Lemma 3.11 (see [21], Lemma 3.3)**.** *Asume that for all $t \geq t_0$, some $\alpha > 0$ and a constant $K > 0$*

$$\int_t^\infty \|u_t\|_H^2 \leq K t^{-2\alpha-1}.$$

Then, we have

$$\|u(t) - u(\tau)\|_H \leq \frac{\sqrt{K}}{1 - 2^{-\alpha}} t^{-\alpha} \quad \text{for all } \tau \geq t \geq t_0.$$

Consequently,

$$\|u(t) - \varphi\|_H \leq \frac{\sqrt{K}}{1 - 2^{-\alpha}} t^{-\alpha} \quad \text{for all } t \geq t_0.$$

Theorem 3.4. *Assume that Theorem 3.2 holds for $\theta \in (0, \frac{1}{2})$, then for $\alpha := \dfrac{\theta}{1-2\theta} > 0$ and a constant $K > 0$, we have*

$$\|u(t) - \varphi\|_H \leq \frac{\sqrt{K}}{1 - 2^{-\alpha}} t^{-\alpha} \quad \text{for all } t > 0.$$

Proof. As in the proof of Theorem 3.3, we only need to prove this theorem for function $\bar{u}$ such that $\int_\Omega \bar{u} dx = 0$. Since $\bar{u}(x,t)$ is smooth for all $t > 0$, we have

$$\frac{d}{dt}(E(\bar{u}) - E(\varphi)) = \langle E'(\bar{u}), \bar{u}_t \rangle = -\langle E'(\bar{u}), E'(\bar{u}) \rangle = -\|E'(\bar{u})\|_H^2. \tag{3.25}$$

Since $\bar{u}(t)$ tends to φ as $t \to \infty$. Therefore there exists $T_0 > 0$ such that for all $t \geq T_0$

$$\|\bar{u}(t) - \varphi\| \leq \sigma \quad (\sigma \text{ in Theorem 3.2}).$$

It follows that for all $t \geq T_0$

$$C\|E'(\bar{u})\|_{V^*} \geq |E(\bar{u}) - E(\varphi)|^{1-\theta}.$$

Therefore, by using the continuous embedding $H \hookrightarrow V^*$, we obtain

$$C_1\|E'(\bar{u})\|_H \geq |E(\bar{u}) - E(\varphi)|^{1-\theta} = (E(\bar{u}) - E(\varphi))^{1-\theta}. \tag{3.26}$$

Combining (3.27) and (3.28), we get

$$\frac{d}{dt}(E(\bar{u}) - E(\varphi)) \leq -C_2(E(\bar{u}) - E(\varphi))^{2(1-\theta)} \text{ for all } t \geq T_0,$$

where $C_2 := 1/C_1^2$.

Note that $y(t) := \left((E(\bar{u}(T_0) - E(\varphi))^{2\theta-1} + C_2(1-2\theta)(t - T_0) \right)^{-1/(1-2\theta)}$ is the unique solution of the differential equation

$$\begin{cases} \dfrac{d}{dt} y(t) = -C_2 y^{2(1-\theta)} \text{ for } t \geq T_0, \\ \\ y(T_0) = E(\bar{u}(T_0) - E(\varphi)). \end{cases}$$

We use a differential inequality in [20, Lemma 2.7, p.53] to deduce for all $t \geq T_0$ that

$$\begin{aligned} E(\bar{u}(t)) - E(\varphi) &\leq \left((E(\bar{u}(T_0) - E(\varphi))^{2\theta-1} + C_2(1-2\theta)(t-T_0) \right)^{-1/(1-2\theta)} \\ &= \left((E(\bar{u}(T_0) - E(\varphi))^{2\theta-1} - C_2(1-2\theta)T_0 + C_2(1-2\theta)t \right)^{-1/(1-2\theta)} \\ &= \left((E(\bar{u}(T_0) - E(\varphi))^{2\theta-1} - C_2(1-2\theta)T_0 + C_2(1-2\theta)\frac{t}{2} + C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)} \\ &\leq \left(C_2(1-2\theta)\frac{t}{2} \right)^{-1/(1-2\theta)} \quad \text{for all } t \geq \overline{T_0}, \text{ with some } \overline{T_0} > T_0 \text{ large enough.} \end{aligned}$$

It follows that for all $t \geq \overline{T_0}$

$$\int_t^\infty \|\bar{u}_t(s)\|^2 \, ds \leq K t^{-2\alpha-1}$$

Here, $K := \left(\dfrac{C_2(1-2\theta)}{2} \right)^{-1/(1-2\theta)}$ and $\alpha := \dfrac{\theta}{1-2\theta} > 0$. Now, according to Lemma 3.11, we obtain

$$\|\bar{u}(t) - \varphi\|_H \leq \frac{\sqrt{K}}{1-2^{-\alpha}} t^{-\alpha} \quad \text{for all } t \geq \overline{T_0}.$$

□

3.3.2 When Lojasiewicz exponent $\theta = \frac{1}{2}$

Lemma 3.12 (see [22], Lemma 2.2)**.** *Assume that there exists two constants $\gamma > 0$ and $a > 0$ such that for all $t \in [0, T]$,*

$$\int_t^{+\infty} \|u_t\|_H^2 \, ds \leq a \exp(-\gamma t).$$

Then for all $\tau \geq t \geq 0$, we have

$$\|u(t) - u(\tau)\|_H \leq \sqrt{a} b \exp(-\frac{\gamma t}{2}).$$

Theorem 3.5. *Assume that Theorem 3.2 holds for $\theta \in (0, \frac{1}{2})$, then for constants $K, \delta > 0$, we have*

$$\|u(t) - \varphi\|_H \leq K \exp(-\delta t) \quad \text{for all } t > 0.$$

Proof. As in the proof of Theorem 3.3, we only need to prove this theorem for function $\bar{u}$ such that $\int_\Omega \bar{u} dx = 0$. Since $\bar{u}(x,t)$ is smooth for all $t > 0$, we have

$$\frac{d}{dt}(E(\bar{u}) - E(\varphi)) = \langle E'(\bar{u}), \bar{u}_t \rangle = -\langle E'(\bar{u}), E'(\bar{u}) \rangle = -\|E'(\bar{u})\|_H^2. \tag{3.27}$$

Since $\bar{u}(t)$ tends to φ as $t \to \infty$. Therefore there exists $T_0 > 0$ such that for all $t \geq T_0$

$$\|\bar{u}(t) - \varphi\| \leq \sigma \quad (\sigma \text{ in Theorem 3.2}).$$

It follows that for all $t \geq T_0$
$$C\|E'(\bar{u})\|_{V^*} \geq |E(\bar{u}) - E(\varphi)|^{\frac{1}{2}}.$$
Therefore, by using the continuous embedding $H \hookrightarrow V^*$, we obtain
$$C_1\|E'(\bar{u})\|_H \geq |E(\bar{u}) - E(\varphi)|^{\frac{1}{2}} = (E(\bar{u}) - E(\varphi))^{\frac{1}{2}}. \tag{3.28}$$
Combining (3.27) and (3.28), we get
$$\frac{d}{dt}(E(\bar{u}) - E(\varphi)) \leq -C_2(E(\bar{u}) - E(\varphi)) \text{ for all } t \geq T_0,$$
where $C_2 := 1/C_1^2$.

Note that $y(t) := \Bigg(E(\bar{u}(T_0) - E(\varphi)\Bigg) \exp(-C_2(t - T_0))$ is the unique solution of the differential equation
$$\begin{cases} \dfrac{d}{dt}y(t) = -C_2 y \text{ for } t \geq T_0, \\ \\ y(T_0) = E(\bar{u}(T_0) - E(\varphi)). \end{cases}$$
We use a differential inequality in [20, Lemma 2.7, p.53] to deduce for all $t \geq T_0$ that
$$E(\bar{u}(t)) - E(\varphi) \leq \Bigg(E(\bar{u}(T_0) - E(\varphi)\Bigg) \exp(-C_2(t - T_0)).$$
In particular for all $t \geq T_0$,
$$\int_t^\infty \|\bar{u}_t(s)\|_H^2\, ds \leq \Bigg(E(\bar{u}(T_0) - E(\varphi)\Bigg) \exp(-C_2(t - T_0)).$$
Now, Using Lemma 3.12, we deduce the result of the theorem. □

Chapter II

The mass conserved Allen Cahn problem with a logarithmic nonlinearity

1 The problem

We propose to prove the existence of a global attractor for the mass conserved Allen Cahn problem. This problem was proposed by J. Rubinstein and P. Sternberg [33], as a model of a binary mixture undergoing phase separation. Whereas they choose a smooth reaction term f, we propose a singular form of the reaction term. More precisely, we study the problem

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f(u)dx, & x \in \Omega, t > 0, \quad (1.1)\\ \partial_\nu u = 0 & x \in \partial\Omega, t \geq 0 \quad (1.2)\\ u(x,0) = u_0(x) & x \in \Omega, \quad (1.3)\end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with outer unit normal ν and total volume $|\Omega|$. This model is mass conserved, namely

$$\frac{1}{|\Omega|}\int_\Omega u(x,t)dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx = M.$$

The nonlinearity $f = -F'$, with $0 < \theta < \theta_c$ a critical temperature, is such that

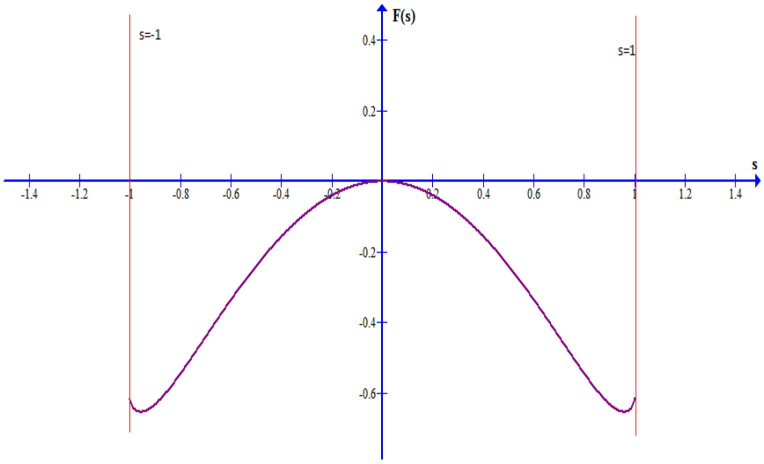
$$F(s) = -\frac{\theta_c}{2}s^2 + \frac{\theta}{2}\Phi(s),$$
$$\Phi(s) = (1+s)\ln(1+s) + (1-s)\ln(1-s), \text{ for } s \in (-1,1),$$

so that

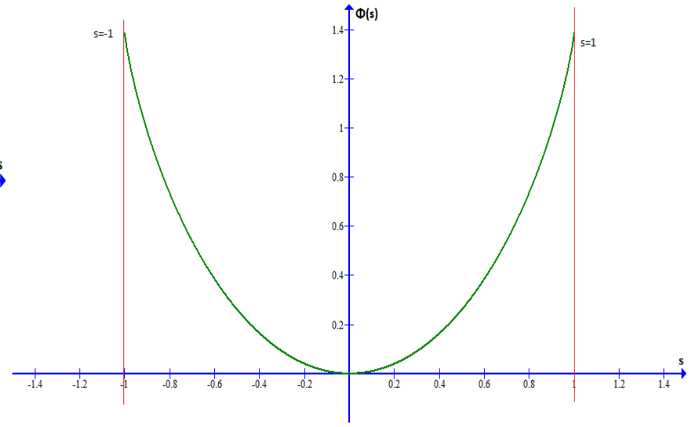
$$f(s) = \theta_c s - \frac{\theta}{2}\varphi(s),$$
$$\varphi(s) = \ln\left(\frac{1+s}{1-s}\right), \text{ for } s \in (-1,1),$$

and then problem (P) will have the form

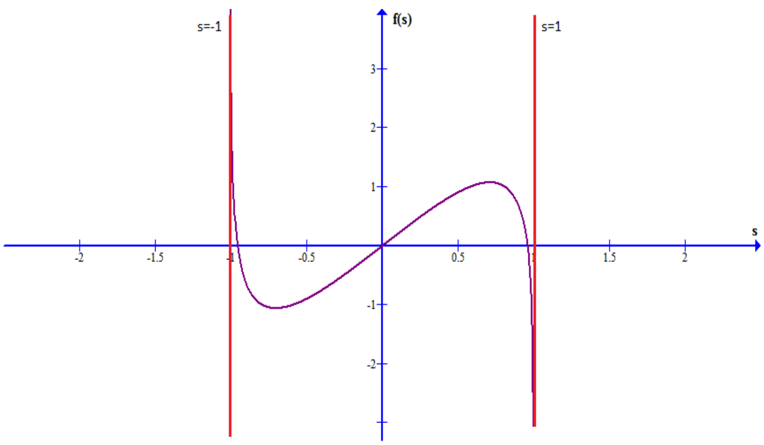
$$(P)\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + \theta_c(u-M) - \dfrac{\theta}{2}\left[\varphi(u) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \varphi(u)dx\right], & x \in \Omega, t > 0 \quad (1.4)\\ \partial_\nu u = 0 & x \in \partial\Omega, t \geq 0. \quad (1.5)\\ u(x,0) = u_0(x) & x \in \Omega \quad (1.6)\end{cases}$$



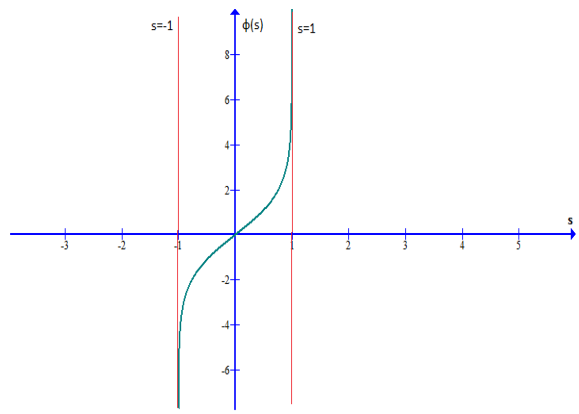
Graphic of the logarithmic potential F



Graphic of Φ



Graphic of the nonlinearity f



Graphic of ϕ

First we prove the global existence and uniqueness of solution for problem (P). We will prove that if u_0 satisfies the hypothesis

$$(H_0) : u_0 - M \in L^2(\Omega), ||u_0||_{L^\infty(\Omega)} \leq 1 \text{ a.e on } \Omega, \text{ and } |M| < 1,$$

then there exists a unique global solution u satisfying $||u(t)||_{L^\infty(\Omega)} \leq 1$. Moreover, the set $\{x \in \Omega, |u(x,t)| = 1\}$ has measure zero. The technique used here is inspired by A. Debussche and L. Dettori [15] and C. Dupaix [16] where in order to avoid the singularities in ± 1, we approximate φ by the sequence of functions

$$\varphi_N(s) = 2\sum_{k=0}^{N} \frac{s^{2k+1}}{2k+1}, N \geq 1, s \in (-1,1) \tag{1.7}$$

which converges to

$$\varphi(s) = 2\sum_{k=0}^{+\infty} \frac{s^{2k+1}}{2k+1} = \ln\left(\frac{1+s}{1-s}\right), s \in (-1,1)$$

as $N \to +\infty$.
Also we define the sequence of functions

$$\Phi_N(s) = 2\sum_{k=0}^{N} \frac{s^{2k+2}}{(2k+2)(2k+1)}, N \geq 2, s \in (-1,1) \tag{1.8}$$

which converges to the function

$$\Phi(s) = 2\sum_{k=0}^{+\infty} \frac{s^{2k+2}}{(2k+2)(2k+1)}, s \in (-1,1)$$

as $N \to +\infty$, and remark that for $s \in (-1,1)$

$$0 \leq \Phi_N(s) \leq \Phi(s) \leq 2\ln 2. \tag{1.9}$$

The regularized problem is then

$$(P_N)\begin{cases} \dfrac{\partial u_N}{\partial t} - \Delta u_N = \theta_c\left(u_N - M\right) - \dfrac{\theta}{2}\left[\varphi_N(u_N) - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \varphi_N(u_N)dx\right] & x \in \Omega, t > 0 \quad (1.10)\\ \partial_\nu u_N = 0 & x \in \partial\Omega, t \geq 0, \quad (1.11)\\ u_N(x,0) = u_0(x) & x \in \Omega, \quad (1.12)\end{cases}$$

with $M = \frac{1}{|\Omega|}\int_\Omega u_N(x,t)dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$, and the solution of problem (P) is obtained by letting $N \to +\infty$ in problem (P_N), where the limit function u has to stay in the interval $(-1,1)$.
We define the Lyapunov functional as

$$E_N(u_N) = \int_\Omega \left(\frac{1}{2}|\nabla u_N|^2 - \frac{\theta_c}{2}u_N^2 + \frac{\theta}{2}\Phi_N(u_N)\right)dx. \tag{1.13}$$

2 Existence of a unique solution to problem (P)

The proof of the existence of a unique global solution of problem (P) is based on several estimates independent of N on the solution u_N of problem (P_N), this will enable us to let N go to $+\infty$ and enforces the limiting phase variable u to belong to $(-1,1)$. More precisely we want to prove the following result

Theorem 2.1. *(i) For each u_0 satisfying (H_0), problem (P) possesses a unique solution u which satisfies*

$$u - M \in L^\infty\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right) \text{ for all } T>0, \text{ and } u - M \in C\left(\mathbb{R}_+;L^2(\Omega)\right).$$

(ii) Moreover, if $u_0 - M \in H^1(\Omega)$, with $||u_0||_{L^\infty(\Omega)} \leq 1$ a.e on Ω and $|M| < 1$, the solution u satisfies

$$u - M \in L^\infty\left(0,T;H^1(\Omega)\right) \cap L^2\left(0,T;H^2(\Omega)\right) \text{ for all } T>0, \text{ and } u - M \in C\left([0,T];H^1(\Omega)\right).$$

Furthermore for all $t > 0$ and both in cases (i) and (ii), $||u(t)||_{L^\infty(\Omega)} \leq 1$, and the set $\{x \in \Omega, |u(x,t)| = 1\}$ has measure zero. The mapping $S(t) : u_0 - M \to u(t) - M$ is Lipschitz continuous on $L^2(\Omega)$ and $(S(t))_{t\geq 0}$ is a semigroup on $L^2(\Omega)$.

2.1 Existence of a unique solution to problem (P_N)

We will give here a complete proof of the existence of a unique solution to the problem

$$(P_N)\begin{cases}\dfrac{\partial u_N}{\partial t}-\Delta u_N=\theta_c\,(u_N-M)-\dfrac{\theta}{2}\left[\varphi_N(u_N)-\dfrac{1}{|\Omega|}\displaystyle\int_\Omega \varphi_N(u_N)dx\right] & x\in\Omega, t>0 \quad (2.1)\\ \partial_\nu u_N=0 & x\in\partial\Omega, t\geq 0 \quad (2.2)\\ u_N(x,0)=u_0(x) & x\in\Omega. \quad (2.3)\end{cases}$$

Theorem 2.2. *(i) For $u_0-M\in L^2(\Omega)$, there exists a unique solution u_N of problem (P_N) satisfying*

$$u_N-M\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap L^{2N+2}(0,T;L^{2N+2}(\Omega)), N\geq 1,$$

with

$$(u_N-M)_t\in L^2(0,T;(H^1(\Omega))')\ \textit{for all}\ T>0,$$

and

$$u_N-M\in C([0,+\infty);L^2(\Omega)).$$

The mapping $u_0\to u_N(t)$ is Lipschitz continuous on $L^2(\Omega)$.

(ii) If furthermore $u_0-M\in H^1(\Omega)\cap L^{2N+2}(\Omega)$, then $u_N-M\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))$, and $u_N-M\in C([0,T];H^1(\Omega))$.

Before proving this result, we remark that the polynomial $-\frac{\theta}{2}\varphi_N(s), s\in(-1,1)$ is such that $-\frac{\theta}{2}\varphi_N(s)=\sum_{k=0}^{N}a_k s^{2k+1}$, with $a_N<0, N\geq 1$ and $s\in(-1,1)$ so that it satisfies
$\exists C_i>0,$ for $i=1,\cdots,7$, such that

(P_1) $-C_2.(s-M)^{2N+2}-C_3\leq -\frac{\theta}{2}(s-M)\varphi_N(s)\leq -C_1.(s-M)^{2N+2}+C_3,$ for $N\geq 1, s\in(-1,1)$

(P_2) $-\frac{\theta}{2}\varphi_N'(s)\leq C_4,$ for $N\geq 1, s\in(-1,1),$

(P_3) $-C_5((s-M)^{2N+2}+1)\leq -\frac{\theta}{2}\Phi_N(s)\leq -C_6((s-M)^{2N+2}-1),$ with $N\geq 1, s\in(-1,1).$

(P_4) $|\frac{\theta}{2}\varphi_N(s)|\leq C_7(|s-M|^{2N+1}+1),$ for $N\geq 1, s\in(-1,1).$

Proof. (i) The proof relies on a Galerkin method, where we denote by $0=\lambda_1<\lambda_2\leq\cdots\leq\lambda_i\leq\cdots$ the eigenvalues of the operator $A=-\Delta:H^1(\Omega)\to(H^1(\Omega))'$ associated to the bilinear form

$$a(u,v)=\int_\Omega \nabla u.\nabla v dx,$$

with a homogenous Neumann condition, and denote by $\omega_i \in H^1(\Omega) \cap L^{2N+2}(\Omega), i = 1, \cdots$ the corresponding unit eigenfunctions. For each integer m we look for an approximate solution $u_N^m - M$ of the form

$$u_N^m(t) - M = \sum_{i=1}^{m} d_{im}(t)\omega_i,$$

satisfying

$$(\frac{\partial}{\partial t}(u_N^m - M), \omega_i) + a(u_N^m - M, \omega_i) = \theta_c(u_N^m - M, \omega_i) - \frac{\theta}{2}(\left[\varphi_N(u_N^m) - \frac{1}{|\Omega|}(\varphi_N(u_N^m), 1)\right], \omega_i), \tag{2.4}$$

for $i = 1, ..., m$

and $u_N^m(0) = u_{m0} \to u_0 \in L^2(\Omega)$ as $m \to +\infty$. (2.5)

Problem (2.4) is an initial value problem for m ordinary differential equations, so by standard argument we can state the existence of a unique solution on $(0, T_m), T_m > 0$.
We multiply (2.4) by $d_{im}(t)$ and sum on $i = 1, \cdots, m$ to obtain

$$(\frac{\partial}{\partial t}(u_N^m - M), u_N^m - M) + a(u_N^m - M, u_N^m - M) = \theta_c(u_N^m - M, u_N^m - M) - \frac{\theta}{2}(\varphi_N(u_N^m), u_N^m - M),$$

or

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N^m - M)^2 dx\right) + \int_\Omega |\nabla(u_N^m - M)|^2 \, dx = \theta_c \int_\Omega (u_N^m - M)^2 dx - \frac{\theta}{2}\int_\Omega (u_N^m - M)\varphi_N(u_N^m) dx, \tag{2.6}$$

where

$$(\frac{1}{|\Omega|}, u_N^m - M) = \frac{1}{|\Omega|}\int_\Omega u_N^m dx - M = 0.$$

We know that

$$\theta_c \int_\Omega (u_N^m - M)^2 dx = \theta_c \int_\Omega (u_N^m)^2 dx - \theta_c|\Omega|M^2,$$

and in view of the definition of the function Φ_N we have that

$$\int_\Omega (u_N^m)^2 dx \leq 2\sum_{k=0}^{N} \frac{1}{(2k+2)(2k+1)} \int_\Omega (u_N^m)^{2k+2} dx = \int_\Omega \Phi_N(u_N^m) dx, \tag{2.7}$$

where we have estimated Φ_N from below by the first term of its expansion; thus

$$\theta_c \int_\Omega (u_N^m)^2 dx \leq \theta_c \int_\Omega \Phi_N(u_N^m) dx;$$

and by (1.9) equation (2.6) yields

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N^m - M)^2 dx\right) + \int_\Omega |\nabla(u_N^m - M)|^2 \, dx \leq 2\theta_c|\Omega| \ln 2 - \theta_c|\Omega|M^2 - \frac{\theta}{2}\int_\Omega (u_N^m - M)\varphi_N(u_N^m) dx, \tag{2.8}$$

then by property (P_1) it will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N^m - M)^2 dx\right) + \int_\Omega |\nabla(u_N^m - M)|^2 \, dx + C_1 \int_\Omega (u_N^m - M)^{2N+2} dx \leq C_3, \tag{2.9}$$

where $C_3 = C_3|\Omega| + 2\theta_c|\Omega| \ln 2 - \theta_c|\Omega| M^2$, integrating (2.9) from 0 to T gives

$$\frac{1}{2}\int_\Omega (u_N^m(x,T) - M)^2 dx + \int_0^T \int_\Omega |\nabla(u_N^m - M)|^2 \, dxds + C_1 \int_0^T \int_\Omega (u_N^m - M)^{2N+2} dxds \leq$$
$$\leq \frac{1}{2}\int_\Omega (u_0 - M)^2 dx + C_3 T, \tag{2.10}$$

so that, if we set $\kappa = \frac{1}{2}\int_\Omega (u_0 - M)^2 dx + C_3 T$, there holds

$$\sup_{t\in[0,T]} \left(\int_\Omega (u_N^m(x,t) - M)^2 dx\right) \leq 2\kappa, \int_0^T \int_\Omega |\nabla(u_N^m - M)|^2 \, dxds \leq \kappa, \int_0^T \int_\Omega (u_N^m - M)^{2N+2} dxds \leq \kappa/C_1,$$

so $u_N^m - M$ is bounded independently of m in $L^\infty(0,T;L^2(\Omega))$, $L^2(0,T;H^1(\Omega))$ and in $L^{2N+2}(0,T;L^{2N+2}(\Omega))$. Hence there exists a subsequence of $\{u_N^m\}_{m\geq 1}$ denoted $\{u_N^\mu\}_{\mu\geq 1}$ such that

$$u_N^\mu - M \rightharpoonup u_N - M \text{ in } L^2(0,T;H^1(\Omega)) \text{ and in } L^{2N+2}(0,T;L^{2N+2}(\Omega)) \text{ weakly}, \tag{2.11}$$
$$u_N^\mu - M \rightharpoonup u_N - M \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ weak star}, \tag{2.12}$$
$$\varphi_N(u_N^\mu) \rightharpoonup \chi \text{ in } L^{(2N+2)'}(0,T;L^{(2N+2)'}(\Omega)) \text{ weakly, where } (2N+2)' = \frac{2N+2}{2N+1}, N \geq 1. \tag{2.13}$$

The last result is given by properties (P_1) and (P_4) where

$$\begin{aligned}
||\varphi_N(u_N^\mu)||^{(2N+2)'}_{L^{(2N+2)'}(0,T;L^{(2N+2)'}(\Omega))} &= \int_0^T \int_\Omega |\varphi_N(u_N^\mu)|^{(2N+2)'} dxds \leq \\
&\leq C_7 \int_0^T \left(\int_\Omega |u_N^\mu - M|^{(2N+1)} + 1\right)^{(2N+2)'} dxds \\
&\leq C \int_0^T \int_\Omega |u_N^\mu - M|^{2N+2} dxds,
\end{aligned}$$

which means that the bound on $(u_N^\mu - M)$ in $L^{(2N+2)}(0,T;L^{(2N+2)}(\Omega))$ gives a bound on $\varphi_N(u_N^\mu)$ in $L^{(2N+2)'}(0,T;L^{(2N+2)'}(\Omega))$.
Thus passing to the limit in (2.4), we find

$$((u_N - M)_t, v) + a(u_N - M, v) = \theta_c(u_N - M, v) - \frac{\theta}{2}(\chi - \frac{1}{|\Omega|}(\chi,1), v), \tag{2.14}$$
$$\forall v \in H^1(\Omega) \cap L^{(2N+2)}(\Omega)$$

which shows that

$$\frac{\partial(u_N - M)}{\partial t} = -A(u_N - M) + \theta_c(u_N - M) - \frac{\theta}{2}(\chi - \frac{1}{|\Omega|}(\chi,1)).$$

Moreover $(u_N - M)_t$ is in $L^2(0,T;(H^1(\Omega))')$ which is in duality with $L^2(0,T;H^1(\Omega))$, this allows us to apply the compactness theorem (see [35] Lemma 3.2 p 71) to deduce that $u_N - M \in C([0,T];L^2(\Omega))$.
It remains to check that $\chi = \varphi_N(u_N)$.
Besides the results (2.11), (2.12) and (2.13), we can state by theorem 8.1 p 214 in [32] that the subsequence $\{u_N^\mu\}_{\mu\geq 1}$ is relatively compact in $L^2(0,T;L^2(\Omega))$, so there exists a subsequence of $\{u_N^\mu\}_{\mu\geq 1}$ denoted $\{u_N^{\mu'}\}_{\mu'\geq 1}$ such that

$$u_N^{\mu'} \to u_N \text{ in } L^2(0,T;L^2(\Omega)),$$

so by Corollary 1.2 p 27 in [32] we can say that

$$u_N^{\mu'} \to u_N \text{ a.e. in } \Omega\times(0,+\infty).$$

Then, as φ_N is continuous

$$\varphi_N(u_N^{\mu'}) \to \varphi_N(u_N) \text{ in } \Omega\times(0,+\infty),$$

where $\{\varphi_N(u_N^{\mu'})\}_{\mu'\geq 1}$ is bounded in $L^{(2N+2)'}(0,T;L^{(2N+2)'}(\Omega))$ with $(2N+2)' = \frac{2N+2}{2N+1}$, so applying lemma 8.3 p 218 in [32] we obtain that

$$\varphi_N(u_N^{\mu'}) \rightharpoonup \varphi_N(u_N) \text{ in } L^{(2N+2)'}(0,T;L^{(2N+2)'}(\Omega)),$$

but in (2.13) we had

$$\varphi_N(u_N^{\mu}) \rightharpoonup \chi \text{ in } L^{(2N+2)'}(0,T;L^{(2N+2)'}(\Omega)),$$

so by the uniqueness of the weak limit $\chi = \varphi_N(u_N)$ a.e. in $\Omega\times(0,+\infty)$.
To check that $u_N(0) = u_0$, let's choose $\phi \in C^1(0,T;H^1(\Omega)\cap L^{2N+2}(\Omega))$ with $\phi(T) = 0$ and so $\phi \in L^{2N+2}(0,T;H^1(\Omega))\cap L^{2N+2}(0,T;L^{2N+2}(\Omega))$, and using

$$(\frac{\partial u_N}{\partial t}, v) + a(u_N, v) = \theta_c(u_N - M, v) - \frac{\theta}{2}([\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)], v), \forall v \in H^1(\Omega)\cap L^{(2N+2)}(\Omega)$$

which we integrate by parts in the t variable in $[0,T]$ and for $v = \phi$ we know that

$$\begin{aligned}\int_0^T (\frac{\partial u_N}{\partial t}, \phi) &= (u_N, \phi)|_0^T - \int_0^T (u_N, \phi')ds \\ &= -(u_N(0), \phi(0)) - \int_0^T (u_N, \phi')ds,\end{aligned}$$

yields

$$\begin{aligned}\int_0^T -(u_N, \phi') + \int_0^T a(u_N, \phi)ds = \theta_c \int_0^T (u_N - M, \phi)ds - \\ -\frac{\theta}{2}\int_0^T ([\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)], \phi)ds + (u_N(0), \phi(0)).\end{aligned}$$

Doing the same in (2.4) yields

$$\int_0^T -(u_N^m,\phi') + \int_0^T a(u_N^m,\phi)ds = \theta_c \int_0^T (u_N^m - M,\phi)ds-$$
$$-\frac{\theta}{2}\int_0^T ([\varphi_N(u_N^m) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N^m)],\phi)ds + (u_N(0),\phi(0)),$$

passing to the limit for m gives

$$\int_0^T -(u_N,\phi') + \int_0^T a(u_N,\phi)ds = \theta_c \int_0^T (u_N - M,\phi)ds-$$
$$-\frac{\theta}{2}\int_0^T ([\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)],\phi)ds + (u_0,\phi(0)),$$

by identification we can conclude that $u_N(0) = u_0$.
To prove the uniqueness of the solution u_N of problem (P_N), suppose the existence of two solutions u_N and $\widetilde{u}_N$ satisfying problem (P_N), then take $v_N = u_N - \widetilde{u}_N$ where $\frac{1}{|\Omega|}\int_\Omega v_N dx = \frac{1}{|\Omega|}\int_\Omega u_0 dx - \frac{1}{|\Omega|}\int_\Omega \widetilde{u_0} dx = M' = M - \widetilde{M}$, v_N satisfies then

$$\frac{\partial v_N}{\partial t} - \Delta v_N = \theta_c(v_N - M') - \frac{\theta}{2}\left[\{\varphi_N(u_N) - \varphi_N(\widetilde{u}_N)\} - \frac{1}{|\Omega|}\int_\Omega \{\varphi_N(u_N) - \varphi_N(\widetilde{u}_N)\}dx\right]. \quad (2.15)$$

Multiplying equation (2.15) by v_N and integrating over Ω, then applying the Green's formula with boundary condition, the result will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega v_N^2 dx\right) + \int_\Omega |\nabla v_N|^2 dx = \theta_c \int_\Omega v_N^2 dx - \theta_c|\Omega|M'^2-$$
$$-\frac{\theta}{2}\int_\Omega v_N(\varphi_N(u_N) - \varphi_N(\widetilde{u}_N))dx + \frac{\theta}{2}M'\int_\Omega (\varphi_N(u_N) - \varphi_N(\widetilde{u}_N))dx,$$

and if we set $\psi(t) = \int_\Omega (\varphi_N(u_N) - \varphi_N(\widetilde{u}_N))dx$ and then use property (P_2) it will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega v_N^2 dx\right) + \int_\Omega |\nabla v_N|^2 dx \le (\theta_c + C_4)\int_\Omega v_N^2 dx - \theta_c|\Omega|M'^2 + \frac{\theta}{2}M'\psi(t),$$

thus

$$\frac{d}{dt}\left(\int_\Omega v_N^2 dx\right) \le \gamma \int_\Omega v_N^2 dx - 2\theta_c|\Omega|M'^2 + \theta M'\psi(t),$$

where $\gamma = 2(\theta_c + C_4)$, then applying Gronwall's lemma to this inequality yields

$$\int_\Omega v_N^2 dx \le \left(\int_\Omega v_0^2 dx\right)e^{\gamma t} + M'\int_0^t (\theta\psi(t) - 2\theta_c|\Omega|M')e^{\gamma(t-s)}ds,$$

and so the uniqueness is given for $u_0 = \widetilde{u_0}$.

(ii) To prove the second part of this theorem, let's multiply equation (2.4) by $\lambda_i d_{im}$ and summing on $i = 1, \cdots, m$, the result will be thanks to property (P_2)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_N|^2 dx + \int_\Omega (\Delta u_N)^2 dx = \theta_c \int_\Omega |\nabla u_N|^2 dx - \frac{\theta}{2}\int_\Omega \varphi_N'(u_N)|\nabla u_N|^2 dx \leq$$
$$\leq (\theta_c + C_4)\int_\Omega |\nabla u_N|^2 dx = \mu \int_\Omega |\nabla u_N|^2 dx, \tag{2.16}$$

for $\mu = \theta_c + C_4$. So by Gronwall's lemma this yields

$$\int_\Omega |\nabla u_N|^2 dx \leq \left(\int_\Omega |\nabla u_0|^2 dx\right) e^{2\mu t}.$$

Again from relation (2.16), we can assert that

$$\int_0^T \int_\Omega (\Delta u_N)^2 dx ds \leq e^{2\mu T}\int_\Omega |\nabla u_0(x)|^2 dx + \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2 dx.$$

□

2.2 Uniform a priori estimates for the solution of problem (P_N)

In what follows we will obtain a uniform in N a priori estimates to problem (P_N), which will enable us to show that the complete trajectories of $(S(t)u_0)_{t\in\mathbb{R}}$ of problem (P) enter a complete set. In this first result we will prove that the energy $E_N(u_N)$ enters an absorbing ball independently of N.

Lemma 2.1. *(i) For each u_0 satisfying hypothesis (H_0), there exist a positive constant D_0 and a time $t_0 = t_0(M, ||u_0 - M||_{L^2(\Omega)})$ which do not depend on N such that*

$$||u_N(t) - M||^2_{L^2(\Omega)} \leq D_0 \text{ for all } t \geq t_0. \tag{2.17}$$

(ii) For each $u_0 - M \in H^1(\Omega)$ with $||u_0||_{L^\infty(\Omega)} \leq 1$ a.e. on Ω, and $|M| < 1$, there exist a positive constant D_1 and a time t_1 which do not depend on N such that

$$||u_N(t)||^2_{H^1(\Omega)} \leq D_1 \text{ and } E_N(u_N)(t) \leq D_1 \text{ for all } t \geq t_1, \tag{2.18}$$

and also

$$\int_{t_1}^{+\infty}\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx ds \leq D_1. \tag{2.19}$$

Proof. (i) Let's multiply equation (2.1) by $u_N - M$ and integrate in space, the result will be due to the mass conservation and an application of the Green's formula

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N - M)^2 dx\right) + \int_\Omega |\nabla(u_N - M)|^2 dx = \theta_c \int_\Omega (u_N - M)^2 dx - \frac{\theta}{2}\int_\Omega (u_N - M)\varphi_N(u_N) dx, \tag{2.20}$$

and using the fact that $\int_\Omega |\nabla(u_N - M)|^2 dx \geq 0$ gives

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N - M)^2 dx\right) - \theta_c \int_\Omega (u_N - M)^2 dx + \frac{\theta}{2}\int_\Omega (u_N - M)\varphi_N(u_N) dx \leq 0,$$

where

$$\int_\Omega (u_N - M)\varphi_N(u_N)dx = \int_\Omega u_N\varphi_N(u_N)dx - \int_\Omega M\varphi_N(u_N)dx$$

so the result will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N - M)^2 dx\right) - \theta_c \int_\Omega (u_N - M)^2 dx + \frac{\theta}{2}\int_\Omega u_N\varphi_N(u_N)dx \le \frac{\theta}{2}\int_\Omega M\varphi_N(u_N)dx, \quad (2.21)$$

but

$$M\varphi_N(u_N) = 2\sum_{k=0}^{N}\frac{Mu_N^{2k+1}}{2k+1}$$

then (2.21) will have the form

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N - M)^2 dx\right) - \theta_c \int_\Omega (u_N - M)^2 dx + \frac{\theta}{2}\int_\Omega u_N\varphi_N(u_N)dx \le \theta\sum_{k=0}^{N}\frac{M}{2k+1}\int_\Omega u_N^{2k+1}dx.$$

We apply the Hölder's inequality with $\frac{1}{p} = \frac{2k+1}{2k+2}$ and $\frac{1}{q} = \frac{1}{2k+2}$ to obtain

$$\int_\Omega u_N^{2k+1}dx \le \left[\int_\Omega u_N^{(2k+1)\frac{2k+2}{2k+1}}dx\right]^{\frac{2k+1}{2k+2}} |\Omega|^{\frac{1}{2k+2}},$$

so that

$$\theta\sum_{k=0}^{N}\frac{M}{2k+1}\int_\Omega u_N^{2k+1}dx \le \theta\sum_{k=0}^{N}\frac{M|\Omega|^{\frac{1}{2k+2}}}{2k+1}\left(\int_\Omega u_N^{2k+2}\right)^{\frac{2k+1}{2k+2}}. \quad (2.22)$$

Next we apply the Young's inequality with $p = 2k+2$ and $q = \frac{2k+2}{2k+1}$ to deduce that

$$M|\Omega|^{\frac{1}{2k+2}}\left(\int_\Omega u_N^{2k+2}dx\right)^{\frac{2k+1}{2k+2}} \le \frac{M^{2k+2}|\Omega|}{2k+2} + \frac{2k+1}{2k+2}\int_\Omega u_N^{2k+2}dx.$$

Thus

$$\theta\sum_{k=0}^{N}\frac{M|\Omega|^{\frac{1}{2k+2}}}{2k+1}\left(\int_\Omega u_N^{2k+2}dx\right)^{\frac{2k+2}{2k+1}} \le \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+1)(2k+2)} + \theta\sum_{k=0}^{N}\frac{1}{2k+2}\int_\Omega u_N^{2k+2}dx,$$

so that

$$\theta\sum_{k=0}^{N}\frac{M}{2k+1}\int_\Omega u_N^{2k+1}dx \le \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+1)(2k+2)} + \theta\sum_{k=0}^{N}\frac{1}{2k+2}\int_\Omega u_N^{2k+2}dx, \quad (2.23)$$

this development yields

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N - M)^2 dx\right) - \theta_c \int_\Omega (u_N - M)^2 dx + \frac{\theta}{2}\int_\Omega u_N\varphi_N(u_N)dx \le \theta\sum_{k=0}^{N}\frac{1}{2k+2}\int_\Omega u_N^{2k+2}dx + $$
$$+ \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+2)(2k+1)},$$

but

$$u_N\varphi_N(u_N) = 2u_N\sum_{k=0}^{N}\frac{u_N^{2k+1}}{2k+1} = 2\sum_{k=0}^{N}\frac{u_N^{2k+2}}{2k+1},$$

and so

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right) - \theta_c\int_\Omega (u_N-M)^2dx \le -\theta\sum_{k=0}^{N}\frac{1}{(2k+2)(2k+1)}\int_\Omega u_N^{2k+2}dx + \\ + \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+2)(2k+1)},$$

which we rewrite as

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right) - \theta_c\int_\Omega (u_N-M)^2dx + \frac{\theta}{2}\int_\Omega \Phi_N(u_N)dx \le \frac{\theta}{2}|\Omega|\Phi_N(M),$$

or as

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right) - \theta_c\int_\Omega (u_N-M)^2dx + \frac{\theta}{4}\int_\Omega \Phi_N(u_N)dx + \frac{\theta}{4}\int_\Omega \Phi_N(u_N)dx \le \frac{\theta}{2}|\Omega|\Phi_N(M). \tag{2.24}$$

But

$$\frac{\theta}{4}\int_\Omega (u_N-M)^2dx = \frac{\theta}{4}\int_\Omega u_N^2dx - \frac{\theta}{4}|\Omega|M^2 = \frac{\theta}{2}\left(\frac{1}{2}\int_\Omega u_N^2dx - \frac{M^2}{2}|\Omega|\right),$$

and due to the positivity of the function Φ we can state that

$$\frac{1}{2}\int_\Omega u_N^2dx \le \sum_{k=0}^{N}\frac{1}{(2k+2)(2k+1)}\int_\Omega u_N^{2k+2}dx,$$

thus

$$\frac{\theta}{4}\int_\Omega (u_N-M)^2dx \le \frac{\theta}{4}\int_\Omega \Phi_N(u_N)dx - \frac{\theta}{4}M^2|\Omega|. \tag{2.25}$$

And then (2.24) implies

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right) - \theta_c\int_\Omega (u_N-M)^2dx + \frac{\theta}{4}\int_\Omega (u_N-M)^2dx + \frac{\theta}{4}\int_\Omega \Phi_N(u_N)dx \le \\ \le \frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right) - \theta_c\int_\Omega (u_N-M)^2dx + \frac{\theta}{4}\int_\Omega \Phi_N(u_N)dx + \frac{\theta}{4}\int_\Omega \Phi_N(u_N)dx - \frac{\theta}{4}M^2|\Omega| \le \\ \le \frac{\theta}{2}|\Omega|\Phi_N(M) - \frac{\theta}{4}M^2|\Omega|. \tag{2.26}$$

So if we set $C_1 = \frac{\theta}{2}|\Omega|\left(\Phi_N(M) - \frac{1}{2}M^2\right)$, (2.26) will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right) + \frac{\theta}{4}\int_\Omega (u_N-M)^2dx - \theta_c\int_\Omega (u_N-M)^2dx + \frac{\theta}{4}\int_\Omega \Phi_N(u_N)dx \le C_1, \tag{2.27}$$

where by (2.25) and the fact that $0 \le \Phi_N(s) \le \Phi(s) \le 2\ln 2$, we can say that for $C_2 = 2\theta_c|\Omega|\ln 2$

$$-\theta_c\int_\Omega (u_N-M)^2dx + \frac{\theta}{4}\int_\Omega \Phi_N(u_N)dx \ge -C_2,$$

thus

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right)+\frac{\theta}{4}\int_\Omega (u_N-M)^2dx - C_2 \le \frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right)+$$
$$+\frac{\theta}{4}\int_\Omega (u_N-M)^2dx-\theta_c\int_\Omega (u_N-M)^2dx+\frac{\theta}{4}\int_\Omega \Phi_N(u_N)dx \le C_1$$

that means that for $C_3 = 2(C_1+C_2)$

$$\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right) \le -\frac{\theta}{2}\int_\Omega (u_N-M)^2dx + C_3. \tag{2.28}$$

Applying the Gronwall's lemma to (2.28) will give the following result

$$\int_\Omega (u_N-M)^2dx \le e^{-\frac{\theta}{2}t}\int_\Omega (u_0(x)-M)^2dx+\frac{2}{\theta}C_3\left[1-e^{-\frac{\theta}{2}t}\right]. \tag{2.29}$$

We deduce from (2.29) that any ball of $L^2(\Omega)$ centered at 0 and of radius $\rho_2 > \rho_1 = \left(\frac{2}{\theta}C_3\right)^{1/2}$ is an absorbing set in $L^2(\Omega)$. Indeed if $\mathcal{B}_0$ is a bounded set of $L^2(\Omega)$, included in a ball $B(0,R)$ of $L^2(\Omega)$ centered at 0 of radius R, then $S(t)\mathcal{B}_0 \subset B(0,\rho_2)$ for $t \ge t_0 = \frac{2}{\theta}\ln\left(\frac{R^2}{\rho_2^2-\rho_1^2}\right)$.

(ii) In the beginning let's multiply equation (2.1) by $-\Delta u_N$ and integrate it on Ω, the resulting equation will be after application of the Green's formula

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_N|^2dx+\int_\Omega (\Delta u_N)^2dx = \theta_c\int_\Omega |\nabla u_N|^2dx - \frac{\theta}{2}\int_\Omega \varphi_N'(u_N)|\nabla u_N|^2dx.$$

But $\varphi_N'(s) \ge 2,$ for all $s \in (-1,1)$, and $0<\theta<\theta_c$, so

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_N|^2dx+\int_\Omega (\Delta u_N)^2dx \le (\theta_c-\theta)\int_\Omega |\nabla u_N|^2dx, \tag{2.30}$$

we want to apply the uniform Gronwall's lemma to

$$\frac{d}{dt}\int_\Omega |\nabla u_N|^2dx \le 2(\theta_c-\theta)\int_\Omega |\nabla u_N|^2dx, \tag{2.31}$$

for $g(t) = 2\left(\theta_c-\theta\right)$, $y(t)=\int_\Omega |\nabla u_N|^2dx$ and $h(t)=0$, where we had from part (i) of this lemma

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right)+\int_\Omega |\nabla(u_N-M)|^2dx = \theta_c\int_\Omega (u_N-M)^2dx-\frac{\theta}{2}\int_\Omega (u_N-M)\varphi_N(u_N)dx,$$

with $t \ge t_0$ where $t_0=\frac{2}{\theta}\ln\left(\frac{R^2}{\rho_2^2-\rho_1^2}\right)$ and $\rho_2>\rho_1=\left(\frac{2}{\theta}C_3\right)^{1/2}$, such that

$$\int_\Omega (u_N-M)^2dx \le \rho_2^2, \text{ for } u_0 \in B(0,R),$$

thus

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right)+\int_\Omega |\nabla(u_N-M)|^2dx \le \theta_c\rho_2^2-\frac{\theta}{2}\int_\Omega (u_N-M)\varphi_N(u_N)dx,$$

and

$$-\frac{\theta}{2}\int_\Omega (u_N-M)\varphi_N(u_N)dx = -\frac{\theta}{2}\int_\Omega u_N\varphi_N(u_N)dx + \theta\sum_{k=0}^{N}\frac{M}{2k+1}\int_\Omega u_N^{2k+1}dx,$$

which with (2.23) yields

$$-\frac{\theta}{2}\int_\Omega (u_N-M)\varphi_N(u_N)dx \le -\frac{\theta}{2}\int_\Omega u_N\varphi_N(u_N)dx+\theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+2)(2k+1)}+\theta\sum_{k=0}^{N}\frac{1}{2k+2}\int_\Omega u_N^{2k+2}dx,$$

and since

$$u_N\varphi_N(u_N) = 2u_N\sum_{k=0}^{N}\frac{u_N^{2k+1}}{2k+1} = 2\sum_{k=0}^{N}\frac{u_N^{2k+2}}{2k+1}, \tag{2.32}$$

it will be

$$-\frac{\theta}{2}\int_\Omega (u_N-M)\varphi_N(u_N)dx \le \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+2)(2k+1)} - \theta\sum_{k=0}^{N}\frac{1}{(2k+2)(2k+1)}\int_\Omega u_N^{2k+2}dx =$$
$$= \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+2)(2k+1)} - \theta\int_\Omega \Phi_N(u_N)dx,$$

but $\Phi_N(u_N)\ge 0$ thus

$$-\frac{\theta}{2}\int_\Omega (u_N-M)\varphi_N(u_N)dx \le \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+2)(2k+1)},$$

so finally

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right) + \int_\Omega |\nabla(u_N-M)|^2dx \le \theta_c\rho_2^2 - \frac{\theta}{2}\int_\Omega (u_N-M)\varphi_N(u_N)dx \le$$
$$\le \theta_c\rho_2^2 + \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+2)(2k+1)},$$

for $t\ge t_0$ and then

$$\int_\Omega |\nabla(u_N-M)|^2dx \le \theta_c\rho_2^2 + \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+2)(2k+1)} - \frac{1}{2}\frac{d}{dt}\left(\int_\Omega (u_N-M)^2dx\right),$$

a uniform integration in time asserts the existence of a_3 such that

$$\int_t^{t+r}\int_\Omega |\nabla(u_N-M)|^2dxds \le \left(\theta_c\rho_2^2 + \theta\sum_{k=0}^{N}\frac{M^{2k+2}|\Omega|}{(2k+2)(2k+1)}\right)r + \frac{1}{2}\left(\int_\Omega (u_N-M)^2(t)dx\right) -$$
$$-\frac{1}{2}\left(\int_\Omega (u_N-M)^2(t+r)dx\right) \le a_3,$$

and the existence of a_1 such that

$$\int_t^{t+r} 2(\theta_c-\theta)ds \le a_1.$$

So applying the uniform Gronwall's lemma to (2.31) gives

$$\int_\Omega |\nabla(u_N - M)(t+r)|^2 dx \leq \frac{a_3}{r} e^{a_1} = D_1, \forall t \geq t_1 = t_0 + r.$$

So for $t \geq t_1$, $u_N - M$ is in $B(0, D_1^{1/2})$, a bounded set in $H^1(\Omega)$. Then for $t \geq t_1$, $S_N(t)$ transforms the bounded sets on $L^2(\Omega)$ in a bounded sets in $H^1(\Omega)$. But $H^1(\Omega)$ is compactly imbedded in $L^2(\Omega)$, these bounded sets are relatively compact in $L^2(\Omega)$. Thus $\mathcal{A}_N = \omega\left(B(0, D_1^{1/2})\right)$ is a global attractor.

To prove the second part, we multiply equation (2.1) by $\dfrac{\partial u_N}{\partial t}$ and then integrate the result in Ω, after an application of the Green's formula, the use of the mass conservation and the fact that $\varphi = \Phi'$ the result will be

$$\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx + \frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\nabla u_N|^2 dx\right) = \frac{\theta_c}{2}\frac{d}{dt}\left(\int_\Omega u_N^2 dx\right) - \frac{\theta}{2}\frac{d}{dt}\left(\int_\Omega \Phi_N(u_N) dx\right),$$

so

$$\frac{d}{dt}E_N(u_N) = -\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx \leq 0,$$

then we can state that

$$\frac{d}{dt}E_N(u_N) \leq 0.E_N(u_N) + 0,$$

where to apply the uniform Gronwall's lemma, we need to to prove that

$$\int_t^{t+r} E_N(u_N) ds \leq a_3, \forall t \geq t_0.$$

Remark that

$$E_N(u_N) = \int_\Omega \left(\frac{1}{2}|\nabla u_N|^2 - \frac{\theta_c}{2}u_N^2 + \frac{\theta}{2}\Phi_N(u_N)\right) dx \leq \frac{1}{2}\int_\Omega |\nabla u_N|^2 dx + \frac{\theta_c}{2}\int_\Omega u_N^2 dx + \frac{\theta}{2}\int_\Omega \Phi_N(u_N) dx,$$

but $0 \leq \Phi_N(s) \leq \Phi(s) \leq \ln(2), \forall s \in [-1, 1]$. And so

$$E_N(u_N) = \int_\Omega \left(\frac{1}{2}|\nabla u_N|^2 - \frac{\theta_c}{2}u_N^2 + \frac{\theta}{2}\Phi_N(u_N)\right) dx \leq \frac{1}{2}\int_\Omega |\nabla u_N|^2 dx + \frac{\theta_c}{2}\int_\Omega u_N^2 dx + \theta|\Omega| \ln 2,$$

and so the proof is given by the preceding results of this lemma.

Finally to prove that

$$\int_{t_1}^{+\infty} \int_\Omega \left(\frac{du_N}{dt}\right)^2 dx ds \leq D_1,$$

remark that we had

$$\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx = -\frac{d}{dt}E_N(u_N)(t),$$

and so the last result can be given.

□

The following results are given, to show later that the trajectories of the attractor are compact in $C([-T, T]; H^2(\Omega))$ for all $T > 0$.

Lemma 2.2. *There exist a positive constant D_2 and a time $t_2 > t_1$ which do not depend on N such that*

(i)

$$\| (u_N)_t (t)\|^2_{L^2(\Omega)} \leq D_2 \text{ for all } t \geq t_2, \tag{2.33}$$

(ii)

$$\int_{t_2}^{+\infty} \|(u_N)_t\|^2_{H^1(\Omega)} dt \leq D_2 \tag{2.34}$$

Proof. (i) We differentiate equation (2.1) with respect to t, multiply the result by $\frac{\partial u_N}{\partial t}$ and then integrate it on Ω, and after an application of the Green's formula the result will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx\right) + \int_\Omega \left|\nabla \frac{du_N}{dt}\right|^2 dx = \theta_c \int_\Omega \left(\frac{du_N}{dt}\right)^2 dx - \frac{\theta}{2}\int_\Omega \varphi_N'(u_N)\left(\frac{du_N}{dt}\right)^2 dx,$$

but

$$\varphi_N'(s) \geq 2, \forall s \in (-1,1),$$

thus

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx\right) + \int_\Omega \left|\nabla \frac{du_N}{dt}\right|^2 dx \leq \theta_c \int_\Omega \left(\frac{du_N}{dt}\right)^2 dx - \theta \int_\Omega \left(\frac{du_N}{dt}\right)^2 dx =$$

$$= (\theta_c - \theta)\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx,$$

where $0 < \theta < \theta_c$, and so

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx\right) + \int_\Omega \left|\nabla \frac{du_N}{dt}\right|^2 dx \leq (\theta_c - \theta)\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx, \tag{2.35}$$

thus

$$\frac{d}{dt}\left(\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx\right) \leq 2(\theta_c - \theta)\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx,$$

to which we will apply the uniform Gronwall's lemma, where thanks to (2.19) we can state that

$$\exists a_3 > 0 \text{ such that } \int_t^{t+r}\int_\Omega \left(\frac{du_N}{dt}\right)^2 dxds \leq a_3, \forall t \geq t_2 > t_1,$$

thus

$$\int_\Omega \left(\frac{du_N}{dt}\right)^2 (t+r)dx \leq \frac{a_3}{r}e^{a_1}, \forall t \geq t_2,$$

where $\int_t^{t+r} 2(\theta_c - \theta)ds = 2(\theta_c - \theta)r \leq a_1$.

So $\displaystyle\int_\Omega \left(\frac{du_N}{dt}\right)^2 (t)dx \leq D_2, \forall t \geq t_2.$

(ii) If we integrate relation (2.35) between t_2 and t we will have

$$\frac{1}{2}\int_\Omega\left(\frac{du_N}{dt}\right)^2(t)dx-\frac{1}{2}\int_\Omega\left(\frac{du_N}{dt}\right)^2(t_2)dx+\int_{t_2}^t\int_\Omega\left|\nabla\frac{du_N}{dt}\right|^2dxds\le(\theta_c-\theta)\int_{t_2}^t\int_\Omega\left(\frac{du_N}{dt}\right)^2dxds,$$

thus for $t\longrightarrow+\infty$

$$\int_{t_2}^{+\infty}\int_\Omega\left|\nabla\frac{du_N}{dt}\right|^2dxds\le D_2,$$

from the preceding result of this lemma.

□

Corollary 2.1. *There exist a positive constant D_3 and a time $t_2>t_1$ which do not depend on N such that*

$$||u_N(t)||^2_{H^2(\Omega)}\le D_3 \text{ for all } t\ge t_2, \tag{2.36}$$

Proof. From the relation (2.30), we can state that

$$\int_\Omega(\Delta u_N)^2dx\le-\frac{1}{2}\frac{d}{dt}\left(\int_\Omega|\nabla u_N|^2dx\right)+(\theta_c-\theta)\int_\Omega|\nabla u_N|^2dx,$$

and so the result is given after a uniform in time integration and application of the preceding lemma.

□

The next results are used to prove the existence of a unique solution to problem (P).

Lemma 2.3. *For any u_0 satisfying hypothesis (H_0), there exists a positive constant D_4, such that*

$$||\varphi_N(u_N(t))-\frac{1}{|\Omega|}\int_\Omega\varphi_N(u_N)||^2_{L^2(\Omega)}\le D_4,\forall t\ge t_3, \tag{2.37}$$

Proof. From the problem form

$$\frac{\partial u_N}{\partial t}-\Delta u_N=\theta_c(u_N-M)-\frac{\theta}{2}\left[\varphi_N(u_N)-\frac{1}{|\Omega|}\int_\Omega\varphi_N(u_N)dx\right],$$

let's define

$$\overline{\varphi_N}(u_N)=\varphi_N(u_N)-\frac{1}{|\Omega|}\int_\Omega\varphi_N(u_N)dx=-\frac{2}{\theta}\frac{\partial u_N}{\partial t}+\frac{2}{\theta}\Delta u_N+\frac{2\theta_c}{\theta}(u_N-M),$$

so

$$(\overline{\varphi_N}(u_N))^2=\left[-\frac{2}{\theta}\frac{\partial u_N}{\partial t}+\frac{2}{\theta}\Delta u_N+\frac{2\theta_c}{\theta}(u_N-M)\right]^2,$$

thus by Cauchy's inequality we can state that there exists $\alpha>0$ such that

$$(\overline{\varphi_N}(u_N))^2\le\alpha\left[\left(\frac{\partial u_N}{\partial t}\right)^2+(\Delta u_N)^2+(u_N-M)^2\right],$$

where by lemma 2.1, lemma 2.2 and corollary 2.1 we can state that

$$\int_\Omega \left(\overline{\varphi_N}(u_N)\right)^2 dx \leq \alpha \left[\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx + \int_\Omega (\Delta u_N)^2 dx + \int_\Omega (u_N - M)^2 dx\right] \leq D_4, \forall t \geq t_3, \tag{2.38}$$

where $D_4 = \alpha(D_0 + D_2 + D_3)$ and $t_3 = \max(t_0, t_2)$.
Moreover we can state that there exists a constant C' such that

$$t||\varphi_N(u_N)||_{L^\infty(0,T;L^2(\Omega))} \leq C'. \tag{2.39}$$

We infer from (2.37), the existence of a constant C, such that for all $t \in (0, T)$

$$t||\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx||_{L^\infty(0,T;L^2(\Omega))} \leq C.$$

Therefore in order to prove (2.39) it suffices to prove that $t|\frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx|$ is bounded.
By contradiction suppose that $t|\frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx|$ is unbounded; then there exit two sequences $(t_k)_{k\in\mathbb{N}}$ and $(N_k)_{k\in\mathbb{N}}$ such that

$$t_k \to t^* \in [0, T], \text{ as } k \to +\infty$$

and

$$t_k \frac{1}{|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k})dx \to +\infty, \text{ as } k \to +\infty \tag{2.40}$$

which will lead to
$\lim_{k\to+\infty} \frac{1}{|\Omega|} \sup \int_\Omega u_{N_k}(x, t_k)dx \geq 1$,
and to
$\lim_{k\to+\infty} \frac{1}{|\Omega|} \sup \int_\Omega u_{N_k}(x, t^*)dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$.
Let's introduce the set

$$E_k = \left\{x \in \Omega, \varphi_{N_k}(u_{N_k}(x)) \geq \frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k})dx\right\},$$

and $F_k = \complement_\Omega^{E_k}$.
But for k large enough, we can state by the relations (2.38) and (2.40) that

$$\begin{aligned}
D_4 &\geq \int_\Omega \left[\overline{\varphi}_{N_k}(u_{N_k}(x, t_k))\right]^2 dx \geq \int_{F_k} \left[\overline{\varphi}_{N_k}(u_{N_k}(x, t_k))\right]^2 dx = \\
&= \int_{F_k} \left[\frac{1}{|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x, t_k))dx - \varphi_{N_k}(u_{N_k}(x, t_k))\right]^2 dx = \\
&= \int_{F_k} \left[\frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x, t_k))dx + \frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x, t_k))dx - \varphi_{N_k}(u_{N_k}(x, t_k))\right]^2 dx,
\end{aligned}$$

but in the set F_k

$$-\varphi_{N_k}(u_{N_k}(x, t_k)) > -\frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x, t_k))dx,$$

so that

$$D_4 \geq \int_{F_k} \left[\frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x, t_k))dx\right]^2 dx = \left(\frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x, t_k))dx\right)^2 \times \int_{F_k} dx,$$

that is

$$D_4 \geq \left(\frac{1}{|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx\right)^2 . \frac{|F_k|}{4},$$

then

$$|F_k| \leq 4D_4 / \left(\frac{1}{|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx\right)^2,$$

thus

$$\lim_{k\to+\infty} |F_k| = 0, \tag{2.41}$$

which yields

$$\lim_{k\to+\infty} |E_k| = |\Omega|.$$

Moreover the function φ is continuous and increasing, and using the definition of the set E_k we can say that

$$x \in E_k \Leftrightarrow \varphi_{N_k}(u_{N_k}(x,t_k)) \geq \frac{1}{2}\left(\frac{1}{|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx\right). \tag{2.42}$$

Since φ is one-to-one, (2.42) implies that

$$u_{N_k}(x,t_k) \geq \varphi^{-1}\left(\frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx\right),$$

so that

$$\begin{aligned}\int_{E_k} u_{N_k}(x,t_k)dx &\geq \varphi_{N_k}^{-1}\left(\frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx\right)\int_{E_k} dx = \varphi_{N_k}^{-1}\left(\frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx\right)|E_k| = \\ &= \varphi_{N_k}^{-1}\left(\frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx\right)(|\Omega| - |F_k|),\end{aligned}$$

and thus, in view of (2.41),

$$\lim_{k\to+\infty}\sup \int_{E_k} u_{N_k}(x,t_k)dx = \lim_{k\to+\infty}\sup \int_\Omega u_{N_k}(x,t_k)dx \geq |\Omega|\left[\lim_{k\to+\infty} \varphi_{N_k}^{-1}\left(\frac{1}{2|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx\right)\right].$$

It follows from the assumption (2.40) that

$$\lim_{k\to+\infty}\sup \frac{1}{|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx = +\infty,$$

which implies that

$$\lim_{k\to+\infty} \varphi^{-1}\left(\frac{1}{|\Omega|}\int_\Omega \varphi_{N_k}(u_{N_k}(x,t_k))dx\right) = \varphi^{-1}(+\infty) = 1.$$

We also have that

$$\lim_{k\to+\infty}\sup\int_{E_k} u_{N_k}(x,t_k)dx \geq |\Omega| \tag{2.43}$$

so that

$$\int_\Omega u_{N_k}(x,t_k)dx = \int_{E_k} u_{N_k}(x,t_k)dx + \int_{F_k} u_{N_k}(x,t_k)dx,$$

and we remark that by Hölder's inequality and Lemma (2.1)

$$\left|\int_{F_k} u_{N_k}(x,t_k)dx\right|^2 \leq \left[\int_{F_k} \left(u_{N_k}(x,t_k)\right)^2 dx\right]|F_k| \leq \left[\int_\Omega \left(u_{N_k}(x,t_k)\right)^2 dx\right]|F_k| \leq C|F_k|,$$

so that

$$\lim_{k\to+\infty}\sup\int_{F_k} u_{N_k}(x,t_k)dx = 0.$$

Thus, also using (2.43), we deduce that

$$\lim_{k\to+\infty}\sup\left(\frac{1}{|\Omega|}\int_\Omega u_{N_k}(x,t_k)dx\right) \geq 1,$$

while by the mass conservation

$$\frac{1}{|\Omega|}\int_\Omega u_0(x)dx = \lim_{k\to+\infty}\sup\left(\frac{1}{|\Omega|}\int_\Omega u_{N_k}(x,t_k)dx\right) \geq 1,$$

which contradicts the hypothesis (H_0).

□

Lemma 2.4. *For each $u_0 - M \in H^1(\Omega)$ with $||u_0||_{L^\infty(\Omega)} \leq 1$ a.e. on Ω and $|M| < 1$, and all $T > 0$ there exists a positive constant $C_0 = C_0\left(T, ||u_0||_{H^1(\Omega)}\right)$ independent of N such that*

(i)

$$||u_N(t)||_{H^1(\Omega)} \leq C_0 \text{ for all } t \in (0,T); \tag{2.44}$$

(ii)

$$||u_N||_{L^2(0,T;H^2(\Omega))} + ||(u_N)_t||_{L^2(0,T;L^2(\Omega))} + ||\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx||_{L^2(0,T;L^2(\Omega))} \leq C_0. \tag{2.45}$$

(iii) Moreover, for an arbitrary δ satisfying $0 < \delta < T$ there exists a positive constant $C_\delta = C_\delta\left(\delta, T, ||u_0||_{H^1(\Omega)}\right)$ which does not depend on N such that

$$||u_N(t)||_{H^2(\Omega)} \leq C_\delta \text{ for all } t \in (\delta, T). \tag{2.46}$$

Proof. (i) From relation (2.30) we have

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega |\nabla u_N|^2 dx\right) \leq (\theta_c - \theta)\int_\Omega |\nabla u_N|^2 dx,$$

thus

$$\frac{d}{dt}\left(\int_\Omega |\nabla u_N|^2 dx\right) \leq 2(\theta_c - \theta)\int_\Omega |\nabla u_N|^2 dx,$$

and so (2.44) is given after application of the Gronwall's lemma, where $C_0^2 = e^{2(\theta_c-\theta)T}\int_\Omega |\nabla u_0(x)|^2 dx$.

(ii) From equation (2.1) we remark that

$$\frac{\partial u_N}{\partial t} - \Delta u_N + \frac{\theta}{2}\left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right] = \theta_c\left(u_N - M\right),$$

thus

$$\left(\frac{\partial u_N}{\partial t} - \Delta u_N + \frac{\theta}{2}\left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right]\right)^2 = \theta_c^2\left(u_N - M\right)^2,$$

which yields after integration on Ω

$$\begin{aligned}\theta_c^2\int_\Omega \left(u_N - M\right)^2 dx = &\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx + \int_\Omega (\Delta u_N)^2 dx + \frac{\theta^2}{4}\int_\Omega \left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right]^2 dx- \\ &- 2\int_\Omega \frac{du_N}{dt}\Delta u_N dx - \theta\int_\Omega \Delta u_N\left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right] dx+ \\ &+ \theta\int_\Omega \frac{du_N}{dt}\left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right] dx,\end{aligned}$$

and so

$$\begin{aligned}\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx + \int_\Omega (\Delta u_N)^2 dx + \frac{\theta^2}{4}\int_\Omega \left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right]^2 dx = \theta_c^2\int_\Omega \left(u_N - M\right)^2 dx+ \\ + 2\int_\Omega \frac{du_N}{dt}\Delta u_N dx + \theta\int_\Omega \Delta u_N\left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right] dx- \\ - \theta\int_\Omega \frac{du_N}{dt}\left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right] dx,\end{aligned}$$

after application of the Green's formula with the boundary condition and the use of the mass conservation property and theorem (2.2), the resulting inequality will be

$$\begin{aligned}\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx + \int_\Omega (\Delta u_N)^2 dx + \frac{\theta^2}{4}\int_\Omega \left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right]^2 dx \leq \theta_c^2 K- \\ - \frac{d}{dt}\left(\int_\Omega |\nabla u_N|^2 dx\right) - \theta\int_\Omega \varphi_N'(u_N)|\nabla u_N|^2 dx - \theta\frac{d}{dt}\left(\int_\Omega \Phi_N(u_N)dx\right),\end{aligned}$$

using the fact that $\varphi_N'(s) \geq 2, \forall s \in (-1,1)$, we obtain the following inequality

$$\begin{aligned}\int_\Omega \left(\frac{du_N}{dt}\right)^2 dx + \int_\Omega (\Delta u_N)^2 dx + \frac{\theta^2}{4}\int_\Omega \left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right]^2 dx \leq \theta_c^2 K- \\ - \frac{d}{dt}\left(\int_\Omega |\nabla u_N|^2 dx\right) + 2\theta\int_\Omega |\nabla u_N|^2 dx - \theta\frac{d}{dt}\left(\int_\Omega \Phi_N(u_N)dx\right),\end{aligned} \tag{2.47}$$

which we integrate in time for $t \in (0,T)$ to have

$$\int_0^T \int_\Omega \left(\frac{du_N}{dt}\right)^2 dxds + \int_0^T \int_\Omega (\Delta u_N)^2 dxds + \frac{\theta^2}{4} \int_0^T \int_\Omega \left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right]^2 dxds \leq$$
$$\leq \theta_c^2 KT - \int_\Omega |\nabla u_N(x,T)|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx + 2\theta \int_0^T \int_\Omega |\nabla u_N|^2 dxds -$$
$$- \theta \int_\Omega \Phi_N(u_N(x,T))dx + \theta \int_\Omega \Phi_N(u_0(x))dx,$$

and using the fact that $0 \leq \Phi_N(s) \leq \Phi(s) \leq 2\ln 2$, with the result (2.44) yields

$$\int_0^T \int_\Omega \left(\frac{du_N}{dt}\right)^2 dxds + \int_0^T \int_\Omega (\Delta u_N)^2 dxds + \frac{\theta^2}{4} \int_0^T \int_\Omega \left[\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N)dx\right]^2 dxds \leq C_0,$$

where $C_0^2 = \theta_c^2 KT + \int_\Omega |\nabla u_0(x)|^2 dx + 2\theta T C_0 + 2\ln 2\theta|\Omega|$.

(iii) From the preceding result we can state that

$$u_N(t) \in H^2(\Omega) \text{ for a.e. } t \in (0,T),$$

thus

$$\exists \delta \in (0,T) \text{ such that } u_N(\delta) \in H^2(\Omega),$$

and then applying the mean value theorem for integrals, yields

$$\exists t \in (\delta, T), \exists C_\delta = C_\delta(\delta, T, ||u_0||_{H^1(\Omega)}) \text{ such that } ||u_N(t)||_{H^2(\Omega)} \leq C_\delta.$$

□

2.3 Proof of the existence of a solution to problem (P)

After the elaboration of a several a priori estimates on problem (P_N) independent of N, we are now ready to prove theorem (2.1).

Proof. First we will consider the case (ii), that is $u_0 - M \in H^1(\Omega)$ with $||u_0||_{L^\infty(\Omega)} \leq 1$, a.e. on Ω and $|M| < 1$.
Remark that from inequality (2.45) in lemma (2.4), we can state that

$$\{u_N - M\}_{N\geq 1} \text{ is bounded in } L^2(0,T;H^2(\Omega))$$
$$\{(u_N - M)_t\}_{N\geq 1} \text{ is bounded in } L^2(0,T;L^2(\Omega))$$
$$\{\varphi_N(u_N) - \frac{1}{\Omega}\int_\Omega \varphi_N(u_N)dx\}_{N\geq 1} \text{ is bounded in } L^2(0,T;L^2(\Omega))$$

so there exist a subsequences $\{u_{N_j} - M\}_{j\geq 1}$,$\{(u_{N_j} - M)_t\}_{j\geq 1}$ and $\{\varphi_{N_j}(u_{N_j}) - \frac{1}{\Omega}\int_\Omega \varphi_{N_j}(u_{N_j})dx\}_{j\geq 1}$ respectively satisfying

$$u_{N_j} - M \rightharpoonup u - M \text{ in } L^2(0,T;H^2(\Omega)) \text{ weakly}$$
$$(u_{N_j} - M)_t \rightharpoonup (u - M)_t \text{ in } L^2(0,T;L^2(\Omega)) \text{ weakly}$$
$$\varphi_{N_j}(u_{N_j}) - \frac{1}{\Omega}\int_\Omega \varphi_{N_j}(u_{N_j})dx \rightharpoonup \chi - \frac{1}{\Omega}\int_\Omega \chi dx \text{ in } L^2(0,T;L^2(\Omega)) \text{ weakly}$$

thus with the same arguments as in page 4, we say that as $j \to +\infty$ u, u_t, χ satisfy

$$u_t = \Delta u + \theta_c(u - M) - \frac{\theta}{2}\left[\chi - \frac{1}{\Omega}\int_\Omega \chi dx\right], \text{ in } L^2(0,T;L^2(\Omega))$$

with boundary condition

$$\partial_\nu u = 0, \text{ in } \partial\Omega \times (0,+\infty).$$

Moreover using the result in (2.44) we deduce that

$$u_N - M \to u - M \text{ in } L^2(0,T;L^2(\Omega)) \text{ as } N \to +\infty,$$

and that

$$u - M \in C([0,T];L^2(\Omega)).$$

We now want to prove that the set $\{x \in \Omega, |u(x,t)| = 1\}$ has measure zero.
For an arbitrary small $\eta \in (0,1)$ and for $t \in (0,T)$ we introduce the set

$$E_N^\eta(t) = \{x \in \Omega, |u_N(x,t)| > 1 - \eta\}.$$

We infer from (2.39) that

$$t^2|E_N^\eta| \left(2\sum_{k=0}^N \frac{(1-\eta)^{2k+1}}{2k+1}\right)^2 \leq C'^2,$$

or

$$|E_N^\eta| \leq \frac{C'^2}{t^2\left(2\sum_{k=0}^N \frac{(1-\eta)^{2k+1}}{2k+1}\right)^2},$$

and by the Fatou's lemma we have

$$|\{x \in \Omega, |u(x,t)| > 1 - \eta\}| \leq \lim_{N\to+\infty} \inf |E_N^\eta|,$$

thus

$$|\{x \in \Omega, |u(x,t)| > 1 - \eta\}| \leq \frac{C'^2}{t^2 \ln^2\left(\frac{2-\eta}{\eta}\right)},$$

where

$$\lim_{\eta\to 0} \ln\left(\frac{2-\eta}{\eta}\right) = +\infty$$

which gives

$$|\{x \in \Omega, |u(x,t)| \geq 1\}| = 0.$$

To prove that $\chi = \varphi(u)$ that is $\varphi_N(u_N(x,t)) \to \varphi(u(x,t))$ a.e. in $\Omega \times (0,T)$ as $N \to +\infty$, we proceed as in page 4.
Thus $u - M \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \forall T > 0$.
It remains to prove that $u - M \in C([0,T];H^1(\Omega))$.
We already know that $u - M \in C([0,T];L^2(\Omega))$, and from the results (2.45) and (2.46), we can state that $u - M \in C([\delta,T];H^1(\Omega)), \forall\delta > 0$, so that $u - M \in C((0,T];H^1(\Omega))$.
So it suffices to study the continuity for $t = 0$.
Let $t_0 \in [0,T]$ and let $(t_k)_{k\in\mathbb{N}}$ be a sequence in $[0,T]$ converging to t_0 as $k \to +\infty$. From the properties

of Φ we have $0 \leq \Phi(u(x,t_k)) \leq 2\ln 2$, and from the fact that $u_N - M \to u - M \in C([0,T];L^2(\Omega))$ and that $|\{x \in \Omega, |u(x,t)| \geq 1\}| = 0$, we can apply the dominated convergence theorem to state that $\Phi(u(x,t_k))$ converges to $\Phi(u(x,t_0))$ as $k \to +\infty$ in $L^1(\Omega)$, thus

$$\Phi(u) \in C([0,T];L^1(\Omega)) \tag{2.48}$$

besides from (1.13), E_N is a Lyapunov functional for problem (P_N) and so

$$E_N(u_N) \leq E_N(u_0), \forall t \in [0,T],$$

which implies that

$$\lim_{t\to 0}\sup E(u)(t) \leq E(u_0).$$

And from an other side, since $u - M \in L^\infty(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$; $u - M$ is weakly continuous on $[0,T]$ with values in $H^1(\Omega)$, thus there holds

$$E(u_0) \leq \lim_{t\to 0}\inf E(u)(t),$$

which implies the continuity of $E(u)$ at $t = 0$, and so the continuity of $u - M$ at $t = 0$ in $H^1(\Omega)$. This together with (2.48) leads to the fact that $u - M \in C([0,T];H^1(\Omega))$ since $u - M \in C([0,T];L^2(\Omega))$. And so part (ii) is proven.
Let's proceed now to the prove of part (i) of the theorem of existence of a unique solution to problem (P).
Let u_0 satisfies hypothesis (H_0), and consider a sequence $(u_0^k)_{k\geq 1}$ such that u_0^k satisfies (H_0) and also $u_0^k - M \in H^1(\Omega)$; $M = \frac{1}{|\Omega|}\int_\Omega u_0^k(x)dx, \forall k \geq 1$, where

$$u_0^k - M \to u_0 - M \text{ as } k \to +\infty \text{ in } L^2(\Omega).$$

Then consider u_p and u_q two solutions of problem (P) with initial data u_0^p and u_0^q respectively. And let $\omega_{pq} = u_p - u_q$ which satisfies

$$(\omega_{pq})_t - \Delta\omega_{pq} = \theta_c(\omega_{pq} - M') - \frac{\theta}{2}\left[\{\varphi(u_p) - \varphi(u_q)\} - \frac{1}{|\Omega|}\int_\Omega\{\varphi(u_p) - \varphi(u_q)\}dx\right], \tag{2.49}$$

where $M' = M_p - M_q$ for $\frac{1}{|\Omega|}\int_\Omega u_0^p dx = M_p$, and $\frac{1}{|\Omega|}\int_\Omega u_0^q dx = M_q$.
Multiplying equation (2.49) by ω_{pq} and integrating over Ω, then applying the Green's formula with boundary condition, the result will be

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega_{pq}^2 dx\right) + \int_\Omega |\nabla\omega_{pq}|^2 dx = \theta_c\int_\Omega \omega_{pq}^2 dx - \theta_c|\Omega|M'^2 - \\ -\frac{\theta}{2}\int_\Omega \omega_{pq}(\varphi(u_p) - \varphi(u_q))dx + \frac{\theta}{2}M'\int_\Omega(\varphi(u_p) - \varphi(u_q))dx,\end{aligned}$$

and if we set $\psi(t) = \int_\Omega(\varphi(u_p) - \varphi(u_q))dx$ and apply property (P_2) it will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega_{pq}^2 dx\right) + \int_\Omega |\nabla\omega_{pq}|^2 dx \leq (\theta_c + C_4)\int_\Omega \omega_{pq}^2 dx - \theta_c|\Omega|M'^2 + \frac{\theta}{2}M'\psi(t),$$

thus

$$\frac{d}{dt}\left(\int_\Omega \omega_{pq}^2 dx\right) \leq \gamma \int_\Omega \omega_{pq}^2 dx - 2\theta_c|\Omega|M'^2 + \theta M'\psi(t), \tag{2.50}$$

where $\gamma = 2(\theta_c + C_4)$.
With the results (2.45) and (2.46) we know that the sequence $(u_k - M)_{k\geq 1}$ satisfies equation (1.4) in $L^2(\Omega \times (\delta, T)), \forall \delta > 0$.
Then applying Gronwall's lemma to (2.50) yields

$$\int_\Omega \omega_{pq}^2 dx \leq \left(\int_\Omega \omega_{pq}^2(x,\delta)dx\right) e^{\gamma(t-\delta)} + M' \int_\delta^t (\theta\psi(t) - 2\theta_c|\Omega|M')e^{\gamma(t-s)}ds, \tag{2.51}$$

letting $\delta \to 0$, gives

$$\int_\Omega \omega_{pq}^2 dx \leq \left(\int_\Omega (u_0^p - u_0^q)^2 dx\right) e^{\gamma t} + M' \int_0^t (\theta\psi(t) - 2\theta_c|\Omega|M')e^{\gamma(t-s)}ds,$$

and so for $M' = 0$, that is $u_0^p = u_0^q$, we can say that $(u_k)_{k\geq 1}$ is a Cauchy sequence in $C([0,T]; L^2(\Omega))$.
Thus there exists $u \in C([0,T]; L^2(\Omega))$ such that as $k \to +\infty$

$$u_k \to u \text{ in } C([0,T]; L^2(\Omega)). \tag{2.52}$$

And as proved in the first part, the results (2.39) and (2.52) imply that $\forall t \in (0,T)$ the sets $\{x \in \Omega, |u_k(x,t)| \geq 1\}$ and $\{x \in \Omega, |u(x,t)| \geq 1\}$ have measure zero, therefore by (2.52)

$$\varphi(u_k) \to \varphi(u) \text{ a.e. in } \Omega, \forall t \in (0,T) \text{ as } k \to +\infty. \tag{2.53}$$

It remains to prove that $\varphi(u_k) \to \varphi(u)$ in $L^\infty(0,T; L^1(\Omega))$, as $k \to +\infty$.
For arbitrary small $\eta \in (0,1)$ and all $t \in (0,T)$ we introduce the sets

$$E_\eta^k(t) = \{x \in \Omega, |u_k(x,t)| > 1-\eta\}, F_\eta^k(t) = \Omega - E_\eta^k(t),$$

$$E_\eta(t) = \{x \in \Omega, |u(x,t)| > 1-\eta\}, F_\eta(t) = \Omega - E_\eta(t),$$

and the associated functions χ_η^k and χ_η defined as

$$\chi_\eta^k(x,t) = \begin{cases} 1 & \text{if } x \in F_\eta^k(t) \\ 0 & \text{otherwise} \end{cases}, \qquad \chi_\eta(x,t) = \begin{cases} 1 & \text{if } x \in F_\eta(t) \\ 0 & \text{otherwise} \end{cases}.$$

Note that (2.39) implies that for all $t \in (0,T)$

$$|E_\eta^k(t)| \leq \frac{C'^2}{t^2 \ln\left(\frac{2-\eta}{\eta}\right)}.$$

Moreover, for all $t \in (0,T)$ we have that

$$\varphi(u_k) - \varphi(u) = \varphi(u_k)(1-\chi_\eta^k) + \varphi(u_k)\chi_\eta^k - \varphi(u)\chi_\eta - \varphi(u)(1-\chi_\eta),$$

thus

$$\int_\Omega |\varphi(u_k(x,t)) - \varphi(u(x,t))|dx \leq \int_\Omega |\varphi(u_k(x,t))(1-\chi_\eta^k(x,t))|dx + \\ + \int_\Omega |\varphi(u_k(x,t))\chi_\eta^k(x,t) - \varphi(u(x,t))\chi_\eta(x,t)|dx + \\ + \int_\Omega |\varphi(u(x,t))(1-\chi_\eta(x,t))|dx.$$

For $I_1 = \int_\Omega |\varphi(u_k(x,t))(1-\chi_\eta^k(x,t))|dx$, the use of the Hölder's inequality yields

$$I_1 \leq \left[\int_\Omega \varphi^2(u_k(x,t))dx\right]^{1/2} \left[\int_\Omega (1-\chi_\eta^k(x,t))^2 dx\right]^{1/2},$$

and using the fact tha $\Omega = E_\eta^k \cup F_\eta^k$

$$\int_\Omega (1-\chi_\eta^k(x,t))^2 dx = \int_{E_\eta^k} (1-\chi_\eta^k(x,t))^2 dx + \int_{F_\eta^k} (1-\chi_\eta^k(x,t))^2 dx = \\ = |E_\eta^k| \leq \frac{C'^2}{t^2 \ln\left(\frac{2-\eta}{\eta}\right)}.$$

On the other side $\int_\Omega \varphi^2(u_k(x,t))dx = \int_{E_\eta^k} \varphi^2(u_k(x,t))dx + \int_{F_\eta^k} \varphi^2(u_k(x,t))dx$, where in the set F_η^k

$$-(1-\eta) \leq u_k(x,t) \leq 1-\eta$$

with the monotonicity of φ yields

$$\varphi(-(1-\eta)) \leq \varphi(u_k(x,t)) \leq \varphi(1-\eta),$$

so there exists a constant C' such that

$$\varphi^2(u_k(x,t)) \leq C'^2.$$

Again in the set E_η^k

$$1-\eta \leq u_k(x,t) \leq -(1-\eta)$$

where the monotonicity of φ will insure the existence of a constant C'^2 such that

$$\varphi^2(u_k(x,t)) \leq C'^2,$$

then we can conclude that there exists a constant C'^2 such that

$$I_1 \leq \frac{C'^2}{t^2 \ln\left(\frac{2-\eta}{\eta}\right)}. \tag{2.54}$$

If $I_2 = \int_\Omega |\varphi(u_k(x,t))\chi_\eta^k(x,t) - \varphi(u(x,t))\chi_\eta(x,t)|dx$, due to the monotonicity of φ and thanks to the results (2.52), (2.53) we can say that

$$\varphi(u_k(x,t))\chi_\eta^k(x,t) \longrightarrow \varphi(u(x,t))\chi_\eta(x,t) \text{ a.e in } \Omega \text{ as } k \to +\infty,$$

where

$$\varphi(u_k(x,t))\chi_\eta^k(x,t) = \begin{cases} \varphi(u_k(x,t)) & \text{if } x \in F_\eta^k(t) \\ 0 & \text{otherwise} \end{cases},$$

and for $x \in F_\eta^k(t)$, $|u_k(x,t)| \leq 1-\eta$, with the monotonicity of φ

$$|\varphi(u_k(x,t))| \leq \varphi(1-\eta).$$

Therefore the dominated convergence theorem implies that for all $t \in (0,T)$

$$\varphi(u_k(x,t))\chi_\eta^k(x,t) \longrightarrow \varphi(u(x,t))\chi_\eta(x,t) \text{ in } L^1(\Omega) \text{ as } k \to +\infty. \tag{2.55}$$

Finally when $I_3 = \int_\Omega |\varphi(u(x,t))(1-\chi_\eta(x,t))|dx$ we have

$$\varphi(u(x,t))(1-\chi_\eta(x,t)) = \begin{cases} 0 & \text{if } x \in F_\eta(t) \\ \varphi(u(x,t)) & \text{otherwise} \end{cases},$$

so $I_3 = \int_{E_\eta(t)} |\varphi(u(x,t))dx|$.
Then due to the monotonicity of φ, for all $x \in E_\eta(t)$

$$\varphi(1-\eta) < \varphi(u(x,t)) < \varphi(-(1-\eta)),$$

which gives

$$I_3 < \int_{E_\eta(t)} \varphi(-(1-\eta))dx = \varphi(-(1-\eta))|E_\eta(t)| \leq \varphi(-(1-\eta))\frac{C'^2}{t^2 \ln\left(\frac{2-\eta}{\eta}\right)}. \tag{2.56}$$

But $\eta \in (0,1)$ being arbitrary, (2.54)-(2.56) imply that for all $t \in (0,1)$

$$\varphi(u_k) \longrightarrow \varphi(u) \text{ in } L^\infty(0,T;L^1(\Omega)) \text{ as } k \to +\infty.$$

To prove the uniqueness of the solution of problem (P), let's consider two initial functions u_0, v_0 of problem (P), satisfying hypothesis (H_0), and let u, v be the corresponding solutions of problem (P) resp. For $\omega = u - v$, the corresponding problem will be

$$\omega_t - \Delta\omega = \theta_c(\omega - M) - \frac{\theta}{2}\left[(\varphi(u)-\varphi(v)) - \frac{1}{|\Omega|}\int_\Omega(\varphi(u)-\varphi(v))\right]$$

where $M = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx - \frac{1}{|\Omega|}\int_\Omega v_0(x)dx = M_u - M_v$.
Multiplying this resulting equation by ω and integrating over Ω, then applying the Green's formula with boundary condition yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2 &= \theta_c \int_\Omega \omega^2 dx - \theta_c|\Omega|M^2 - \frac{\theta}{2}\int_\Omega \omega(\varphi(u)-\varphi(v))dx + \frac{\theta}{2}M\int_\Omega(\varphi(u)-\varphi(v))dx \\ &= \theta_c \int_\Omega \omega^2 dx - \theta_c|\Omega|M^2 - \frac{\theta}{2}\int_\Omega \omega(\varphi(u)-\varphi(v))dx + \frac{\theta}{2}M\psi(t)\end{aligned}$$

and using the monotonicity of the function φ the result will be

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) + \int_\Omega |\nabla\omega|^2 \leq (\theta_c + C)\int_\Omega \omega^2 dx - \theta_c|\Omega|M^2 + \frac{\theta}{2}M\psi(t)$$

or

$$\frac{d}{dt}\left(\int_\Omega \omega^2 dx\right) \leq \gamma \int_\Omega \omega^2 dx - 2\theta_c|\Omega|M^2 + \theta M\psi(t),$$

applying Gronwall's inequality, and as in (2.51) we have for all $\delta > 0$ and $t > \delta$

$$\int_\Omega \omega^2 dx \leq \left[\int_\Omega \omega^2(x,\delta)dx\right] e^{\gamma(t-\delta)} + M\int_\delta^t (\theta_c\psi(t) - 2\theta_c|\Omega|M)e^{\gamma(t-s)}ds,$$

and so when $\delta \longrightarrow 0$ we will have

$$\int_\Omega \omega^2 dx \leq \left[\int_\Omega ((u_0 - M_u) - (v_0 - M_v))^2 dx\right] e^{\gamma t} + M\int_0^t (\theta_c\psi(t) - 2\theta_c|\Omega|M)e^{\gamma(t-s)}ds,$$

which gives the uniqueness for $u_0 = v_0$, and thus the Lipschitz continuity of the mapping $S(t)$ from $L^2(\Omega)$ into itself.

□

3 Existence of Attractors

Theorem (2.2) asserts that the mapping $S_N(t) : u_0 \longmapsto u_N(t)$ is Lipschitz continuous on $L^2(\Omega)$, and thus the solution to problem (P_N) is such that $u_N(x,t) = S_N(t)u_0(x)$.
Note also that from Lemma (2.1), we can say that the semigroup $S_N(t)$ associated to problem (P_N) is such that

(i) there exists absorbing sets in $L^2(\Omega)$ and $H^1(\Omega)$,

(ii) there exists a maximal attractor $\mathcal{A}_N$ which is bounded in $H^1(\Omega)$, compact and connected in $L^2(\Omega)$.

The following theorem proves the existence of an upper-semicontinuous maximal attractor $\mathcal{A}$ to problem (P)

Theorem 3.1. *(i) For any $\alpha < 1$, the restriction of the semigroup $(S(t))_{t\geq 0}$ to the set $\{|u|_{L^\infty(\Omega)} \leq 1\} \cap \{u - M \in H^1(\Omega), |M| \leq \alpha\}$ endowed with the topology of $L^2(\Omega)$ possesses a maximal attractor $\mathcal{A}$ bounded in $H^1(\Omega)$, compact and connected in $L^2(\Omega)$.*

(ii) The Haussdorf semidistance $\delta(\mathcal{A}_N, \mathcal{A})$ converges to zero when $N \longrightarrow +\infty$, i.e.

$$\lim_{N\to+\infty} \sup_{u_N\in\mathcal{A}_N} \inf_{u\in\mathcal{A}} ||u_N - u||_{H^1(\Omega)} = 0.$$

Proof. (i) If $M \in (-1,1)$, we can see from Lemma (2.1) that $\mathcal{A}_N$ is uniformly bounded with respect to N in $H^1(\Omega)$.
Consider a sequence $(u_0^N)_{N\in\mathbb{N}}$ such that, for any N, u_0^N is in $\mathcal{A}_N$, and let $(u_N(t))_{t\in\mathbb{N}}$ be the orbit on $\mathcal{A}_N$ passing through u_0^N at $t = 0$. We can see from Lemma(2.3) that

$$\varphi_N(u_N) - \frac{1}{|\Omega|}\int_\Omega \varphi_N(u_N) \text{ is uniformly bounded in } L^2(\Omega).$$

Therefore $(u_N)_{N\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R};H^1(\Omega))$,
and $((u_N)_t)_{N\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R};(H^1(\Omega))')$.
Hence using classical compactness theorems, there exists a subsequence $(u_{N_k})_{k\in\mathbb{N}}$ and a function u in $C(\mathbb{R},L^2(\Omega))$, such that for all $T>0$, $(u_{N_k})_{k\in\mathbb{N}}$ converges to u in $C([-T,T],L^2(\Omega))$.
Moreover, u is in $L^\infty(\mathbb{R};H^1(\Omega))$ and

$$(u_{N_k})_t \longrightarrow u_t \text{ in } L^\infty(\mathbb{R};(H^1(\Omega))') \text{ weak } *.$$

Also, using similar arguments as in the proof of theorem(2.1), we have

$$\varphi_{N_k}(u_{N_k}) \longrightarrow \varphi_N(u_N) \text{ in } L^\infty(\mathbb{R};L^1(\Omega)) \text{ weak } *.$$

This implies that u satisfies problem (P), defined for all $t\in\mathbb{R}$ where u is bounded in $H^1(\Omega)$. Thus $u(t)\in\mathcal{A}$, for all $t>0$.

(ii) By the convergence of u_{N_k} to u in $C([-T,T],L^2(\Omega))$ we have

$$u^0_{N_k} \longrightarrow u_0 \text{ in } H^1(\Omega)$$

and

$$\inf_{u_0\in\mathcal{A}} ||u^0_{N_k}-u_0||_{H^1(\Omega)} \longrightarrow 0 \text{ as } N\to+\infty.$$

□

Chapter III

Global Solutions to a Nonlocal Reaction-Diffusion System

1 The problem

We want to study the existence of a unique global solution and the asymptotic behavior of the following reaction-diffusion system

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = av - bu\displaystyle\int_\Omega v dx, & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} - \Delta v = h - \alpha u - \beta v\displaystyle\int_\Omega v dx, & x \in \Omega, t > 0, \\ \partial_\nu u = 0, \partial_\nu v = 0 & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0 & x \in \Omega, \end{cases}$$

where the coefficients a, b, h, α and β are supposed to be positive, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with outer unit normal ν and total volume $|\Omega|$. Even if the non linearities are polynomial (linear), the eventual blow up of the time dependent function $\int_\Omega v dx$ makes this study complicated.
Problem (P) is a nonlocal reaction-diffusion system that could arise in physics. It models the effects of an external field on the rheological properties of a dilute suspension of rigid spherical particles containing embedded dipoles [6]. These permanent dipoles may be gravitational, magnetic or electric in nature. Rotary Brownian motion is assumed negligible. Free rotation of the suspended particles resulting from the shear is hindered by the action of the field. This gives rise to a system of body couples and, hence, to a state of antisymmetric stress.
The difficulty for this system is that the reaction terms do not have a constant sign, and this means that none of the equations are good in the sense that neither u nor v is a priori bounded in order to apply the well known regularizing effect to deduce the global existence in time for problem (P).
We will try to prove the global existence of a unique solution to system (P), and to study its asymptotic behavior by the use of the framework of (positively) invariant region $\Sigma \subset \mathbb{R}^2$; which means that if $(u_0(x), v_0(x)) \in \Sigma, \forall x \in \Omega$, then $(u(x,t), v(x,t)) \in \Sigma, \forall t > 0$. Due to the problem form this invariant region is a rectangle (see Smoller [34]). The technique used here to determinate Σ is inspired by Pao [29] where we will find the upper and lower solutions to problem (P).
The region Σ can likewise be thought as an attracting region for the problem (P), which provides a compactness argument, leading to the proof of the existence of a global solution, and to establish a

global attractor.
It is classical (see [27])that under the assumption that $u_0(x)$ and $v_0(x)$ are bounded and measurable to say that there exist $T_{\max} > 0$ and $N_i \in C([0, T_{\max}), \mathbb{R})$ such that

$i-$ (P) has a unique classical; non continuable solution $(u(x,t), v(x,t))$ on $(\overline{\Omega} \times [0, T_{\max}))^2$, and

$ii-$

$$\sup_{x\in\Omega} |u(x,t)| \leq N_1(t), \sup_{x\in\Omega} |v(x,t)| \leq N_2(t), \forall 0 \leq t < T_{\max}$$

Moreover, if $T_{\max} < \infty$ then

$$\lim_{t\to T_{\max}} \sup_{x\in\Omega} |u(x,t)| = +\infty, \lim_{t\to T_{\max}} \sup_{x\in\Omega} |v(x,t)| = +\infty.$$

However we remark that if we integrate in space the system (P) the result will be

$$(E) \begin{cases} \dfrac{d}{dt}\displaystyle\int_\Omega u dx = a \int_\Omega v dx - b \int_\Omega u dx \int_\Omega v dx \\ \dfrac{d}{dt}\displaystyle\int_\Omega v dx = h|\Omega| - \alpha \int_\Omega u dx - \beta \left(\int_\Omega v dx\right)^2, \\ \displaystyle\int_\Omega u(x,0) dx = \int_\Omega u_0(x) dx \geq 0, \int_\Omega v(x,0) dx = \int_\Omega v_0(x) dx \geq 0 \end{cases}$$

which is an ODE system of unknown $(\int_\Omega u dx, \int_\Omega v dx)$.
Set for simplicity $(U, V) = (\int_\Omega u dx, \int_\Omega v dx)$ to have

$$(E) \begin{cases} \dfrac{dU}{dt} = aV - bUV \\ \dfrac{dV}{dt} = h|\Omega| - \alpha U - \beta V^2. \\ U(0) = U_0 \geq 0, V(0) = V_0 \geq 0 \end{cases}$$

The nonlinearities $aV - bUV$ and $h|\Omega| - \alpha U - \beta V^2$ are of C^∞ class towards U and V, thus they are locally Lipschitz in Ω, on the other side these non linearities are time continuous, which insures to the Cauchy problem (E) the existence of a unique maximal solution in $[0, T_{\max}), T_{\max} > 0$ for $(U_0, V_0) \in \mathbb{R}^2$. And because equilibria are an important special class of invariant regions, where the stability is the most commonly studied property of them, a complete study will follow.

1.1 Study of the stability of the ODE (E)

The stability or not of the equilibrium points of the system (E) depends on the sign of $h|\Omega| - \alpha\dfrac{a}{b}$, so

I- If $h|\Omega| - \alpha\dfrac{a}{b} > 0 \left(\dfrac{a}{b} < \dfrac{h}{\alpha}|\Omega|\right)$, system (E) admits three equilibrium points $(U_1, V_1) = \left(\dfrac{h}{\alpha}|\Omega|, 0\right)$, $(U_i, V_i)_{i=2}^3 = \left(\dfrac{a}{b}, \pm A\right)$ where $A = \left\{\dfrac{1}{\beta}\left[h|\Omega| - \alpha\dfrac{a}{b}\right]\right\}^{(1/2)}$.

i- For $(U_1, V_1) = \left(\frac{h}{\alpha}|\Omega|, 0\right)$, the linearized system corresponding to (E) is

$$\begin{cases} \omega_t = \left(a - \frac{bh|\Omega|}{\alpha}\right)\mu \\ \mu_t = -\alpha\omega \end{cases},$$

and the Jacobian matrix is

$$J_1 = \begin{pmatrix} 0 & a - \frac{bh|\Omega|}{\alpha} \\ -\alpha & 0 \end{pmatrix},$$

it has two real eigenvalues with opposite signs $\lambda_1 = -\lambda_2 = (bh|\Omega| - \alpha a)^{\frac{1}{2}}$, so this equilibrium point is a *saddle point*. The corresponding eigenvectors are $v_1 = \left(\frac{-1}{\alpha}(bh|\Omega| - \alpha a)^{\frac{1}{2}}, 1\right)$ and $v_2 = \left(\frac{1}{\alpha}(bh|\Omega| - \alpha a)^{\frac{1}{2}}, 1\right)$ respectively. In the phase portrait, the trajectories given by the eigenvectors of the negative eigenvalue initially start at infinite-distant away, move toward and eventually converge at (U_1, V_1). The trajectories that represent the eigenvectors of the positive eigenvalues move exactly in the opposite way: start at (U_1, V_1) then diverge to infinite distant out.

ii- For $(U_2, V_2) = \left(\frac{a}{b}, A\right)$, where $A = \left\{\frac{1}{\beta}\left[h|\Omega| - \alpha\frac{a}{b}\right]\right\}^{(1/2)}$ the linearized system is

$$\begin{cases} \omega_t &= -bA\omega - b\omega\mu \\ \mu_t &= -\alpha\omega - 2\beta A\mu \end{cases},$$

and the Jacobian matrix is

$$J_2 = \begin{pmatrix} -bA & 0 \\ -\alpha & -2\beta A \end{pmatrix},$$

it has two negative eigenvalues $\lambda_1 = -bA$ and $\lambda_2 = -2\beta A$, which give an asymptotic stable equilibrium point called a *sink point*, the corresponding eigenvectors are $v_1 = \left(-\frac{1}{\alpha}(2\beta A + \lambda_1), 1\right)$ and $v_2 = \left(-\frac{1}{\alpha}(2\beta A + \lambda_2), 1\right)$ respectively. The phase portrait shows trajectories moving directly toward and converge to (U_2, V_2).

iii- For $(U_3, V_3) = \left(\frac{a}{b}, -A\right)$, the Jacobian matrix of the linearized system is

$$J_3 = \begin{pmatrix} bA & 0 \\ -\alpha & 2\beta A \end{pmatrix}.$$

It has two positive eigenvalues $\lambda_1 = bA$ and $\lambda_2 = 2\beta A$, which yields an unstable equilibrium point called a *source point*, and the corresponding eigenvectors are $v_1 = \left(-\frac{1}{\alpha}(b - 2\beta)A, 1\right)$ and $v_2 = (0, 1)$ respectively. The phase portrait shows trajectories moving away from (U_3, V_3) to infinite distant away.

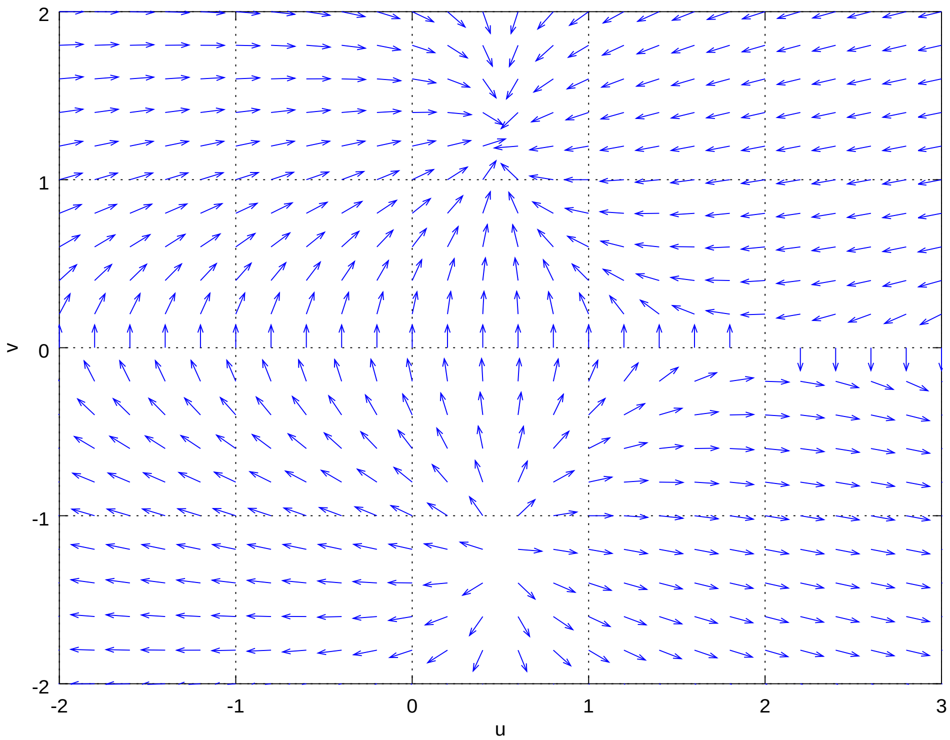


Figure III.1: Phase portrait of (E) when $h|\Omega| - \alpha\dfrac{a}{b} > 0$ $\left(\dfrac{a}{b} < \dfrac{h}{\alpha}|\Omega|\right)$

II- If $h|\Omega| - \alpha\dfrac{a}{b} \leq 0$ $\left(\dfrac{a}{b} \geq \dfrac{h}{\alpha}|\Omega|\right)$, there exists a unique equilibrium point $(U^*, V^*) = \left(\dfrac{h}{\alpha}|\Omega|, 0\right)$, and the linearized system will be then

$$\begin{cases} \omega_t &= \left(a - b\frac{h}{\alpha}|\Omega|\right)\mu \\ \mu_t &= -\alpha\omega \end{cases},$$

the Jacobian matrix is

$$J = \begin{pmatrix} 0 & a - \frac{bh|\Omega|}{\alpha} \\ -\alpha & 0 \end{pmatrix}.$$

It has two pure imaginary eigenvalues $\lambda_1 = -\lambda_2 = (\alpha a - bh|\Omega|)^{\frac{1}{2}}\, i$, and so this equilibrium point is a *center* for the linearized system.

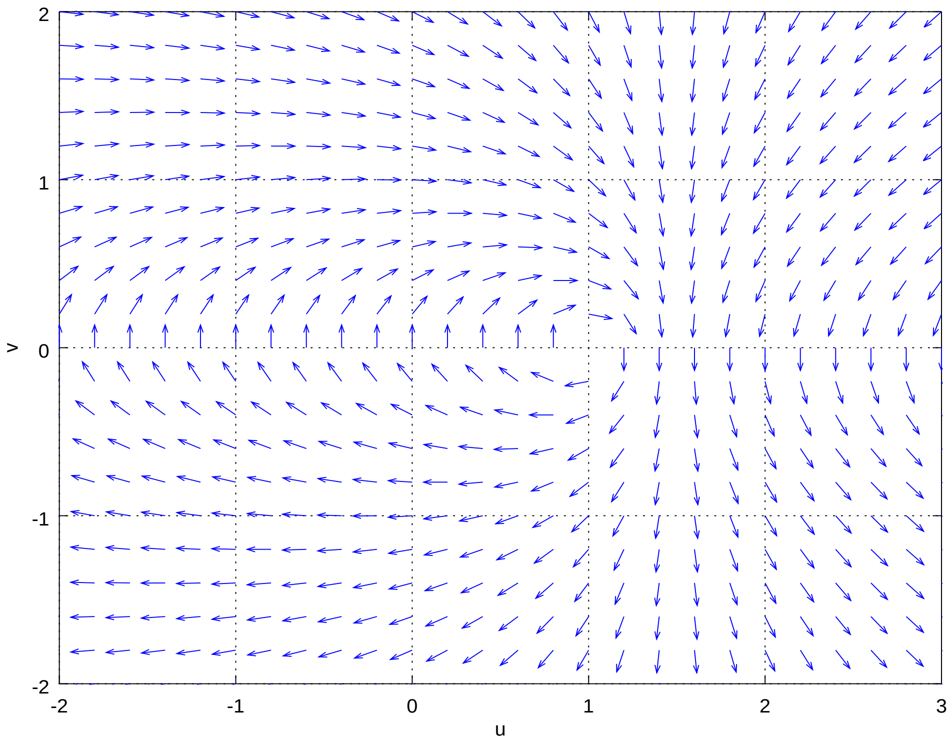


Figure III.2: Phase portrait of (E) when $h|\Omega| - \alpha\frac{a}{b} \leq 0$ $\left(\frac{a}{b} \geq \frac{h}{\alpha}|\Omega|\right)$

Thus we can conclude that when $h|\Omega| - \alpha\frac{a}{b} > 0$ $\left(\frac{a}{b} < \frac{h}{\alpha}|\Omega|\right)$, we have an asymptotic stability of the steady state of (E), where $(U_2, V_2) = \left(\frac{a}{b}, A\right)$ is an attractive point. And so in this case as $t \to \infty$

$$\lim_{t\to\infty} \int_\Omega u dx = U_\infty < \infty \text{ and } \lim_{t\to\infty} \int_\Omega v dx = V_\infty < \infty,$$

where $U_\infty = \frac{a}{b}$ and $V_\infty = A$.

2 Study of a reaction diffusion system

Let's take the following reaction-diffusion system

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = av - b\displaystyle\int_\Omega v dx u, & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} - \Delta v = h - \alpha u - \beta\displaystyle\int_\Omega v dx v, & x \in \Omega, t > 0, \\ \partial_\nu u = 0, \partial_\nu v = 0 & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0 & x \in \Omega, \end{cases}$$

with the condition that $h|\Omega| - \alpha\frac{a}{b} > 0$ $\left(\frac{a}{b} < \frac{h}{\alpha}|\Omega|\right)$, and in the case where $\int_\Omega u dx \to U_\infty$ as $t \to \infty$ and $\int_\Omega v dx \to V_\infty$ as $t \to \infty$. Thus we are dealing with the system

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = av - bAu, & x \in \Omega, t > 0, \\ \dfrac{\partial v}{\partial t} - \Delta v = h - \alpha u - \beta A v, & x \in \Omega, t > 0, \\ \partial_\nu u = 0, \partial_\nu v = 0 & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0 & x \in \Omega, \end{cases}$$

where $A = \left\{\frac{1}{\beta}\left[h|\Omega| - \alpha\frac{a}{b}\right]\right\}^{(1/2)}$. For simplicity this system equation could be writen as

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} - \Delta\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} av - b'u \\ h - \alpha u - \beta' v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix}$$

2.1 The invariant region Σ

For $f(r,s) = as - b'r$ and $g(r,s) = h - \alpha r - \beta' s$, we are dealing with a mixed quasimonotone functions, that is

$$\partial_s f(r,s) = a > 0, \partial_r g(r,s) = -\alpha < 0,$$

where we have the definition

Definition 2.1. $\overline{W} = (\overline{u}, \overline{v})$ *and* $\underline{W} = (\underline{u}, \underline{v})$ *in* $(C^{2,1}(\Omega \times [0,T]))^2 \cap \left(C(\overline{\Omega} \times [0,T])\right)^2$ *are called upper, resp. lower solutions if* $\overline{W} \geq \underline{W}$,

$$\partial_t \overline{u} - \Delta \overline{u} \geq f(\overline{u}, \overline{v}), \partial_t \underline{u} - \Delta \underline{u} \leq f(\underline{u}, \underline{v}) \tag{2.1}$$
$$\partial_t \overline{v} - \Delta \overline{v} \geq g(\underline{u}, \overline{v}), \partial_t \underline{v} - \Delta \underline{v} \leq g(\overline{u}, \underline{v}) \tag{2.2}$$

furthermore

$$\partial_\nu \overline{u} \geq 0 \geq \partial_\nu \underline{u}, \partial_\nu \overline{v} \geq 0 \geq \partial_\nu \underline{v} \tag{2.3}$$

and finally

$$\overline{u}(x,0) \geq u_0(x) \geq \underline{u}(x,0), \overline{v}(x,0) \geq v_0(x) \geq \underline{v}(x,0). \tag{2.4}$$

Then follows the theorem

Theorem 2.1 (Pao (1992))**.** *Let* $\overline{W} = (\overline{u}, \overline{v})$ *and* $\underline{W} = (\underline{u}, \underline{v})$ *be a pair of ordered upper and lower solutions and let* f *and* g *be mixed quasimonotone. Then there exists a unique solution* $W = (u, v)$, *and it is in the sector*

$$< \underline{W}, \overline{W} >= \left\{ Z \in \left(C^{2,1}(\Omega \times [0,T])\right)^2 \cap \left(C(\overline{\Omega} \times [0,T])\right)^2 / \underline{W} \leq Z \leq \overline{W} \right\}.$$

Thus to apply this theorem we have to find a pair of ordered upper and lower solutions.
We choose $\underline{v} = 0$, then $\underline{u}$ has to solve the inequality

$$\partial_t \underline{u} - \Delta \underline{u} \leq f(\underline{u}, 0) = -b' \underline{u},$$

with $\partial_\nu \underline{u} \leq 0$ and $\underline{u} \leq u_0(x)$. We are guided to look for solutions independent of the spatial variable; the solution to the ordinary differential equation

$$\begin{cases} y_t = -b' y \\ y_0 = \inf_\Omega u_0 \end{cases}$$

Thus $\underline{u} = \inf_\Omega u_0 = m$, and then we can look for upper solutions, where
For $\overline{v}$ we use the inequality

$$\partial_t \overline{v} - \Delta \overline{v} \geq g(0, \overline{v}) = h - \beta' \overline{v} \tag{2.5}$$
$$\partial_\nu \overline{v} \geq 0 \geq \partial_\nu \underline{v}, \overline{v} \geq v_0(x) \geq \underline{v}$$

which yields $\overline{v} = \max \left(\dfrac{h - \alpha \underline{u}}{\beta'}, \sup\limits_\Omega v_0 + \dfrac{h - \alpha \underline{u}}{\beta'} \right) = N$ with the condition that $\underline{u} \leq h/\alpha$.
Finally $\overline{v}$ solves

$$\partial_t \overline{u} - \Delta \overline{u} \geq f(\overline{u}, \overline{v}) \tag{2.6}$$
$$\partial_\nu \overline{u} \geq 0 \geq \partial_\nu \underline{u}, \overline{u} \geq u_0(x) \geq \underline{u}$$

and so $\overline{u} = \max \left(\sup_\Omega u_0, \dfrac{a \overline{v}}{b'}, \dfrac{h}{\alpha} \right) = M$, then the invariant region is $[m, M] \times [0, N]$.

2.2 Existence of a global attractor

Let's recall our system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} - \Delta \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} av - b'u \\ h - \alpha u - \beta' v \end{pmatrix} = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix},$$

with Neumann homogeneous boundary conditions, and the assumption that $u_0(x) \geq 0, v_0(x) \geq 0$, where for $W = (u,v), F = (f,g)$ it will be written as

$$\partial_t W - \Delta W = F(W), \tag{2.7}$$

supplemented with the boundary condition $\partial_\nu W = 0$ for $x \in \Omega, t \geq 0$, and the initial condition $W_0 = (u_0, v_0)$.
Let's denote by

$$H = L^2(\Omega, \Sigma) = \Big\{ W \in \left(L^2(\Omega)\right)^2 \text{ s.t. } (u(x), v(x)) \in \Sigma \text{ for a.e. } x \in \Omega \Big\}, \text{ and } V = \left(H^1(\Omega)\right)^2,$$

endowed with the usual norm in $(L^2(\Omega))^2$ and in $(H^1(\Omega))^2$ respectively.
The existence of an invariant region Σ to the system (P) insures that (P) is well set for $W_0 \in H$, in the sens that:
Suppose $W_0 \in H$, then problem (P) admits a unique solution W for all time t, $W(t) \in H$, $\forall t$, $W \in L^2(0,T;V), \forall T > 0$.
The mapping $W_0 \to W(t)$ is continuous in H. If moreover $W_0 \in V$, then $W \in L^2(0,T;(H^2(\Omega))^2), \forall T > 0$

The next result asserts the existence of absorbing sets and of a global attractor

Theorem 2.2. *The semigroup associated to problem (P) is such that*

i) there exist absorbing sets in H and $H \cap V$,

ii) there exists a global attractor $\mathcal{A}$ which is bounded in V, compact in H. $\mathcal{A}$ attracts the bounded sets of H. Furthermore convexity of Σ yields then $\mathcal{A}$ is connected in H.

Proof. $i1)$ Multiplying (2.7) by W and integrating it in space yields thanks to the Green's formula

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |W|^2 dx + \int_\Omega |\nabla W|^2 dx \leq C_1 \int_\Omega |W| dx, \tag{2.8}$$

where

$$C_1 = \sup_{(x,t)\in\overline{\Omega}\times\overline{\Sigma}} |F(W(x,t))|.$$

Because Σ is a connected subset of $\mathbb{R}^2$, it follows from Poincré-Wirtinger inequality that there exists a constant $C_4 = C_4(\Omega)$ s.t.

$$\int_\Omega |W|^2 dx \leq C_4 \int_\Omega |\nabla W|^2 dx$$

and applying the Cauchy's inequality to (2.8) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |W|^2 dx + \int_\Omega |\nabla W|^2 dx \leq C_1 \frac{\varepsilon}{2}\int_\Omega |W|^2 dx + \frac{C_1}{2\varepsilon}|\Omega|$$

therefore

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|W|^2dx \le (C_1\frac{\varepsilon}{2}-\frac{1}{C_4})\int_{\Omega}|W|^2dx + \frac{C_1}{2\varepsilon}|\Omega|,$$

we choose ε small enough to deduce that there exist $C_5, C_6 > 0$ s.t.

$$\frac{d}{dt}\int_{\Omega}|W|^2dx \le -C_5\int_{\Omega}|W|^2dx + C_6,$$

so

$$\int_{\Omega}|W|^2dx \le \left(\int_{\Omega}|W_0|^2dx\right)e^{-C_5t} + \frac{C_6}{C_5}\left(1-e^{-C_5t}\right),$$

thus any ball of $(L^2(\Omega))^2$ centered at $(0,0)$ and of radius $\rho_2 > \rho_1 = \left(\frac{C_6}{C_5}\right)^{1/2}$ is an absorbing set in $(L^2(\Omega))^2$. Indeed, if $\mathcal{B}_0$ is a bounded set of $(L^2(\Omega))^2$ included in a ball $B((0,0),R)$ of $(L^2(\Omega))^2$, then $S(t)\mathcal{B}_0 \subset B(0,\rho_2)$ for $t \ge t_0 = t_0(\mathcal{B}_0)$, $t_0 = \frac{1}{C_5}\ln\left(\frac{R^2}{\rho_2^2-\rho_1^2}\right)$.
Also from (2.8) we will have

$$\int_t^{t+r}\frac{d}{ds}\int_{\Omega}|W|^2dx + 2\int_t^{t+r}\int_{\Omega}|\nabla W|^2dx \le rC_6, \tag{2.9}$$

and then for $W_0 \in \mathcal{B}' \subset B((0,0),R)$ and $t \ge t_0$

$$\int_t^{t+r}\int_{\Omega}|\nabla W|^2dx \le rC_6. \tag{2.10}$$

*i*2) Multiplying (2.7), by $\partial_t W$ and integrating the result in space gives

$$\frac{d}{dt}\left[\frac{1}{2}\int_{\Omega}|\nabla W|^2dx - \int_{\Omega}F(W)W_t\right] \le 0,$$

then $B((0,0),C_6^{1/2})$ is an absorbing set in $(H^1(\Omega))^2$.

ii) From *i*2), it exists an absorbing set in $(H^1(\Omega))^2$, relatively compact in $(L^2(\Omega))^2$ which is connected, and so $\mathcal{A} = \omega\left(B\left((0,0),C_6^{1/2}\right)\right)$ is a global attractor.

□

Conclusion

This thesis was about the proof of the existence of a unique global solution, to three different nonlocal reaction-diffusion problems arising in chemistry and physics. The presence of an integral in space in these non linearities makes this study complicated.
Our main contributions were the proof of the existence of global solutions and a study of their long time behavior by the global attractor concept.

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