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Stabilisation robuste des systèmes stochastiques de dimension infinie

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Abstract

This thesis is mainly concerned with the studies of the robust stabilization of stochastic differential equations in Hilbert spaces. Based on the stability radius approach, we investigate the robust stabilization of infinite dimensional systems subjected to stochastic structured bounded and unbounded perturbations with bounded and unbounded input operator. The maximization of the stability radius is studied by state feedback and by dynamic output feedback. Conditions for the existence of suboptimal controllers are introduced in terms of Riccati equations satisfying some operator inequalities. Moreover, lower bounds for the supremal achievable stability radius are obtained.

Keywords: Stability radius, Stochastic systems, Stochastic stability, Robust stabilization, Dynamic output feedback, Lyapunov equation, Riccati equation, unbounded input operator, Analytic semigroup.

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GENERAL INTRODUCTION

In most dynamical systems which describe processes in engineering, physics and economics, stochastic components and random noise are included. The stochastic aspects of the models are used to capture the uncertainty about the environment in which the system is operating and the structure and parameters of the models of physical processes being studied.

Stochastic evolution equations in infinite dimensions are natural generalizations of stochastic ordinary differential equations. Their theory has motivations coming both from mathematics and the natural sciences: physics, chemistry and biology. Several motivating examples of stochastic evolution equations have been presented in Da Prato and Zabczyk [9], such as examples of purely mathematical motivations (lifts of diffusion processes, Markovian lifting of stochastic delay equations and Zakai's equation); examples from physics (random motion of a string, stochastic equation of the free field and equation of stochastic quantization); examples from chemistry (reaction diffusion equation); and examples from biology (the cable equation arising in neurophysiology and equation of population genetics)

In order to design a controller for a plant with complicated dynamic behaviour the engineer usually considers a simplified model. Then there arises the question of whether the controller designed on the basis of the model is robust enough to achieve the required performance when applied to the real system. Since a mathematical model never exactly

represents the dynamics of a physical plant, the robustness issue is not only important in the context of model reduction but is a fundamental problem for the application of control theory in general. In fact, one of the basic goals of feedback control is to enhance the robustness of the system. Roughly speaking, a controller is said to be robust if it works well for a large class of perturbed models. If this class reflects the uncertain or neglected features of the plant then one can expect good performance of the controlled system.

For the past decades, there have been extensive works on studying of robust measures, where one of the most powerful ideas is the concept of stability radii, introduced by Hinrichsen and Pritchard [30]. The so-called stability radius is defined as the norm of the smallest perturbations destabilizing the equation, and it answers how robust is a stability property of a system when the system comes under the effect of uncertain perturbations. For a known asymptotically stable system $\dot{x}(t) = Ax(t)dt$, its stability radius is the maximal $\rho > 0$ such that all the systems of the form $\dot{x}(t) = (A + D(\Delta)E)x(t)$, $\|\Delta\| < \rho$ are asymptotically stable. Both D and E are known matrices defining the structure of the perturbations, but Δ is an unknown disturbance matrix.

For finite-dimensional time-invariant systems there are formulas available for the stability radius with respect to different classes of perturbations, see ([33, 35]). Similar problems have been considered for many other types of linear dynamical systems, including time-varying and time delay systems, implicit systems, positive systems, linear systems in infinite-dimensional spaces as well as linear systems with respect to stochastic perturbations (see, e.g., [21, 19, 39, 54, 16]).

For stochastic systems a number of papers have been published which deal with robust stabilization problems in the spirit of H^∞ -control or the stability radius approach. Among them we cite([14],[15],[12],[17],[24],[31],[32],[43]) but all these papers consider the case of finite dimensional systems. The aim of this thesis is to study the stabilization problem for infinite dimensional systems subjected to stochastic perturbations, using the stability radius approach. We mainly focus on two problems:

- The maximization problem with structured bounded perturbations; we investigate the maximization by dynamic output feedback and by state feedback.

- The maximization problem with structured unbounded perturbation; we consider the maximization by state feedback for systems with bounded input operator and systems with unbounded input operator.

The design of suboptimal compensators has been studied for finite and infinite dimensional deterministic systems and for finite dimensional stochastic systems. In [31], they consider continuous finite dimensional systems controlled by dynamic output feedback and subjected to stochastic perturbations. Necessary and sufficient conditions which guarantee a prescribed stability radius were given in terms of matrix inequalities. The corresponding results for discrete-time systems can be found in the paper of El Bouhtouri [19]. For infinite dimensional systems subjected to stochastic perturbations, they studied in [41] the maximization of the stability radius by static state feedback. This problem was studied assuming that we can measure the entire state. However, for some control systems this assumption is not always true. Therefore, it is interesting to examine the optimization problem by dynamic output feedback. This will be the first objective of the thesis.

The problem of robust stability with respect to stochastic perturbations of unbounded structure has been studied in [42] according to the assumption that the semigroup is analytic. Our second objective is to extend their results to the multi-perturbations case. We give sufficient conditions providing the stability of the perturbed system. These conditions are used to determine a lower bound for the stability radius. Then we study the maximization of the stability radius by state feedback.

Boundary control is preferable for controlling PDE systems since actuation and sensing are only through the boundary conditions. The problem of boundary stabilization of parabolic type equations has been widely studied both in the deterministic case (see, for instance [55, 45]) as well as in the stochastic case (see, for instance [38]). Our aim is to present a unique methodology to deal with stabilization problems for stochastic partial differential equation with boundary control and boundary noise. These problems arise naturally as models to describe chemical reactions, parabolic Anderson model (see Carmona & Molchanov, 1994), and the Kardar–Parisi–Zhang equation (see

Bertini & Giacomini, 1997; Funaki & Quastel, 2015).

We study the maximization problem in the case where the input operator is unbounded. Two problems are investigated. First we consider systems subjected to bounded structured multi-perturbations, then we consider systems subjected to unbounded structured perturbations. In the bounded case we extend the results of [40] under some conditions.

A brief description of the organization of this work follows.

In **Chapter 1**, we recall basic concepts of the theory of differential equations in infinite dimensional spaces, mainly Hilbert spaces. In this way, such notions as semigroup, analytic semigroup, exponential stability, stochastic stability, L^2 -stability, stabilization and stability radius concept.

Chapter 2 is devoted to an investigation of the maximization of the stability radius by dynamic output feedback. First we give the system description. We recall some useful results. Some characterizations of the stability radius are given in terms of Lyapunov inequalities. Necessary and sufficient conditions for the existence of suboptimal controllers are established. These conditions are given in terms of some linear operator inequalities.

In **Chapter 3** we consider a system subjected to n perturbation with unbounded structure. First, we study the robust stability problem, we derive characterizations of the stability radius. Then we investigate the maximization of the radius by state feedback. We establish some technical lemmas. Based on these lemmas, conditions are derived for the existence of a stabilizing controller ensuring that the norm of the closed loop operator below a prespecified bound.

Chapter 4 is an extensive study of the maximization problem when the input operator is unbounded. We consider bounded and unbounded structured perturbations. In both cases we establish conditions for the existence of suboptimal controllers and we obtain lower bounds for the supremal achievable stability radius via Riccati equations.

We illustrate the results obtained in the thesis on several examples. We consider the stochastic heat equation with: Internal control and boundary noise, bounded multi-perturbations and boundary control, boundary control and point-wise noise, point-wise

control and boundary noise.

Results of chapter 2 are published in "Dynamics of Continuous, Discrete and Impulsive Systems, Series B: Applications and algorithms".

Results of chapter 3 are published in "NONLINEAR STUDIES".

Results of chapter 4 are accepted for publication in "Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis".

NOTATIONS

We list the notations that will be used in the thesis. Let \mathcal{H}, \mathcal{U} and \mathcal{Y} be Hilbert spaces.

$\mathcal{L}(\mathcal{U}, \mathcal{H})$	The space of bounded linear operators from \mathcal{U} to \mathcal{H} .
$\mathcal{L}(\mathcal{H})$	The space of bounded linear operators from \mathcal{H} to \mathcal{H} .
$\ \cdot\ $	The norm in \mathcal{H} .
$\langle \cdot, \cdot \rangle$	The inner product in \mathcal{H} .
$P \succ 0$	$P \in \mathcal{L}(\mathcal{H})$ is positive $\langle Pz, z \rangle > 0$, for all $z \in \mathcal{H}, z \neq 0$.
$P \succeq 0$	$P \in \mathcal{L}(\mathcal{H})$ is nonnegative $\langle Pz, z \rangle \geq 0$, for all $z \in \mathcal{H}$.
$\mathcal{L}^+(\mathcal{H})$	The set of self-adjoint linear bounded operators $P \in \mathcal{L}(\mathcal{H})$ such that $P \succeq 0$.
$Lip(\mathcal{Y}, \mathcal{U})$	The set of lipschitzian function Δ , such that $\Delta : \mathcal{Y} \rightarrow \mathcal{U}$, and $\Delta(0) = 0$.
$C([0, T]; \mathcal{H})$	the space of strongly continuously differentiable functions on $[0, T]$ with values in \mathcal{H} .
$L^2(\Omega, \mathcal{H})$	The space of square integrable \mathcal{H} -valued functions.
$L^2_{\omega}(\mathbb{R}^+; L^2(\Omega, H))$	The space of non-anticipative stochastic processes.
$(\Omega; \mathcal{F}; \mu)$	The probability space with nondecreasing family of σ -algebras.
$\mathcal{E}(z)$	The expectation of z .

CHAPTER 1

PRELIMINARIES

1.1 Introduction

First we recall some fundamental concepts that are necessary to the comprehension of our work. Then, we present stability criteria for stochastic systems in Hilbert space.

The following Lemma is an extension of the Schur Lemma for operators, and it is presented in this chapter with proof.

Lemma 1.1. *Let $Q, S, T \in \mathcal{L}(\mathcal{H})$, such that Q and S are self-adjoint operators and Q is coercive.*

Let $X := \begin{bmatrix} S & T \\ T^ & Q \end{bmatrix}$. The following result holds*

$$X \succ 0 \text{ if and only if } Q \succ 0 \text{ and } S - TQ^{-1}T^* \succ 0.$$

Proof. We begin by demonstrating that if $X \succ 0$, then $S \succ 0$, $Q \succ 0$ and $S - TQ^{-1}T^* \succ 0$ follow. To this end, assume that $X \succ 0$. Three instances are distinguished.

1. For $x = \begin{pmatrix} y \\ 0 \end{pmatrix}$, $y \in \mathcal{H}$, we have

$$\begin{aligned} \langle Xx, x \rangle > 0 &\iff \left\langle \begin{bmatrix} S & T \\ T^* & Q \end{bmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix} \right\rangle > 0 \\ &\iff \left\langle \begin{pmatrix} Sy \\ T^*y \end{pmatrix}, \begin{pmatrix} y \\ 0 \end{pmatrix} \right\rangle > 0. \end{aligned}$$

Hence, for all $y \in \mathcal{H}$, $\langle Sy, y \rangle > 0$.

2. For $x = \begin{pmatrix} 0 \\ y \end{pmatrix}$, $y \in \mathcal{H}$, we have

$$\begin{aligned} \langle Xx, x \rangle > 0 &\iff \left\langle \begin{bmatrix} S & T \\ T^* & Q \end{bmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle > 0 \\ &\iff \left\langle \begin{pmatrix} Ty \\ Qy \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle > 0. \end{aligned}$$

Then, for all $y \in \mathcal{H}$, $\langle Qy, y \rangle > 0$.

3. For $x = \begin{pmatrix} y \\ -Q^{-1}T^*y \end{pmatrix}$, $y \in \mathcal{H}$, we have

$$\begin{aligned} \langle Xx, x \rangle > 0 &\implies \left\langle \begin{bmatrix} S & T \\ T^* & Q \end{bmatrix} \begin{pmatrix} y \\ -Q^{-1}T^*y \end{pmatrix}, \begin{pmatrix} y \\ -Q^{-1}T^*y \end{pmatrix} \right\rangle > 0 \\ &\implies \left\langle \begin{pmatrix} Sy - TQ^{-1}T^*y \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ -Q^{-1}T^*y \end{pmatrix} \right\rangle > 0 \end{aligned}$$

So, for all $y \in \mathcal{H}$, $\langle (S - TQ^{-1}T^*)y, y \rangle > 0$.

Secondly, we show that there exists a linear operator R such that

$$\begin{bmatrix} I & R^* \\ 0 & I \end{bmatrix} \begin{bmatrix} S & T \\ T^* & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ R & I \end{bmatrix} = \begin{bmatrix} S - TQ^{-1}T^* & 0 \\ 0 & Q \end{bmatrix}.$$

We have

$$\begin{bmatrix} I & R^* \\ 0 & I \end{bmatrix} \begin{bmatrix} S & T \\ T^* & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ R & I \end{bmatrix} = \begin{bmatrix} S + R^*T^* + TR + R^*QR & T + R^*Q \\ T^* + QR & Q \end{bmatrix}.$$

For $R = -Q^{-1}T^*$ we have $T^* + QR = 0$. Thus, we get

$$\begin{bmatrix} I & R^* \\ 0 & I \end{bmatrix} \begin{bmatrix} S & T \\ T^* & Q \end{bmatrix} \begin{bmatrix} I & 0 \\ R & I \end{bmatrix} = \begin{bmatrix} S - TQ^{-1}T^* & 0 \\ 0 & Q \end{bmatrix}.$$

It follows from $Q \succ 0$ and $S - TQ^{-1}T^* \succ 0$ that $X \succ 0$. □

1.2 Semigroup concepts

1.2.1 Definition and properties

Let \mathcal{H} be a Hilbert space.

Definition 1.1. *A strongly continuous semigroup is an operator-valued function $S(t)$ from \mathbb{R}^+ to $\mathcal{L}(\mathcal{H})$ that satisfies the following properties:*

- 1 . $S(t + s) = S(t)S(s)$ for any $s, t > 0$.
- 2 . $S(0) = I_{\mathcal{H}}$.
- 3 . $\lim_{t \rightarrow 0} \|S(t)z - z\| = 0$, for any $z \in \mathcal{H}$.

We shall use the standard abbreviation C_0 -semigroup for a strongly continuous semigroup.

Definition 1.2. *Let $(S(t))_{t \geq 0}$ be a C_0 -semigroup defined on \mathcal{H} . The infinitesimal generator A of $S(t)$ is the linear operator defined by*

$$Az = \lim_{h \rightarrow 0} \frac{S(h)z - z}{h}, \quad z \in \mathcal{D}(A)$$

where

$$\mathcal{D}(A) = \left\{ z \in \mathcal{H}, \lim_{h \rightarrow 0} \frac{S(h)z - z}{h} \text{ exists in } \mathcal{H} \right\}.$$

1.2.2 Analytic semigroup

In this section we recall the definition of analytic semigroups and some basic properties of fractional powers of closed linear operators.

For any $w \in \mathbb{R}$ and $\theta \in]0; \frac{\pi}{2}[$ we denote by $S_{w,\theta}$ the sector in \mathbb{C} :

$$S_{w,\theta} = \{ \lambda \in \mathbb{C} \setminus \{w\} : \theta \leq |\arg(\lambda - w)| \leq \pi \}$$

Definition 1.3. We call a linear operator A in a Hilbert space \mathcal{H} a sectorial operator if it is closed densely defined operator such that, for some $\theta \in (0; \frac{\pi}{2})$ and some $M \geq 1$ and a real w , the sector $S_{w,\theta}$ is in the resolvent set of A and

$$\|R(\lambda, A)\| \leq \frac{M}{|\lambda - w|} \text{ for all } \lambda \in S_{w,\theta}$$

A particular important class of strongly continuous semigroups is analytic semigroups.

Definition 1.4. An analytic semigroup on a Hilbert space \mathcal{H} is a family of continuous linear operators on \mathcal{H} , $(S(t))_{t \geq 0}$, satisfying

- 1 . $S(0) = I_{\mathcal{H}}$ and $S(t+s) = S(t)S(s)$ for any $s, t \geq 0$.
- 2 . The map $t \rightarrow S(t)z$ is real analytic on $0 < t < \infty$ for all $z \in \mathcal{H}$.
- 3 . $\lim_{t \rightarrow 0^+} S(t)z = z$, for any $z \in \mathcal{H}$.

We have the following relation ship between analytic semigroups and sectorial operators.

Theorem 1.1. 1. If $(-A)$ is a sectorial operator, then A is the infinitesimal generator of an analytic semigroup $(S(t))_{t \geq 0}$, where

$$S(t) = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} e^{\lambda t} R(\lambda, -A) d\lambda, \quad t > 0, \quad \theta \in]\frac{\pi}{2}, \pi[$$

with

$$\begin{aligned} \gamma_{\varepsilon,\theta} &= \gamma_{\varepsilon,\theta}^+ \cup \gamma_{\varepsilon,\theta}^- \cup \gamma_{\varepsilon,\theta}^0 \\ \gamma_{\varepsilon,\theta}^+ &= \{z \in \mathbb{C} : z = w + re^{+i\theta}, r \geq \varepsilon\} \\ \gamma_{\varepsilon,\theta}^0 &= \{z \in \mathbb{C} : z = w + re^{+i\eta}, |\eta| \leq \theta\} \end{aligned}$$

2. If A generates an analytic semigroup, then $(-A)$ is sectorial.

Assume that $(-A)$ is a sectorial operator and $Re(\sigma(-A)) > 0$.

Definition 1.5. For $0 < \alpha < 1$, the bounded linear operator $(-A)^{-\alpha}$ is defined as follows

$$(-A)^{-\alpha} = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,\theta}} (-\lambda)^{-\alpha} R(\lambda, A) d\lambda, \quad t > 0, \quad x \in \mathcal{H}$$

We shall denote by $(-A)^\alpha$ the inverse of $(-A)^{-\alpha}$ and by $\mathcal{D}((-A)^\alpha)$ its domain.

Definition 1.6. *The operators $(-A)^\alpha$ are called fractional powers of $(-A)$.*

In the next theorem we collect some simple properties of $(-A)^\alpha$.

Theorem 1.2. *We have*

1. $(-A)^\alpha$ is a closed operator with domain $\mathcal{D}((-A)^\alpha) = \text{rg}((-A)^{-\alpha})$.
2. $\alpha \geq \beta$ implies $\mathcal{D}((-A)^\alpha) \subset \mathcal{D}((-A)^\beta)$.
3. $\overline{\mathcal{D}((-A)^\alpha)} = \mathcal{H}$ for every $\alpha \geq 0$.
4. If α, β are real then

$$(-A)^{\alpha+\beta} = (-A)^\alpha (-A)^\beta$$

on $\mathcal{D}((-A)^\gamma)$ where $\gamma = \max\{\alpha, \beta, \alpha + \beta\}$.

We conclude this section with some results relating $(-A)^\alpha$ and the analytic semi-group $(S(t))_{t \geq 0}$.

Theorem 1.3. [29]

Assume that there exists $\delta > 0$ such that $\text{Re}(\sigma(-A)) > \delta > 0$.

- (1) $S(t) : \mathcal{H} \rightarrow \mathcal{D}((-A)^\alpha)$ for every $t > 0$ and $\alpha > 0$.
- (2) For every $z \in \mathcal{D}((-A)^\alpha)$ we have

$$S(t)(-A)^\alpha z = (-A)^\alpha S(t)z, \quad t > 0.$$

- (3) For $\alpha > 0$, there exists M_α such that

$$\|(-A)^\alpha S(t)\| \leq M_\alpha t^{-\alpha} e^{-\delta t}, \quad t > 0.$$

1.3 Stability concepts

1.3.1 Stability

One of the most important aspects of systems theory is that of stability. Consider the linear system

$$\frac{dz(t)}{dt} = Az(t), \quad z(0) = z_0 \in \mathcal{H} \quad (1.1)$$

where A is the infinitesimal generator of a C_0 -semigroup $S(t)$, $t > 0$. We shall be concerned with the following concept of stability.

Definition 1.7. A C_0 -semigroup $(S(t))_{t \geq 0}$, on a Hilbert space \mathcal{H} is exponentially stable if there exist positive constants M and ω such that

$$\|S(t)\| \leq Me^{-\omega t}, \quad \text{for } t \geq 0$$

ω is called the decay rate, and the supremum over all possible values of ω is the stability margin of $S(t)$.

Theorem 1.4. [47]

Consider the system (1.1) defined on a separable real Hilbert space \mathcal{H} . Then the following relations are equivalent.

- (a) $S(t)$ is exponentially stable,
- (b) There exists a self-adjoint nonnegative operator $P \in \mathcal{L}(\mathcal{H})$ which satisfies the Lyapunov equation

$$\langle Az, Pz \rangle + \langle Pz, Az \rangle = -\langle z, z \rangle, \quad \text{for all } z \in \mathcal{D}(A),$$

- (c) For every $z \in H$ there exists a positive constant γ_z such that

$$\int_0^{+\infty} \|S(t)z\|^2 dt \leq \gamma_z$$

1.3.2 Stabilizability

Let \mathcal{U} be Hilbert space and $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$. Consider the system

$$\frac{dz(t)}{dt} = Az(t) + Bu(t), \quad z(0) = z_0 \in \mathcal{H}$$

where A is the infinitesimal generator of a C_0 -semigroup $(S(t))_{t \geq 0}$, on the Hilbert space \mathcal{H} and $u \in L^2(0, \infty; \mathcal{U})$. We recall now the stabilizability definition.

Definition 1.8. *If there exists an $F \in \mathcal{L}(\mathcal{H}, \mathcal{U})$ such that $A + BF$ generates a C_0 -semigroup $S_F(t)$, then we say that (A, B) is stabilizable.*

1.4 Stochastic processes

Let $(\Omega, \mathcal{F}, \mu)$ a complete probability space. A family $\{\mathcal{F}_t\}, t \geq 0$, for which the \mathcal{F}_t are sub- σ -fields of \mathcal{F} and form an increasing family of σ -fields, is called a filtration if $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $s \leq t$.

Definition 1.9. *An \mathcal{H} -valued stochastic process is a map $x : [t_0; T] \times \Omega \rightarrow \mathcal{H}$ which is measurable in the product measure on $[t_0; T] \times \Omega$.*

Wiener processes are used for modelling white noise disturbances in engineering systems. The following is one of several equivalent definitions.

Definition 1.10. *$(\omega(t))_{t \geq 0}$ is an \mathcal{H} -valued Wiener process on $[0, T]$ if it is an \mathcal{H} -valued process on $[0, T]$, such that*

$$\omega(t) - \omega(s) \in L^2(\Omega, \mathcal{H}) \text{ for all } s, t \in [0, T]$$

and

- 1 . $\mathcal{E}(\omega(t) - \omega(s)) = 0$,
- 2 . $\text{cov}(\omega(t) - \omega(s)) = (t - s)W$, where $W \in \mathcal{L}(\mathcal{H})$ nonnegative and
- 3 . $\omega(s_4) - \omega(s_3)$ and $\omega(s_2) - \omega(s_1)$ are independent whenever $0 \leq s_1 \leq s_2 \leq s_3 \leq s_4 \leq T$,
- 4 . $\omega(t)$ has continuous sample paths on $[0, T]$.

where \mathcal{E} denote the expectation.

This inequality is important and useful in the seequal

Lemma (Burkholder-Davis-Gubdy) 1.1. *For arbitrary $p \geq 0$, then there exists a constant c_p , dependent only on p such that for any $T \geq 0$,*

$$\mathcal{E} \left(\sup_{0 \leq t \leq T} \left\| \int_0^t \Phi(s, w) d\omega(s) \right\|_{\mathcal{H}}^p \right) \leq c_p \mathcal{E} \left(\int_0^T \|\Phi(s, w)\|_{L^2(\Omega, \mathcal{H})}^2 ds \right)^{\frac{p}{2}} \quad (1.2)$$

1.5 Stochastic differential equations

1.5.1 Existence of solution

Let \mathcal{Y}, \mathcal{H} be Hilbert spaces. We consider the stochastic evolution equation.

$$\begin{cases} dx(t) = (Ax(t) + f(x(t))) dt + D(x(t)) d\omega(t), & t > 0 \\ x(0) = x_0 \in H. \end{cases} \quad (1.3)$$

where A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ on \mathcal{H} , $\omega(t)$ is a Wiener process with covariance operator W in \mathcal{Y} , $f : \mathcal{H} \rightarrow \mathcal{H}$ and $D(\cdot) : \mathcal{H} \rightarrow \mathcal{L}(\mathcal{Y}, \mathcal{H})$ satisfy Lipschitz conditions

$$\|f(y) - f(z)\| \leq c_1|y - z|, \quad y, z \in \mathcal{H}, c_1 > 0$$

$$\|D(y) - D(z)\| \leq c_2|y - z|, \quad y, z \in \mathcal{H}, c_2 > 0$$

Let \mathcal{F}_t be the σ -field given above. We introduce two concepts of a solution of (1.3).

Definition 1.11. $x(t)$ is a strong solution of (1.3) on $[t_0; T]$ if $x(t) \in \mathcal{D}(A)$ with probability one for all most all t and satisfies the following:

- 1 . $x(t) \in C((t_0, T), L^2(\Omega, \mu, \mathcal{Y}))$ and $\int_{t_0}^T \|A(x(t))\| dt < +\infty$ w.p.1.
- 2 . $x(t)$ is adapted to F_t and has continuous sample paths.
- 3 . $x(t) = x_0 + \int_{t_0}^T Ax(s) ds + \int_{t_0}^T f(x(s)) ds + \int_{t_0}^T D(x(s)) d\omega(s)$ w.p.1.

This concept is very strong, so we introduce a weaker concept.

Definition 1.12. $x(t)$ is a mild solution on $[t_0; T]$ if $x(t) \in C((t_0, t_1), L^2(\Omega, \mathcal{H}))$ and adapted to F_t such that

$$x(t) = S(t - t_0)x_0 + \int_{t_0}^T S(t - t_0)f(x(s)) ds + \int_{t_0}^T S(t - t_0)D(x(s)) d\omega(s) \quad (1.4)$$

The following theorem has been established in [5].

Theorem 1.5. Let $T > t_0 > 0$ be arbitrary and let x_0 be measurable relative to \mathcal{F}_{t_0} with $\mathcal{E}|x_0|^2 < \infty$. Then there exists a unique mild solution of (1.4) in $C((t_0; t_1), L^p(\Omega, \mu, \mathcal{H}))$, $p = 2; 4$ which is adapted to F_t .

1.5.2 Stochastic stability

For stochastic systems, many definitions of stability have been defined in the literature. We look at L^2 -stability in this study.

Consider the system

$$\begin{cases} dx(t) = Ax(t)dt + D(x(t))d\omega(t), & t > 0 \\ x(0) = x_0 \in H. \end{cases} \quad (1.5)$$

with A and D as above.

Definition 1.13. *The system (1.5) is said to be L^2 -stable if the unique \mathcal{F}_t -adapted solution $x(\cdot)$ on \mathbb{R}^+ with initial value $x(0) = x_0$ satisfies $\int_0^{+\infty} \mathcal{E}(\|x(t)\|^2) dt < +\infty$ for every $x_0 \in H$.*

Definition 1.14. *The system (1.5) is said to be mean square stable if there exist positive constants M and ω such that*

$$\mathcal{E} \|z(t)\|^2 \leq Me^{-\omega t} \|z_0\|^2, \text{ for any } z_0 \in \mathcal{H}, t \geq 0.$$

Theorem 1.6. [47] *The following statements are equivalent.*

(1) *There exist $M > 0$ and $\omega > 0$ such that*

$$\mathcal{E} \|z(t)\|^2 \leq Me^{-\omega t} \|z_0\|^2, \text{ for any } z_0 \in \mathcal{H}, t \geq 0$$

(2) *There exists a nonnegative operator $P \in \mathcal{L}(\mathcal{H})$ such that*

$$2\langle Az, Pz \rangle + \langle \Delta(P)z, z \rangle = -\langle z, z \rangle, \quad z \in \mathcal{D}(A)$$

where

$$\langle \Delta(P)z, z \rangle = \text{tr} D^*(z)PD(z)W$$

(3) *For every $z_0 \in H$, we have*

$$\mathcal{E} \int_0^{+\infty} \|z(t)\|^2 dt < +\infty$$

1.6 Stability radius

Consider the system

$$\begin{cases} dx(t) = Ax(t)dt + \sum_{i=1}^N D_i \Delta_i (E_i x(t)) d\omega_i(t), t > 0, \\ x(0) = x_0 \in \mathcal{H}. \end{cases} \quad (1.6)$$

where A denotes the infinitesimal generator of an exponentially stable semigroup $S(t)$ in a separable real Hilbert space \mathcal{H} .

For a given separable real Hilbert spaces $\mathcal{U}_i, \mathcal{Y}_i, i \in \overline{N}$, we consider the family $(D_i, E_i)_{i \in \overline{N}}$ of operators $D_i \in \mathcal{L}(\mathcal{U}_i, \mathcal{H}), E_i \in \mathcal{L}(\mathcal{H}, \mathcal{Y}_i)$.

$\Delta_1, \dots, \Delta_N$ are unknown Lipschitz nonlinearities and $(w_i(t))_{t \in \mathbb{R}_+}, i \in \overline{N}$, are independent real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$.

Set $\Delta = \bigoplus_{i=1}^N \Delta_i$ and $\overline{N} = \{1, 2, 3, \dots, N\}$. The Lipschitz norm of Δ is given by

$$\|\Delta\|_{Lip} = \max_{i \in \overline{N}} \|\Delta_i\|_{Lip}$$

The stability radius of the system (1.6) is defined as follows.

Definition 1.15. *The stochastic stability radius of A with respect to the perturbation structure (D_i, E_i) and the Wiener process $w_i(t)$ is*

$$r^w(A; (D_i, E_i)_{i \in \overline{N}}) = \inf \left\{ \left\| \bigoplus_{i=1}^N \Delta_i \right\|_{Lip}; \Delta_i \in Lip(\mathcal{Y}_i, \mathcal{U}_i) \text{ such that (1.6) is not } L^2\text{-stable} \right\}$$

In this section we recall some characterizations of the stability radius of system (1.6), established in [40], for later use.

Theorem 1.7. *Let $\sigma > 0$ and assume there exists $P \in \mathcal{L}^+(\mathcal{H})$ satisfying*

$$2\langle Px, Ax \rangle + \langle Ex, Ex \rangle = 0, \quad x \in \mathcal{D}(A), \quad (1.7)$$

$$I - \theta \sigma^2 D^* P D \succeq 0, \quad (1.8)$$

then

$$r^w(A; (D, E)) \geq \sigma.$$

A similar result can be obtained if we replace the Lyapunov equation (1.7) by the corresponding Lyapunov inequality.

Proposition 1.1. *Let $\sigma > 0$. Assume that there exist $\alpha \in (0, +\infty)^N$ and $P \in L^+(\mathcal{H})$ satisfying*

$$2 \langle Px, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \leq 0, \quad \text{for all } x \in \mathcal{D}(A), \quad (1.9)$$

$$I - (\sigma/\alpha_j)^2 \theta_j D_j^* P D_j \succeq 0, \quad (\text{resp. } I - (\sigma/\alpha_j)^2 \theta_j D_j^* P D_j \succ 0), \quad \text{for all } j \in \bar{N}, \quad (1.10)$$

where

$$\langle E(\alpha)x, E(\alpha)x \rangle = \sum_{i=1}^N \alpha_i^2 \langle E_i x, E_i x \rangle.$$

Then

$$r^w(A; (D_i, E_i)_{i \in \bar{N}}) \geq \sigma, \quad (\text{resp. } r^w(A; (D_i, E_i)_{i \in \bar{N}}) > \sigma).$$

Moreover, the Lyapunov equation

$$2 \langle Px, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle = 0 \quad (1.11)$$

has a solution $P_0 \in L^+(\mathcal{H})$ such that $P \succeq P_0$.

The following theorem provides a computable formula for the stability radius.

Theorem 1.8. *Let $(A; (D_i; E_i)_{i \in \bar{N}})$ and $(\omega_i(t))_{i \in \bar{N}}$ be as above. Then the associated stability radius is given by*

$$r^w(A; (D_i, E_i)_{i \in \bar{N}}) = \sup_{\alpha \in (0, +\infty)^N} \left(\max_{j \in \bar{N}} \|(\theta_j/\alpha_j^2) D_j^* P(\alpha) D_j\| \right)^{-1/2}, \quad (1.12)$$

where $P(\alpha)$ is the unique solution of (1.11). If $r^w(A; (D_i, E_i)_{i \in \bar{N}}) < +\infty$ there exists a minimum norm destabilizing perturbation $\Delta = \bigoplus_{i=1}^N \Delta_i$, $\|\Delta\|_{Lip} = r^w(A; (D_i, E_i)_{i \in \bar{N}})$. Moreover there exist a subset $J \subset \bar{N}$ and a scaling vector $\alpha^J \in (0, \infty)^J$ such that

$$r^w(A; (D_i, E_i)_{i \in \bar{N}}) = r^w(A; (D_i, E_i)_{i \in J}) = \left(\max_{j \in \bar{N}} \|(\theta_j/\alpha_j^2) D_j^* P(\alpha) D_j\| \right)^{-1/2} \quad (1.13)$$

where $P(\alpha^J) \in \mathcal{L}^+(\mathcal{H})$ is the unique solution of (1.11)

CHAPTER 2

MAXIMIZATION OF THE STABILITY RADIUS BY DYNAMIC OUTPUT FEEDBACK

2.1 Introduction

The goal of the chapter is to design a dynamic output feedback that stabilizes and reaches a stability radius. We follow Hinrichsen & Pritchard (1996) to generalize the results achieved for finite-dimensional systems to infinite-dimensional ones. We show that the approach used in the finite dimension may be generalized to the infinite dimension. Still, the problem is considerably more difficult in the latter, and the findings are given under additional assumptions such as the coercivity of specific operators.

The following is the outline for the chapter. The system description in the next section. We also recollect a few noteworthy outcomes. Lyapunov inequalities are used to characterize several aspects of the stability radius. The main findings of this chapter are presented in section 3. Necessary and sufficient conditions, for the existence of suboptimal controllers are established. These requirements are expressed as linear operator inequalities. The chapter comes to a close with an illustration.

2.2 System description

Let \mathcal{H} , \mathcal{W} and \mathcal{Z} be a separable real Hilbert spaces. Consider a class of infinite-dimensional uncertain systems, which are described by Itô equations of the form

$$dx(t) = Ax(t)dt + \sum_{i=1}^N D_i \Delta_i (E_i x(t)) dw_i(t) + Bu(t)dt, \quad t \in \mathbb{R}^+, \quad (2.1)$$

$$y(t) = Cx(t), \quad t \in \mathbb{R}^+, \quad (2.2)$$

$$x(0) = x_0 \in \mathcal{H},$$

where A denotes the infinitesimal generator of an exponentially stable semigroup $S(t)$ in \mathcal{H} . The input and output operators, respectively, are $B \in \mathcal{L}(\mathcal{W}, \mathcal{H})$ and $C \in \mathcal{L}(\mathcal{H}, \mathcal{Z})$. For a given separable real Hilbert spaces $\mathcal{U}_i, \mathcal{Y}_i, i \in \bar{N}$, we consider the family $(D_i, E_i)_{i \in \bar{N}}$ of operators $D_i \in \mathcal{L}(\mathcal{U}_i, \mathcal{H})$, $E_i \in \mathcal{L}(\mathcal{H}, \mathcal{Y}_i)$. Moreover, we assume that (A, B) is stabilizable and $\Delta_1, \dots, \Delta_N$ are unknown Lipschitz nonlinearities and $(w_i(t))_{t \in \mathbb{R}^+}, i \in \bar{N}$, are independent real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ relative to an increasing family $(\mathcal{F}_t)_{t \in \mathbb{R}^+}$ of σ -algebras $(\mathcal{F}_t)_{t \in \mathbb{R}^+} \subset \mathcal{F}$, such that

$$\mathcal{E}(w_i(t)) = 0 \quad \text{and} \quad \mathcal{E}(w_i(t) - w_i(s))(w_j(t) - w_j(s)) = \delta_{ij} \theta_i (t - s), \quad \text{for all } i, j \in \bar{N},$$

where δ_{ij} is the Kronecker symbol, $\theta_i > 0$ and the disturbances Δ_i vary in $\text{Lip}(\mathcal{Y}_i, \mathcal{U}_i)$. The unknown $\Delta_i, i \in \bar{N}$, represents uncertainty in the state-dependent gains that stationary white noise processes $dw_i(t)$ exert on the system's evolution.

The framework of the perturbations is determined by the family $(D_i, E_i)_{i \in \bar{N}}$, and the overall level of stochastic disturbances is identified by $\sigma > 0$. Define

$$\|\Delta_i\|_{Lip} := \inf \left\{ \gamma > 0; \forall y, \hat{y} \in \mathcal{Y}_i : \|\Delta_i(y) - \Delta_i(\hat{y})\|_{\mathcal{U}_i} \leq \gamma \|y - \hat{y}\|_{\mathcal{Y}_i} \right\}.$$

Overall, (2.1) refers to a set of stochastic systems parametrized by $\Delta_i \in \text{Lip}(\mathcal{Y}_i, \mathcal{U}_i)$, $i \in \bar{N}$, with the norm

$$\|\Delta_i\|_{Lip} < \sigma.$$

We note that the Lipschitz norm is used to measure the size of each $\Delta_i \in \text{Lip}(\mathcal{Y}_i, \mathcal{U}_i)$.

The goal of this chapter is to determine conditions under which dynamic compensators of the form

$$d\hat{x}(t) = H\hat{x}(t)dt + Gy(t)dt, \quad u(t) = F\hat{x}(t) + Ky(t) \quad (2.3)$$

can exist, where $H : \mathcal{H} \rightarrow \mathcal{H}$ is a linear operator, $G \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$, $F \in \mathcal{L}(\mathcal{H}, \mathcal{W})$ and $K \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$ which stabilizes the system and reaches a specified stability radius.

By combining (2.1), (2.2) and (2.3), we get the following system

$$\begin{pmatrix} dx(t) \\ d\hat{x}(t) \end{pmatrix} = \begin{bmatrix} A + BKC & BF \\ GC & H \end{bmatrix} \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} dt + \sum_{i=1}^N \begin{bmatrix} D_i \\ 0 \end{bmatrix} \Delta_i \left(\begin{bmatrix} E_i & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} \right) dw_i(t). \quad (2.4)$$

Set

$$\mathbb{A} := \begin{bmatrix} A + BKC & BF \\ GC & H \end{bmatrix}, \quad \mathbb{D}_i := \begin{bmatrix} D_i \\ 0 \end{bmatrix}, \quad \mathbb{E}_i := \begin{bmatrix} E_i & 0 \end{bmatrix} \quad \text{and} \quad \bar{x}(t) := \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix}.$$

The system (2.4) can be expressed as follows:

$$d\bar{x}(t) = \mathbb{A}\bar{x}(t)dt + \sum_{i=1}^N \mathbb{D}_i \Delta_i (\mathbb{E}_i \bar{x}(t)) dw_i(t). \quad (2.5)$$

For all compensators (2.3), arbitrary $\Delta_i \in Lip(\mathcal{Y}_i, \mathcal{U}_i)$, $i \in \bar{N}$, and any $\bar{x}_0 = \begin{pmatrix} x_0 \\ \hat{x}_0 \end{pmatrix}$, $\hat{x}_0 \in \mathcal{H}$, there exists a unique solution \bar{x} of (2.5) such that $\bar{x}(0) = \bar{x}_0$, (see [37]). The proposition that follows will be crucial in establishing suboptimality requirements.

Proposition 2.1. *Let $\sigma > 0$. The statements below are equivalent:*

1. *A generates an exponentially stable semigroup such that*

$$r^w(A; (D_i, E_i)_{i \in \bar{N}}) > \sigma.$$

2. *There exist $\alpha_i > 0$, $1 \leq i \leq N$, and a coercive operator $P \in \mathcal{L}(\mathcal{H})$ satisfying*

$$2 \langle Px, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle < 0, \quad \text{for all } x \in \mathcal{D}(A), \quad (2.6)$$

and

$$I - (\sigma/\alpha_j)^2 \theta_j D_j^* P D_j \succeq 0, \quad \text{for all } j \in \bar{N}.$$

Proof. Assume that $r^w(A; (D_i, E_i)_{i \in \bar{N}}) > \sigma$. Following Theorem 1.8, we have

$$r^w(A; (D_i, E_i)_{i \in \bar{N}}) = \sup_{\alpha \in (0, +\infty)^N} \left(\max_{j \in \bar{N}} \|(\theta_j/\alpha_j^2) D_j^* P(\alpha) D_j\| \right)^{-1/2}, \quad (2.7)$$

where $P(\alpha)$ solves

$$2 \langle Px, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle = 0.$$

Hence, there exist $\sigma' \in (\sigma, r^w(A; (D_i, E_i)_{i \in \bar{N}}))$ and $\alpha \in (0, +\infty)^N$ such that

$$\sigma' \leq \left(\max_{j \in \bar{N}} \|(\theta_j/\alpha_j^2) D_j^* P(\alpha) D_j\| \right)^{-1/2}.$$

Hence

$$(\sigma')^{-2} \geq \|(\theta_j/\alpha_j^2) D_j^* P(\alpha) D_j\|, \text{ for all } j \in \bar{N},$$

and hence

$$\left\langle (\sigma')^2 (\theta_j/\alpha_j^2) D_j^* P(\alpha) D_j x, x \right\rangle \leq \|x\|^2, \text{ for all } j \in \bar{N} \text{ and } x \neq 0.$$

This means

$$\left\langle \sigma^2 (\theta_j/\alpha_j^2) D_j^* P(\alpha) D_j x, x \right\rangle < \|x\|^2, \quad x \neq 0.$$

Thus

$$\sigma^2 (\theta_j/\alpha_j^2) x^* D_j^* \left(\int_0^{+\infty} S^*(t) E^*(\alpha) E(\alpha) S(t) dt \right) D_j x < \|x\|^2, \quad x \neq 0.$$

Pick ε for which

$$\varepsilon \sigma^2 (\theta_j/\alpha_j^2) \|D_j^* M D_j\| < 1 - \sigma^2 (\theta_j/\alpha_j^2) \|D_j^* P(\alpha) D_j\|,$$

where M solves the following Lyapunov equation

$$2 \langle Px, Ax \rangle + \varepsilon \langle x, x \rangle = 0.$$

We get

$$\varepsilon \sigma^2 (\theta_j/\alpha_j^2) \frac{\langle x, D_j^* M D_j x \rangle}{\|x\|^2} < 1 - \sigma^2 (\theta_j/\alpha_j^2) \frac{\langle x, D_j^* P(\alpha) D_j x \rangle}{\|x\|^2}, \quad x \neq 0.$$

Thus

$$\sigma^2 (\theta_j/\alpha_j^2) \left\langle \left(\int_0^{+\infty} S^*(t) (E^*(\alpha) E(\alpha) + \varepsilon I) S(t) dt \right) D_j x, D_j x \right\rangle < \|x\|^2, \quad x \neq 0.$$

Consequently, the bounded operator

$$X := \int_0^{+\infty} S^*(t) (E^*(\alpha)E(\alpha) + \varepsilon I) S(t) dt$$

is coercive over \mathcal{H} and satisfies the Lyapunov equation:

$$\langle XAx, x \rangle + \langle Xx, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon \langle x, x \rangle = 0.$$

Thus,

$$\langle XAx, x \rangle + \langle Xx, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle < 0, \quad x \in \mathcal{D}(A).$$

Finally, to obtain the converse propriety, we employ Theorem 1.7 and Proposition 1.1. \square

2.3 Suboptimality conditions

The following theorem gives necessary conditions for the existence of suboptimal controllers.

Theorem 2.1. *Let $\sigma > 0$. Given an exponentially stable compensator (2.3) such that $r^w(\mathbb{A}, (\mathbb{D}_i, \mathbb{E}_i)_{i \in \overline{N}}) > \sigma$, then there exist $\alpha \in (0, \infty)^N$, $\gamma, \delta > 0$, $R \in H_n^+(\mathcal{H})$ and $S \in H_n^+(\mathcal{H})$ coercive operators such that*

$$2 \langle \hat{x}, AR\hat{x} \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)R\hat{x} \rangle - \left\langle \frac{B^*}{\gamma} \hat{x}, \frac{B^*}{\gamma} \hat{x} \right\rangle < 0, \quad \hat{x} \in \mathcal{H}, \quad R\hat{x} \in \mathcal{D}(A), \quad (2.8)$$

$$2 \langle Sx, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle - \left\langle \frac{C}{\delta} x, \frac{C}{\delta} x \right\rangle < 0, \quad x \in \mathcal{D}(A), \quad (2.9)$$

$$R \succ 0 \text{ and } S \succeq R^{-1}, \quad (2.10)$$

$$I - (\sigma/\alpha_j)^2 \theta_j D_j^* S D_j \succ 0, \quad j \in \overline{N}. \quad (2.11)$$

Proof. According to Proposition 2.1, there exist $\alpha_i > 0$, $i \in \{1 \cdots N\}$ and a coercive operator $X \in \mathcal{L}(\mathcal{H} \times \mathcal{H})$ satisfying

$$2 \langle Xz, \mathbb{A}z \rangle + \langle \mathbb{E}(\alpha)z, \mathbb{E}(\alpha)z \rangle < 0, \quad z \in \mathcal{D}(\mathbb{A}), \quad (2.12)$$

$$I - (\sigma/\alpha_j)^2 \theta_j \mathbb{D}_j^* X \mathbb{D}_j \succ 0, \quad j \in \overline{N}. \quad (2.13)$$

Assume the partitioning of X and X^{-1} is as follows:

$$X = \begin{bmatrix} S & T \\ T^* & Q \end{bmatrix}, \quad X^{-1} = \begin{bmatrix} R & M \\ M^* & P \end{bmatrix},$$

where R, S, P and Q are self-adjoint operators. We get

$$\begin{bmatrix} S & T \\ T^* & Q \end{bmatrix} \begin{bmatrix} R & M \\ M^* & P \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & I_{\mathcal{H}} \end{bmatrix}.$$

Or equivalently,

$$\begin{cases} SR + TM^* & = I_{\mathcal{H}}, \\ SM + TP & = 0, \\ QM^* + T^*R & = 0, \\ T^*M + QP & = I_{\mathcal{H}}. \end{cases}$$

By using the fact that $X \succ 0$, and Lemma 1.1 we get

$$Q \succ 0, \quad S - TQ^{-1}T^* \succ 0.$$

We have

$$\begin{aligned} \begin{cases} SR + TM^* = I_{\mathcal{H}} \\ QM^* + T^*R = 0 \end{cases} &\implies \begin{cases} SR + TM^* = I_{\mathcal{H}} \\ M^* = -Q^{-1}T^*R \end{cases} \\ &\implies SR - TQ^{-1}T^*R = I_{\mathcal{H}} \end{aligned}$$

Using the assumption that $(S - TQ^{-1}T^*)R = I_{\mathcal{H}}$, we obtain that

$$R^{-1} = S - TQ^{-1}T^* \succ 0.$$

Thus,

$$R^{-1} - S = -TQ^{-1}T^* \preceq 0.$$

But, for any $j \in \bar{N}$, we have

$$\begin{aligned} I - (\sigma/\alpha_j)^2 \theta_j \mathbb{D}_j^* X \mathbb{D}_j &= I - (\sigma/\alpha_j)^2 \theta_j \begin{bmatrix} D_j^* & 0 \end{bmatrix} \begin{bmatrix} S & N \\ N^* & Q \end{bmatrix} \begin{bmatrix} D_j \\ 0 \end{bmatrix} \\ &= I - (\sigma/\alpha_j)^2 \theta_j \begin{bmatrix} D_j^* S & D_j^* N \end{bmatrix} \begin{bmatrix} D_j \\ 0 \end{bmatrix} \\ &= I - (\sigma/\alpha_j)^2 \theta_j D_j^* S D_j \end{aligned}$$

So,

$$I - (\sigma/\alpha_j)^2 \theta_j D_j^* S D_j \succ 0.$$

Let

$$J := \begin{bmatrix} I & R \\ 0 & M^* \end{bmatrix}, \quad z := J \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \quad x \in \mathcal{D}(A),$$

and pick $\hat{x} \in \mathcal{H}$ such that $R\hat{x} \in \mathcal{D}(A)$, we have

$$2 \langle Xz, \mathbb{A}z \rangle + \langle \mathbb{E}(\alpha)z, \mathbb{E}(\alpha)z \rangle < 0, \quad z \in \mathcal{D}(\mathbb{A}).$$

Alternatively,

$$2 \left\langle XJ \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \mathbb{A}J \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \right\rangle + \left\langle \mathbb{E}(\alpha)J \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \mathbb{E}(\alpha)J \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \right\rangle < 0.$$

Note that since

$$XJ = \begin{bmatrix} S & I \\ T^* & 0 \end{bmatrix}, \quad \mathbb{A}J = \begin{bmatrix} A + BKC & (A + BKC)R + BFM^* \\ GC & GCR + HM^* \end{bmatrix},$$

and since

$$\mathbb{E}(\alpha)J = \begin{bmatrix} E(\alpha) & E(\alpha)R \\ 0 & 0 \end{bmatrix},$$

we have

$$\begin{aligned} & 2 \left\langle \begin{bmatrix} S & I \\ T^* & 0 \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \begin{bmatrix} A + BKC & (A + BKC)R + BFM^* \\ GC & GCR + HM^* \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \right\rangle \\ & + \left\langle \begin{bmatrix} E(\alpha) & E(\alpha)R \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \begin{bmatrix} E(\alpha) & E(\alpha)R \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \right\rangle < 0, \end{aligned}$$

Hence

$$\begin{aligned} & 2 \langle (A + BKC)x, Sx \rangle + 2 \langle GCx, T^*x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + 2 \langle (A + BKC)x, \hat{x} \rangle \\ & + \langle E(\alpha)x, E(\alpha)R\hat{x} \rangle + 2 \langle [(A + BKC)R + BFM^*]\hat{x}, Sx \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)x \rangle \\ & + 2 \langle (GCR + HM^*)\hat{x}, T^*x \rangle + 2 \langle [(A + BKC)R + BFM^*]\hat{x}, \hat{x} \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)R\hat{x} \rangle < 0. \end{aligned}$$

For $\hat{x} = 0$, we get

$$2 \langle (A + BKC)x, Sx \rangle + 2 \langle GCx, T^*x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle < 0.$$

This gives

$$2 \langle Ax, Sx \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + 2 \langle BKCx, Sx \rangle + 2 \langle GCx, T^*x \rangle < 0$$

Also,

$$2 \langle Ax, Sx \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + 2 \langle (SBK + TG)Cx, x \rangle < 0.$$

Thus,

$$\begin{aligned} & 2 \langle Ax, Sx \rangle + \langle E(\alpha)x, E(\alpha)x \rangle - \left\langle \frac{C}{\delta}x, \frac{C}{\delta}x \right\rangle - \delta^2 \langle (SBK + TG)^*x, (SBK + TG)^*x \rangle \\ & + \left\langle \left(\frac{C}{\delta} + \delta(SBK + TG)^* \right)x, \left(\frac{C}{\delta} + \delta(SBK + TG)^* \right)x \right\rangle < 0. \end{aligned}$$

For δ is small enough, we get

$$2 \langle Ax, Sx \rangle + \langle E(\alpha)x, E(\alpha)x \rangle - \left\langle \frac{C}{\delta}x, \frac{C}{\delta}x \right\rangle < 0.$$

Now, for $x = 0$ we have

$$2 \langle [(A + BKC)R + BFM^*]\hat{x}, \hat{x} \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)R\hat{x} \rangle < 0,$$

that means

$$2 \langle AR\hat{x}, \hat{x} \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)R\hat{x} \rangle + 2 \langle B(KCR + FM^*)\hat{x}, \hat{x} \rangle < 0,$$

which gives

$$\begin{aligned} & 2 \langle \hat{x}, AR\hat{x} \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)R\hat{x} \rangle - \gamma^2 \langle (KCR + FM^*)\hat{x}, (KCR + FM^*)\hat{x} \rangle \\ & - \left\langle \frac{B^*}{\gamma}\hat{x}, \frac{B^*}{\gamma}\hat{x} \right\rangle + \left\langle \left(\frac{B^*}{\gamma} + \gamma(KCR + FM^*) \right)\hat{x}, \left(\frac{B^*}{\gamma} + \gamma(KCR + FM^*) \right)\hat{x} \right\rangle < 0. \end{aligned}$$

For γ is small enough, we receive

$$2 \langle \hat{x}, AR\hat{x} \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)R\hat{x} \rangle - \left\langle \frac{B^*}{\gamma}\hat{x}, \frac{B^*}{\gamma}\hat{x} \right\rangle < 0.$$

□

Now, we can prove that there is a suboptimal compensator by establishing sufficient conditions for its existence.

Theorem 2.2. *Assume that (2.8)-(2.11) hold for some $\alpha \in (0, \infty)^N$, $\gamma > 0$, $\delta > 0$, $\sigma > 0$, and for some $R, S \in \mathcal{H}_n^+(\mathbb{K})$ coercive operators, such that $R^{-1} - S$ is coercive. Then there exists a compensator (H, G, F, K) such that \mathbb{A} generates an exponentially stable semigroup with $r^w(\mathbb{A}, (\mathbb{D}_i, \mathbb{E}_i)_{i \in \bar{N}}) > \sigma$.*

Proof. We aim to demonstrate the existence of a positive operator Π and a compensator (H, G, F, K) such that

$$2 \langle Xz, \mathbb{A}z \rangle + \langle \mathbb{E}(\alpha)z, \mathbb{E}(\alpha)z \rangle + \langle \Pi z, z \rangle = 0, \quad (2.14)$$

for

$$z := J \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \in \mathcal{D}(\mathbb{A}), \quad x \in \mathcal{D}(A) \quad \text{and} \quad \hat{x} \in \mathcal{H} \quad \text{such that} \quad R\hat{x} \in \mathcal{D}(A).$$

Equation (2.14) reads as

$$\begin{aligned} & 2 \langle (A + BKC)x, Sx \rangle + 2 \langle GCx, T^*x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \langle \Pi_{11}x, x \rangle \\ & + 2 \langle (A + BKC)x, \hat{x} \rangle + \langle E(\alpha)x, E(\alpha)R\hat{x} \rangle + \langle \Pi_{11}x, R\hat{x} \rangle + \langle \Pi_{12}x, M^*\hat{x} \rangle \\ & + 2 \langle [(A + BKC)R + BFM^*]\hat{x}, Sx \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)x \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)R\hat{x} \rangle \\ & + 2 \langle (GCR + HM^*)\hat{x}, T^*x \rangle + \langle (\Pi_{11}R + \Pi_{12}M^*)\hat{x}, x \rangle + 2 \langle [(A + BKC)R + BFM^*]\hat{x}, \hat{x} \rangle \\ & + \langle (\Pi_{11}R + \Pi_{12}M^*)\hat{x}, R\hat{x} \rangle + \langle (\Pi_{12}R + \Pi_{22}M^*)\hat{x}, M^*\hat{x} \rangle = 0, \end{aligned} \quad (2.15)$$

where

$$\Pi := \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} \succ 0.$$

Choose

$$\begin{aligned} F &= -\frac{1}{\gamma^2} B^* R^{-1}, \quad G = -\frac{1}{\delta^2} T^{-1} C^*, \\ K &= 0, \quad T = R^{-1} - S, \quad M = R, \quad \Pi_{12} = -\Pi_{11}, \\ \Pi_{11} &= -\left(2SA + E^*(\alpha)E(\alpha) - \frac{2}{\delta^2} C^* C \right), \\ \Pi_{22} &= \Pi_{11} - R^{-1} \left(2AR + RE^*(\alpha)E(\alpha)R - \frac{2}{\gamma^2} BB^* \right) R^{-1}, \end{aligned}$$

Where T is coercive.

We will establish that equality (2.15) holds for this selection. In fact,

$$\begin{aligned}
 & 2 \langle Ax, Sx \rangle + 2 \left\langle \left(-\frac{1}{\delta^2} T^{-1} C^*\right) Cx, T^* x \right\rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \langle -\Pi_{11}x, R\hat{x} \rangle \\
 & + \left\langle -\left(2SA + E^*(\alpha)E(\alpha) - \frac{2}{\delta^2} C^*C\right) x, x \right\rangle + 2 \langle Ax, \hat{x} \rangle + \langle E(\alpha)x, E(\alpha)R\hat{x} \rangle \\
 & + \langle \Pi_{11}x, R\hat{x} \rangle + 2 \left\langle \left[AR + B\left(-\frac{1}{\gamma^2} B^* R^{-1}\right)R\right]\hat{x}, Sx \right\rangle + \langle E(\alpha)R\hat{x}, E(\alpha)x \rangle \\
 & + 2 \left\langle \left[\left(-\frac{1}{\delta^2} T^{-1} C^*\right)CR + HR\right]\hat{x}, T^* x \right\rangle + \langle (\Pi_{11}R - \Pi_{11}R)\hat{x}, x \rangle \\
 & + 2 \left\langle \left[AR + B\left(-\frac{1}{\gamma^2} B^* R^{-1}\right)R\right]\hat{x}, \hat{x} \right\rangle + \langle (\Pi_{11}R - \Pi_{11}R)\hat{x}, R\hat{x} \rangle + \langle E(\alpha)R\hat{x}, E(\alpha)R\hat{x} \rangle \\
 & + \left\langle \left[-\Pi_{11}R + \left(\Pi_{11} - R^{-1} \left(2AR + RE^*(\alpha)E(\alpha)R - \frac{2}{\gamma^2} BB^*\right) R^{-1}\right)R\right]\hat{x}, R\hat{x} \right\rangle \\
 & = 2 \langle A\hat{x}, x \rangle + \langle E^*(\alpha)E(\alpha)R\hat{x}, x \rangle + 2 \left\langle S\left[AR - \frac{BB^*}{\gamma^2}\right]\hat{x}, x \right\rangle \\
 & + \langle E^*(\alpha)E(\alpha)R\hat{x}, x \rangle + 2 \left\langle \left[\left(-\frac{C^*C}{\delta^2}\right) + TH\right] R\hat{x}, x \right\rangle.
 \end{aligned}$$

Choice

$$H = -T^{-1}\left(SA - \frac{C^*C}{\delta^2} + E^*(\alpha)E(\alpha) + TA\right)$$

yields

$$\begin{aligned}
 & 2 \left\langle XJ \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \mathbb{A}J \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \right\rangle + \left\langle \mathbb{E}(\alpha)J \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \mathbb{E}(\alpha)J \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \right\rangle \\
 & + \left\langle \Pi J \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, J \begin{pmatrix} x \\ \hat{x} \end{pmatrix} \right\rangle = 0, \quad x \in \mathcal{D}(A), \hat{x} \in \mathcal{H}.
 \end{aligned}$$

Also, pick

$$\begin{pmatrix} x \\ \hat{x} \end{pmatrix} = \begin{bmatrix} I & -I \\ 0 & R^{-1} \end{bmatrix} z, \quad z \in \mathcal{D}(\mathbb{A}),$$

produces

$$2 \langle Xz, \mathbb{A}z \rangle + \langle \mathbb{E}(\alpha)z, \mathbb{E}(\alpha)z \rangle + \langle \Pi z, z \rangle = 0, \quad z \in \mathcal{D}(\mathbb{A}).$$

Since $\Pi \succ 0$, we may deduce that

$$2 \langle Xz, \mathbb{A}z \rangle + \langle \mathbb{E}(\alpha)z, \mathbb{E}(\alpha)z \rangle < 0, \quad z \in \mathcal{D}(\mathbb{A}).$$

Finally, we apply Proposition 2.1 to achieve the required outcome. \square

Example 2.1. *In this example, let us consider the following heat equation:*

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) = \frac{\partial^2 y}{\partial x^2}(x, t) + \frac{1}{2\epsilon} 1_{[x_0-\epsilon, x_0+\epsilon]}(x)u(t) + \Delta y(x, t)\xi(t), & x \in [0, 1], \\ y(x, 0) = y_0(x), & x \in [0, 1], \quad \frac{\partial y}{\partial x}(0, t) = \frac{\partial y}{\partial x}(1, t) = 0, & t \geq 0, \\ z(t) = \frac{1}{2\nu} \int_{x_1-\nu}^{x_1+\nu} y(x, t)dx, \end{cases} \quad (2.16)$$

where

$$\xi(t)dt = dw(t), \quad x_0, x_1 \in [0, 1], \quad \text{and} \quad \epsilon, \nu \in [0, 1],$$

with

$$0 < x_0 - \epsilon < x_0 + \epsilon < 1, \quad \text{and} \quad 0 < x_1 - \nu < x_1 + \nu < 1.$$

To abstract this problem, we introduce the following operators:

- The self-adjoint operator $Ah = \frac{d^2 h}{dx^2}$ in the Hilbert space $\mathcal{H} = L^2(0, 1)$ with

$$\mathcal{D}(A) = \left\{ h \in L^2(0, 1), \frac{d^2 h}{dx^2} \in L^2(0, 1), \frac{dh}{dx}(0) = \frac{dh}{dx}(1) = 0 \right\}$$

The operator A generates an exponentially stable semigroup $S(t)$. Moreover, its eigenvalues and the corresponding eigenfunctions are given by (cf. [8])

$$\lambda_n = -n^2\pi^2, \quad \phi_n(x) = \sqrt{2} \cos(n\pi x), \quad n \geq 1, \quad \lambda_0 = 0, \quad \phi_0 = 1$$

- $B : \mathcal{W} \rightarrow \mathcal{H}$ defined by

$$Bu(t) = \frac{1}{2\epsilon} 1_{[x_0-\epsilon, x_0+\epsilon]}(x)u(t);$$

- $C : \mathcal{H} \rightarrow \mathcal{Z}$ defined by

$$Cf = \frac{1}{2\nu} \int_{x_1-\nu}^{x_1+\nu} f(x)dx;$$

-

$$D = I : \mathcal{U} \rightarrow \mathcal{H}, \quad \Delta : \mathcal{Y} \rightarrow \mathcal{U}, \quad E = I : \mathcal{H} \rightarrow \mathcal{Y}, \quad \text{with} \quad \mathcal{U} = \mathcal{Y} = \mathcal{H}.$$

The abstract version of the system (2.16) is as follows:

$$\begin{cases} d\mathbb{Y}(t) = A\mathbb{Y}(t)dt + D\Delta(E\mathbb{Y}(t))dw(t) + Bu(t)dt, & t > 0, \\ \mathbb{Y}(0) = \mathbb{Y}_0, \\ z(t) = C\mathbb{Y}(t), & t \geq 0. \end{cases} \quad (2.17)$$

The compensator is shaped as follows

$$d\widehat{\mathbb{Y}}(t) = H\widehat{\mathbb{Y}}(t)dt + Gz(t)dt, \quad u(t) = F\widehat{\mathbb{Y}}(t) + Kz(t), \quad (2.18)$$

where $H : \mathcal{H} \rightarrow \mathcal{H}$ is a linear operator; $G \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$, $F \in \mathcal{L}(\mathcal{H}, \mathcal{W})$, $K \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$. Theorem 2.2 is applied to the compensator design. To begin, we identify the operators R and S that satisfy the requirements (2.8)-(2.11).

Let $\sigma > 0$ be given, let \mathbb{Y} and $\widehat{\mathbb{Y}} \in \mathcal{H}$. For $f \in D(A)$ we have

$$Af = \sum_{n=1}^{+\infty} - (n\pi)^2 \langle f, \phi_n(x) \rangle \phi_n(x),$$

$$\langle E(\alpha)f, E(\alpha)f \rangle = \alpha^2 \langle Ef, Ef \rangle = \alpha^2 \langle f, f \rangle.$$

Inequality (2.8) reads as

$$2 \langle \widehat{\mathbb{Y}}(t), AR\widehat{\mathbb{Y}}(t) \rangle + \alpha^2 \langle R\widehat{\mathbb{Y}}(t), R\widehat{\mathbb{Y}}(t) \rangle - \left\langle \frac{B^*}{\gamma} \widehat{\mathbb{Y}}(t), \frac{B^*}{\gamma} \widehat{\mathbb{Y}}(t) \right\rangle < 0. \quad (2.19)$$

Letting

$$R\widehat{\mathbb{Y}}(t) := \sum_{n=1}^{+\infty} r_n \langle \widehat{\mathbb{Y}}(t), \phi_n(x) \rangle \phi_n(x), \quad \text{with } \widehat{\mathbb{Y}}(t) \in \mathcal{H}, \quad r_n := \frac{2(n\pi)^2}{\alpha^2},$$

we find

$$\alpha^2 r_k^2 - 2(k\pi)^2 r_k - \frac{2}{(\gamma \epsilon k \pi)^2} (\cos k\pi \epsilon \sin k\pi x_0)^2 < 0, \quad k \geq 1.$$

Through using the knowledge that

$$\widehat{\mathbb{Y}}(t) = \sum_{n=1}^{+\infty} \langle \widehat{\mathbb{Y}}(t), \phi_n(x) \rangle \phi_n(x),$$

and since

$$B^* \phi_k(x) = \frac{\sqrt{2}}{(\epsilon k \pi)} \cos k\pi \epsilon \sin k\pi x_0$$

we derive

$$B^* \widehat{\mathbb{Y}}(t) = \sum_{n=1}^{+\infty} \langle \widehat{\mathbb{Y}}(t), \phi_n(x) \rangle \frac{\sqrt{2}}{(\epsilon n \pi)} \cos n\pi \epsilon \sin n\pi x_0.$$

Thus,

$$\begin{aligned}
 & 2 \left\langle \widehat{\mathbb{Y}}(t), AR\widehat{\mathbb{Y}}(t) \right\rangle + \alpha^2 \left\langle R\widehat{\mathbb{Y}}(t), R\widehat{\mathbb{Y}}(t) \right\rangle - \left\langle \frac{B^*}{\gamma} \widehat{\mathbb{Y}}(t), \frac{B^*}{\gamma} \widehat{\mathbb{Y}}(t) \right\rangle \\
 &= 2 \left\langle \widehat{\mathbb{Y}}(t), \sum_{k=1}^{+\infty} - (k\pi)^2 \left\langle \sum_{n=1}^{+\infty} r_n \left\langle \widehat{\mathbb{Y}}(t), \phi_n(x) \right\rangle \phi_n(x), \phi_k(x) \right\rangle \phi_k(x) \right\rangle \\
 &+ \alpha^2 \left\langle \sum_{k=1}^{+\infty} r_k \left\langle \widehat{\mathbb{Y}}(t), \phi_k(x) \right\rangle \phi_k(x), \sum_{n=1}^{+\infty} r_n \left\langle \widehat{\mathbb{Y}}(t), \phi_n(x) \right\rangle \phi_n(x) \right\rangle \\
 &- \frac{1}{\gamma^2} \left\langle \sum_{k=1}^{+\infty} \left\langle \widehat{\mathbb{Y}}(t), \phi_k(x) \right\rangle \frac{\sqrt{2}}{(\epsilon k \pi)} \cos k\pi\epsilon \sin k\pi x_0, \sum_{n=1}^{+\infty} \left\langle \widehat{\mathbb{Y}}(t), \phi_n(x) \right\rangle \frac{\sqrt{2}}{(\epsilon n \pi)} \cos n\pi\epsilon \sin n\pi x_0 \right\rangle \\
 &= 2 \left\langle \widehat{\mathbb{Y}}(t), \sum_{k=1}^{+\infty} - (k\pi)^2 r_k \left\langle \widehat{\mathbb{Y}}(t), \phi_k(x) \right\rangle \phi_k(x) \right\rangle + \alpha^2 \sum_{k=1}^{+\infty} r_k^2 \left\langle \widehat{\mathbb{Y}}(t), \phi_k(x) \right\rangle^2 \\
 &- \frac{1}{\gamma^2} \left\langle \sum_{k=1}^{+\infty} \left\langle \widehat{\mathbb{Y}}(t), \phi_k(x) \right\rangle \frac{\sqrt{2}}{(\epsilon k \pi)} \cos k\pi\epsilon \sin k\pi x_0, \sum_{n=1}^{+\infty} \left\langle \widehat{\mathbb{Y}}(t), \phi_n(x) \right\rangle \frac{\sqrt{2}}{(\epsilon n \pi)} \cos n\pi\epsilon \sin n\pi x_0 \right\rangle
 \end{aligned}$$

Inequality

$$-2(k\pi)^2 r_k + \alpha^2 r_k^2 - \frac{2}{(\gamma \epsilon k \pi)^2} (\cos k\pi\epsilon \sin k\pi x_0)^2 < 0,$$

give (2.19).

Now, inequality (2.9) is equivalent to

$$2 \langle S\mathbb{Y}(t), A\mathbb{Y}(t) \rangle + \langle E(\alpha)\mathbb{Y}(t), E(\alpha)\mathbb{Y}(t) \rangle - \left\langle \frac{C}{\delta} \mathbb{Y}(t), \frac{C}{\delta} \mathbb{Y}(t) \right\rangle < 0. \quad (2.20)$$

Letting

$$S\mathbb{Y}(t) := \sum_{k=1}^{+\infty} s_k \langle \mathbb{Y}(t), \phi_k(x) \rangle \phi_k(x) \quad \text{and} \quad \mathbb{Y}(t) \in \mathcal{H}, \quad s_k = \frac{\alpha^2}{(k\pi)^2}.$$

We have

$$-2s_k(k\pi)^2 + \alpha^2 - \left(\frac{\sqrt{2}}{(\delta \nu k \pi)} \cos k\pi\nu \sin k\pi x_1 \right)^2 < 0, \quad k \geq 1.$$

Using the observation that

$$\mathbb{Y}(t) = \sum_{n=1}^{+\infty} \langle \mathbb{Y}(t), \phi_n(x) \rangle \phi_n(x) \quad \text{and} \quad C\phi_k(x) = \frac{\sqrt{2}}{(\nu k \pi)} \cos k\pi\nu \sin k\pi x_1,$$

we get

$$C\mathbb{Y}(t) = \sum_{n=1}^{+\infty} \langle \mathbb{Y}(t), \phi_n(x) \rangle \frac{\sqrt{2}}{(\nu n \pi)} \cos n\pi\nu \sin n\pi x_1.$$

It follows that

$$\begin{aligned} & 2 \left\langle S\mathbb{Y}(t), A\mathbb{Y}(t) \right\rangle + \left\langle E(\alpha)\mathbb{Y}(t), E(\alpha)\mathbb{Y}(t) \right\rangle - \left\langle \frac{C}{\delta}\mathbb{Y}(t), \frac{C}{\delta}\mathbb{Y}(t) \right\rangle \\ &= 2 \left\langle \sum_{k=1}^{+\infty} s_k \langle \mathbb{Y}(t), \phi_k(x) \rangle \phi_k(x), \sum_{n=1}^{+\infty} - (n\pi)^2 \langle \mathbb{Y}(t), \phi_n(x) \rangle \phi_n(x) \right\rangle \\ &+ \alpha^2 \langle \mathbb{Y}(t), \mathbb{Y}(t) \rangle \\ &- \frac{1}{\delta^2} \left\langle \sum_{k=1}^{+\infty} \frac{\sqrt{2}}{(\nu k \pi)} \cos k\pi\nu \sin k\pi x_1 \langle \mathbb{Y}(t), \phi_k(x) \rangle, \right. \\ &\quad \left. \sum_{n=1}^{+\infty} \frac{\sqrt{2}}{(\nu n \pi)} \cos n\pi\nu \sin n\pi x_1 \langle \mathbb{Y}(t), \phi_n(x) \rangle \right\rangle \end{aligned}$$

For any $k \geq 1$ we have $s_k > r_k^{-1}$, then $S \succ R^{-1}$.

Thus for $\sigma < \frac{\pi}{\sqrt{\theta}}$ we get $1 - (\frac{\sigma}{\alpha})^2 \theta s_1 > 0$, and hence $I - (\frac{\sigma}{\alpha})^2 \theta S \succ 0$.

Consequently,

$$I - \left(\frac{\sigma}{\alpha}\right)^2 \theta D^* S D \succ 0.$$

Inequality (2.20) follows from the fact that

$$s_k (k\pi)^2 + \alpha^2 - \frac{1}{\delta^2} \frac{2}{(\nu k \pi)^2} (\cos k\pi\nu \sin k\pi x_1)^2 < 0.$$

Because the conditions of Theorem 2.2 are met, we select the compensator (H, G, F, K) such that

$$\begin{aligned} K &= 0, \quad F = -\frac{1}{\gamma^2} B^* R^{-1}, \quad M = R, \\ G &= -\frac{1}{\delta^2} N^{-1} C^*, \quad N = R^{-1} - S, \\ H &= -N^{-1} \left(SA - \frac{C^* C}{\delta^2} + E^*(\alpha) E(\alpha) + NA \right). \end{aligned}$$

with

$$\begin{aligned} Rz &= \sum_{n=1}^{+\infty} r_n \langle z, \phi_n(x) \rangle \phi_n(x), \quad \text{where } r_n := \frac{2(n\pi)^2}{\alpha^2}, \quad z \in \mathcal{H}, \\ Sz &= \sum_{n=1}^{+\infty} s_n \langle z, \phi_n(x) \rangle \phi_n(x), \quad \text{where } s_n := \frac{\alpha^2}{(n\pi)^2}, \quad z \in \mathcal{H}. \end{aligned}$$

CHAPTER 3

ROBUST STABILITY AND STABILIZATION OF SYSTEMS WITH UNBOUNDED MULTI-PERTURBATIONS

3.1 Introduction

The problems of robust stability and robust stabilization of an infinite dimensional system subjected to stochastic bounded structured perturbations have been studied in [41] using the stability radius approach. In the applications it is important to study these problems in the case where the perturbations operators structure are unbounded, because it covers the case of partial differential equations with boundary and pointwise noise.

Kada and Rebiai [42] considered the problem of robust stability with respect to single stochastic perturbations of unbounded structure for a class of infinite dimensional linear systems described by analytic semigroups. Our first objective in this chapter is to extend their results to the multiperturbations case. We give sufficient conditions pro-

viding the stability of the perturbed system. These conditions are used to determine a computational formula for the stability radius.

The second objective is to study the maximization of the stability radius by state feedback. First, we establish some technical lemmas. Based on these lemmas, conditions are derived for the existence of a stabilizing controller ensuring that the norm of the closed loop operator below a prespecified bound. Such controllers will be called suboptimal controllers. The suboptimality conditions are obtained in terms of a Riccati equation which satisfies an operator inequality. The supremal achievable stability radius is characterized via the Riccati equation. Two examples illustrating the results are treated at the end.

3.2 Robust stability

3.2.1 System description

Let A be the infinitesimal generator of an analytic exponentially stable semigroup $(S(t))_{t>0}$ on the real separable Hilbert space \mathcal{H} .

We will consider the system

$$dx(t) = Ax(t)dt + \sum_{i=1}^N D_i \Delta_i (E_i x(t)) d\omega_i(t), t > 0 \quad (3.1)$$

$$\| \Delta_i \|_{Lip} < \sigma, i \in \bar{N} = \{1, \dots, N\},$$

$$x(0) = x_0 \in \mathcal{H}$$

where, for $i \in \bar{N} = \{1, \dots, N\}$, we have

- D_i is a linear operator from \mathcal{U}_i to \mathcal{H} , (D_i is generally unbounded as an operator from \mathcal{U}_i to \mathcal{H}), such that $(-A)^{-\gamma} D_i \in \mathcal{L}(\mathcal{U}_i, \mathcal{H})$ for some fixed γ , $0 \leq \gamma < \frac{1}{2}$, where \mathcal{U}_i is a real separable Hilbert space.
- $E_i \in \mathcal{L}(D((-A)^{-\delta}), \mathcal{Y}_i)$ with $\delta < \min\{\frac{1}{2} - \gamma, \frac{1}{2}\}$, such that $E_i(-A)^{-\delta} \in \mathcal{L}(\mathcal{H}, \mathcal{Y}_i)$, where \mathcal{Y}_i is a real separable Hilbert spaces.
- $\Delta_i \in Lip(\mathcal{Y}_i, \mathcal{U}_i)$.

- $(w_i(t))_{t \in \mathbb{R}_+}$, $i \in \overline{N}$, are independent real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ relative to an increasing family $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ of σ -algebras $(\mathcal{F}_t)_{t \in \mathbb{R}_+} \subset \mathcal{F}$, such that

$$\begin{cases} \mathcal{E}(w_i(t)) = 0, \\ \mathcal{E}(w_i(t) - w_i(s))(w_j(t) - w_j(s)) = \delta_{ij}\theta_i(t - s), \quad i, j \in \overline{N}, \end{cases}$$

where δ_{ij} is the Kronecker symbol and $\theta_i > 0$.

Existence and uniqueness

In this theorem we establish the existence and uniqueness of the solution to the problem (3.1). We proceed as in [42].

Theorem 3.1. *For any $T > 0$, there exists a unique mild solution of the equation (3.1) in $\mathbb{C}([0, T], L^2(\Omega, \mathcal{H}))$, satisfying the initial condition $x(0) = x_0$*

Proof. The approach adopted to prove this theorem is based on the classical fixed point Theorem for contractions and on the analytic estimates.

Set

$$\Sigma = \mathbb{C}([0, T], L^2(\Omega, \mathcal{H}))$$

and define the corresponding norm by

$$\|x\|_{\Sigma} = \left(\sup_{t \in [0, T]} (\mathcal{E} \|x(t)\|^2) \right)^{\frac{1}{2}} < +\infty$$

The solution of the system (3.1) is given by

$$x(t) = S(t)x_0 + \sum_{i=1}^N \int_0^t S(t-s) D_i \Delta_i (E_i x(s)) d\omega_i(s)$$

We have

$$\begin{aligned} x(t) &= S(t)x_0 + \sum_{i=1}^N \int_0^t S(t-s) (-A)^{\gamma} (-A)^{-\gamma} D_i \Delta_i \left(E_i (-A)^{-\delta} (-A)^{\delta} x(s) \right) d\omega_i(s) \\ &= S(t)x_0 + \sum_{i=1}^N \int_0^t S(t-s) (-A)^{\gamma} \tilde{D}_i \tilde{\Delta}_i \left(\tilde{E}_i (-A)^{\delta} x(s) \right) d\omega_i(s) \end{aligned}$$

where $\tilde{D}_i = (-A)^{-\gamma} D_i \in \mathcal{L}(\mathcal{U}_i, \mathcal{H})$ and $\tilde{E}_i = E_i (-A)^{-\delta} \in \mathcal{L}(\mathcal{H}, \mathcal{Y}_i)$.

It follows that

$$(-A)^\delta x(t) = (-A)^\delta S(t)x_0 + \sum_{i=1}^N \int_0^t S(t-s) (-A)^\delta (-A)^\gamma \tilde{D}_i \Delta_i \left(\tilde{E}_i (-A)^\delta x(s) \right) d\omega_i(s)$$

By setting $z(t) = (-A)^\delta x(t)$ and $z_0 = z(0)$, we obtain

$$z(t) = S(t)z_0 + \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \quad (3.2)$$

where $\beta = \gamma + \delta$.

In order to establish existence and uniqueness for (3.1), we establish the existence and uniqueness for (3.2). We proceed in three steps.

Step 1: Let

$$\Gamma : \Sigma \rightarrow \Sigma$$

defined by

$$\Gamma(z(t)) = S(t)z_0 + \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s)$$

Γ is well defined as a mapping from Σ to Σ

Since $(S(t))_{t \geq 0}$ is an analytic exponentially stable semigroup there exist positive constants M , M_β , β and ω such that

$$\|S(t)\| \leq M e^{-\omega t}, \quad t > 0, \quad \omega > 0 \quad \text{and} \quad \left\| (-A)^\beta S(t) \right\| \leq M_\beta t^{-\beta} e^{-\omega t}.$$

Now since $\tilde{D}_i = (-A)^{-\gamma} D_i \in \mathcal{L}(\mathcal{U}_i, \mathcal{H})$ and $\tilde{E}_i = E_i (-A)^{-\delta} \in \mathcal{L}(\mathcal{H}, \mathcal{Y}_i)$, there exist constants M_γ and M_δ such that

$$\left\| \tilde{D}_i \right\|_{\mathcal{L}(\mathcal{U}_i, \mathcal{H})} \leq M_\gamma \quad \text{and} \quad \left\| \tilde{E}_i \right\|_{\mathcal{L}(\mathcal{H}, \mathcal{Y}_i)} \leq M_\delta$$

we have

$$\begin{aligned} \|\Gamma(z(t))\|^2 &= \langle \Gamma(z(t)), \Gamma(z(t)) \rangle \\ &= \|S(t)z_0\|^2 + \left\langle S(t)z_0, \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \right\rangle \\ &\quad + \left\langle \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s), S(t)z_0 \right\rangle \end{aligned}$$

$$+ \left\langle \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s), \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s) \right\rangle$$

We have

$$\mathcal{E} (\|S(t)z_0\|^2) = \|S(t)z_0\|^2$$

$$\mathcal{E} \left(\left\langle S(t)z_0, \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s) \right\rangle \right) = 0$$

Let $G_i(s) = S(t-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s))$, then

$$\mathcal{E} \left(\left\| \sum_{i=1}^N \int_0^t G_i(s) d\omega_i(s) \right\|^2 \right) = \mathcal{E} \left(\left\langle \sum_{i=1}^N \int_0^t G_i(s) d\omega_i(s), \sum_{j=1}^N \int_0^t G_j(s) d\omega_j(s) \right\rangle \right)$$

Since the Wiener process are independant, we have for $i = j$:

$$\mathcal{E} \left(\left\| \sum_{i=1}^N \int_0^t G_i(s) d\omega_i(s) \right\|^2 \right) = \mathcal{E} \left(\sum_{i=1}^N \left\langle \int_0^t G_i(s) d\omega_i(s), \int_0^t G_i(s) d\omega_i(s) \right\rangle \right)$$

By Burkholder-Davis-Gundy inequality 1.2, there exist positive constants c_i such that

$$\mathcal{E} \left(\left\| \sum_{i=1}^N \int_0^t G_i(s) d\omega_i(s) \right\|^2 \right) \leq \sum_{i=1}^N c_i \int_0^t \mathcal{E} \|G_i(s)\|^2 ds$$

Then

$$\begin{aligned} \mathcal{E} (\|\Gamma(z(t))\|^2) &\leq \|S(t)z_0\|^2 + \sum_{i=1}^N c_i \int_0^t \mathcal{E} \|G_i(s)\|^2 ds \\ &\leq \|S(t)z_0\|^2 + \sum_{i=1}^N c_i \int_0^t \mathcal{E} \left(\|S(t-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s))\|^2 \right) ds \\ &\leq \|S(t)z_0\|^2 + \sum_{i=1}^N c_i \int_0^t \mathcal{E} \left\| S(t-s) (-A)^\beta \right\|^2 \left\| \tilde{D}_i \right\|^2 \left\| \Delta_i (\tilde{E}_i z(s)) \right\|^2 ds \\ &\leq M^2 e^{-2\omega t} \|z_0\|^2 + \sum_{i=1}^N c_i \int_0^t M_\beta^2 (t-s)^{-2\beta} e^{-2\omega(t-s)} M_\gamma^2 K_i^2 \mathcal{E} \left\| \tilde{E}_i z(s) \right\|^2 ds \\ &\leq M^2 \|z_0\|^2 + \sum_{i=1}^N c_i M_\beta^2 M_\gamma^2 K_i^2 M_\delta^2 \int_0^t (t-s)^{-2\beta} e^{-2\omega(t-s)} \mathcal{E} \|z(s)\|^2 ds \end{aligned}$$

$$\begin{aligned} &\leq M^2 \|z_0\|^2 + \sum_{i=1}^N c_i M_i \int_0^t (t-s)^{-2\beta} e^{-2\omega(t-s)} \mathcal{E} \|z(s)\|^2 ds \text{ where } M_i = M_\beta^2 M_\gamma^2 K_i^2 M_\delta^2 \\ &\leq M^2 \|z_0\|^2 + \sum_{i=1}^N M'_i \int_0^t (t-s)^{-2\beta} e^{-2\omega(t-s)} \mathcal{E} \|z(s)\|^2 ds \text{ where } M'_i = c_i M_i \end{aligned}$$

But

$$\int_0^t (t-s)^{-2\beta} e^{-2\omega(t-s)} ds \leq \frac{1}{1-2\beta} t^{1-2\beta} \leq \frac{1}{1-2\beta} T^{1-2\beta}$$

hence

$$\begin{aligned} \mathcal{E} (\|\Gamma(z(t))\|^2) &\leq M^2 \|z_0\|^2 + \sum_{i=1}^N M'_i \frac{1}{1-2\beta} T^{1-2\beta} \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \\ &\leq M^2 \|z_0\|^2 + \frac{1}{1-2\beta} T^{1-2\beta} \left(N \max_{i \in \bar{N}} M'_i \right) \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 < +\infty \end{aligned}$$

We conclude that Γ is well defined on Σ .

Step 2: Now we show that Γ maps Σ into Σ .

For $h \in [0, T]$ and $t \in [0, T-h]$, we have

$$\begin{aligned} \Gamma(z)(t+h) - \Gamma(z)(t) &= S(t+h)z_0 + \sum_{i=1}^N \int_0^{t+h} S(t+h-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \\ &\quad - \left(S(t)z_0 + \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \right) \\ &= (S(t+h)z_0 - S(t)z_0) + \sum_{i=1}^N \int_0^t S(t+h-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \\ &\quad - \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \\ &\quad + \sum_{i=1}^N \int_t^{t+h} S(t+h-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \\ &= (S(t+h)z_0 - S(t)z_0) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^N \int_0^t (S(t+h-s) - S(t-s)) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \\
 & + \sum_{i=1}^N \int_t^{t+h} S(t+h-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s)
 \end{aligned}$$

From a version of Burkholder-Davis-Gundy inequality 1.2, there exist $c_i > 0$ and $c'_i > 0$ such that

$$\begin{aligned}
 & \mathcal{E} \left(\|\Gamma(z)(t+h) - \Gamma(z)(t)\|^2 \right) \leq \|S(t+h)z_0 - S(t)z_0\|^2 \\
 & + \sum_{i=1}^N c_i \int_0^t \varepsilon \left\| (S(t+h-s) - S(t-s)) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 ds \\
 & + \sum_{i=1}^N c'_i \int_t^{t+h} \mathcal{E} \left\| S(t+h-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 ds
 \end{aligned}$$

Since $(S(t))_{t \geq 0}$ is strongly continuous, we have

$$\lim_{h \rightarrow 0} \|S(t+h)z_0 - S(t)z_0\| = 0$$

and we have

$$S(t+h-s) = S(t-s)S(h)$$

then

$$S(t+h-s) - S(t-s) = S(t-s)S(h) - S(t-s) = S(t-s)[S(h) - I]$$

and

$$\begin{aligned}
 & \int_0^t \mathcal{E} \left\| (S(t+h-s) - S(t-s)) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 ds \\
 & = \int_0^t \mathcal{E} \left\| S(t-s) (S(h) - I) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 ds \\
 & \leq \int_0^t \mathcal{E} \left(\|S(h) - I\|^2 \left\| S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 \right) ds \\
 & \leq \|S(h) - I\|^2 \int_0^t M_i(t-s)^{-2\beta} e^{-2\omega(t-s)} \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 ds
 \end{aligned}$$

But

$$\int_0^t (t-s)^{-2\beta} e^{-2\omega(t-s)} ds \leq \frac{1}{1-2\beta} T^{1-2\beta}$$

It follows that

$$\begin{aligned} & \int_0^t \mathcal{E} \left\| (S(t+h-s) - S(t-s)) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 ds \\ & \leq M_i \|S(h) - I\|^2 \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \frac{1}{1-2\beta} T^{1-2\beta} \end{aligned}$$

Then

$$\begin{aligned} & \sum_{i=1}^N c_i \int_0^t \mathcal{E} \left\| (S(t+h-s) - S(t-s)) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 ds \\ & \leq \sum_{i=1}^N c_i M_i \|S(h) - I\|^2 \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \frac{1}{1-2\beta} T^{1-2\beta} \\ & \leq \|S(h) - I\|^2 \frac{1}{1-2\beta} T^{1-2\beta} \sum_{i=1}^N M'_i \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \\ & \leq \|S(h) - I\|^2 \frac{1}{1-2\beta} T^{1-2\beta} \left(N \max_{i \in \mathcal{N}} M'_i \right) \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \text{ where } M'_i = c_i M_i \end{aligned}$$

Now for the last term we have

$$\begin{aligned} & \sum_{i=1}^N c'_i \int_t^{t+h} \mathcal{E} \left\| S(t+h-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 ds \\ & \leq \sum_{i=1}^N c'_i M_i \int_t^{t+h} (t+h-s)^{-2\beta} e^{-2\omega(t+h-s)} \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 ds \\ & \leq \sum_{i=1}^N M''_i \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \int_t^{t+h} (t+h-s)^{-2\beta} e^{-2\omega(t+h-s)} ds \text{ where } M''_i = c'_i M_i \end{aligned}$$

But

$$\begin{aligned} & \int_t^{t+h} (t+h-s)^{-2\beta} e^{-2\omega(t+h-s)} ds \leq \frac{h^{1-2\beta}}{1-2\beta} \\ & \sum_{i=1}^N c'_i \int_t^{t+h} \mathcal{E} \left\| S(t+h-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 ds \\ & \leq \sum_{i=1}^N M''_i \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \frac{h^{1-2\beta}}{1-2\beta} \end{aligned}$$

$$\leq \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \frac{h^{1-2\beta}}{1-2\beta} \left(N \max_{i \in \bar{N}} M_i'' \right)$$

Therefore

$$\begin{aligned} \mathcal{E} (\|\Gamma(z)(t+h) - \Gamma(z)(t)\|^2) &\leq \|S(t+h)z_0 - S(t)z_0\|^2 \\ + \|S(h) - I\|^2 \frac{1}{1-2\beta} T^{1-2\beta} N \max_{i \in \bar{N}} M_i' \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 &+ \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \frac{h^{1-2\beta}}{1-2\beta} \left(N \max_{i \in \bar{N}} M_i'' \right) \end{aligned}$$

Then

$$\lim_{h \rightarrow 0} \mathcal{E} (\|\Gamma(z)(t+h) - \Gamma(z)(t)\|^2) = 0$$

In order to prove the left continuity of $\Gamma(z)$ we have, for all $h \in [0, t]$, $t \in [h, T]$

$$\begin{aligned} \Gamma(z)(t-h) - \Gamma(z)(t) &= S(t-h)z_0 + \sum_{i=1}^N \int_0^{t-h} S(t-h-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s) \\ &\quad - S(t)z_0 - \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s) \\ &= (S(t-h)z_0 - S(t)z_0) + \sum_{i=1}^N \int_0^{t-h} S(t-h-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s) \\ &\quad - \sum_{i=1}^N \int_{t-h}^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s) \\ &= (S(t-h)z_0 - S(t)z_0) + \sum_{i=1}^N \int_0^{t-h} (S(t-h-s) - S(t-s)) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s) \\ &\quad - \sum_{i=1}^N \int_{t-h}^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s) \end{aligned}$$

$(S(t))_{t \geq 0}$ is strongly continuous then

$$\lim_{h \rightarrow 0} \|S(t-h)z_0 - S(t)z_0\| = 0$$

and

$$\sum_{i=1}^N \int_0^{t-h} (S(t-h-s) - S(t-s)) (-A)^\beta \tilde{D}_i \Delta_i (\tilde{E}_i z(s)) d\omega_i(s)$$

$$\begin{aligned}
 &= \sum_{i=1}^N \int_0^{t-h} S(t-h-s) (I - S(h)) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \\
 &= - \sum_{i=1}^N \int_0^{t-h} S(t-h-s) (S(h) - I) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s)
 \end{aligned}$$

Then

$$\begin{aligned}
 &\mathcal{E} \left(\left\| - \sum_{i=1}^N \int_0^{t-h} S(t-h-s) (S(h) - I) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) d\omega_i(s) \right\|^2 \right) \\
 &\leq \|S(h) - I\|^2 \sum_{i=1}^N \tilde{c}_i'' M_i \int_0^{t-h} (t-h-s)^{-2\beta} e^{-2\omega(t-h-s)} \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 ds \\
 &\leq \|S(h) - I\|^2 \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \frac{1}{1-2\beta} T^{1-2\beta} \sum_{i=1}^N \tilde{c}_i'' M_i \\
 &\leq \|S(h) - I\|^2 \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \frac{1}{1-2\beta} T^{1-2\beta} \left(N \max_{i \in \bar{N}} \tilde{c}_i'' M_i \right)
 \end{aligned}$$

For the last term

$$\begin{aligned}
 &\sum_{i=1}^N \left(\mathcal{E} \left\| \int_{t-h}^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) ds \right\|^2 \right) \\
 &\leq \sum_{i=1}^N \tilde{c}_i \int_{t-h}^t \mathcal{E} \left(\left\| S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z(s) \right) \right\|^2 \right) ds \\
 &\leq \sum_{i=1}^N \tilde{c}_i M_i \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \left| \int_{t-h}^t (t-s)^{-2\beta} e^{-2\omega(t-s)} ds \right| \\
 &\leq \sum_{i=1}^N \tilde{c}_i M_i \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \frac{h^{1-2\beta}}{1-2\beta} \\
 &\leq \sup_{s \in [0, T]} \mathcal{E} \|z(s)\|^2 \frac{h^{1-2\beta}}{1-2\beta} \left(N \max_{i \in \bar{N}} \tilde{c}_i M_i \right)
 \end{aligned}$$

Then

$$\lim_{h \rightarrow 0} \mathcal{E} \left(\|\Gamma(z)(t-h) - \Gamma(z)(t)\|^2 \right) = 0$$

Step 3: It remains to verify that Γ is a contraction.

Let $z_1, z_2 \in \Sigma$, then

$$\begin{aligned} \Gamma(z_1(t)) - \Gamma(z_2(t)) &= S(t)z_0 + \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z_1(s) \right) d\omega_i(s) \\ &\quad - S(t)z_0 - \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \Delta_i \left(\tilde{E}_i z_2(s) \right) d\omega_i(s) \\ &= \sum_{i=1}^N \int_0^t S(t-s) (-A)^\beta \tilde{D}_i \left(\Delta_i \left(\tilde{E}_i z_1(s) \right) - \Delta_i \left(\tilde{E}_i z_2(s) \right) \right) d\omega_i(s) \end{aligned}$$

Then

$$\begin{aligned} &\mathcal{E} \left(\|\Gamma(z_1(t)) - \Gamma(z_2(t))\|^2 \right) \\ &\leq \sum_{i=1}^N q_i \int_0^t \left\| S(t-s) (-A)^\beta \right\|^2 \mathcal{E} \left\| \tilde{D}_i \left(\Delta_i \left(\tilde{E}_i z_1(s) \right) - \Delta_i \left(\tilde{E}_i z_2(s) \right) \right) \right\|^2 ds \\ &\leq \sum_{i=1}^N q_i M_i \int_0^t (t-s)^{-2\beta} e^{-2\omega(t-s)} \sup_{s \in [0, T]} (\mathcal{E} \|z_1(s) - z_2(s)\|^2) ds \\ &\leq \sum_{i=1}^N q_i M_i \frac{1}{1-2\beta} T^{1-2\beta} \sup_{s \in [0, T]} (\mathcal{E} \|z_1(s) - z_2(s)\|^2) \\ &\leq \frac{1}{1-2\beta} T^{1-2\beta} \left(N \max_{i \in \bar{N}} (q_i M_i) \right) \sup_{s \in [0, T]} (\mathcal{E} \|z_1(s) - z_2(s)\|^2) \\ &\leq M(T) \sup_{s \in [0, T]} (\mathcal{E} \|z_1(s) - z_2(s)\|^2) \end{aligned}$$

Therefore Γ is contractive for enough small $T > 0$. For large T we can proceed in a usual way by considering the equation on intervals $[0, \tilde{T}]$, $[\tilde{T}, 2\tilde{T}]$, ... with \tilde{T} enough small. \square

3.2.2 Characterizations of the stability radius

Let $\mathcal{Y} = \bigoplus_{i=1}^N \mathcal{Y}_i$ and $\mathcal{U} = \bigoplus_{i=1}^N \mathcal{U}_i$.

The approach used in this work to characterize the stochastic stability radius $r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right)$ is based on the following lemma.

Lemma 3.1. Let $E \in \mathcal{L}(D((-A)^\delta), \mathcal{Y})$, and

$$\begin{aligned} y(t) &= ES(t)x_0 + E \sum_{j=1}^N \int_0^t S(t-s) D_j v_j(s) d\omega_j(s) \\ &= ES(t)x_0 + \sum_{j=1}^N E \int_0^t S(t-s) D_j v_j(s) d\omega_j(s) \end{aligned} \quad (3.3)$$

where $v_j \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{U}_j))$, $j \in \overline{N}$, and $x_0 \in \mathcal{H}$.

Then

$$\mathcal{E}(\|y(t)\|_{\mathcal{Y}}^2) = \|ES(t)x_0\|^2 + \sum_{j=1}^N \theta_j \int_0^t \mathcal{E} \left(\left\| \tilde{E}S(t-s) (-A)^\beta \tilde{D}_j v_j(s) \right\|^2 \right) ds$$

where $\tilde{D}_j = (-A)^{-\gamma} D_j$, $\tilde{E} = E(-A)^{-\delta}$ and $\beta = \gamma + \delta$

Moreover, $y(\cdot) \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{Y}))$ and

$$\begin{aligned} \|y\|_{L_\omega^2}^2 &= \int_0^{+\infty} \mathcal{E}(\|y(t)\|^2) dt \\ &= \int_0^{+\infty} \|ES(t)x_0\|^2 dt + \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E}(\langle D_j v_j(s), P D_j v_j(s) \rangle) ds \end{aligned}$$

where $P \in \mathcal{L}(\mathcal{H})$ is a self-adjoint nonnegative operator satisfying

$$2 \langle Px, Ax \rangle + \langle Ex, Ex \rangle = 0, x \in \mathcal{D}(A). \quad (3.4)$$

Proof. For any $j \in \overline{N}$, we have

$$\begin{aligned} E \int_0^t S(t-s) D_j v_j(s) d\omega_j(s) &= E \int_0^t (-A)^{-\delta} (-A)^\delta S(t-s) D_j v_j(s) d\omega_j(s) \\ &= E (-A)^{-\delta} \int_0^t (-A)^\delta S(t-s) D_j v_j(s) d\omega_j(s) \\ &= \tilde{E} \int_0^t (-A)^\delta S(t-s) D_j v_j(s) d\omega_j(s) \\ &= \tilde{E} \int_0^t (-A)^\delta S(t-s) (-A)^\gamma (-A)^{-\gamma} D_j v_j(s) d\omega_j(s) \\ &= \tilde{E} \int_0^t (-A)^\delta S(t-s) (-A)^\gamma \tilde{D}_j v_j(s) d\omega_j(s) \\ &= \int_0^t \tilde{E} S(t-s) (-A)^\beta \tilde{D}_j v_j(s) d\omega_j(s) \end{aligned}$$

and we have

$$\begin{aligned}
 \|y(t)\|^2 &= \langle y(t), y(t) \rangle \\
 &= \|ES(t)x_0\|^2 + \left\langle ES(t)x_0, \sum_{j=1}^N \int_0^t \tilde{E}S(t-s)(-A)^\beta \tilde{D}_j v_j(s) d\omega_j(s) \right\rangle \\
 &\quad + \left\langle \sum_{j=1}^N \int_0^t \tilde{E}S(t-s)(-A)^\beta \tilde{D}_j v_j(s) d\omega_j(s), ES(t)x_0 \right\rangle \\
 &\quad + \left\| \sum_{j=1}^N \int_0^t \tilde{E}S(t-s)(-A)^\beta \tilde{D}_j v_j(s) d\omega_j(s) \right\|^2
 \end{aligned}$$

Hence

$$\mathcal{E}(\|y(t)\|^2) = \|ES(t)x_0\|^2 + \sum_{j=1}^N \theta_j \int_0^t \mathcal{E} \left(\left\| \tilde{E}S(t-s)(-A)^\beta \tilde{D}_j v_j(s) \right\|^2 \right) ds. \quad (3.5)$$

Thus

$$\begin{aligned}
 \|y\|_{L_w^2}^2 &= \int_0^{+\infty} \mathcal{E}(\|y(t)\|^2) dt = \int_0^{+\infty} \|ES(t)x_0\|^2 dt \\
 &\quad + \int_0^{+\infty} \left(\sum_{j=1}^N \theta_j \int_0^t \mathcal{E} \left(\left\| \tilde{E}S(t-s)(-A)^\beta \tilde{D}_j v_j(s) \right\|^2 \right) ds \right) dt
 \end{aligned}$$

we have

$$\|ES(t)x_0\|^2 = \left\| \tilde{E}(-A)^\delta S(t)x_0 \right\|^2$$

Since $\tilde{E} \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ there exists M_δ and M'_δ positive constants such that

$$\begin{aligned}
 \int_0^{+\infty} \|ES(t)x_0\|^2 dt &= \int_0^{+\infty} \left\| \tilde{E}(-A)^\delta S(t)x_0 \right\|^2 dt \\
 &\leq M_\delta^2 M'_\delta \|x_0\|^2 \int_0^{+\infty} t^{-2\delta} e^{-2\omega t} dt
 \end{aligned}$$

From the fact that $\delta < \frac{1}{2}$ it follows that $\int_0^{+\infty} t^{-2\delta} e^{-2\omega t} dt < +\infty$.

Therefore

$$\int_0^{+\infty} \|ES(t)x_0\|^2 dt < +\infty.$$

Now we have

$$\begin{aligned}
 I &= \int_0^{+\infty} \left(\sum_{j=1}^N \theta_j \int_0^t \mathcal{E} \left(\left\| \tilde{E}S(t-s) (-A)^\beta \tilde{D}_j v_j(s) \right\|^2 \right) ds \right) dt \\
 &= \sum_{j=1}^N \theta_j \int_0^{+\infty} \left(\int_0^t \mathcal{E} \left(\left\| \tilde{E}S(t-s) (-A)^\beta \tilde{D}_j v_j(s) \right\|^2 \right) ds \right) dt \\
 &= \sum_{j=1}^N \theta_j \int_0^{+\infty} \left[\int_s^{+\infty} \mathcal{E} \left(\left\| \tilde{E}S(t-s) (-A)^\beta \tilde{D}_j v_j(s) \right\|^2 \right) dt \right] ds \\
 &= \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} \left(\int_s^{+\infty} \left\langle \tilde{E}S(t-s) (-A)^\beta \tilde{D}_j v_j(s), \tilde{E}S(t-s) (-A)^\beta \tilde{D}_j v_j(s) \right\rangle dt \right) ds \\
 &= \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} \left(\left\langle \tilde{D}_j v_j(s), \int_s^{+\infty} \left(\tilde{E}S(t-s) (-A)^\beta \right)^* \tilde{E}S(t-s) (-A)^\beta \tilde{D}_j v_j(s) \right\rangle dt \right) ds \\
 &= \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} \left(\left\langle \begin{array}{c} (-A)^{-\gamma} D_j v_j(s), \int_s^{+\infty} \left(S^*(t-s) \left((-A)^\beta \right)^* \left(\tilde{E} \right)^* \right) \\ \tilde{E}S(t-s) (-A)^\beta \tilde{D}_j v_j(s) \end{array} \right\rangle dt \right) ds \\
 &= \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} \left(\left\langle \begin{array}{c} D_j v_j(s), \int_s^{+\infty} \left(S^*(t-s) \left((-A)^\beta \right)^* \left((-A)^{-\gamma} \right)^* \left(\tilde{E} \right)^* \right) \\ \tilde{E}S(t-s) (-A)^\beta (-A)^{-\gamma} D_j v_j(s) \end{array} \right\rangle dt \right) ds \\
 &= \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} \left(\left\langle \begin{array}{c} D_j v_j(s), \\ \left(\int_s^{+\infty} S^*(t-s) \left((-A)^\delta \right)^* \left(\tilde{E} \right)^* \tilde{E} (-A)^\delta S(t-s) dt \right) D_j v_j(s) \end{array} \right\rangle \right) ds \\
 &= \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} \left(\left\langle D_j v_j(s), \left(\int_s^{+\infty} S^*(t-s) \left(\tilde{E} (-A)^\delta \right)^* \tilde{E} (-A)^\delta S(t-s) dt \right) D_j v_j(s) \right\rangle \right) ds \\
 &= \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} \left(\left\langle D_j v_j(s), \left(\int_s^{+\infty} S^*(t-s) E^* E S(t-s) dt \right) D_j v_j(s) \right\rangle \right) ds
 \end{aligned}$$

Therefore

$$I = \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} (\langle D_j v_j(s), P D_j v_j(s) \rangle) ds$$

where

$$Pz = \int_0^{+\infty} S^*(r) E^* E S(r) z dr, z \in \mathcal{H}$$

We show that P is a bounded operator.

Let $z \in \mathcal{H}$, we have

$$\begin{aligned} \langle Pz, z \rangle &= \left\langle \int_0^{+\infty} S^*(r) E^* E S(r) z dr, z \right\rangle = \left\langle \int_0^{+\infty} S^*(r) \left((-A)^\delta \right)^* \tilde{E}^* \tilde{E} (-A)^\delta S(r) z dr, z \right\rangle \\ &= \int_0^{+\infty} \left\langle \tilde{E} (-A)^\delta S(r) z, \tilde{E} (-A)^\delta S(r) z \right\rangle dr \quad (3.6) \\ &= \int_0^{+\infty} \left\| \tilde{E} (-A)^\delta S(r) z \right\|^2 dr \geq 0 \end{aligned}$$

Therefore

$$\langle Pz, z \rangle \leq M_\delta^2 M'_\delta \|z\|^2 \int_0^{+\infty} r^{-2\delta} e^{-2\omega r} dr,$$

Since $\delta < \frac{1}{2}$ the integral $\int_0^{+\infty} r^{-2\delta} e^{-2\omega r} dr$ is bounded. We deduce then that $P \in \mathcal{L}^+(\mathcal{H})$.

Then

$$\begin{aligned} \int_0^{+\infty} \mathcal{E} (\|y(t)\|_Y^2) dt &= \int_0^{+\infty} \left\| \tilde{E} (-A)^\delta S(t) x_0 \right\|^2 dt + \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} (\langle D_j v_j(s), P D_j v_j(s) \rangle) ds \\ &= \int_0^{+\infty} \|E S(t) x_0\|^2 dt + \sum_{j=1}^N \theta_j \int_0^{+\infty} \mathcal{E} (\langle D_j v_j(s), P D_j v_j(s) \rangle) ds \end{aligned}$$

Now we show that $\int_0^{+\infty} \mathcal{E} (\langle D_j v_j(s), P D_j v_j(s) \rangle) ds < \infty$. From the previous, we have for $z \in \mathcal{U}$, $j \in \bar{N}$,

$$\langle D_j z, P D_j z \rangle = \left\langle D_j z, \int_0^{+\infty} S^*(r) E^* E S(r) D_j z dr \right\rangle$$

$$\begin{aligned}
 &= \left\langle (-A)^\gamma \tilde{D}_j z, \int_0^{+\infty} S^*(r) \left((-A)^\delta \right)^* \tilde{E}^* \tilde{E} (-A)^\delta S(r) (-A)^\gamma \tilde{D}_j z dr \right\rangle \\
 &= \left\langle (-A)^\gamma \tilde{D}_j z, \int_0^{+\infty} S^*(r) \left((-A)^{-\gamma} \right)^* \left((-A)^\beta \right)^* \tilde{E}^* \tilde{E} (-A)^\beta (-A)^{-\gamma} S(r) (-A)^\gamma \tilde{D}_j z dr \right\rangle \\
 &= \left\langle (-A)^\gamma \tilde{D}_j z, \int_0^{+\infty} \left((-A)^{-\gamma} \right)^* S^*(r) \left((-A)^\beta \right)^* \tilde{E}^* \tilde{E} (-A)^\beta S(r) (-A)^{-\gamma} (-A)^\gamma \tilde{D}_j z dr \right\rangle \\
 &= \left\langle \tilde{D}_j z, \int_0^{+\infty} S^*(r) \left((-A)^\beta \right)^* \tilde{E}^* \tilde{E} (-A)^\beta S(r) \tilde{D}_j z dr \right\rangle \\
 &= \int_0^{+\infty} \left\langle \tilde{E} (-A)^\beta S(r) \tilde{D}_j z, \tilde{E} (-A)^\beta S(r) \tilde{D}_j z \right\rangle dr \\
 &= \int_0^{+\infty} \left\| \tilde{E} (-A)^\beta S(r) \tilde{D}_j z \right\|^2 dr
 \end{aligned}$$

where $\beta = \delta + \gamma$. Therefore

$$\langle D_j z, P D_j z \rangle \leq M_\beta^2 M_\delta' M_\gamma' \|z\|^2 \int_0^{+\infty} r^{-2\beta} e^{-2\omega r} dr,$$

Since $\beta < \frac{1}{2}$ the integral $\int_0^{+\infty} r^{-2\beta} e^{-2\omega r} dr$ is bounded. We deduce then that $\int_0^{+\infty} \mathcal{E} (\langle D_j v_j(s), P D_j v_j(s) \rangle) ds < \infty$

□

Now consider the input-output linear map

$$\tilde{L} : L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{U})) \rightarrow L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{Y}))$$

defined by

$$\begin{aligned}
 (\tilde{L}v)(t) &= E \sum_{i=1}^N \int_0^t S(t-s) D_i v_i(s) d\omega_i(s) \\
 &= \sum_{i=1}^N E \int_0^t S(t-s) D_i v_i(s) d\omega_i(s)
 \end{aligned} \tag{3.7}$$

By Lemma 3.1, $\tilde{L}v \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{Y}))$ for any $v \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{U}))$.

Lemma 3.2. *The linear map \tilde{L} defined by (3.7) has the operator norm*

$$\left\| \tilde{L} \right\|_{L_\omega^2} = \left(\max_{i \in \bar{N}} (\theta_i \|D_i^* P D_i\|) \right)^{\frac{1}{2}}$$

where P satisfies (3.4).

Proof. Let $v \in L^2_\omega(\mathbb{R}^+, L^2(\Omega, \mathcal{U}))$, we have

$$\begin{aligned} \|\tilde{L}v\|_{L^2_\omega}^2 &= \int_0^{+\infty} \mathcal{E} \left(\left\| (\tilde{L}v)(t) \right\|^2 \right) dt \\ &= \sum_{i=1}^N \theta_i \int_0^{+\infty} \mathcal{E} (\langle D_i v_i(s), PD_i v_i(s) \rangle) ds \\ &\leq \sum_{i=1}^N \theta_i \int_0^{+\infty} \mathcal{E} (\|D_i^* PD_i\| \|v_i(s)\|^2) ds \\ &\leq \max_{i \in \bar{N}} (\theta_i \|D_i^* PD_i\|) \|v\|_{L^2_\omega}^2 \end{aligned}$$

Thus

$$\|\tilde{L}\|_{L^2_\omega} \leq \left(\max_{i \in \bar{N}} (\theta_i \|D_i^* PD_i\|) \right)^{\frac{1}{2}}$$

Now we will show that there exists $v^0 \in L^2_\omega(\mathbb{R}^+, L^2(\Omega, \mathcal{U}))$ such that

$$\|\tilde{L}v^0\|_{L^2_\omega} = \left(\max_{i \in \bar{N}} (\theta_i \|D_i^* PD_i\|) \right)^{\frac{1}{2}}$$

Or

$$\sum_{i=1}^N \theta_i \int_0^{+\infty} \mathcal{E} (\langle D_i v_i^0(s), PD_i v_i^0(s) \rangle) ds = \left(\max_{i \in \bar{N}} (\theta_i \|D_i^* PD_i\|) \right)^{\frac{1}{2}}$$

Now suppose that

$$\max_{i \in \bar{N}} (\theta_i \|D_i^* PD_i\|) = \theta_{i_0} \|D_{i_0}^* PD_{i_0}\|.$$

Since $D_{i_0}^* PD_{i_0}$ is self-adjoint, then there exists v_{i_0} such that $\|v_{i_0}\| = 1$ and

$$\|D_{i_0}^* PD_{i_0}\| = \sup_{\|v\|=1} \langle v, D_{i_0}^* PD_{i_0} v \rangle = \langle v_{i_0}, D_{i_0}^* PD_{i_0} v_{i_0} \rangle$$

We define v^0 as follows

$$\begin{cases} v_i^0(t) = 0, t \in \mathbb{R}^+, i \neq i_0 \\ v_{i_0}^0(t) = \beta(t) v_{i_0} \end{cases}$$

where $\beta(\cdot) \in L^2(\mathbb{R}^+, \mathbb{R})$ and $\|\beta(\cdot)\|_{L^2(\mathbb{R}^+, \mathbb{R})} = 1$.

Then $v^0(\cdot) = (v_i^0(\cdot))_{i \in \bar{N}} \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{U}))$. We have

$$\begin{aligned}
 \|v^0(\cdot)\|_{L_\omega^2}^2 &= \sum_{i=1}^N \int_0^{+\infty} \mathcal{E} \|v_i^0(s)\|^2 ds \\
 &= \sum_{i=1}^N \int_0^{+\infty} \|v_i^0(s)\|^2 ds \\
 &= \int_0^{+\infty} \|\beta(s) v_{i_0}\|^2 ds \\
 &= \|v_{i_0}\|^2 \int_0^{+\infty} |\beta(s)|^2 ds \\
 &= \|v_{i_0}\|^2 = 1
 \end{aligned}$$

and

$$\begin{aligned}
 \|\tilde{L}v^0\|_{L_\omega^2}^2 &= \sum_{i=1}^N \theta_i \int_0^{+\infty} \mathcal{E} (\langle D_i v_i^0(s), PD_i v_i^0(s) \rangle) ds \\
 &= \sum_{i=1}^N \theta_i \int_0^{+\infty} (\langle D_i v_i^0(s), PD_i v_i^0(s) \rangle) ds \\
 &= \theta_{i_0} \int_0^{+\infty} (\langle \beta(s) v_{i_0}, D_{i_0}^* PD_{i_0} \beta(s) v_{i_0} \rangle) ds \\
 &= \theta_{i_0} \int_0^{+\infty} |\beta(s)|^2 \|D_{i_0}^* PD_{i_0}\| ds \\
 &= \theta_{i_0} \|D_{i_0}^* PD_{i_0}\| \int_0^{+\infty} |\beta(s)|^2 ds \\
 \|\tilde{L}v^0\|_{L_\omega^2}^2 &= \theta_{i_0} \|D_{i_0}^* PD_{i_0}\| = \max_{i \in \bar{N}} (\theta_i \|D_i^* PD_i\|)
 \end{aligned}$$

Therefore

$$\|\tilde{L}\|_{L_\omega^2} = \frac{\|\tilde{L}v^0\|_{L_\omega^2}}{\|v^0\|_{L_\omega^2}} = \left(\max_{i \in \bar{N}} (\theta_i \|D_i^* PD_i\|) \right)^{\frac{1}{2}}$$

□

Let $\alpha \in (0, \infty)^N$, and

$$D_i^{\alpha_i} = \alpha_i^{-1} D_i, E_i^{\alpha_i} = \alpha_i E_i, \Delta_i^{\alpha_i}(\cdot) = \alpha_i \Delta_i(\alpha_i^{-1} \cdot), i \in \bar{N}. \quad (3.8)$$

Then every solution of the perturbed system (3.1) is also a solution of the scaled perturbed system

$$x(t) = S(t)x_0 + \sum_{i=1}^N \int_0^t S(t-s) D_i^{\alpha_i} \Delta_i^{\alpha_i}(E_i^{\alpha_i} x(s)) d\omega_i(s), t > 0 \quad (3.9)$$

The input-output operator $L^\alpha : L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{U})) \rightarrow L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{Y}))$ of the system $(A; (D_i^{\alpha_i}, E_i^{\alpha_i}))$ is given by

$$\begin{aligned} (L^\alpha v)(t) &= \left(L^\alpha \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \right)(t) \\ &= \sum_{i=1}^N \begin{pmatrix} E_1^{\alpha_1} \\ \vdots \\ E_N^{\alpha_N} \end{pmatrix} \int_0^t S(t-s) D_i^{\alpha_i}(v_i(s)) d\omega_i(s), t > 0 \end{aligned}$$

In the following theorem we give an important characterisation of the stability radius $r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right)$ in terms of the solution of the Lyapunov equation (3.4).

Theorem 3.2. *Suppose that there exist $\alpha = (\alpha_i)_{i \in \bar{N}} \in (0, +\infty)^N$ and $P(\alpha) \in \mathcal{L}^+(\mathcal{H})$ satisfying*

$$2 \langle P(\alpha)x, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle = 0, x \in \mathcal{D}(A) \quad (3.10)$$

$$I - (\sigma/\alpha_j)^2 \theta_j D_j^* P(\alpha) D_j \succeq 0, j \in \bar{N} \quad (3.11)$$

where $\langle E(\alpha)x, E(\alpha)x \rangle = \sum_{i=1}^N \alpha_i^2 \langle E_i x, E_i x \rangle$.

Then $r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right) \geq \sigma$.

Proof. Let $\Delta = \bigoplus_{i=1}^N \Delta_i$ such that $\Delta_i \in Lip(\mathcal{Y}_i, \mathcal{U}_i), i \in \bar{N}$ and $\|\Delta\|_{Lip} < \sigma$, and suppose that $P(\alpha) \in \mathcal{L}^+(\mathcal{H})$ is such that (3.10) and (3.11) hold. The unique solution of (3.1) with initial condition $x(0) = x_0$ satisfies the scaled equation (3.9).

Let

$$E(\alpha) = \begin{pmatrix} E_1^{\alpha_1} \\ \vdots \\ E_N^{\alpha_N} \end{pmatrix}, \quad y^\alpha(t) = \begin{pmatrix} y_1^{\alpha_1}(t) \\ \vdots \\ y_N^{\alpha_N}(t) \end{pmatrix}, \quad u^\alpha(t) = \begin{pmatrix} u_1^{\alpha_1}(t) \\ \vdots \\ u_N^{\alpha_N}(t) \end{pmatrix}$$

where

$$u_i^{\alpha_i}(t) = \Delta_i^{\alpha_i}(y_i^{\alpha_i}(t)) \quad \text{and} \quad y_i^{\alpha_i}(t) = E_i^{\alpha_i}(x(t)), \quad t > 0$$

For any $i \in \bar{N}$, we have

$$E_i^{\alpha_i} = \alpha_i E_i = \alpha_i E_i (-A)^{-\delta} (-A)^\delta = \alpha_i \tilde{E}_i (-A)^\delta = \tilde{E}_i^{\alpha_i} (-A)^\delta$$

$$D_i^{\alpha_i} = \alpha_i^{-1} D_i = \alpha_i^{-1} (-A)^\gamma (-A)^{-\gamma} D_i = \alpha_i^{-1} (-A)^\gamma \tilde{D}_i = (-A)^\gamma \alpha_i^{-1} \tilde{D}_i = (-A)^\gamma \tilde{D}_i^{\alpha_i}$$

$$y^\alpha(t) = \begin{pmatrix} E_1^{\alpha_1} \\ \vdots \\ E_N^{\alpha_N} \end{pmatrix} S(t)x_0 + \sum_{i=1}^N \begin{pmatrix} E_1^{\alpha_1} \\ \vdots \\ E_N^{\alpha_N} \end{pmatrix} \int_0^t S(t-s) (-A)^\gamma \tilde{D}_i^{\alpha_i}(u_i^{\alpha_i}(s)) d\omega_i(s)$$

For any $i \in \bar{N}$ we have

$$\begin{aligned} y_i^{\alpha_i}(t) &= E_i^{\alpha_i} S(t)x_0 + \sum_{i=1}^N E_i^{\alpha_i} \int_0^t S(t-s) D_i^{\alpha_i}(u_i^{\alpha_i}(s)) d\omega_i(s) \\ &= E_i^{\alpha_i} (-A)^{-\delta} (-A)^\delta S(t)x_0 + \sum_{i=1}^N E_i^{\alpha_i} \int_0^t (-A)^{-\delta} (-A)^\delta S(t-s) D_i^{\alpha_i}(u_i^{\alpha_i}(s)) d\omega_i(s) \\ &= \tilde{E}_i^{\alpha_i} (-A)^\delta S(t)x_0 + \sum_{i=1}^N E_i^{\alpha_i} (-A)^{-\delta} \int_0^t (-A)^\delta S(t-s) D_i^{\alpha_i}(u_i^{\alpha_i}(s)) d\omega_i(s) \\ &= \tilde{E}_i^{\alpha_i} (-A)^\delta S(t)x_0 + \sum_{i=1}^N \tilde{E}_i^{\alpha_i} \int_0^t (-A)^\delta S(t-s) (-A)^\gamma \tilde{D}_i^{\alpha_i}(u_i^{\alpha_i}(s)) d\omega_i(s) \\ &= \tilde{E}_i^{\alpha_i} (-A)^\delta S(t)x_0 + \sum_{i=1}^N \int_0^t \tilde{E}_i^{\alpha_i} (-A)^\delta S(t-s) (-A)^\gamma \tilde{D}_i^{\alpha_i}(u_i^{\alpha_i}(s)) d\omega_i(s) \\ &= \tilde{E}_i^{\alpha_i} (-A)^\delta S(t)x_0 + \sum_{i=1}^N \int_0^t \tilde{E}_i^{\alpha_i} (-A)^{\delta+\gamma} S(t-s) \tilde{D}_i^{\alpha_i}(u_i^{\alpha_i}(s)) d\omega_i(s) \\ y_i^{\alpha_i}(t) &= \tilde{E}_i^{\alpha_i} (-A)^\delta S(t)x_0 + \sum_{i=1}^N \int_0^t \tilde{E}_i^{\alpha_i} (-A)^\beta S(t-s) \tilde{D}_i^{\alpha_i}(u_i^{\alpha_i}(s)) d\omega_i(s) \quad (3.12) \end{aligned}$$

where $\beta = \delta + \gamma$.

For every $T > 0$, define the truncations $u_{i,T}^{\alpha_i} \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{U}_i))$, $i \in \bar{N}$, and $u_T^\alpha \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{U}_i))$ by

$$u_{i,T}^{\alpha_i} = \begin{cases} u_i^{\alpha_i}(t) = \Delta_i^{\alpha_i}(y_i^{\alpha_i}(t)) & \text{if } t \in [0, T] \\ 0 & \text{if } t > T \end{cases}, u_T^\alpha(t) = \begin{pmatrix} u_{1,T}^{\alpha_1}(t) \\ \vdots \\ u_{N,T}^{\alpha_N}(t) \end{pmatrix}$$

Then

$$\begin{aligned} \|u_T^\alpha\|_{L_\omega^2}^2 &= \int_0^{+\infty} \mathcal{E}(\|u_T^\alpha(t)\|^2) dt \\ &= \int_0^T \mathcal{E}(\|u_T^\alpha(t)\|^2) dt \\ &= \int_0^T \mathcal{E}\left(\sum_{i=1}^N \|u_{i,T}^{\alpha_i}(t)\|^2\right) dt \\ &= \int_0^T \left(\sum_{i=1}^N \int_\Omega \|\Delta_i^{\alpha_i}(y_i^{\alpha_i}(t, \omega))\|^2 d\mu(\omega)\right) dt \\ &= \int_0^T \left(\sum_{i=1}^N \|\Delta_i^{\alpha_i}\|_{Lip}^2 \int_\Omega (y_i^{\alpha_i}(t, \omega))^2 d\mu(\omega)\right) dt \\ &\leq \max_{i \in \bar{N}} \|\Delta_i^{\alpha_i}\|_{Lip}^2 \int_0^T \sum_{i=1}^N \mathcal{E}(\|y_i^{\alpha_i}(t)\|^2) dt \\ &\leq \|\Delta^\alpha\|_{Lip}^2 \int_0^T \mathcal{E}(\|y^\alpha(t)\|^2) dt \end{aligned}$$

Hence

$$\|u_T^\alpha\|_{L_\omega^2}^2 \leq \|\Delta^\alpha\|_{Lip}^2 \int_0^T \mathcal{E}(\|y^\alpha(t)\|^2) dt \quad (3.13)$$

Now define y_T^α as the output of the system $(A, (D_i^{\alpha_i}, E_i^{\alpha_i}))$ generated by the input u_T^α with initial condition $x(0) = x_0$.

Then

$$y_T^\alpha(t) = E(\alpha) S(t)x_0 + L^\alpha u_T^\alpha(t), t > 0 \quad (3.14)$$

and

$$\left(\int_0^T \mathcal{E}(\|y^\alpha(t)\|^2) dt\right)^{\frac{1}{2}} \leq \|y_T^\alpha\|_{L_\omega^2}$$

$$\leq \|E(\alpha) S(t)x_0\| + \|L^\alpha\| \|u_T^\alpha\|_{L_\omega^2}$$

Using (3.13), we obtain

$$\left(\int_0^T \mathcal{E}(\|y^\alpha(t)\|^2) dt \right)^{\frac{1}{2}} \leq \|E(\alpha) S(t)x_0\| + \|L^\alpha\| \|\Delta^\alpha\|_{Lip} \left(\int_0^T \mathcal{E}(\|y^\alpha(t)\|^2) dt \right)^{\frac{1}{2}} \quad (3.15)$$

The second condition implies that

$$1 - \left(\frac{\sigma}{\alpha_j} \right)^2 \theta_j \|D_j^* P D_j\| \geq 0, \quad \text{for any } j \in \bar{N}$$

Thus

$$\max_{j \in \bar{N}} \theta_j \|\alpha_j^{-2} D_j^* P D_j\| \leq \sigma^{-2}$$

hence

$$\max_{j \in \bar{N}} \theta_j \|(\alpha_j^{-1} D_j^*) P (\alpha_j^{-1} D_j)\| \leq \sigma^{-2}$$

Therefore

$$\max_{j \in \bar{N}} \theta_j \|(D_j^{\alpha_j})^* P D_j^{\alpha_j}\| \leq \sigma^{-2}$$

By Lemma 3.2 we obtain

$$\|L^\alpha\|^2 \leq \sigma^{-2}$$

Now since $\|\Delta\|_{Lip} = \|\Delta^\alpha\|_{Lip} < \sigma$, the operator $L^\alpha \Delta^\alpha$ is a truncation on $L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{Y}))$ with $\xi := \|L^\alpha\| \|\Delta^\alpha\|_{Lip} < 1$. By the inequality (3.15) for all $T > 0$, we get that

$$\left(\int_0^T \mathcal{E}(\|y^\alpha(t)\|^2) dt \right)^{\frac{1}{2}} \leq (1 - \xi)^{-1} \|E(\alpha) S(t)x_0\|$$

Then $y_i^{\alpha_i} \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{Y}_i))$ and $u_i^{\alpha_i} = \Delta^{\alpha_i}(y_i^{\alpha_i}) \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{U}_i))$.

By Lemma 3.1 the solution $x(\cdot)$ of (3.9) belongs to $L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{H}))$.

We conclude then that

$$r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right) \geq \sigma$$

□

As a consequence of this theorem we have the following corollary which enables us to obtain a lower bound for the stability radius

Corollary 3.1.

$$r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right) \geq \sup_{\alpha \in (0, \infty)^N} \left(\max_{i \in \bar{N}} \left(\frac{\theta_i}{\alpha_i^2} \right) \|D_i^* P(\alpha) D_i\| \right)^{-\frac{1}{2}}$$

Proof. 1. For $i \in \bar{N}$, if $\|D_i^* P(\alpha) D_i\| = 0$ then $I - \left(\frac{\sigma}{\alpha_i} \right)^2 \theta_i D_i^* P(\alpha) D_i \succ 0$, for all $\sigma > 0$ and $\alpha_i \in (0, \infty)$, $i \in \bar{N}$. From Theorem 3.2 we obtain that $r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right) = +\infty$.

2. Assume that $\|D_i^* P(\alpha) D_i\| \neq 0$, for any $u \in \mathcal{U}_i$, $\alpha \in (0, \infty)^N$, $i \in \bar{N}$, we obtain that

$$\|D_i^* P(\alpha) D_i\| \geq \frac{\langle D_i^* P(\alpha) D_i u, u \rangle}{\|u\|^2}$$

Then we get that

$$\langle D_i^* P(\alpha) D_i u, u \rangle \|D_i^* P(\alpha) D_i\|^{-1} \leq \|u\|^2$$

and

$$\|u\|^2 - \left(\frac{\left(\frac{\theta_i}{\alpha_i^2} \|D_i^* P(\alpha) D_i\| \right)^{-\frac{1}{2}}}{\alpha_i} \right)^2 \theta_i \langle D_i^* P(\alpha) D_i u, u \rangle \geq 0$$

thus

$$1 - \left(\frac{\left(\frac{\theta_i}{\alpha_i^2} \|D_i^* P(\alpha) D_i\| \right)^{-\frac{1}{2}}}{\alpha_i} \right)^2 \theta_i \|D_i^* P(\alpha) D_i\| \geq 0$$

Therefore

$$1 - \left(\frac{\left(\frac{\theta_i}{\alpha_i^2} \|D_i^* P(\alpha) D_i\| \right)^{-\frac{1}{2}}}{\alpha_i} \right)^2 \theta_i \|D_i^* P(\alpha) D_i\| \geq 0$$

which implies that

$$r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right) \geq \left(\frac{\theta_i}{\alpha_i^2} \|D_i^* P(\alpha) D_i\| \right)^{-\frac{1}{2}} \geq \left(\max_{i \in \bar{N}} \left(\frac{\theta_i}{\alpha_i^2} \|D_i^* P(\alpha) D_i\| \right) \right)^{-\frac{1}{2}}$$

Hence

$$r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right) \geq \sup_{\alpha \in (0, \infty)^N} \left(\max_{i \in \bar{N}} \frac{\theta_i}{\alpha_i^2} \|D_i^* P(\alpha) D_i\| \right)^{-\frac{1}{2}}$$

□

A similar result can be obtained if we replace the Lyapunov equation (3.10) by the corresponding Lyapunov inequality.

Corollary 3.2. *Suppose that there exist $\alpha \in (0, +\infty)^N$, $P \in \mathcal{L}^+(\mathcal{H})$ satisfying*

$$2 \langle Px, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \leq 0, \quad x \in \mathcal{D}(A) \quad (3.16)$$

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0 \quad (\text{resp } I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succ 0), \quad j \in \bar{N} \quad (3.17)$$

Then $r^w(A; (D_i, E_i)_{i \in \bar{N}}) \geq \sigma$ (resp $r^w(A; (D_i, E_i)_{i \in \bar{N}}) > \sigma$).

In this case the Lyapunov equation

$$2 \langle Px, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle = 0, \quad x \in \mathcal{D}(A) \quad (3.18)$$

has a solution $P_0 \in \mathcal{L}^+(\mathcal{H})$ such that $P \succeq P_0$.

Proof. Suppose that there exist $\alpha \in (0, +\infty)^N$, $P \in \mathcal{L}^+(\mathcal{H})$ satisfying

$$2 \langle Px, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \leq 0, \quad x \in \mathcal{D}(A)$$

or

$$2 \langle Px, Ax \rangle + \sum_{i=1}^n \alpha_i \langle E_i x, E_i x \rangle \leq 0, \quad x \in \mathcal{D}(A)$$

since

$$E_i = E_i(-A)^{-\delta}(-A)^\delta = \tilde{E}_i(-A)^\delta$$

then

$$2 \langle Px, Ax \rangle + \sum_{i=1}^n \alpha_i \langle \tilde{E}_i(-A)^\delta x, \tilde{E}_i(-A)^\delta x \rangle \leq 0, \quad x \in \mathcal{D}(A)$$

Now, because A is the infinitesimal generator of an analytic exponentially stable semi-group $S(t)$, there exists $P_0 \in \mathcal{L}^+(\mathcal{H})$ such that

$$2 \langle P_0 x, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle = 0$$

Or

$$2 \langle P_0 x, Ax \rangle + \sum_{i=1}^n \alpha_i \langle E_i x, E_i x \rangle = 0$$

Then for $x \in \mathcal{D}(A)$, we have

$$2 \langle P_0 x, Ax \rangle + \sum_{i=1}^n \alpha_i \langle \tilde{E}_i(-A)^\delta x, \tilde{E}_i(-A)^\delta x \rangle = 0,$$

Setting $X = P - P_0$, we get

$$2 \langle Xx, Ax \rangle \leq 0, \quad x \in \mathcal{D}(A)$$

Applying the Lyapunov Theorem we obtain that $X \succeq 0$, thus $P \succeq P_0$.

Hence conditions (3.10) and (3.11) are satisfied. By applying Theorem 3.2 we deduce that $r^w(A; (D_i, E_i)_{i \in \bar{N}}) \geq \sigma$.

□

For $J \subset \bar{N}$ and $\alpha^J \in (0, \infty)^J$, we define $P(\alpha^J) = \int_0^{+\infty} S^*(t) \left(\sum_{i \in J} \alpha_i^2 E_i^* E_i \right) S(t) dt$ the solution of the inequality (3.16).

Then

$$\begin{aligned} \theta_j / \alpha_j^2 D_j^* P(\alpha^J) D_j &= \sum_{i \in J} (\alpha_i^2 / \alpha_j^2) \theta_j \int_0^{+\infty} D_j^* S^*(t) E_i^* E_i S(t) D_j dt \\ &= \sum_{i \in J} (\alpha_i^2 / \alpha_j^2) H_{ij}, \quad j \in J \end{aligned}$$

where $H_{ij} = \theta_j \int_0^{+\infty} D_j^* S^*(t) E_i^* E_i S(t) D_j dt$, $i, j \in \bar{N}$.

We will show that $H_{ij} \in \mathcal{L}^+(\mathcal{H})$.

Let $u_j \in \mathcal{H}$, we have

$$\begin{aligned} \langle H_{ij} u_j, u_j \rangle &= \left\langle \theta_j \int_0^{+\infty} D_j^* S^*(t) E_i^* E_i S(t) D_j u_j dt, u_j \right\rangle \\ &= \theta_j \int_0^{+\infty} \langle D_j^* S^*(t) E_i^* E_i S(t) D_j u_j, u_j \rangle dt \\ &= \theta_j \int_0^{+\infty} \langle E_i S(t) D_j u_j, E_i S(t) D_j u_j \rangle dt \\ &= \theta_j \int_0^{+\infty} \|E_i S(t) D_j u_j\|^2 dt \geq 0 \end{aligned}$$

But

$$\begin{aligned} \int_0^{+\infty} \|E_i S(t) D_j u_j\|^2 dt &= \int_0^{+\infty} \left\| \tilde{E}_i (-A)^\delta S(t) (-A)^\gamma \tilde{D}_j u_j \right\|^2 dt \\ &= \int_0^{+\infty} \left\| \tilde{E}_i S(t) (-A)^\beta \tilde{D}_j u_j \right\|^2 dt, \quad \text{where } \beta = \delta + \gamma \end{aligned}$$

hence

$$\begin{aligned} \langle H_{ij}u_j, u_j \rangle &\leq \theta_j \int_0^{+\infty} \left\| \tilde{E}_i \right\|^2 \left\| S(t) (-A)^\beta \right\|^2 \left\| \tilde{D}_j \right\|^2 \|u_j\|^2 dt \\ &\leq \theta_j \left\| \tilde{E}_i \right\|^2 \left\| \tilde{D}_j \right\|^2 \|u_j\|^2 \int_0^{+\infty} \frac{e^{-2\omega t}}{t^{2\beta}} dt \\ &\leq \theta_j M_{i\delta}^2 M_{j\gamma}^2 \left(\int_0^{+\infty} \frac{e^{-2\omega t}}{t^{2\beta}} dt \right) \|u_j\|^2 \end{aligned}$$

Since $\beta < \frac{1}{2}$, the integral $\left(\int_0^{+\infty} \frac{e^{-2\omega t}}{t^{2\beta}} dt \right)$ is bounded. We deduce that $H_{ij} \in \mathcal{L}^+(\mathcal{H})$.

Lemma 3.3. [40] Let $\hat{\mu} = \inf_{\alpha \in (0, +\infty)^N} \max_{j \in \bar{N}} \left\| \sum_{i \in \bar{N}} (\alpha_i / \alpha_j)^2 H_{ij} \right\|$. There exist a subset $J \subset \bar{N}$ and for every $\delta > 0$, a vector $\alpha \in (0, +\infty)^N$, such that

$$\begin{aligned} \left\| \sum_{i \in \bar{N}} (\alpha_i / \alpha_j)^2 H_{ij} \right\| &\leq \hat{\mu} + \delta, \quad j \in \bar{N} \\ \left\| \sum_{i \in J} (\alpha_i / \alpha_j)^2 H_{ij} \right\| &= \hat{\mu}, \quad j \in J \end{aligned}$$

Now we establish a computational formula for the stability radius

Theorem 3.3. Let $(A; (D_i, E_i)_{i \in \bar{N}})$ and $(\omega_i(t))_{i \in \bar{N}}$ be as above. Then the associated stability radius is given by

$$r^\omega(A; (D_i, E_i)_{i \in \bar{N}}) = \sup_{\alpha \in (0, \infty)^N} \left(\max_{j \in \bar{N}} \left\| (\theta_j / \alpha_j^2) D_j^* P(\alpha) D_j \right\| \right)^{-\frac{1}{2}} \quad (3.19)$$

where $P(\alpha)$ is the unique solution of (3.10). If $r^\omega(A; (D_i, E_i)_{i \in \bar{N}}) < +\infty$, there exists a minimum norm destabilizing perturbation $\Delta = \bigoplus_{i=1}^N \Delta_i$,

$$\|\Delta\|_{Lip} = r^\omega(A; (D_i, E_i)_{i \in \bar{N}}).$$

Moreover there exist a subset $J \subset \bar{N}$ and a scaling vector $\alpha^J \in (0, +\infty)^J$ such that

$$\begin{aligned} r^\omega(A; (D_i, E_i)_{i \in \bar{N}}) &= r^\omega(A; (D_i, E_i)_{i \in J}) \\ &= \left(\max_{j \in J} \left\| (\theta_j / \alpha_j^2) D_j^* P(\alpha^J) D_j \right\| \right)^{-\frac{1}{2}} \end{aligned} \quad (3.20)$$

where $P(\alpha^J) \in \mathcal{L}^+(\mathcal{H})$ is the unique solution of (3.16).

Proof. We have $\hat{\mu} = \inf_{\alpha \in (0, \infty)^N} (\max_{j \in \bar{N}} \left\| \left(\frac{\theta_j}{\alpha_j^2} \right) D_j^* P(\alpha) D_j \right\|)$

1. If $\hat{\mu} = 0$, it follows from Theorem 3.2 that $r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right) = +\infty$ and the equality (3.19) is satisfied. Moreover

$$E_j S(t) D_j = 0, t > 0, j \in \bar{N}$$

so (3.20) is satisfied for every singleton $J = \{j\} \subset \bar{N}$ and all $\alpha^J \in (0, +\infty)^J$.

2. If $\hat{\mu} > 0$, we show that $r^\omega \left(A; (D_i, E_i)_{i \in \bar{N}} \right) \leq \hat{\mu}^{\frac{1}{2}}$ by constructing a destabilizing perturbation Δ .

For every $\alpha \in (0, +\infty)^N$, let $\sigma(\alpha)$ be the largest σ for which (3.16) has a solution $P(\alpha) \in \mathcal{L}^+(\mathcal{H})$ satisfying (3.17).

For all $\sigma \leq \sigma(\alpha)$, we have

$$I - \left(\frac{\sigma}{\alpha_j} \right)^2 \theta_j D_j^* P(\alpha) D_j \succeq 0, j \in \bar{N}$$

this implies that

$$\sigma \leq \left(\max_{j \in \bar{N}} \left(\theta_j / \alpha_j^2 \right) \left\| D_j^* P(\alpha) D_j \right\| \right)^{\frac{1}{2}}, \forall j \in \bar{N}$$

$$\text{Let } \sigma(\alpha) = \left(\max_{j \in \bar{N}} \left(\frac{\theta_j}{\alpha_j^2} \right) \left\| D_j^* P(\alpha) D_j \right\| \right)^{\frac{1}{2}}$$

then $\sup_{\alpha \in (0, \infty)^N} \sigma^2(\alpha) = \hat{\mu}^{-1}$.

By Lemma 3.3, there exist $J \subset \bar{N}$ and a vector $\alpha^J \in (0, +\infty)^J$ such that

$$\hat{\mu} = \left\| \sum_{i \in J} (\alpha_i / \alpha_j)^2 H_{ij} \right\| = \left\| (\theta_j / \alpha_j^2) D_j^* P(\alpha^J) D_j \right\|, j \in J$$

where $P(\alpha^J) \in \mathcal{L}^+(\mathcal{H})$ is the unique solution of (3.16).

Let $v_j^0 \in \mathcal{U}_j, \|v_j^0\| = 1, j \in J$, such that

$$\begin{aligned} \left\| \sum_{i \in J} (\alpha_i / \alpha_j)^2 H_{ij} \right\| &= \left\langle v_j^0, \sum_{i \in J} (\alpha_i / \alpha_j)^2 H_{ij} v_j^0 \right\rangle \\ &= \left\langle v_j^0, (\theta_j / \alpha_j^2) D_j^* P(\alpha^J) D_j v_j^0 \right\rangle = \hat{\mu} \end{aligned}$$

It follows that

$$\left\langle v_j^0, (\theta_j / \alpha_j^2) \hat{\mu}^{-1} D_j^* P(\alpha^J) D_j v_j^0 \right\rangle = 1, \forall j \in J,$$

Setting $\hat{\sigma} = \hat{\mu}^{-\frac{1}{2}}$, we obtain

$$\langle v_j^0, (\theta_j/\alpha_j^2) \hat{\sigma}^2 D_j^* P(\alpha^J) D_j v_j^0 \rangle = 1, \forall j \in J, \quad (3.21)$$

Define for $j \in \bar{N}$ the perturbation $\Delta_j \in Lip(\mathcal{Y}_j, \mathcal{U}_j)$ by

$$\begin{cases} \Delta_j(y_j) = \hat{\sigma} \|y_j\| v_j^0, & j \in J, \quad y_j \in Y_j \\ \Delta_j(y_j) = 0, & j \in \bar{N} \setminus J. \end{cases}$$

Then $\|\Delta_j\|_{Lip} = \hat{\sigma}$, $j \in J$, it follows that $\|\Delta\|_{Lip} = \hat{\sigma}$ where $\Delta = \bigoplus_{i=1}^N \Delta_i$. We will show that for this Δ the system (3.1) cannot be stable.

Assume that this is not the case. Let $x_0 \in \mathcal{H}$, the solution $x(\cdot)$ of (3.1) satisfies

$$x(t) = S(t)x_0 + \sum_{j \in J} \int_0^t S(t-s) D_j^{\alpha_j} \Delta_j^{\alpha_j} (E_j^{\alpha_j} x(s)) d\omega_j(s), t > 0, \quad (3.22)$$

where $D_j^{\alpha_j}$, $E_j^{\alpha_j}$, and $\Delta_j^{\alpha_j}$ are defined in (3.8). Set $y_j^{\alpha_j} = E_j^{\alpha_j} x$, then $y_j^{\alpha_j} \in L_\omega^2(\mathbb{R}^+, L^2(\Omega, \mathcal{Y}_j))$, $j \in J$.

We have for any $j \in J$

$$\begin{aligned} E_j^{\alpha_j} &= \alpha_j E_j = \alpha_j E_j (-A)^{-\delta} (-A)^\delta = \alpha_j \tilde{E}_j (-A)^\delta = \tilde{E}_j^{\alpha_j} (-A)^\delta \\ \Delta_j^{\alpha_j}(y_j) &= \alpha_j \Delta_j(\alpha_j^{-1} y_j) = \hat{\sigma} \|y_j\| v_j^0, j \in J, y_j \in \mathcal{Y}_j. \end{aligned}$$

define y^{α_J} and E^{α_J} by

$$y^{\alpha_J} = (y_j^{\alpha_j})_{j \in J}, \quad E^{\alpha_J} x = (E_j^{\alpha_j} x)_{j \in J},$$

Then

$$\begin{aligned} y^{\alpha_J}(t) &= E^{\alpha_J} S(t)x_0 + \sum_{j \in J} E^{\alpha_J} \int_0^t S(t-s) D_j^{\alpha_j} \Delta_j^{\alpha_j} (y_j^{\alpha_j}(s)) d\omega_j(s), t > 0 \\ &= E^{\alpha_J} S(t)x_0 + \sum_{j \in J} E^{\alpha_J} \int_0^t (-A)^{-\delta} (-A)^\delta S(t-s) D_j^{\alpha_j} \Delta_j^{\alpha_j} (y_j^{\alpha_j}(s)) d\omega_j(s), t > 0 \\ &= E^{\alpha_J} S(t)x_0 + \sum_{j \in J} E^{\alpha_J} (-A)^{-\delta} \int_0^t (-A)^\delta S(t-s) D_j^{\alpha_j} \Delta_j^{\alpha_j} (y_j^{\alpha_j}(s)) d\omega_j(s), t > 0 \\ &= E^{\alpha_J} S(t)x_0 + \sum_{j \in J} \tilde{E}^{\alpha_J} \int_0^t (-A)^\delta S(t-s) D_j^{\alpha_j} \Delta_j^{\alpha_j} (y_j^{\alpha_j}(s)) d\omega_j(s), t > 0 \end{aligned}$$

$$\begin{aligned}
 &= E^{\alpha_J} S(t)x_0 + \sum_{j \in J} \int_0^t \tilde{E}^{\alpha_J} (-A)^\delta S(t-s) D_j^{\alpha_j} \Delta_j^{\alpha_j} (y_j^{\alpha_j}(s)) d\omega_j(s), t > 0 \\
 &= E^{\alpha_J} S(t)x_0 + \hat{\sigma} \sum_{j \in J} \int_0^t \tilde{E}^{\alpha_J} (-A)^\delta S(t-s) D_j^{\alpha_j} v_j^0 \|y_j^{\alpha_j}(s)\| d\omega_j(s), t > 0 \\
 &= E^{\alpha_J} S(t)x_0 + \hat{\sigma} \sum_{j \in J} \int_0^t E^{\alpha_J} S(t-s) D_j^{\alpha_j} v_j^0 \|y_j^{\alpha_j}(s)\| d\omega_j(s), t > 0
 \end{aligned}$$

By applying Lemma 3.1, we obtain

$$\begin{aligned}
 &\int_0^{+\infty} \mathcal{E} (\|y^{\alpha_J}(t)\|_Y^2) dt = \int_0^{+\infty} \|E^{\alpha_J} S(t)x_0\|^2 dt \\
 &+ \hat{\sigma}^2 \sum_{i \in J} \theta_j \int_0^{+\infty} \mathcal{E} \langle D_j^{\alpha_j} v_j^0 \|y_j^{\alpha_j}(s)\|, P(\alpha^J) D_j^{\alpha_j} v_j^0 \|y_j^{\alpha_j}(s)\| \rangle ds \\
 &= \int_0^{+\infty} \|E^{\alpha_J} S(t)x_0\|^2 dt \\
 &+ \hat{\sigma}^2 \sum_{i \in J} \theta_j / \alpha_j^2 \langle v_j^0, D_j^* P(\alpha^J) D_j v_j^0 \rangle \int_0^{+\infty} \mathcal{E} (\|y_j^{\alpha_j}(s)\|_Y^2) ds
 \end{aligned}$$

By (3.21) we get

$$\int_0^{+\infty} \mathcal{E} (\|y^{\alpha_J}(t)\|_Y^2) dt = \int_0^{+\infty} \|E^{\alpha_J} S(t)x_0\|^2 dt + \int_0^{+\infty} \mathcal{E} (\|y^{\alpha_J}(s)\|_Y^2) ds$$

for all $x_0 \in \mathcal{H}$, this identity implies that $E_j = 0$ for every $j \in J$ hence $P(\alpha^J) = 0$ thus $\hat{\mu} = 0$.

Therefore neither of the stochastic system (3.1) nor (3.22) is L^2 -stable. It follows that

$$\begin{aligned}
 r^\omega (A; (D_i, E_i)_{i \in \bar{N}}) &\leq r^\omega (A; (D_i, E_i)_{i \in J}) \\
 &\leq \sup_{\alpha \in (0, \infty)^N} (\max_{j \in J} (\theta_j / \alpha_j^2) \|D_j^* P(\alpha) D_j\|)^{\frac{-1}{2}} \\
 &= \hat{\mu}^{-\frac{1}{2}}
 \end{aligned}$$

By Corollary 3.1 we obtain the desired result. \square

Corollary 3.3. *Let $\sigma > 0$ such that $r^\omega (A; (D_i, E_i)_{i \in \bar{N}}) > \sigma$.*

Then there exist $\alpha_i > 0, i \in \bar{N}$, and $P(\alpha) \in \mathcal{L}^+ (\mathcal{H})$ such that

$$2 \langle P(\alpha)x, Ax \rangle + \langle E(\alpha)x, E(\alpha)x \rangle = 0, \quad x \in \mathcal{D}(A)$$

$$I - (\sigma/\alpha_j)^2 \theta_j D_j^* P(\alpha) D_j \succ 0, \quad j \in \bar{N} \quad (3.23)$$

Proof. Let $\sigma' \in]\sigma, r^\omega (A; (D_i, E_i)_{i \in \bar{N}})[$.

It follows from Theorem 3.3, that there exist $\alpha \in (0, +\infty)^N$ such that $\sigma' < (\max_{j \in \bar{N}} (\theta_j/\alpha_j^2) \|D_j^* P(\alpha) D_j\|)^{-\frac{1}{2}}$ where $P(\alpha)$ is the solution of the equation (3.10).

This implies that

$$\sigma < \sigma' \leq (\max_{j \in \bar{N}} (\theta_j/\alpha_j^2) \|D_j^* P(\alpha) D_j\|)^{-\frac{1}{2}}$$

\implies

$$\sigma^{-2} > \max_{j \in \bar{N}} (\theta_j/\alpha_j^2) \|D_j^* P(\alpha) D_j\|$$

\implies

$$\sigma^{-2} > (\theta_j/\alpha_j^2) \|D_j^* P(\alpha) D_j\| \quad \forall j \in \bar{N}$$

\implies

$$\theta_j (\sigma/\alpha_j)^2 D_j^* P(\alpha) D_j < I, \quad j \in \bar{N}$$

Then $P(\alpha)$ satisfies the equation (3.10) and the condition (3.23). \square

3.2.3 Example

In this section, we present a practical example to illustrate the above theoretical results. Denoting by δ the Dirac Delta function at zero and by δ_ξ the Dirac Delta function at ξ .

Example 3.1. *Consider the stochastic parabolic equation with Newman boundary condition*

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) = \frac{\partial^2 y}{\partial x^2}(x, t) + c_1 \delta_\xi(x) \langle f, y(x, t) \rangle \dot{w}_1(t); & 0 \leq x, \xi \leq 1, \quad t \geq 0. \\ y(x, 0) = y_0(x), \\ \frac{\partial y}{\partial x}(0, t) = -c_2 \langle g, y(0, t) \rangle \dot{w}_2(t), \quad \frac{\partial y}{\partial x}(1, t) = 0, \end{cases} \quad (3.24)$$

where $(c_1, c_2) \in \mathbb{R}^2$, $f \in L^2(0, 1)$, $g \in L^2(0, 1)$.

To put the problem (3.24) into the abstract setting we introduce the self adjoint operator

$Ah = \frac{d^2h}{dx^2}$ in the Hilbert space $\mathcal{H} = L^2(0, 1)$ with

$$\mathcal{D}(A) = \{z \in L^2(0, 1), \quad \frac{d^2z}{dx^2} \in L^2(0, 1), \quad \frac{dz}{dx}(0) = \frac{dz}{dx}(1) = 0\}$$

The operator A generates an analytic semigroup $S(t)$, the eigenvalues of A are $\lambda_n = -n^2\pi^2$ and the corresponding eigenfunction are $\phi_n(x) = \sqrt{2} \cos n\pi x$, $n \geq 1$ and $\phi_0(x) = 1$

In this example we have $D_1u = \delta_\xi u$ such that $D_1 \in \mathcal{L}(\mathbb{R}, H^{-\frac{1}{2}-\beta}(0, 1))$, $\beta > 0$, and $E_1z = \langle f, z \rangle$ with $E_1 \in \mathcal{L}(\mathcal{H}, \mathbb{R})$, $\Delta_1 = c_1 \in \mathbb{R}$ and $D_2 \in \mathcal{L}(\mathbb{R}, H^{-\frac{1}{2}-\beta}(0, 1))$, $\beta > 0$ is defined by $D_2u = -\delta u$. Set $E_2z = \langle g, z \rangle$ with $E_2 \in \mathcal{L}(\mathcal{H}, \mathbb{R})$, $\Delta_2 = c_2 \in \mathbb{R}$.

In the abstract form the problem (3.24) can be formulated as follows

$$\begin{cases} dz(t) = Az(t)dt + \sum_{i=1}^2 D_i \Delta_i E_i(z(t)) dw_i(t), \\ z(0) = z_0 \end{cases} \quad (3.25)$$

For this system we have $\gamma = \frac{1}{4} + \frac{\beta}{2}$, and $\delta = 0$ such that $0 < \beta < \frac{1}{2}$. then there exists a unique solution of (3.24). In order to get an explicit formula for the stability radius we need at first to solve the following Lyapunov equation

$$2 \langle Pz, Az \rangle + \langle E(\alpha)z, E(\alpha)z \rangle = 0, \quad z \in \mathcal{D}(A) \quad (3.26)$$

where

$$\langle E(\alpha)z, E(\alpha)z \rangle = \sum_{i=1}^2 \alpha_i^2 \langle E_i z, E_i z \rangle = \alpha_1^2 \langle E_1 z, E_1 z \rangle + \alpha_2^2 \langle E_2 z, E_2 z \rangle$$

Suppose that we can express the solution P of (3.26) by

$$Pz = \sum_{n,j=1}^{+\infty} P_{nj} \langle z, \phi_n \rangle \phi_j, \quad z \in \mathcal{H}$$

Then since $Az = \sum_{n=1}^{+\infty} \lambda_n \langle z, \phi_n \rangle \phi_n$, $z \in \mathcal{D}(A)$, it follows that

$$\langle Pz, Az \rangle = \left\langle \sum_{n=1}^{+\infty} \lambda_n \langle z, \phi_n \rangle \phi_n, \sum_{n,j=1}^{+\infty} P_{nj} \langle z, \phi_n \rangle \phi_j \right\rangle = \sum_{n=1}^{+\infty} \lambda_n P_{nn} (\langle z, \phi_n \rangle)^2$$

Now since $E_1z = \langle f, z \rangle$ and $E_2z = \langle g, z \rangle$, and because

$$\langle f, z \rangle = \left\langle f, \sum_{n=1}^{+\infty} \langle z, \phi_n \rangle \phi_n \right\rangle = \sum_{n=1}^{+\infty} \langle z, \phi_n \rangle \langle f, \phi_n \rangle$$

$$\langle g, z \rangle = \left\langle g, \sum_{n=1}^{+\infty} \langle z, \phi_n \rangle \phi_n \right\rangle = \sum_{n=1}^{+\infty} \langle z, \phi_n \rangle \langle g, \phi_n \rangle$$

It follows that

$$\begin{aligned} \langle E(\alpha)z, E(\alpha)z \rangle &= \alpha_1^2 \langle E_1 z, E_1 z \rangle + \alpha_2^2 \langle E_2 z, E_2 z \rangle \\ &= \alpha_1^2 \left(\sum_{n=1}^{+\infty} \langle z, \phi_n \rangle \langle f, \phi_n \rangle \right)^2 + \alpha_2^2 \left(\sum_{n=1}^{+\infty} \langle z, \phi_n \rangle \langle g, \phi_n \rangle \right)^2 \end{aligned}$$

Equation (3.26) becomes

$$2 \sum_{n=1}^{+\infty} \lambda_n P_{nn} (\langle z, \phi_n \rangle)^2 + \alpha_1^2 \left(\sum_{n=1}^{+\infty} \langle z, \phi_n \rangle \langle f, \phi_n \rangle \right)^2 + \alpha_2^2 \left(\sum_{n=1}^{+\infty} \langle z, \phi_n \rangle \langle g, \phi_n \rangle \right)^2 = 0$$

For $z = \phi_k$, $k \in \mathbb{N}^*$, we get

$$2\lambda_k P_{kk} + \alpha_1^2 (\langle f, \phi_k \rangle)^2 + \alpha_2^2 (\langle g, \phi_k \rangle)^2 = 0$$

Therefore

$$P_{kk} = \frac{-\alpha_1^2 (\langle f, \phi_k \rangle)^2 - \alpha_2^2 (\langle g, \phi_k \rangle)^2}{2\lambda_k} = \frac{\alpha_1^2 (\langle f, \phi_k \rangle)^2 + \alpha_2^2 (\langle g, \phi_k \rangle)^2}{2k^2\pi^2}$$

we deduce that the solution of (3.26) is given by

$$Pz = \sum_{k=1}^{+\infty} P_k \langle z, \phi_k \rangle \phi_k, \quad z \in \mathcal{H} \quad \text{where} \quad P_k = \frac{\alpha_1^2 (\langle f, \phi_k \rangle)^2 + \alpha_2^2 (\langle g, \phi_k \rangle)^2}{2k^2\pi^2}.$$

For all $u \in \mathcal{U}$, we have

$$\begin{aligned} PD_1 u &= \sum_{k=1}^{+\infty} P_k \langle D_1 u, \phi_k \rangle \phi_k \\ &= \sum_{k=1}^{+\infty} P_k \langle u, D_1^* \phi_k \rangle \phi_k \\ &= \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(\xi) \rangle \phi_k \end{aligned}$$

Or

$$\langle PD_1 u, D_1 u \rangle = \left\langle \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(\xi) \rangle \phi_k, D_1 u \right\rangle$$

$$\begin{aligned}
 &= \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(\xi) \rangle \langle \phi_k, D_1 u \rangle \\
 &= \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(\xi) \rangle \langle D_1^* \phi_k, u \rangle \\
 &= \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(\xi) \rangle^2
 \end{aligned}$$

Then

$$\begin{aligned}
 \|D_1^* P(\alpha) D_1\| &= 2 \sum_{k=0}^{+\infty} P_k (\cos^2(k\pi\xi)) \\
 &\leq 2 \sum_{k=1}^{+\infty} P_k = 2 \sum_{k=1}^{+\infty} \frac{\alpha_1^2 (\langle f, \phi_k \rangle)^2 + \alpha_2^2 (\langle g, \phi_k \rangle)^2}{2k^2\pi^2}
 \end{aligned}$$

If we assume that $\langle f, \phi_k \rangle = 1$ and $\langle g, \phi_k \rangle = 1$, for $k \geq 0$, we get

$$\|D_1^* P(\alpha) D_1\| \leq 2 \sum_{k=1}^{+\infty} \frac{\alpha_1^2 + \alpha_2^2}{2k^2\pi^2} = (\alpha_1^2 + \alpha_2^2) \sum_{k=1}^{+\infty} \frac{1}{k^2\pi^2} = \frac{\alpha_1^2 + \alpha_2^2}{6}$$

For all $u \in \mathcal{U}$, we have

$$\begin{aligned}
 PD_2 u &= \sum_{k=1}^{+\infty} P_k \langle D_2 u, \phi_k \rangle \phi_k \\
 &= \sum_{k=1}^{+\infty} P_k \langle u, D_2^* \phi_k \rangle \phi_k \\
 &= - \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(0) \rangle \phi_k
 \end{aligned}$$

Or

$$\begin{aligned}
 \langle PD_2 u, D_2 u \rangle &= \left\langle - \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(0) \rangle \phi_k, D_2 u \right\rangle \\
 &= - \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(0) \rangle \langle \phi_k, D_2 u \rangle \\
 &= - \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(0) \rangle \langle D_2^* \phi_k, u \rangle
 \end{aligned}$$

$$= \sum_{k=1}^{+\infty} P_k \langle u, \phi_k(0) \rangle^2$$

Then

$$\|D_2^* P(\alpha) D_2\| = 2 \sum_{k=1}^{+\infty} P_k = 2 \sum_{k=1}^{+\infty} \frac{\alpha_1^2 (\langle f, \phi_k \rangle)^2 + \alpha_2^2 (\langle g, \phi_k \rangle)^2}{2k^2 \pi^2}$$

If we assume that $\langle f, \phi_k \rangle = 1$ and $\langle g, \phi_k \rangle = 1$ for $k \geq 0$

Then

$$\|D_2^* P(\alpha) D_2\| = 2 \sum_{k=1}^{+\infty} \frac{\alpha_1^2 + \alpha_2^2}{2k^2 \pi^2} = (\alpha_1^2 + \alpha_2^2) \sum_{k=1}^{+\infty} \frac{1}{k^2 \pi^2} = \frac{\alpha_1^2 + \alpha_2^2}{6}$$

Therefore

$$\begin{aligned} r^w(A, (D_i, E_i)_{i \in \bar{N}}) &= \sup_{\alpha \in (0, +\infty)^2} \left(\max_{j \in \{1, 2\}} \left\| \frac{\theta_j}{\alpha_j^2} D_j^* P(\alpha) D_j \right\| \right)^{-\frac{1}{2}} \\ &= \sup_{\alpha \in (0, +\infty)^2} \left(\max_{j \in \{1, 2\}} \frac{\theta_j}{\alpha_j^2} \|D_j^* P(\alpha) D_j\| \right)^{-\frac{1}{2}} \\ &= \sup_{\alpha \in (0, +\infty)^2} \left(\min_{j \in \{1, 2\}} \frac{\alpha_j}{\sqrt{\theta_j}} \left(\frac{\sqrt{6}}{\sqrt{\alpha_1^2 + \alpha_2^2}} \right) \right) \\ &= \sup_{\alpha \in (0, +\infty)^2} \left(\min_{j \in \{1, 2\}} \frac{\alpha_j}{\sqrt{\alpha_1^2 + \alpha_2^2}} \left(\frac{\sqrt{6}}{\sqrt{\theta_j}} \right) \right) \\ &= \min_{j \in \{1, 2\}} \left(\frac{\sqrt{6}}{\sqrt{\theta_j}} \right) \end{aligned}$$

Or

$$r^w(A, (D_i, E_i)_{i \in \{1, 2\}}) = \min \left(\frac{\sqrt{6}}{\sqrt{\theta_1}}, \frac{\sqrt{6}}{\sqrt{\theta_2}} \right)$$

3.3 Robust stabilization

3.3.1 System description

We consider controlled stochastic systems described by Itô equation of the form:

$$\begin{cases} dx(t) = Ax(t)dt + \sum_{i=1}^N D_i \Delta_i(E_i(x(t))) dw_i(t) + Bu(t)dt, t \in \mathbb{R}^+ \\ x(0) = x_0, x_0 \in \mathcal{H} \\ \|\Delta_i\|_{Lip} < \sigma \end{cases} \quad (3.27)$$

where u takes its values in the real separable Hilbert space Z , $B \in \mathcal{L}(Z, \mathcal{H})$ and the other operators are as in the previous part. In addition we assume that (A, B) is stabilizable.

Our aim is to characterize the supremum of the stability radii which can be achieved by linear state feedback $u = Fx$, where $F \in \mathcal{L}(\mathcal{H}, Z)$.

Let

$$\overline{\mathcal{F}} = \left\{ \begin{array}{l} F \in L(H, Z); A + BF \text{ is the infinitesimal generator of an} \\ \text{exponentially stable analytic semigroup} \end{array} \right\}.$$

and define

$$\overline{r^w}(A; (D_i, E_i)_{i \in \overline{N}}) = \sup \{ r^w(A + BF, (D_i, E_i)_{i \in \overline{N}}); F \in \overline{\mathcal{F}} \}.$$

For all $F \in \overline{\mathcal{F}}$, $\alpha \in (0, +\infty)^N$, and $\varepsilon > 0$, consider the Lyapunov inequality

$$2 \langle P(A + BF)x, x \rangle + \sum_{i=1}^N \alpha_i^2 \langle E_i x, E_i x \rangle + \varepsilon^2 \langle Fx, Fx \rangle \leq 0, \quad x \in \mathcal{D}(A) \quad (3.28)$$

with

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0, \quad j \in \overline{N} \quad (3.29)$$

3.3.2 Suboptimality Conditions

In order to establish conditions for the existence of suboptimal controllers $u(t) = Fx(t)$ such that $F \in \overline{\mathcal{F}}$ and $\sigma \leq r^w(A + BF; (D_i, E_i)_{i \in \overline{N}})$, for $\sigma > 0$, we need the following lemmas.

This lemma is of technical interest.

Lemma 3.4. *Let $\alpha \in (0, +\infty)^N$ and $\varepsilon > 0$. If there exists $P \in \mathcal{L}^+(\mathcal{H})$, such that*

$$2 \langle Px, (A - \varepsilon^{-2}BB^*P)x \rangle + \varepsilon^{-2} \langle PBB^*Px, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \leq 0, \quad x \in \mathcal{D}(A) \quad (3.30)$$

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0, \quad j \in \overline{N} \quad (3.31)$$

then $A_\varepsilon = A - \varepsilon^{-2}BB^*P$ generates an exponentially stable analytic semigroup and $\sigma \leq r^w(A_\varepsilon; (D_i, E_i)_{i \in \overline{N}})$.

Proof. Consider the initial value problem

$$\begin{cases} \frac{d}{dt}x(t) = A_\varepsilon x(t) & t \in \mathbb{R}^+ \\ x(0) = x_0, x_0 \in \mathcal{H} \end{cases} \quad (3.32)$$

The solution $x(t)$ of the system (3.32) is given by

$$x(t) = S(t)x_0 - \varepsilon^{-2} \int_0^t S(t-s)BB^*Px(s)ds$$

We have

$$\begin{aligned} \|x(t)\| &\leq \|S(t)x_0\| + \varepsilon^{-2} \left\| \int_0^t S(t-s)BB^*Px(s)ds \right\| \\ &\leq Me^{-wt} \|x_0\| + \varepsilon^{-2}M \|B\| \int_0^t e^{-w(t-s)} \|B^*Px(s)\| ds \end{aligned}$$

From which we get

$$\begin{aligned} \|x(t)\|^2 &\leq 2M^2 \|x_0\|^2 e^{-2wt} + 2\varepsilon^{-4}M^2 \|B\|^2 \left[\int_0^t e^{-w(t-s)} \|B^*Px(s)\| ds \right]^2 \\ &\leq K_1 e^{-2wt} + K_2 \int_0^t e^{-2w(t-s)} \|B^*Px(s)\|^2 ds \end{aligned}$$

Where $K_1 = 2M^2 \|x_0\|^2$ and $K_2 = 2\varepsilon^{-4}M^2 \|B\|^2$.

It follows then that

$$\int_0^{+\infty} \|x(t)\|^2 dt \leq K_1 \int_0^{+\infty} e^{-2wt} dt + K_2 \int_0^{+\infty} \left(\int_0^t e^{-2w(t-s)} \|B^*Px(s)\|^2 ds \right) dt$$

Thus

$$\begin{aligned} \int_0^{+\infty} \|x(t)\|^2 dt &\leq \frac{K_1}{2w} + K_2 \int_0^{+\infty} \left(\int_s^{+\infty} e^{-2w(t-s)} \|B^*Px(s)\|^2 dt \right) ds \\ &\leq \frac{K_1}{2w} + K_2 \int_0^{+\infty} \|B^*Px(s)\|^2 e^{2ws} \left(\int_s^{+\infty} e^{-2wt} dt \right) ds \end{aligned}$$

which implies that

$$\int_0^{+\infty} \|x(t)\|^2 dt \leq \frac{K_1}{2w} + \frac{K_2}{2w} \int_0^{+\infty} \|B^*Px(s)\|^2 ds$$

Now we show that $B^*Px(t) \in L^2(\mathbb{R}^+, \mathcal{Z})$.

We have for $x_0 \in \mathcal{D}(A_\varepsilon)$, $V(x) = \langle x, Px \rangle$ is differentiable and

$$\frac{d}{dt}V(x(t)) = 2 \langle PA_\varepsilon x, x \rangle$$

From the inequality (3.30) we obtain

$$\begin{aligned} \frac{d}{dt}V(x(t)) &\leq -\varepsilon^{-2} \langle PBB^*Px, x \rangle - \langle E(\alpha)x, E(\alpha)x \rangle \\ &\leq -\varepsilon^{-2} \langle PBB^*Px, x \rangle \end{aligned}$$

Thus

$$\int_0^T \frac{d}{dt}V(x(t))dt \leq -\varepsilon^{-2} \int_0^T \langle PBB^*Px(t), x(t) \rangle dt$$

Hence

$$V(x(T)) - V(x(0)) \leq -\varepsilon^{-2} \int_0^T \|B^*Px(t)\|^2 dt$$

Using the fact that $P \geq 0$ we get

$$\varepsilon^{-2} \int_0^T \|B^*Px(t)\|^2 dt \leq V(x_0), \quad \text{for all } T > 0$$

Therefore

$$\varepsilon^{-2} \int_0^T \|B^*Px(t)\|^2 dt \leq K \|x_0\|^2$$

which implies that $B^*Px(t) \in L^2(\mathbb{R}^+, \mathcal{Z})$, and $x(t) \in L^2(\mathbb{R}^+, \mathcal{H})$.

Now inequality (3.30) implies that

$$2 \langle PA_\varepsilon x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \leq -\varepsilon^{-2} \|B^*Px(t)\|^2$$

Thus

$$\begin{aligned} 2 \langle PA_\varepsilon x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle &\leq 0, \\ I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j &\succeq 0, \quad j \in \overline{N} \end{aligned}$$

Because A is the infinitesimal generator of an exponentially stable analytic semi-group and B is a bounded operator, then $F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathcal{F}}$ (see [46]). Applying Corollary 3.2 we get that $\sigma \leq r^w(A_\varepsilon; (D_i, E_i)_{i \in \overline{N}})$ and if the inequality (3.31) is strict, we obtain $\sigma < r^w(A_\varepsilon; (D_i, E_i)_{i \in \overline{N}})$. \square

The following lemma is of basic importance for the approach used to investigate the maximisation problem.

Lemma 3.5. *Let $\alpha \in (0, +\infty)^N$, $\varepsilon > 0$ and $F \in \overline{\mathcal{F}}$. If the inequality (3.30) has a solution $P_0 \in \mathcal{L}^+(\mathcal{H})$ satisfying condition (3.31) then $F_0 = -\varepsilon^{-2}B^*P_0 \in \overline{\mathcal{F}}$ and $\sigma \leq r^w(A + BF_0; (D_i, E_i)_{i \in \overline{N}})$.*

Moreover, there exists $P_1 \in \mathcal{L}^+(\mathcal{H})$ such that

$$\begin{aligned} 2\langle P_1(A + BF_0)x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2}\langle P_0BB^*P_0x, x \rangle &= 0, \quad x \in \mathcal{D}(A) \\ I - \sigma^2(\theta_j/\alpha_j^2)D_j^*P_1D_j &\succeq 0, \quad j \in \overline{N} \\ P_1 &\preceq P_0. \end{aligned}$$

Proof. For any $x \in \mathcal{D}(A)$, we have

$$\begin{aligned} &2\langle P_0(A + BF)x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^2\langle Fx, Fx \rangle \\ &= 2\langle P_0Ax, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + 2\langle P_0BFx, x \rangle + \varepsilon^2\langle Fx, Fx \rangle \end{aligned}$$

Set $F' = \varepsilon F + \varepsilon^{-1}B^*P_0$, then

$$\begin{aligned} \langle F'x, F'x \rangle &= \langle (\varepsilon F + \varepsilon^{-1}B^*P_0)x, (\varepsilon F + \varepsilon^{-1}B^*P_0)x \rangle \\ &= \varepsilon^2\langle Fx, Fx \rangle + \langle Fx, B^*P_0x \rangle + \langle B^*P_0x, Fx \rangle + \varepsilon^{-2}\langle B^*P_0x, B^*P_0x \rangle \\ &= \varepsilon^2\langle Fx, Fx \rangle + 2\langle B^*P_0x, Fx \rangle + \varepsilon^{-2}\langle B^*P_0x, B^*P_0x \rangle \end{aligned}$$

hence

$$\langle F'x, F'x \rangle - \varepsilon^{-2}\langle B^*P_0x, B^*P_0x \rangle = \varepsilon^2\langle Fx, Fx \rangle + 2\langle B^*P_0x, Fx \rangle$$

Since P_0 is a solution of the inequality (3.28) it follows that

$$2\langle P_0Ax, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + 2\langle P_0BFx, x \rangle + \varepsilon^2\langle Fx, Fx \rangle \leq 0$$

with

$$\langle P_0BFx, x \rangle = \langle Fx, B^*P_0x \rangle = \langle B^*P_0x, Fx \rangle$$

Then

$$2\langle P_0Ax, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle - \varepsilon^{-2}\langle B^*P_0x, B^*P_0x \rangle + \langle F'x, F'x \rangle \leq 0 \quad (3.33)$$

Set $A_0 = A + BF_0$ where $F_0 = -\varepsilon^{-2}B^*P_0$ then

$$2\langle P_0A_0x, x \rangle = 2\langle P_0Ax, x \rangle - 2\varepsilon^{-2}\langle P_0BB^*P_0x, x \rangle$$

The inequality (3.33) implies that

$$2 \langle P_0 A_0 x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle P_0 B B^* P_0 x, x \rangle \leq - \langle F' x, F' x \rangle$$

Thus

$$2 \langle P_0 A_0 x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle P_0 B B^* P_0 x, x \rangle \leq 0 \quad (3.34)$$

Applying Lemma 3.4 we conclude that $F_0 \in \overline{\mathcal{F}}$ and $\sigma \leq r^w(A_0; (D_i, E_i)_{i \in \overline{N}})$.

Now since P_0 is a solution of the inequality (3.34), then it satisfies the following inequality

$$2 \langle P_0 A_0 x, x \rangle + \langle \widehat{E}(\alpha)x, \widehat{E}(\alpha)x \rangle \leq 0$$

where

$$\widehat{E}(\alpha) = \begin{pmatrix} E(\alpha) \\ \varepsilon^{-1} B^* P_0 \end{pmatrix}$$

By Corollary 3.2 there exists $P_1 \in \mathcal{L}^+(\mathcal{H})$ such that

$$2 \langle P_1 A_0 x, x \rangle + \langle \widehat{E}(\alpha)x, \widehat{E}(\alpha)x \rangle \leq 0$$

with $P_1 \preceq P_0$.

Therefore

$$2 \langle P_1 A_0 x, x \rangle + \varepsilon^{-2} \langle x, P_0 B B^* P_0 x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \leq 0$$

and

$$I - \sigma^2 (\theta_j / \alpha_j^2) D_j^* P D_j \succeq 0, \quad j \in \overline{N}.$$

□

Applying this lemma iteratively we show in the following theorem that there exists $P \in \mathcal{L}^+(\mathcal{H})$ such that

$$2 \langle Ax, Px \rangle + \langle E(\alpha)x, E(\alpha)x \rangle - \varepsilon^{-2} \langle x, P B B^* P x \rangle = 0, \quad x \in \mathcal{D}(A) \quad (3.35)$$

$$I - \sigma^2 (\theta_j / \alpha_j^2) D_j^* P D_j \succeq 0, \quad j \in \overline{N}$$

Theorem 3.4. *Let $F \in \overline{\mathcal{F}}$. Suppose that there exist $\alpha \in (0, +\infty)^N$ and $\varepsilon > 0$ such that the Lyapunov inequality (3.30) has a solution $P_0 \in \mathcal{L}^+(\mathcal{H})$ which satisfies condition (3.31) then the Riccati equation (3.35) has a solution $P \in \mathcal{L}^+(\mathcal{H})$ satisfying*

$$I - \sigma^2 (\theta_j / \alpha_j^2) D_j^* P D_j \succeq 0, \quad j \in \overline{N}$$

$$F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathcal{F}}$$

$$\sigma \leq r^w(A - \varepsilon^{-2}BB^*P; (D_i, E_i)_{i \in \overline{N}})$$

Proof. Applying the above lemma iteratively we construct a sequence of linear operators $(P_k)_{k \in \mathbb{N}} \in \mathcal{L}^+(\mathcal{H})$ which satisfies

$$2 \langle P_{k+1}A_k x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle x, P_k B B^* P_k x \rangle \leq 0, \quad x \in \mathcal{D}(A)$$

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*P_{k+1}D_j \succeq 0, \quad j \in \overline{N}$$

$$P_{k+1} \preceq P_k$$

where P_0 solves the inequality (3.30) and $A_k = A - \varepsilon^{-2}BB^*P_k$.

Since $(P_k)_{k \in \mathbb{N}}$ is a decreasing sequence and it is bounded from below by 0, the limit as $k \rightarrow +\infty$ exists. Let $P = \lim_{k \rightarrow +\infty} P_k$. Then

$$2 \langle PA_\varepsilon x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle x, P B B^* P x \rangle = 0, \quad x \in \mathcal{D}(A)$$

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*P D_j \succeq 0, \quad j \in \overline{N}$$

where $A_\varepsilon = A - \varepsilon^{-2}BB^*P$.

Using Lemma 3.5 we deduce that $F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathcal{F}}$ and $\sigma \leq r^w(A - \varepsilon^{-2}BB^*P; (D_i, E_i)_{i \in \overline{N}})$.

Finally since

$$\begin{aligned} & 2 \langle PA_\varepsilon x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle x, P B B^* P x \rangle \\ &= 2 \langle P A x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle - \varepsilon^{-2} \langle x, P B B^* P x \rangle \end{aligned}$$

Then P satisfies the Riccati equation (3.35) □

Now we will use the above result to establish conditions for the existence of suboptimal controllers.

Proposition 3.1. *Let $\sigma, \varepsilon > 0$, Suppose that the Riccati equation (3.35) has a solution $P \in \mathcal{L}^+(\mathcal{H})$ such that $I - \sigma^2(\theta_j/\alpha_j^2)D_j^*P D_j \succeq 0$, $j \in \overline{N}$, for some $\alpha \in (0, +\infty)^N$ then $F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathcal{F}}$ and $\sigma \leq r^w(A + B F_\varepsilon; (D_i, E_i)_{i \in \overline{N}})$.*

Proof. Since P is a solution of the Riccati equation (3.35) then

$$2 \langle P(A - \varepsilon^{-2}BB^*P)x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle x, PBB^*Px \rangle = 0, \quad x \in \mathcal{D}(A)$$

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0, \quad j \in \bar{N}$$

From Lemma 3.5 we obtain $F_\varepsilon = -\varepsilon^{-2}B^*P \in \bar{\mathcal{F}}$ and $\sigma \leq r^w(A + BF_\varepsilon; (D_i, E_i)_{i \in \bar{N}})$ \square

As in [40], we can obtain

Proposition 3.2. *Let $\sigma > 0$. Suppose that there exists $F \in \bar{\mathcal{F}}$ such that $\sigma \leq r^w(A + BF; (D_i, E_i)_{i \in \bar{N}})$. Then there exist $\alpha \in (0, \infty)^N$, $\varepsilon > 0$ such that the Riccati equation (3.35) has a solution $P \in \mathcal{L}^+(\mathcal{H})$ satisfying*

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0; \quad j \in \bar{N}$$

$$F_\varepsilon = -\varepsilon^{-2}B^*P \in \bar{\mathcal{F}}$$

Using Proposition 3.1 and Proposition 3.2 we get the following result

Corollary 3.4. *We have*

$$\bar{r}^w(A; (D_i, E_i)_{i \in \bar{N}}) = \sup \left\{ \begin{array}{l} \sigma > 0; \text{ there exist } \alpha \in (0, +\infty)^N \text{ and } \varepsilon > 0 \\ \text{such that (3.35) has a solution } P \in \mathcal{L}^+(\mathcal{H}) \\ \text{with } I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0 \text{ for all } j \in \bar{N} \end{array} \right\}.$$

3.3.3 Examples

Example 3.2. *Consider the parabolic equation with Newman boundary condition*

$$\left\{ \begin{array}{l} \frac{\partial y}{\partial t}(x, t) = \frac{\partial^2 y}{\partial x^2}(x, t) - y(x, t) + u(t), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T \\ y(x, 0) = y_0(x) \\ \frac{\partial y}{\partial x}(0, t) = -c \langle f, y(0, t) \rangle \dot{w}(t), \quad c \in \mathbb{R}, \quad f \in L^2(0, 1) \\ \frac{\partial y}{\partial x}(1, t) = 0 \end{array} \right. \quad (3.36)$$

To put the problem (3.36) into the abstract setting we introduce the self-adjoint operator $Ah = \frac{d^2 h}{dx^2} - h$ in the Hilbert space $\mathcal{H} = L^2(0, 1)$ with

$$\mathcal{D}(A) = \{z \in L^2(0, 1), \quad \frac{d^2 z}{dx^2} \in L^2(0, 1), \quad \frac{dz}{dx}(0) = \frac{dz}{dx}(1) = 0\}$$

The operator A generates an analytic semigroup $S(t)$, the eigenvalues of A are $\lambda_n = -(1 + n^2\pi^2)$ and the corresponding eigenfunction are $\phi_n(x) = \sqrt{2} \cos n\pi x$, $n \geq 1$, $\phi_0(x) = 1$ (see [8]). We define $Dy = -\delta y$ such that $D \in \mathcal{L}(\mathbb{R}, H^{-\frac{1}{2}-\beta}(0, 1))$. Setting $Ez = \langle f, z \rangle$, $B = I$ and $\Delta = c \in \mathbb{R}$, we can present (3.36) as follows

$$\begin{cases} dz(t) = Az(t)dt + D\Delta E(z(t))dw(t) + Bu(t)dt \\ z(0) = z_0 \end{cases} \quad (3.37)$$

For this system we have $\gamma = \frac{1}{4} + \frac{\beta}{2}$ and $\delta = 0$ such that $0 < \beta < \frac{1}{2}$.

We solve the following Riccati equation

$$2\langle Az, Pz \rangle + \langle Ez, Ez \rangle - \varepsilon^{-2} \langle B^*Pz, B^*Pz \rangle = 0, \quad z \in \mathcal{D}(A) \quad (3.38)$$

Suppose that we can express the solution P of (3.38) by

$$Pz = \sum_{n,j=0}^{+\infty} P_{nj} \langle z, \phi_n \rangle \phi_j, \quad z \in \mathcal{H}$$

Then since $Az = \sum_{n=0}^{+\infty} \lambda_n \langle z, \phi_n \rangle \phi_n$, $z \in \mathcal{D}(A)$, it follows that

$$\langle Az, Pz \rangle = \sum_{n=0}^{+\infty} \lambda_n P_{nn} (\langle z, \phi_n \rangle)^2$$

We have

$$\langle f, z \rangle = \left\langle f, \sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \phi_n \right\rangle = \sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \langle f, \phi_n \rangle$$

and

$$\langle Pz, Pz \rangle = \sum_{n=0}^{+\infty} P_{nn}^2 (\langle z, \phi_n \rangle)^2$$

Equation (3.38) becomes

$$2 \sum_{n=0}^{+\infty} \lambda_n P_{nn} (\langle z, \phi_n \rangle)^2 + \left(\sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \langle f, \phi_n \rangle \right)^2 - \varepsilon^{-2} \sum_{n=0}^{+\infty} P_{nn}^2 (\langle z, \phi_n \rangle)^2 = 0$$

Set $z = \phi_k$, $k \in \mathbb{N}$, then

$$2\lambda_k P_{kk} + (\langle f, \phi_k \rangle)^2 - \varepsilon^{-2} P_{kk}^2 = 0$$

Or

$$-\varepsilon^{-2} P_{kk}^2 + 2\lambda_k P_{kk} + (\langle f, \phi_k \rangle)^2 = 0$$

Therefore $P_{kk} = \frac{\lambda_k + \sqrt{\Delta}}{\varepsilon^{-2}}$ where $\Delta = \lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2} > 0$.

Or

$$Pz = \sum_{k=0}^{+\infty} P_k \langle z, \phi_k \rangle \phi_k, \quad z \in \mathcal{H}$$

Where

$$P_k = \frac{\lambda_k + \sqrt{\Delta}}{\varepsilon^{-2}}$$

$$\Delta = \lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}$$

Now we choose σ such that

$$\langle z, z \rangle - \sigma^2 \theta \langle PDz, Dz \rangle \geq 0 \quad (3.39)$$

Let $z \in \mathcal{H}$, we have

$$\begin{aligned} PDz &= \sum_{k=0}^{+\infty} P_k \langle Dz, \phi_k \rangle \phi_k \\ &= \sum_{k=0}^{+\infty} P_k \langle z, D^* \phi_k \rangle \phi_k \\ &= - \sum_{k=0}^{+\infty} P_k \langle z, \phi_k(0) \rangle \phi_k \end{aligned}$$

Or

$$\begin{aligned} \langle PDz, Dz \rangle &= \left\langle - \sum_{k=0}^{+\infty} P_k \langle z, \phi_k(0) \rangle \phi_k, Dz \right\rangle \\ &= - \sum_{k=0}^{+\infty} P_k \langle z, \phi_k(0) \rangle \langle \phi_k, Dz \rangle \\ &= - \sum_{k=0}^{+\infty} P_k \langle z, \phi_k(0) \rangle \langle D^* \phi_k, z \rangle \\ &= \sum_{k=0}^{+\infty} P_k \langle z, \phi_k(0) \rangle^2 \end{aligned}$$

Hence (3.39) will be

$$\langle z, z \rangle - \sigma^2 \theta \sum_{k=0}^{+\infty} P_k \langle z, \phi_k(0) \rangle^2 \geq 0$$

For $z = \phi_k$ we obtain

$$1 - \sigma^2 \theta \sum_{k=0}^{+\infty} P_k \phi_k^2(0) \geq 0$$

$$1 - 2\sigma^2 \theta \sum_{k=0}^{+\infty} P_k \geq 0$$

$$\sum_{k=0}^{+\infty} P_k \leq \frac{1}{2\sigma^2 \theta}$$

Or

$$\begin{aligned} \sum_{k=0}^{+\infty} \frac{\lambda_k + \sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}}}{\varepsilon^{-2}} &\leq \frac{1}{2\sigma^2 \theta} \\ \sum_{k=0}^{+\infty} \lambda_k + \sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} &\leq \frac{\varepsilon^{-2}}{2\sigma^2 \theta} \end{aligned} \quad (3.40)$$

Since

$$\begin{aligned} \lambda_k + \sqrt{\Delta} &= \frac{\lambda_k^2 - \Delta}{\lambda_k - \sqrt{\Delta}} \\ &= \frac{\lambda_k^2 - (\lambda_k^2 + \varepsilon^{-2} \langle f, \phi_k \rangle^2)}{\lambda_k - \sqrt{\Delta}} \\ &= \frac{\varepsilon^{-2} \langle f, \phi_k \rangle^2}{\sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} - \lambda_k} \end{aligned}$$

Then (3.40) will be

$$\sum_{k=0}^{+\infty} \frac{\varepsilon^{-2} (\langle f, \phi_k \rangle)^2}{\sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} - \lambda_k} \leq \frac{\varepsilon^{-2}}{2\sigma^2 \theta}$$

Or

$$\frac{1}{2\theta \sigma^2} \geq \sum_{k=0}^{+\infty} \frac{(\langle f, \phi_k \rangle)^2}{\sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} - \lambda_k} \quad (3.41)$$

We have

$$\sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} \geq \sqrt{\lambda_k^2} = |\lambda_k| = -\lambda_k, \quad \forall k \in \mathbb{N}$$

Then

$$\sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} - \lambda_k \geq -2\lambda_k, \quad \forall k \in \mathbb{N}$$

Or

$$\sum_{k=0}^{+\infty} \frac{(\langle f, \phi_k \rangle)^2}{\sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} - \lambda_k} \leq \sum_{k=0}^{+\infty} \frac{(\langle f, \phi_k \rangle)^2}{-2\lambda_k} \quad (3.42)$$

Assume $f(x)$ is such that $f(x) > 0$ for all $0 \leq x \leq 1$, we obtain

$f(x)\phi_n(x) \leq \sqrt{2}f(x)\phi_0$. Then

$$\int_0^1 f(x)\phi_n(x)dx \leq \sqrt{2} \int_0^1 f(x)\phi_0 dx$$

Or

$$\langle f, \phi_n \rangle \leq \sqrt{2} \langle f, \phi_0 \rangle$$

Therefore

$$\langle f, \phi_n \rangle^2 \leq 2 \langle f, \phi_0 \rangle^2$$

Then (3.42) becomes

$$\begin{aligned} \sum_{k=0}^{+\infty} \frac{(\langle f, \phi_k \rangle)^2}{\sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} - \lambda_k} &\leq \sum_{k=0}^{+\infty} \frac{2(\langle f, \phi_0 \rangle)^2}{-2\lambda_k} \\ \sum_{k=0}^{+\infty} \frac{(\langle f, \phi_k \rangle)^2}{\sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} - \lambda_k} &\leq \sum_{k=0}^{+\infty} \frac{(\langle f, \phi_0 \rangle)^2}{-\lambda_k} \end{aligned}$$

We have

$$\sum_{k=0}^{+\infty} \frac{(\langle f, \phi_0 \rangle)^2}{-\lambda_k} = \sum_{k=0}^{+\infty} \frac{(\langle f, \phi_0 \rangle)^2}{1 + n^2\pi^2} = 1.1565(\langle f, \phi_0 \rangle)^2$$

Then (3.41) will be

$$1.1565(\langle f, \phi_0 \rangle)^2 \leq \frac{1}{2\sigma^2\theta}$$

Let σ such that

$$\sigma \leq \frac{1}{1.5208\sqrt{\theta}(\langle f, \phi_0 \rangle)}$$

Or

$$\sigma^2 \leq \frac{1}{2.313\theta(\langle f, \phi_0 \rangle)^2}$$

Then

$$\frac{1}{\sigma^2} \geq 2.313\theta(\langle f, \phi_0 \rangle)^2$$

hence

$$\frac{1}{2\theta\sigma^2} \geq 1.1565(\langle f, \phi_0 \rangle)^2$$

Or

$$\frac{1}{2\theta\sigma^2} \geq \sum_{k=0}^{+\infty} \frac{(\langle f, \phi_k \rangle)^2}{\sqrt{\lambda_k^2 + (\langle f, \phi_k \rangle)^2 \varepsilon^{-2}} - \lambda_k}$$

Thus for

$$\frac{1}{1.5208\sqrt{\theta} \langle f, \phi_0 \rangle} \geq \sigma$$

the Riccati equation (3.38) has a solution $P \in \mathcal{L}^+(H)$ with

$$I - \theta\sigma^2 D^* P D \succeq 0.$$

By Corollary 3.4, we get

$$\overline{r^w}(A; (D, E)) \geq \frac{1}{1.5208\sqrt{\theta} \langle f, \phi_0 \rangle}.$$

Example 3.3. Consider the following stochastic parabolic equation with Newman boundary condition

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) = \frac{\partial^2 y}{\partial x^2}(x, t) + c_1 \delta_\xi(x) \langle f, y(x, t) \rangle \dot{w}_1(t) + u(t), & 0 \leq x, \xi \leq 1, \quad t \geq 0. \\ y(x, 0) = y_0(x) \\ \frac{\partial y}{\partial x}(0, t) = -c_2 \langle g, y(0, t) \rangle \dot{w}_2(t), \quad \frac{\partial y}{\partial x}(1, t) = 0 \end{cases} \quad (3.43)$$

where $(c_1, c_2) \in \mathbb{R}^2$, $f \in L^2(0, 1)$, $g \in L^2(0, 1)$.

To put the problem (3.43) into the abstract setting we introduce the self adjoint operator $Ah = \frac{d^2 h}{dx^2}$ in the Hilbert space $\mathcal{H} = L^2(0, 1)$ with

$$\mathcal{D}(A) = \left\{ z \in L^2(0, 1), \quad \frac{d^2 z}{dx^2} \in L^2(0, 1), \quad \frac{dz}{dx}(0) = \frac{dz}{dx}(1) = 0 \right\}$$

The operator A generates an analytic semigroup $S(t)$, the eigenvalues of A are $\lambda_n = -n^2\pi^2$ and the corresponding eigenfunction are $\phi_n(x) = \sqrt{2} \cos n\pi x$, $n \geq 1$, and $\phi_0(x) = 1$.

In this example we have $B = I$, and $D_1 u = \delta_\xi u$ such that $D_1 \in \mathcal{L}(\mathbb{R}, H^{-\frac{1}{2}-\beta}(0, 1))$, $\beta > 0$, and $E_1 z = \langle f, z \rangle$ with $E_1 \in \mathcal{L}(\mathcal{H}, \mathbb{R})$, $\Delta_1 = c_1 \in \mathbb{R}$ and $D_2 \in \mathcal{L}(\mathbb{R}, H^{-\frac{1}{2}-\beta}(0, 1))$, $\beta > 0$ is defined by $D_2 u = -\delta u$. Set $E_2 z = \langle g, z \rangle$ with $E_2 \in \mathcal{L}(\mathcal{H}, \mathbb{R})$, $\Delta_2 = c_2 \in \mathbb{R}$.

In the abstract form the problem (3.43) can be formulated as follows

$$\begin{cases} dz(t) = Az(t)dt + Bu(t)dt + \sum_{i=1}^2 D_i \Delta_i E_i(z(t)) dw_i(t) \\ z(0) = z_0 \end{cases} \quad (3.44)$$

For this system we have $\gamma = \frac{1}{4} + \frac{\beta}{2}$, and $\delta = 0$ such that $0 < \beta < \frac{1}{2}$.

The corresponding Riccati equation reads

$$2 \langle Az, Pz \rangle + \langle E(\alpha)z, E(\alpha)z \rangle - \varepsilon^{-2} \langle B^* Pz, B^* Pz \rangle = 0 \quad (3.45)$$

Suppose that we can express the solution P of (3.45) by

$$Pz = \sum_{n,j=0}^{+\infty} P_{nj} \langle z, \phi_n \rangle \phi_j, \quad z \in \mathcal{H}$$

Then since $Az = \sum_{n=0}^{+\infty} \lambda_n \langle z, \phi_n \rangle \phi_n$, $z \in \mathcal{D}(A)$, it follows that

$$\langle Az, Pz \rangle = \left\langle \sum_{n=0}^{+\infty} \lambda_n \langle z, \phi_n \rangle \phi_n, \sum_{n,j=0}^{+\infty} P_{nj} \langle z, \phi_n \rangle \phi_j \right\rangle = \sum_{n=0}^{+\infty} \lambda_n P_{nn} (\langle z, \phi_n \rangle)^2$$

Now since $E_1 z = \langle f, z \rangle$ and $E_2 z = \langle g, z \rangle$, and because

$$\langle f, z \rangle = \left\langle f, \sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \phi_n \right\rangle = \sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \langle f, \phi_n \rangle$$

$$\langle g, z \rangle = \left\langle g, \sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \phi_n \right\rangle = \sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \langle g, \phi_n \rangle$$

It follows that

$$\begin{aligned} \langle E(\alpha)z, E(\alpha)z \rangle &= \alpha_1^2 \langle E_1 z, E_1 z \rangle + \alpha_2^2 \langle E_2 z, E_2 z \rangle \\ &= \alpha_1^2 \left(\sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \langle f, \phi_n \rangle \right)^2 + \alpha_2^2 \left(\sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \langle g, \phi_n \rangle \right)^2 \end{aligned}$$

and

$$\langle Pz, Pz \rangle = \sum_{n=0}^{+\infty} P_{nn}^2 (\langle z, \phi_n \rangle)^2$$

Equation (3.45) becomes

$$\begin{aligned} &2 \sum_{n=0}^{+\infty} \lambda_n P_{nn} (\langle z, \phi_n \rangle)^2 - \varepsilon^{-2} \sum_{n=0}^{+\infty} P_{nn}^2 (\langle z, \phi_n \rangle)^2 \\ &+ \alpha_1^2 \left(\sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \langle f, \phi_n \rangle \right)^2 + \alpha_2^2 \left(\sum_{n=0}^{+\infty} \langle z, \phi_n \rangle \langle g, \phi_n \rangle \right)^2 = 0 \end{aligned}$$

Set $z = \phi_k$, $k \in \mathbb{N}$, then

$$2\lambda_k P_{kk} + \alpha_1^2(\langle f, \phi_k \rangle)^2 + \alpha_2^2(\langle g, \phi_k \rangle)^2 - \varepsilon^{-2} P_{kk}^2 = 0$$

Or

$$-\varepsilon^{-2} P_{kk}^2 + 2\lambda_k P_{kk} + \alpha_1^2(\langle f, \phi_k \rangle)^2 + \alpha_2^2(\langle g, \phi_k \rangle)^2 = 0$$

Therefore $P_{kk} = \frac{\lambda_k + \sqrt{\Delta}}{\varepsilon^{-2}}$ where $\Delta = \lambda_k^2 + \varepsilon^{-2}(\alpha_1^2(\langle f, \phi_k \rangle)^2 + \alpha_2^2(\langle g, \phi_k \rangle)^2) > 0$.

Or

$$Pz = \sum_{k=0}^{+\infty} P_k \langle z, \phi_k \rangle \phi_k, \quad z \in \mathcal{H}$$

Where

$$P_k = \frac{\lambda_k + \sqrt{\Delta}}{\varepsilon^{-2}}$$

$$\Delta = \lambda_k^2 + \varepsilon^{-2}(\alpha_1^2(\langle f, \phi_k \rangle)^2 + \alpha_2^2(\langle g, \phi_k \rangle)^2)$$

For $j \in \{1, 2\}$, we have

$$I - \sigma^2(\theta_j/\alpha_j^2) D_j^* P D_j \succeq 0$$

is equivalent to

$$\|z\|^2 - \sigma^2(\theta_j/\alpha_j^2) \langle D_j^* P D_j z, z \rangle \geq 0, \quad \forall z \in \mathcal{U}$$

and we have

$$\begin{aligned} D_j^* P D_j z &= D_j^* \left(\sum_{n=0}^{+\infty} P_n \langle D_j z, \phi_n \rangle \phi_n \right) \\ &= \sum_{n=0}^{+\infty} P_n \langle D_j z, \phi_n \rangle D_j^* \phi_n \\ &= \sum_{n=0}^{+\infty} P_n \langle z, D_j^* \phi_n \rangle D_j^* \phi_n \end{aligned}$$

Or

$$\begin{aligned} \langle D_j^* P D_j z, z \rangle &= \sum_{n=0}^{+\infty} P_n \langle z, D_j^* \phi_n \rangle \langle D_j^* \phi_n, z \rangle \\ &= \sum_{n=0}^{+\infty} P_n \langle z, D_j^* \phi_n \rangle^2 \end{aligned}$$

So

$$\|z\|^2 - \sigma^2(\theta_j/\alpha_j^2) \langle D_j^* P D_j z, z \rangle \geq 0, \quad \forall z \in \mathcal{H}$$

is equivalent to

$$\|z\|^2 - \sigma^2(\theta_j/\alpha_j^2) \sum_{n=0}^{+\infty} P_n \langle z, D_j^* \phi_n \rangle^2 \geq 0, \quad \forall z \in \mathcal{H}$$

Then

$$\|z\|^2 \geq \sigma^2(\theta_j/\alpha_j^2) \sum_{n=0}^{+\infty} P_n \langle z, D_j^* \phi_n \rangle^2, \quad \forall z \in \mathcal{H}$$

Or

$$\frac{\sum_{n=0}^{+\infty} P_n \langle z, D_j^* \phi_n \rangle^2}{\|z\|^2} \leq \frac{\alpha_j^2}{\sigma^2 \theta_j}, \quad \forall z \neq 0$$

is equivalent to

$$\sum_{n=0}^{+\infty} P_n (D_j^* \phi_n)^2 \leq \frac{\alpha_j^2}{\sigma^2 \theta_j}$$

For $j = 1$, we have

$$\sum_{n=0}^{+\infty} P_n (D_1^* \phi_n(x))^2 \leq \frac{\alpha_1^2}{\sigma^2 \theta_1} \quad \text{such that } D_1 = \delta_\xi$$

$$\sum_{n=0}^{+\infty} P_n (\phi_n(\xi))^2 \leq \frac{\alpha_1^2}{\sigma^2 \theta_1}$$

Then

$$\sum_{n=0}^{+\infty} \frac{\lambda_n + \sqrt{\Delta}}{\varepsilon^{-2}} (\phi_n(\xi))^2 \leq \frac{\alpha_1^2}{\sigma^2 \theta_1} \quad (3.46)$$

Since

$$\begin{aligned} \lambda_n + \sqrt{\Delta} &= \frac{\lambda_n^2 - \Delta}{\lambda_n - \sqrt{\Delta}} \\ &= \frac{\lambda_n^2 - (\lambda_n^2 + \varepsilon^{-2}(\alpha_1^2(\langle f, \phi_n \rangle)^2 + \alpha_2^2(\langle g, \phi_n \rangle)^2))}{\lambda_n - \sqrt{\Delta}} \\ &= \frac{-\varepsilon^{-2}(\alpha_1^2(\langle f, \phi_n \rangle)^2 + \alpha_2^2(\langle g, \phi_n \rangle)^2)}{\lambda_n - \sqrt{\Delta}} \\ &= \frac{\varepsilon^{-2}(\alpha_1^2(\langle f, \phi_n \rangle)^2 + \alpha_2^2(\langle g, \phi_n \rangle)^2)}{\sqrt{\Delta} - \lambda_n} \end{aligned}$$

Then (3.46) will be

$$\sum_{n=0}^{+\infty} \frac{(\alpha^2(\langle f, \phi_n \rangle)^2 + \alpha_2^2(\langle g, \phi_n \rangle)^2)(\phi_n(\xi))^2}{\sqrt{\Delta} - \lambda_n} \leq \frac{\alpha_1^2}{\sigma^2 \theta_1}$$

Since $\phi_n(\xi) = \sqrt{2} \cos(n\pi\xi)$, then $\phi_n^2(\xi) = 2 \cos^2(n\pi\xi)$. Or

$$\sum_{n=0}^{+\infty} \frac{2(\alpha^2(\langle f, \phi_n \rangle)^2 + \alpha_2^2(\langle g, \phi_n \rangle)^2) \cos^2(n\pi\xi)}{\sqrt{\Delta} - \lambda_n} \leq \frac{\alpha_1^2}{\sigma^2 \theta_1}$$

But

$$\sum_{n=0}^{+\infty} \frac{2(\alpha^2(\langle f, \phi_n \rangle)^2 + \alpha_2^2(\langle g, \phi_n \rangle)^2) \cos^2(n\pi\xi)}{\sqrt{\Delta} - \lambda_n} \leq \sum_{n=0}^{+\infty} \frac{2(\alpha^2(\langle f, \phi_n \rangle)^2 + \alpha_2^2(\langle g, \phi_n \rangle)^2) \cos^2(n\pi\xi)}{-2\lambda_n} \quad (3.47)$$

and we have $\phi_0 = 1$, $\phi_n(x) = \sqrt{2} \cos(n\pi x)$, since $\cos(n\pi x) \leq 1$, $\forall n \geq 1$, and $0 \leq x \leq 1$, then $\phi_n(x) \leq \sqrt{2}\phi_0$.

Assume $f(x)$ is such that $f(x) > 0$ for all $0 \leq x \leq 1$, we obtain

$f(x)\phi_n(x) \leq \sqrt{2}f(x)\phi_0$. Then

$$\int_0^1 f(x)\phi_n(x)dx \leq \sqrt{2} \int_0^1 f(x)\phi_0 dx$$

Or

$$\langle f, \phi_n \rangle \leq \sqrt{2} \langle f, \phi_0 \rangle$$

Therefore

$$\langle f, \phi_n \rangle^2 \leq 2 \langle f, \phi_0 \rangle^2$$

Assume $g(x)$ is such that $g(x) > 0$ for all $0 \leq x \leq 1$, we obtain $g(x)\phi_n(x) \leq \sqrt{2}g(x)\phi_0$. Then

$$\int_0^1 g(x)\phi_n(x)dx \leq \sqrt{2} \int_0^1 g(x)\phi_0 dx$$

Or

$$\langle g, \phi_n \rangle \leq \sqrt{2} \langle g, \phi_0 \rangle$$

Therefore

$$\langle g, \phi_n \rangle^2 \leq 2 \langle g, \phi_0 \rangle^2$$

Then (3.47) becomes

$$\sum_{n=0}^{+\infty} \frac{2(\alpha_1^2(\langle f, \phi_n \rangle)^2 + \alpha_2^2(\langle g, \phi_n \rangle)^2) \cos^2(n\pi\xi)}{\sqrt{\Delta} - \lambda_n} \leq \sum_{n=0}^{+\infty} \frac{4(\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2)}{-2\lambda_n}$$

$$\sum_{n=0}^{+\infty} \frac{2(\alpha_1^2(\langle f, \phi_n \rangle)^2 + \alpha_2^2(\langle g, \phi_n \rangle)^2) \cos^2(n\pi\xi)}{\sqrt{\Delta} - \lambda_n} \leq 2 \sum_{n=0}^{+\infty} \frac{(\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2)}{-\lambda_n}$$

We have

$$2 \sum_{n=0}^{+\infty} \frac{\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2}{-\lambda_n} = 2 \sum_{n=0}^{+\infty} \frac{\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2}{n^2\pi^2}$$

$$= 2(\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2) \sum_{n=0}^{+\infty} \frac{1}{n^2\pi^2} = \frac{(\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2)}{3}$$

Let σ such that

$$\frac{(\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2)}{3} \leq \frac{\alpha_1^2}{\sigma^2\theta_1}$$

$$\frac{\theta_1(\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2)}{3\alpha_1^2} \leq \frac{1}{\sigma^2}$$

$$\sigma^2 \leq \frac{3\alpha_1^2}{\theta_1(\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2)}$$

$$\sigma \leq \sqrt{\frac{3\alpha_1^2}{\theta_1(\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2)}}$$

For $j = 2$ and using the same steps, we obtain

$$\sigma \leq \sqrt{\frac{3\alpha_2^2}{\theta_2(\alpha_1^2(\langle f, \phi_0 \rangle)^2 + \alpha_2^2(\langle g, \phi_0 \rangle)^2)}}$$

Therefore, for $\alpha_1 = \alpha_2$

$$\bar{r}^w(A; (D_i, E_i)_{i \in \{1,2\}}) \geq \max \left(\sqrt{\frac{3}{\theta_1((\langle f, \phi_0 \rangle)^2 + (\langle g, \phi_0 \rangle)^2)}}, \sqrt{\frac{3}{\theta_2((\langle f, \phi_0 \rangle)^2 + (\langle g, \phi_0 \rangle)^2)}} \right)$$

CHAPTER 4

MAXIMIZATION OF THE STABILITY RADIUS OF SYSTEMS WITH UNBOUNDED INPUT OPERATOR

4.1 Introduction

For deterministic infinite dimensional systems, subjected to stochastic perturbations, the problems of robust stability and robust stabilization have been studied in [41], using the stability radius approach. These problems have been studied in the case where the operators structure perturbations are bounded and the input operator is bounded.

In this chapter, we consider infinite dimensional systems subjected to stochastic perturbations with unbounded input operator. These systems may represent perturbations of partial differential equations with stochastic uncertainties and boundary or point control. Our objective is to investigate the maximization of the stability radius by state feedback. We consider two cases, the bounded and unbounded structured perturbations.

In both cases, we proceed with similar arguments but with different assumptions. First we give the system description and some characterizations of the stability radius.

Next, we establish some technical lemmas. Based on these lemmas, conditions are derived for the existence of a stabilizing controller ensuring that the norm of the closed loop operator below a prespecified bound. Such controllers will be called suboptimal controllers. The suboptimality conditions are obtained in terms of a Riccati equation which satisfies an operator inequality. Then, we give a lower bound for the supremal achievable stability radius via a Riccati equation. Finally, two illustrative examples of the theoretical results are given for both cases.

4.2 Systems subjected to stochastic bounded multi-perturbations

4.2.1 System description

Let $\mathcal{H}, \mathcal{Z}, \mathcal{U}_i, \mathcal{Y}_i, i \in \overline{N}$, real separable Hilbert spaces. Consider the controlled stochastic system

$$\begin{cases} dx(t) = Ax(t)dt + \sum_{i=1}^N D_i \Delta_i (E_i x(t)) dw_i(t) + Bu(t)dt, & t \in \mathbb{R}^+, \\ x(0) = x_0 \in \mathcal{H}, \end{cases} \quad (4.1)$$

where

H1 A is the infinitesimal generator of an exponentially stable analytic semigroup $(S(t))_{t \geq 0}$ on \mathcal{H} .

H2 B is a linear operator from \mathcal{Z} to \mathcal{H} , (B is generally unbounded as an operator from \mathcal{Z} to \mathcal{H}), such that $(-A)^{-\eta} B \in \mathcal{L}(\mathcal{Z}, \mathcal{H})$ for some fixed $\eta, 0 \leq \eta < \frac{1}{2}$.

For $i \in \overline{N}$:

H3 D_i is a linear operator from \mathcal{U}_i to \mathcal{H} .

H4 E_i is a linear operator from \mathcal{H} to \mathcal{Y}_i .

H5 $\Delta_i \in Lip(\mathcal{Y}_i, \mathcal{U}_i)$.

H6 $(w_i(t))_{t \in \mathbb{R}_+}, i \in \overline{N}$, are independent real Wiener processes on a probability space $(\Omega, \mathcal{F}, \mu)$ relative to an increasing family $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ of σ -algebras $(\mathcal{F}_t)_{t \in \mathbb{R}_+} \subset \mathcal{F}$, such that

$$\begin{cases} \mathcal{E}(w_i(t)) = 0 \\ \mathcal{E}(w_i(t) - w_i(s))(w_j(t) - w_j(s)) = \delta_{ij}\theta_i(t - s), \quad i, j \in \overline{N}, \end{cases}$$

where δ_{ij} is the Kronecker symbol, $\theta_i > 0$. u takes its values in the real separable Hilbert space \mathcal{Z} .

We assume that (A, B) is stabilizable. Our aim is to characterize the supremum of the stability radii which can be achieved by linear state feedback $u = Fx$.

Let

$$\overline{\mathcal{F}} = \left\{ \begin{array}{l} F \in \mathcal{L}(\mathcal{H}, \mathcal{Z}); A + BF \text{ is the infinitesimal generator} \\ \text{of an exponentially stable semigroup} \end{array} \right\}$$

and define

$$\overline{r^w}(A; (D, E)) = \sup \{ r^w(A + BF; (D, E)); F \in \overline{\mathcal{F}} \}.$$

4.2.2 Suboptimality Conditions

In order to establish conditions for the existence of suboptimal controllers $u(t) = Fx(t)$ such that $F \in \overline{\mathcal{F}}$ and $\sigma \leq r^w(A + BF; (D, E))$, for $\sigma > 0$, we need the following lemmas.

In this lemma, we establish some elementary results.

Lemma 4.1. *Let $\sigma > 0$ and $\varepsilon > 0$. If there exists $P \in \mathcal{L}^+(\mathcal{H})$, such that*

$$2 \langle Px, (A - \varepsilon^{-2}BB^*P)x \rangle + \varepsilon^{-2} \langle B^*Px, B^*Px \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \leq 0, \quad x \in \mathcal{D}(A) \quad (4.2)$$

and

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0; \quad j \in \overline{N}, \quad (4.3)$$

then $A_\varepsilon := A - \varepsilon^{-2}BB^*P$ generates an exponentially stable analytic semigroup.

Moreover,

$$\sigma \leq r^w(A_\varepsilon; (D_i, E_i)_{i \in \overline{N}}).$$

Proof. By assumptions **H1** and **H2**, it follows, that $A_\varepsilon = A - \varepsilon^{-2}BB^*P$ is the infinitesimal generator of an analytic semigroup on $\mathcal{D}(A_\varepsilon)$ (see[44])

Let us consider in \mathcal{H} the initial value problem

$$\begin{cases} \frac{d}{dt}x(t) = A_\varepsilon x(t), & t > 0, \\ x(0) = x_0, & x_0 \in \mathcal{H}. \end{cases} \quad (4.4)$$

We recall that the solution $x(\cdot)$ of the system (4.4) is given by

$$x(t) = S(t)x_0 - \varepsilon^{-2} \int_0^t S(t-s)BB^*Px(s)ds.$$

This implies that

$$\begin{aligned} \|x(t)\| &\leq \|S(t)x_0\| + \varepsilon^{-2} \left\| \int_0^t S(t-s)BB^*Px(s)ds \right\| \\ &\leq \|S(t)x_0\| + \varepsilon^{-2} \int_0^t \|S(t-s)BB^*Px(s)\| ds \\ &\leq \|S(t)x_0\| + \varepsilon^{-2} \int_0^t \|S(t-s)(-A)^\eta \widehat{B}B^*Px(s)\| ds \\ &\leq \|S(t)x_0\| + \varepsilon^{-2} \int_0^t \|S(t-s)(-A)^\eta\| \|\widehat{B}\| \|B^*Px(s)\| ds \\ &\leq Me^{-wt} \|x_0\| + \varepsilon^{-2} \|\widehat{B}\| M_\eta \int_0^t \frac{e^{-w(t-s)}}{(t-s)^\eta} \|B^*Px(s)\| ds, \end{aligned}$$

where $\widehat{B} = (-A)^{-\eta}B$, and hence

$$\|x(t)\|^2 \leq 2M^2 e^{-2wt} \|x_0\|^2 + 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 \int_0^t \frac{e^{-2w(t-s)}}{(t-s)^{2\eta}} \|B^*Px(s)\|^2 ds.$$

It follows that

$$\int_0^{+\infty} \|x(t)\|^2 dt \leq 2M^2 \|x_0\|^2 \int_0^{+\infty} e^{-2wt} dt$$

$$+ 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 \int_0^{+\infty} \int_0^t \frac{e^{-2w(t-s)}}{(t-s)^{2\eta}} \|B^*Px(s)\|^2 ds dt$$

and the Fubini Theorem gives

$$\begin{aligned} \int_0^{+\infty} \|x(t)\|^2 dt &\leq 2M^2 \|x_0\|^2 \int_0^{+\infty} e^{-2wt} dt \\ &\quad + 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 \int_0^{+\infty} \int_s^{+\infty} \|B^*Px(s)\|^2 \frac{e^{-2w(t-s)}}{(t-s)^{2\eta}} dt ds \\ &\leq \frac{2M^2 \|x_0\|^2}{2w} + 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 \int_0^{+\infty} \|B^*Px(s)\|^2 \left(\int_s^{+\infty} \frac{e^{-2w(t-s)}}{(t-s)^{2\eta}} dt \right) ds. \end{aligned}$$

On other hand, since $\eta < \frac{1}{2}$, there exists K such that

$$\int_0^{+\infty} \|x(t)\|^2 dt \leq \frac{M^2 \|x_0\|^2}{w} + 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 K \int_0^{+\infty} \|B^* Px(s)\|^2 ds.$$

We shall now prove that $B^* Px(t) \in L^2(\mathbb{R}^+, \mathcal{Z})$. For this purpose, we set $V(x) := \langle x, Px \rangle$, where x solves the system (4.4). From (4.2) we obtain,

$$\begin{aligned} \frac{d}{dt} V(x(t)) &\leq -\varepsilon^{-2} \langle PBB^* Px, x \rangle - \langle E(\alpha)x, E(\alpha)x \rangle \\ &\leq -\varepsilon^{-2} \langle PBB^* Px, x \rangle. \end{aligned}$$

This implies that

$$\int_0^T \frac{d}{dt} V(x(t)) dt \leq -\varepsilon^{-2} \int_0^T \langle PBB^* Px(t), x(t) \rangle dt,$$

and therefore

$$V(x(T)) - V(x(0)) \leq -\varepsilon^{-2} \int_0^T \|B^* Px(t)\|^2 dt.$$

Since $P \succeq 0$, we have

$$\varepsilon^{-2} \int_0^T \|B^* Px(t)\|^2 dt \leq V(x_0), \quad \text{for all } T > 0.$$

Hence

$$\varepsilon^{-2} \int_0^T \|B^* Px(t)\|^2 dt \leq K' \|x_0\|^2,$$

and hence $B^* Px(t) \in L^2(\mathbb{R}^+, \mathcal{Z})$. Therefore, $x \in L^2(\mathbb{R}^+, \mathcal{H})$ and $F_\varepsilon := -\varepsilon^{-2} B^* P \in \overline{\mathcal{F}}$.

It follows from (4.2) that

$$2 \langle PA_\varepsilon x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \leq -\varepsilon^{-2} \|B^* Px(t)\|^2$$

Thus,

$$\begin{aligned} 2 \langle PA_\varepsilon x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle &\leq 0, \\ I - \sigma^2 (\theta_j / \alpha_j^2) D_j^* P D_j &\succeq 0, \quad j \in \overline{N} \end{aligned}$$

Applying Proposition 1.1 we get that $\sigma \leq r^w(A_\varepsilon; (D_i, E_i)_{i \in \overline{N}})$ and if the inequality (4.3) is strict, we obtain $\sigma < r^w(A_\varepsilon; (D_i, E_i)_{i \in \overline{N}})$.

□

The following lemma is of basic importance for the approach used to investigate the maximization problem.

Lemma 4.2. *Let $\sigma > 0$, $\varepsilon > 0$ and $F \in \overline{\mathcal{F}}$. If the inequality*

$$2\langle P(A + BF)x, x \rangle + \varepsilon^{-2} \langle B^*Px, B^*Px \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \leq 0, \quad x \in \mathcal{D}(A + BF) \quad (4.5)$$

has a solution $P_0 \in \mathcal{L}^+(\mathcal{H})$ satisfying the condition

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*P_0D_j \succeq 0, \quad j \in \overline{N} \quad (4.6)$$

then

$$F_0 = -\varepsilon^{-2}B^*P_0 \in \overline{\mathcal{F}} \text{ and } \sigma \leq r^w(A + BF_0; (D_i, E_i)_{i \in \overline{N}}).$$

Moreover, there exists $P_1 \in \mathcal{L}^+(\mathcal{H})$ such that for all $x \in \mathcal{D}(A + BF_0)$, we have

$$2\langle P_1(A + BF_0)x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle B^*P_0x, B^*P_0x \rangle = 0,$$

also,

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*P_1D_j \succeq 0, \quad j \in \overline{N} \text{ and } P_1 \preceq P_0.$$

Proof. For $x \in \mathcal{D}(A + BF)$, we have

$$\begin{aligned} 2\langle P_0(A + BF)x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^2 \langle Fx, Fx \rangle &= 2\langle P_0Ax, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle \\ &+ 2\langle P_0BFx, x \rangle + \varepsilon^2 \langle Fx, Fx \rangle. \end{aligned}$$

Setting $F' := \varepsilon F + \varepsilon^{-1}B^*P_0$, we get

$$\begin{aligned} \langle F'x, F'x \rangle &= \langle (\varepsilon F + \varepsilon^{-1}B^*P_0)x, (\varepsilon F + \varepsilon^{-1}B^*P_0)x \rangle \\ &= \varepsilon^2 \langle Fx, Fx \rangle + 2\langle B^*P_0x, Fx \rangle + \varepsilon^{-2} \langle B^*P_0x, B^*P_0x \rangle. \end{aligned}$$

Hence

$$\langle F'x, F'x \rangle - \varepsilon^{-2} \langle B^*P_0x, B^*P_0x \rangle = \varepsilon^2 \langle Fx, Fx \rangle + 2\langle B^*P_0x, Fx \rangle.$$

Since P_0 solves (4.5), it follows that

$$2\langle P_0Ax, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + 2\langle P_0BFx, x \rangle + \varepsilon^2 \langle Fx, Fx \rangle \leq 0,$$

so that

$$2 \langle P_0 A x, x \rangle + \langle E(\alpha) x, E(\alpha) x \rangle - \varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle + \langle F' x, F' x \rangle \leq 0. \quad (4.7)$$

Set $A_0 := A + B F_0$ where $F_0 := -\varepsilon^{-2} B^* P_0$, we obtain

$$2 \langle P_0 A_0 x, x \rangle = 2 \langle P_0 A x, x \rangle - 2 \varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle.$$

From (4.7) it follows that

$$2 \langle P_0 A_0 x, x \rangle + \langle E(\alpha) x, E(\alpha) x \rangle + \varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle \leq - \langle F' x, F' x \rangle.$$

Therefore

$$2 \langle P_0 A_0 x, x \rangle + \langle E(\alpha) x, E(\alpha) x \rangle + \varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle \leq 0. \quad (4.8)$$

Lemma 4.1 shows that $F_0 \in \overline{\mathcal{F}}$ and $\sigma \leq r^w(A_0; (D_i, E_i)_{i \in \overline{N}})$.

Now, since P_0 solves (4.8), it follows that

$$2 \langle P_0 A_0 x, x \rangle + \langle \widehat{E}(\alpha) x, \widehat{E}(\alpha) x \rangle \leq 0,$$

where

$$\widehat{E}(\alpha) := \begin{pmatrix} E(\alpha) \\ \varepsilon^{-1} B^* P_0 \end{pmatrix}.$$

By Proposition 1.1, there exists $P_1 \in \mathcal{L}^+(\mathcal{H})$ such that

$$2 \langle P_1 A_0 x, x \rangle + \langle \widehat{E}(\alpha) x, \widehat{E}(\alpha) x \rangle = 0, \quad \text{with } P_1 \preceq P_0.$$

Consequently,

$$2 \langle P_1 A_0 x, x \rangle + \varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle + \langle E(\alpha) x, E(\alpha) x \rangle = 0,$$

and

$$I - \sigma^2 (\theta_j / \alpha_j^2) D_j^* P_1 D_j \succeq 0, \quad j \in \overline{N}.$$

□

Applying this lemma iteratively we get the following important result.

Theorem 4.1. *Let $F \in \overline{\mathcal{F}}$. Assume that there exist $\sigma > 0$ and $\varepsilon > 0$ such that the Lyapunov inequality (4.5) has a solution $P_0 \in \mathcal{L}^+(\mathcal{H})$ which satisfies the condition (4.6). Then the Riccati equation*

$$2 \langle Ax, Px \rangle + \langle E(\alpha)x, E(\alpha)x \rangle - \varepsilon^{-2} \langle B^*Px, B^*Px \rangle = 0, \quad x \in \mathcal{D}(A) \quad (4.9)$$

has a solution $P \in \mathcal{L}^+(\mathcal{H})$ satisfying

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0, \quad j \in \overline{N}, \quad F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathcal{F}}$$

and

$$r^w(A - \varepsilon^{-2}BB^*P; (D_i, E_i)_{i \in \overline{N}}) \geq \sigma.$$

Proof. Applying Lemma 4.2 iteratively we construct a sequence of linear operators $(P_k)_{k \in \mathbb{N}} \in \mathcal{L}^+(\mathcal{H})$ which satisfies

$$2 \langle P_{k+1}A_kx, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle B^*P_kx, B^*P_kx \rangle = 0, \quad x \in \mathcal{D}(A_k),$$

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*P_{k+1}D_j \succeq 0, \quad j \in \overline{N}, \quad \text{and } P_k \succeq P_{k+1},$$

where P_0 is a solution of (4.5) and $A_k := A - \varepsilon^{-2}BB^*P_k$.

Letting $P := \lim_{k \rightarrow +\infty} P_k$. Hence, for all $x \in \mathcal{D}(A_\varepsilon)$ we have

$$2 \langle PA_\varepsilon x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle B^*Px, B^*Px \rangle = 0, \quad \text{where } A_\varepsilon := A - \varepsilon^{-2}BB^*P,$$

and

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0, \quad j \in \overline{N}.$$

Lemma 4.2 shows that

$$F_\varepsilon = -\varepsilon^{-2}B^*P \in \overline{\mathcal{F}} \quad \text{and} \quad \sigma \leq r^w(A - \varepsilon^{-2}BB^*P; (D_i, E_i)_{i \in \overline{N}}).$$

Since

$$\begin{aligned} & 2 \langle PA_\varepsilon x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle x, PBB^*Px \rangle \\ &= 2 \langle PAx, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle - \varepsilon^{-2} \langle B^*Px, B^*Px \rangle, \end{aligned}$$

it follows that P satisfies the Riccati equation (4.9). This ends the proof. \square

The following result is another consequence of the above Theorem. Here, we establish some conditions for the existence of suboptimal controllers.

Proposition 4.1. *Let $\sigma, \varepsilon > 0$. Assume that the Riccati equation (4.9) has a solution $P \in \mathcal{L}^+(\mathcal{H})$ such that $I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0$, $j \in \bar{N}$. Then*

$$F_\varepsilon \in \bar{\mathcal{F}} \text{ and } \sigma \leq r^w(A + BF_\varepsilon; (D_i, E_i)_{i \in \bar{N}}).$$

Proof. Since P solves the Riccati equation (4.9), we see that

$$2 \langle P(A - \varepsilon^{-2}BB^*P)x, x \rangle + \langle E(\alpha)x, E(\alpha)x \rangle + \varepsilon^{-2} \langle B^*Px, B^*Px \rangle = 0, \quad x \in \mathcal{D}(A).$$

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0, \quad j \in \bar{N}.$$

By Lemma 4.2, we obtain the desired result. \square

As a consequence of the proposition 4.1 we characterize the supremal achievable stability radius via the Riccati equation (4.9). We set

$$S' := \left\{ \begin{array}{l} \sigma > 0; \text{ there exist } \varepsilon > 0 \text{ such that (4.9) has a solution} \\ P \in \mathcal{L}^+(\mathcal{H}) \text{ with } I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0, \quad j \in \bar{N} \end{array} \right\}.$$

Corollary 4.1. *The following inequality holds*

$$\bar{r}^w(A; (D_i, E_i)_{i \in \bar{N}}) \geq \sup S'.$$

Proof. Let $\sigma \in S'$. Then there exist $\varepsilon > 0$ such that the Riccati equation (4.9) has a solution $P \in \mathcal{L}^+(\mathcal{H})$ with

$$I - \sigma^2(\theta_j/\alpha_j^2)D_j^*PD_j \succeq 0, \quad j \in \bar{N}.$$

By Proposition 4.1,

$$\sigma \leq r^w(A + BF_\varepsilon; (D_i, E_i)_{i \in \bar{N}}) \text{ with } F_\varepsilon \in \bar{\mathcal{F}}.$$

Therefore, we have the desired result. \square

4.2.3 Examples

In this section, we present two practical examples to illustrate the above theoretical results. Denoting by δ the Dirac Delta function at zero and by δ_ξ the Dirac Delta function at ξ .

Example 4.1. Consider the stochastic heat equation

$$\begin{cases} dz(t, x) = \left(\frac{\partial^2 z(t, x)}{\partial x^2}\right)dt + \sum_{i=1}^N \Delta_i(z(t, x))dw_i(t) + \delta_\xi(x)u(t)dt, & 0 \leq x \leq 1, \quad t \geq 0 \\ z(0, x) = z_0(x) \\ z(t, 0) = z(t, 1) = 0 \end{cases} \quad (4.10)$$

To put the problem (4.10) into the abstract setting, we introduce the self-adjoint operator $A = \frac{d^2}{dx^2}$ defined on the Hilbert space $\mathcal{H} = L^2(0, 1)$ with $\mathcal{D}(A) = H_0^1(0, 1) \cap H^2(0, 1)$.

The operator A generates an exponentially stable semigroup $S(t)$. According to [8], the eigenvalues of A are

$$\lambda_n = -n^2\pi^2, \quad n \geq 0$$

However, the corresponding eigenfunctions are

$$\phi_n(x) = \sqrt{2} \sin n\pi x, \quad n \geq 1, \quad \phi_0(x) = 1$$

Define the following operators:

$$\begin{aligned} D_i &= E_i = I_{\mathcal{H}} \text{ for all } i \in \{1, \dots, N\} = \overline{N}, \\ \Delta_i &\in Lip(\mathcal{H}, \mathcal{H}), \\ Bu &= \delta_\xi u, \text{ with } B \in \mathcal{L}(\text{Re}, H^{-\frac{1}{2}-\beta}(0, 1)). \end{aligned}$$

The problem (4.10) can be formulated in the abstract form as follows

$$\begin{cases} dZ(t) = AZ(t)dt + \sum_{i=1}^N D_i \Delta_i(E_i Z(t))dw_i(t) + Bu(t)dt \\ Z(0) = Z_0 \end{cases} \quad (4.11)$$

Let $\alpha \in (0; +\infty)^N$ and $\varepsilon > 0$. The Riccati equation corresponding to this system is

$$2 \langle AZ, PZ \rangle + \langle E(\alpha)Z, E(\alpha)Z \rangle - \varepsilon^{-2} \langle B^* PZ, B^* PZ \rangle = 0, \quad Z \in \mathcal{D}(A) \quad (4.12)$$

Expressing the solution P of (4.12) as follows:

$$PZ = \sum_{n,j=1}^{+\infty} P_{nj} \langle Z, \phi_n \rangle \phi_j, \quad Z \in \mathcal{H}$$

and using

$$AZ = \sum_{n=1}^{+\infty} \lambda_n \langle Z, \phi_n \rangle \phi_n, \quad Z \in \mathcal{D}(A),$$

we get

$$\langle AZ, PZ \rangle = \sum_{n=1}^{+\infty} \lambda_n P_{nn} (\langle Z, \phi_n \rangle)^2$$

and

$$\langle E(\alpha)Z, E(\alpha)Z \rangle = \sum_{i=1}^N \alpha_i^2 \langle E_i Z, E_i Z \rangle = \sum_{i=1}^N \alpha_i^2 \langle Z, Z \rangle$$

Also, we have

$$\begin{aligned} B^* PZ &= B^* \left(\sum_{n,j=1}^{+\infty} P_{nj} \langle Z, \phi_n \rangle \phi_j \right) = \left(\sum_{n,j=1}^{+\infty} P_{nj} \langle Z, \phi_n \rangle B^* \phi_j \right) \\ &= \left(\sum_{n,j=1}^{+\infty} P_{nj} \langle Z, \phi_n \rangle \phi_j(\xi) \right) = \sqrt{2} \left(\sum_{n,j=1}^{+\infty} P_{nj} \sin(j\pi\xi) \langle Z, \phi_n \rangle \right) \end{aligned}$$

So,

$$\langle B^* PZ, B^* PZ \rangle = 2 \sum_{n,j=1}^{+\infty} \sum_{m,k=1}^{+\infty} P_{nj} P_{mk} \sin(j\pi\xi) \sin(k\pi\xi) \langle Z, \phi_n \rangle \langle Z, \phi_m \rangle$$

Equation (4.12) becomes

$$\begin{aligned} &2 \sum_{n=1}^{+\infty} P_{nn} \lambda_n \langle Z, \phi_n \rangle^2 + \sum_{i=1}^N \alpha_i^2 \langle Z, Z \rangle \\ &- 2\varepsilon^{-2} \sum_{n,j=1}^{+\infty} \sum_{m,k=1}^{+\infty} P_{nj} P_{mk} \sin(j\pi\xi) \sin(k\pi\xi) \langle Z, \phi_n \rangle \langle Z, \phi_m \rangle = 0 \end{aligned}$$

For $Z = \phi_k$, $k \in \mathbb{N}^*$, we get

$$2\lambda_k P_{kk} + \sum_{i=1}^N \alpha_i^2 - 2\varepsilon^{-2} P_{kk}^2 \sin^2 k\pi\xi = 0$$

Hence

$$-2\varepsilon^{-2}P_{kk}^2 \sin^2 k\pi\xi + 2\lambda_k P_{kk} + a = 0, \quad \text{where } a = \sum_{i=1}^N \alpha_i^2$$

we have

$$\Delta' = \lambda_k^2 + 2a\varepsilon^{-2} \sin^2 k\pi\xi > 0$$

Then

$$P_{kk} = \frac{\lambda_k + \sqrt{\Delta'}}{2\varepsilon^{-2} \sin^2 k\pi\xi} > 0$$

Or

$$Pz = \sum_{k=1}^{+\infty} P_k \langle z, \phi_k \rangle \phi_k, \quad z \in \mathcal{H}$$

Where

$$P_k = \frac{\lambda_k + \sqrt{\Delta'}}{2\varepsilon^{-2} \sin^2 k\pi\xi}$$

$$\Delta' = \lambda_k^2 + 2a\varepsilon^{-2} \sin^2 k\pi\xi$$

Now we show that

$$I - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) D_j^* P D_j \succeq 0, \quad \text{for all } j \in \bar{N} \quad (4.13)$$

For some $\sigma > 0$. Let $Z \in \mathcal{H}$, we have

$$\left\langle \left(I - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P \right) Z, Z \right\rangle = \sum_{n=1}^{+\infty} \left(1 - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P_n \right) \langle Z, \phi_n \rangle^2$$

Hence $I - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P \succeq 0$ if and only if $1 - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P_n \geq 0$, for all $n \geq 1$.

Let $j \in \bar{N}$. For $n \in \mathbb{N}^*$, we have

$$1 - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P_n \geq 0,$$

$$\iff \frac{\alpha_j^2}{\theta_j \sigma^2} \geq \frac{\lambda_n + \sqrt{\Delta'}}{2\varepsilon^{-2} \sin^2 n\pi\xi},$$

$$\iff \frac{\alpha_j^2}{\theta_j a \sigma^2} \geq \frac{1}{\sqrt{\lambda_n^2 + 2a\varepsilon^{-2} \sin^2 n\pi\xi} - \lambda_n}$$

Let σ such that

$$\sigma \leq \frac{\pi\alpha_j\sqrt{2}}{\sqrt{\theta_j a}} \quad \text{with } a = \sum_{i=1}^N \alpha_i^2.$$

then

$$\frac{\alpha_j^2}{\theta_j a \sigma^2} \geq \frac{1}{2\pi^2}.$$

hence

$$\frac{\alpha_j^2}{\theta_j a \sigma^2} \geq \frac{1}{-2\lambda_n} \quad \text{for all } n \in \mathbb{N}^* \quad (4.14)$$

and because we have

$$\sqrt{\lambda_n^2 + 2a\varepsilon^{-2} \sin^2 n\pi\xi} - \lambda_n \geq -2\lambda_n$$

Then

$$\frac{1}{\sqrt{\lambda_n^2 + 2a\varepsilon^{-2} \sin^2 n\pi\xi} - \lambda_n} \leq \frac{1}{-2\lambda_n}$$

Or (4.14) implies that

$$\frac{\alpha_j^2}{\theta_j a \sigma^2} \geq \frac{1}{\sqrt{\lambda_n^2 + 2\varepsilon^{-2} \sin^2 n\pi\xi a} - \lambda_n} \quad \text{for all } n \in \mathbb{N}^* \quad (4.15)$$

Thus for $\sigma \leq \frac{\pi\alpha_j\sqrt{2}}{\sqrt{\theta_j a}}$, the Riccati equation has a solution $P \in \mathcal{L}^+(\mathcal{H})$ with

$$I - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) D_j^* P D_j \succeq 0, \quad \forall j \in \overline{N}$$

Recalling Corollary 4.1 we get

$$\overline{r^w}(A; (D_i, E_i)_{i \in \overline{N}}) \geq \frac{\pi\sqrt{2}}{\min_{j \in \overline{N}}(\sqrt{\theta_j})}$$

Example 4.2. Consider the stochastic parabolic equation

$$\begin{cases} dy(x, t) = \pi^{-2} \left(\frac{\partial^2 y(x, t)}{\partial x^2} \right) dt - y(x, t) dt + \sum_{i=1}^N \Delta_i(y(x, t)) dw_i(t), & 0 \leq x \leq 1, t \geq 0, \\ y(x, 0) = y_0(x), \\ \frac{\partial y}{\partial x}(0, t) = u(t), \quad \frac{\partial y}{\partial x}(1, t) = 0. \end{cases} \quad (4.16)$$

Consider the operator $Ah = \pi^{-2} \frac{d^2 h}{dx^2} - h$ defined on the Hilbert space $\mathcal{H} = L^2(0, 1)$ with

$$D(A) = \{\psi \in H^2(0, 1), \dot{\psi}(0) = \dot{\psi}(1) = 0\}.$$

The operator A generates an exponentially stable semigroup $S(t)$, the eigenvalues of A are

$$\lambda_n = -n^2 - 1, \quad n \geq 0.$$

In addition, the corresponding eigenfunctions are

$$\phi_0 = 1 \text{ and } \phi_n(x) = \sqrt{2} \cos n\pi x, \text{ for all } n \geq 1.$$

Define the following operators:

$$\begin{aligned} D_i &= E_i = I_{\mathcal{H}} \text{ for all } i \in \{1, \dots, N\} = \bar{N}, \text{ with} \\ \Delta_i &\in Lip(\mathcal{H}, \mathcal{H}), \\ B^* \psi(x) &= -\frac{1}{\pi^2} \psi(0), \text{ with } \psi \in \mathcal{D}(B^*). \end{aligned}$$

The problem (4.16) takes the following abstract form

$$\begin{cases} d\mathcal{Y}(t) = A\mathcal{Y}(t)dt + \sum_{i=1}^N D_i \Delta_i (E_i \mathcal{Y}(t)) dw_i(t) + Bu(t)dt \\ \mathcal{Y}(0) = \mathcal{Y}_0 \end{cases} \quad (4.17)$$

Let $\alpha \in (0; +\infty)^N$ and $\varepsilon > 0$. Note that the corresponding Riccati equation is

$$2 \langle A\mathcal{Y}, P\mathcal{Y} \rangle + \langle E(\alpha)\mathcal{Y}, E(\alpha)\mathcal{Y} \rangle - \varepsilon^{-2} \langle B^* P\mathcal{Y}, B^* P\mathcal{Y} \rangle = 0, \quad \mathcal{Y} \in \mathcal{D}(A) \quad (4.18)$$

Assume that we can express the solution P of (4.18) as

$$P\mathcal{Y} = \sum_{n,j=0}^{+\infty} P_{nj} \langle \mathcal{Y}, \phi_n \rangle \phi_j, \quad \mathcal{Y} \in \mathcal{H}.$$

we have

$$A\mathcal{Y} = \sum_{n=0}^{+\infty} \lambda_n \langle \mathcal{Y}, \phi_n \rangle \phi_n, \text{ for } \mathcal{Y} \in \mathcal{D}(A),$$

Hence

$$\langle A\mathcal{Y}, P\mathcal{Y} \rangle = \sum_{n=0}^{+\infty} \lambda_n P_{nn} (\langle \mathcal{Y}, \phi_n \rangle)^2$$

We have

$$\langle E(\alpha)\mathcal{Y}, E(\alpha)\mathcal{Y} \rangle = \sum_{i=1}^N \alpha_i^2 \langle E_i\mathcal{Y}, E_i\mathcal{Y} \rangle = \sum_{i=1}^N \alpha_i^2 \langle \mathcal{Y}, \mathcal{Y} \rangle$$

Since

$$\begin{aligned} B^*P\mathcal{Y} &= B^* \left(\sum_{n,j=0}^{+\infty} P_{nj} \langle \mathcal{Y}, \phi_n \rangle \phi_j \right) = \left(\sum_{n,j=0}^{+\infty} P_{nj} \langle \mathcal{Y}, \phi_n \rangle B^* \phi_j \right) \\ &= -\frac{1}{\pi^2} \left(\sum_{n,j=0}^{+\infty} P_{nj} \langle \mathcal{Y}, \phi_n \rangle \phi_j(0) \right) = -\frac{\sqrt{2}}{\pi^2} \left(\sum_{n,j=0}^{+\infty} P_{nj} \langle \mathcal{Y}, \phi_n \rangle \right), \end{aligned}$$

We get

$$\langle B^*P\mathcal{Y}, B^*P\mathcal{Y} \rangle = \frac{2}{\pi^4} \sum_{n,j=0}^{+\infty} \sum_{m,k=0}^{+\infty} P_{nj} P_{mk} \langle \mathcal{Y}, \phi_n \rangle \langle \mathcal{Y}, \phi_m \rangle$$

Equation (4.18) takes the form

$$2 \sum_{n=0}^{+\infty} P_{nn} \lambda_n \langle \mathcal{Y}, \phi_n \rangle^2 + \sum_{i=1}^N \alpha_i^2 \langle \mathcal{Y}, \mathcal{Y} \rangle - \frac{2\varepsilon^{-2}}{\pi^4} \sum_{n,j=0}^{+\infty} \sum_{m,k=0}^{+\infty} P_{nj} P_{mk} \langle \mathcal{Y}, \phi_n \rangle \langle \mathcal{Y}, \phi_m \rangle = 0.$$

For $\mathcal{Y} = \phi_k$, $k \in \mathbb{N}$, we obtain

$$2\lambda_k P_{kk} + \sum_{i=1}^N \alpha_i^2 - \frac{2\varepsilon^{-2}}{\pi^4} P_{kk}^2 = 0.$$

Or

$$-\frac{2\varepsilon^{-2}}{\pi^4} P_{kk}^2 + 2\lambda_k P_{kk} + a = 0, \quad \text{where } a = \sum_{i=1}^N \alpha_i^2$$

we have

$$\Delta' = \lambda_k^2 + \frac{2\varepsilon^{-2}a}{\pi^4} > 0$$

Then

$$P_{kk} = \frac{\pi^4(\lambda_k + \sqrt{\Delta'})}{2\varepsilon^{-2}} > 0$$

hence

$$P\mathcal{Y} = \sum_{n=0}^{+\infty} P_n \langle \mathcal{Y}, \phi_n \rangle \phi_n, \quad \mathcal{Y} \in \mathcal{H}$$

where $P_n = \frac{\pi^4(\lambda_n + \sqrt{\Delta'})}{2\varepsilon^{-2}}$. Now we show that

$$I - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) D_j^* P D_j \succeq 0, \quad \text{for all } j \in \bar{N}, \quad (4.19)$$

for some $\sigma > 0$. Let $Z \in \mathcal{H}$, we have

$$\left\langle \left(I - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P \right) \mathcal{Y}, \mathcal{Y} \right\rangle = \sum_{n=1}^{+\infty} \left(1 - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P_n \right) \langle \mathcal{Y}, \phi_n \rangle^2$$

Hence $I - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P \succeq 0$ if and only if $1 - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P_n \geq 0$, for all $n \geq 0$.

Let $j \in \bar{N}$. For $n \in \mathbb{N}$, we have

$$\begin{aligned} 1 - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) P_n &\geq 0, \\ \iff \frac{\alpha_j^2}{\theta_j \sigma^2} &\geq \frac{\pi^4(\lambda_n + \sqrt{\Delta'})}{2\varepsilon^{-2}}, \\ \iff \frac{\alpha_j^2}{\theta_j a \sigma^2} &\geq \frac{1}{\sqrt{\lambda_n^2 + \frac{2\varepsilon^{-2}a}{\pi^4}} - \lambda_n}. \end{aligned}$$

Let σ such that

$$\sigma \leq \frac{\alpha_j \sqrt{2}}{\sqrt{\theta_j a}} \quad \text{with } a = \sum_{i=1}^N \alpha_i^2.$$

hence

$$\frac{\alpha_j^2}{\theta_j a \sigma^2} \geq \frac{1}{2} \geq \frac{1}{-2(-n^2 - 1)}$$

Then

$$\frac{\alpha_j^2}{\theta_j a \sigma^2} \geq \frac{1}{-2\lambda_n} \quad (4.20)$$

and because we have

$$\sqrt{\lambda_n^2 + \frac{2\varepsilon^{-2}a}{\pi^4}} - \lambda_n \geq -2\lambda_n$$

Then

$$\frac{1}{\sqrt{\lambda_n^2 + \frac{2\varepsilon^{-2}a}{\pi^4} - \lambda_n}} \leq \frac{1}{-2\lambda_n}$$

Or (4.20) implies that

$$\frac{\alpha_j^2}{\theta_j a \sigma^2} \geq \frac{1}{\sqrt{\lambda_n^2 + \frac{2\varepsilon^{-2}a}{\pi^4} - \lambda_n}}, \quad (4.21)$$

Thus for $\sigma \leq \frac{\alpha_j \sqrt{2}}{\sqrt{\theta_j a}}$, the Riccati equation has a solution $P \in \mathcal{L}^+(\mathcal{H})$ with

$$I - \sigma^2 \left(\frac{\theta_j}{\alpha_j^2} \right) D_j^* P D_j \succeq 0, \quad \text{for all } j \in \bar{N}$$

Recalling Corollary 4.1 we get

$$\bar{r}^w(A; (D_i, E_i)_{i \in \bar{N}}) \geq \frac{\sqrt{2}}{\min_{i \in \bar{N}}(\sqrt{\theta_j})}$$

4.3 Systems subjected to stochastic unbounded perturbations

4.3.1 System description

Let us consider the controlled stochastic system represented by the equation

$$\begin{cases} dx(t) = Ax(t)dt + D\Delta(E(x(t))dw(t) + Bu(t)dt, & t \geq 0, \\ x(0) = x_0, \quad x_0 \in \mathcal{H}, \\ \|\Delta\|_{Lip} < \sigma. \end{cases} \quad (4.22)$$

We introduce the following assumptions

H1. A is the infinitesimal generator of an exponentially stable analytic semigroup $(S(t))_{t \geq 0}$ on \mathcal{H} ;

H2. B is a linear operator from \mathcal{Z} to \mathcal{H} , such that

$$(-A)^{-\eta} B \in \mathcal{L}(\mathcal{Z}, \mathcal{H}) \text{ for some fixed } \eta, \quad \text{with } 0 \leq \eta < \frac{1}{2};$$

H3. D is generally an unbounded linear operator, in order that,

$$(-A)^{-\gamma}D \in \mathcal{L}(\mathcal{U}, \mathcal{H}) \text{ for some fixed } \gamma, \text{ with } 0 \leq \gamma < \frac{1}{2};$$

H4. E is a linear operator from \mathcal{H} to \mathcal{Y} , such that $E(-A)^{-\delta} \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$, with $\delta < \min\{\frac{1}{2} - \gamma, \frac{1}{2} - \eta\}$;

H5. $(w(t))_{t \in \mathbb{R}_+}$ is a real Wiener process on the probability space $(\Omega, \mathcal{F}, \mu)$ with variance θ ;

u takes its values in the real separable Hilbert space \mathcal{Z} and $\Delta \in \text{Lip}(\mathcal{Y}, \mathcal{U})$.

The goal of this work is to characterize the supremum of the stability radii which can be achieved by linear state feedback $u = Fx$. For this purpose, we set

$$\overline{\mathcal{F}} := \left\{ \begin{array}{l} F : \mathcal{H} \longrightarrow \mathcal{Z}; A + BF \text{ is the infinitesimal generator} \\ \text{of an exponentially stable analytic semigroup} \end{array} \right\}.$$

Define

$$\overline{r^w}(A, (D, E)) = \sup \{ r^w(A + BF, (D, E)), F \in \overline{\mathcal{F}} \}.$$

4.3.2 Suboptimality Conditions

The aim of this section is to establish conditions for the existence of suboptimal controllers $u(t) = Fx(t)$, with

$$F \in \overline{\mathcal{F}} \text{ and } 0 < \sigma \leq r^w(A + BF; (D, E)).$$

To do this, we need the following Lemmas.

Lemma 4.3. *Let $\sigma > 0$ and $\varepsilon > 0$. If there exists $P \in \mathcal{L}^+(\mathcal{H})$, such that*

$$2 \langle Px, (A - \varepsilon^{-2}BB^*P)x \rangle + \varepsilon^{-2} \langle B^*Px, B^*Px \rangle + \langle Ex, Ex \rangle \leq 0, \quad x \in \mathcal{D}((-A)^\delta) \quad (4.23)$$

and

$$I - \theta\sigma^2 D^*PD \succeq 0, \quad (4.24)$$

then $A_\varepsilon := A - \varepsilon^{-2}BB^*P$ generates an exponentially stable analytic semigroup.

Moreover,

$$\sigma \leq r^w(A_\varepsilon; (D, E)).$$

Proof. According to [46, Theorem 2.5.1], it follows from assumptions (H1), (H2) and (H4) that A_ε is the infinitesimal generator of an analytic semigroup on $\mathcal{D}((-A)^\delta)$.

Let us consider in \mathcal{H} the initial value problem

$$\begin{cases} \frac{d}{dt}x(t) = A_\varepsilon x(t), & t \in \mathbb{R}^+, \\ x(0) = x_0, & x_0 \in \mathcal{H}. \end{cases} \quad (4.25)$$

We recall that the solution $x(\cdot)$ of the system (4.25) is given by

$$x(t) = S(t)x_0 - \varepsilon^{-2} \int_0^t S(t-s)BB^*Px(s)ds.$$

This implies that

$$\begin{aligned} \|x(t)\| &\leq \|S(t)x_0\| + \varepsilon^{-2} \left\| \int_0^t S(t-s)BB^*Px(s)ds \right\| \\ &\leq \|S(t)x_0\| + \varepsilon^{-2} \int_0^t \|S(t-s)BB^*Px(s)\| ds \\ &\leq \|S(t)x_0\| + \varepsilon^{-2} \int_0^t \|S(t-s)(-A)^\eta \widehat{B}B^*Px(s)\| ds \\ &\leq \|S(t)x_0\| + \varepsilon^{-2} \int_0^t \|S(t-s)(-A)^\eta\| \|\widehat{B}\| \|B^*Px(s)\| ds \\ &\leq Me^{-wt} \|x_0\| + \varepsilon^{-2} \|\widehat{B}\| M_\eta \int_0^t \frac{e^{-w(t-s)}}{(t-s)^\eta} \|B^*Px(s)\| ds, \end{aligned}$$

where $\widehat{B} = (-A)^{-\eta}B$, and hence

$$\|x(t)\|^2 \leq 2M^2 e^{-2wt} \|x_0\|^2 + 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 \int_0^t \frac{e^{-2w(t-s)}}{(t-s)^{2\eta}} \|B^*Px(s)\|^2 ds.$$

It follows that

$$\begin{aligned} \int_0^{+\infty} \|x(t)\|^2 dt &\leq 2M^2 \|x_0\|^2 \int_0^{+\infty} e^{-2wt} dt \\ &\quad + 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 \int_0^{+\infty} \int_0^t \frac{e^{-2w(t-s)}}{(t-s)^{2\eta}} \|B^*Px(s)\|^2 ds dt \end{aligned}$$

and the Fubini Theorem gives

$$\begin{aligned} \int_0^{+\infty} \|x(t)\|^2 dt &\leq 2M^2 \|x_0\|^2 \int_0^{+\infty} e^{-2wt} dt \\ &\quad + 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 \int_0^{+\infty} \int_s^{+\infty} \|B^*Px(s)\|^2 \frac{e^{-2w(t-s)}}{(t-s)^{2\eta}} dt ds \\ &\leq \frac{2M^2 \|x_0\|^2}{2w} + 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 \int_0^{+\infty} \|B^*Px(s)\|^2 \left(\int_s^{+\infty} \frac{e^{-2w(t-s)}}{(t-s)^{2\eta}} dt \right) ds. \end{aligned}$$

On other hand, since $\eta < \frac{1}{2}$, there exists K such that

$$\int_0^{+\infty} \|x(t)\|^2 dt \leq \frac{M^2 \|x_0\|^2}{w} + 2\varepsilon^{-4} \|\widehat{B}\|^2 M_\eta^2 K \int_0^{+\infty} \|B^* Px(s)\|^2 ds.$$

We shall now prove that $B^* Px(\cdot) \in L^2(\mathbb{R}^+, \mathcal{Z})$. For this purpose, we set $V(x) := \langle x, Px \rangle$, where x solves the system (4.25). From (4.23) we obtain,

$$\begin{aligned} \frac{d}{dt} V(x(t)) &\leq -\varepsilon^{-2} \langle PBB^* Px, x \rangle - \langle Ex, Ex \rangle \\ &\leq -\varepsilon^{-2} \langle PBB^* Px, x \rangle. \end{aligned}$$

This implies that

$$\int_0^T \frac{d}{dt} V(x(t)) dt \leq -\varepsilon^{-2} \int_0^T \langle PBB^* Px(t), x(t) \rangle dt,$$

and therefore

$$V(x(T)) - V(x(0)) \leq -\varepsilon^{-2} \int_0^T \|B^* Px(t)\|^2 dt.$$

Since $P \succeq 0$, we have

$$\varepsilon^{-2} \int_0^T \|B^* Px(t)\|^2 dt \leq V(x_0), \quad \text{for all } T > 0.$$

Hence

$$\varepsilon^{-2} \int_0^T \|B^* Px(t)\|^2 dt \leq K' \|x_0\|^2,$$

and hence $B^* Px(\cdot) \in L^2(\mathbb{R}^+, \mathcal{Z})$. Therefore, $x(\cdot) \in L^2(\mathbb{R}^+, \mathcal{H})$ and $F_\varepsilon := -\varepsilon^{-2} B^* P \in \overline{\mathcal{F}}$.

It follows from (4.23) that

$$2 \langle PA_\varepsilon x, x \rangle + \langle Ex, Ex \rangle \leq -\varepsilon^{-2} \|B^* Px(t)\|^2.$$

Thus,

$$\begin{aligned} 2 \langle PA_\varepsilon x, x \rangle + \langle Ex, Ex \rangle &\leq 0, \\ I - \theta \sigma^2 D^* P D &\succeq 0, \end{aligned}$$

and so $\sigma \leq r^w(A_\varepsilon; (D, E))$. If the inequality (4.24) is strict, we get $\sigma < r^w(A_\varepsilon; (D, E))$. \square

Now, we prove the following necessary Lemma which is the main ingredient in the proof of next Theorem.

Lemma 4.4. *Let $\sigma > 0$, $\varepsilon > 0$ and $F \in \overline{\mathcal{F}}$. If the inequality*

$$2\langle P(A + BF)x, x \rangle + \varepsilon^{-2} \langle B^*Px, B^*Px \rangle + \langle Ex, Ex \rangle \leq 0, \quad x \in \mathcal{D}(A + BF) \quad (4.26)$$

has a solution $P_0 \in \mathcal{L}^+(\mathcal{H})$ satisfying the condition

$$I - \theta\sigma^2 D^*P_0D \succeq 0, \quad (4.27)$$

then

$$F_0 = -\varepsilon^{-2}B^*P_0 \in \overline{\mathcal{F}} \text{ and } \sigma \leq r^w(A + BF_0; (D, E)).$$

Moreover, there exists $P_1 \in \mathcal{L}^+(\mathcal{H})$ such that for all $x \in \mathcal{D}(A + BF_0)$, we have

$$2\langle P_1(A + BF_0)x, x \rangle + \langle Ex, Ex \rangle + \varepsilon^{-2} \langle B^*P_0x, B^*P_0x \rangle = 0,$$

also,

$$I - \theta\sigma^2 D^*P_1D \succeq 0 \text{ and } P_1 \preceq P_0.$$

Proof. For $x \in \mathcal{D}(A + BF)$, we have

$$\begin{aligned} 2\langle P_0(A + BF)x, x \rangle + \langle Ex, Ex \rangle + \varepsilon^2 \langle Fx, Fx \rangle &= 2\langle P_0Ax, x \rangle + \langle Ex, Ex \rangle \\ &+ 2\langle P_0BFx, x \rangle + \varepsilon^2 \langle Fx, Fx \rangle. \end{aligned}$$

Letting $F' := \varepsilon F + \varepsilon^{-1}B^*P_0$, we get

$$\begin{aligned} \langle F'x, F'x \rangle &= \langle (\varepsilon F + \varepsilon^{-1}B^*P_0)x, (\varepsilon F + \varepsilon^{-1}B^*P_0)x \rangle \\ &= \varepsilon^2 \langle Fx, Fx \rangle + 2\langle B^*P_0x, Fx \rangle + \varepsilon^{-2} \langle B^*P_0x, B^*P_0x \rangle. \end{aligned}$$

Hence

$$\langle F'x, F'x \rangle - \varepsilon^{-2} \langle B^*P_0x, B^*P_0x \rangle = \varepsilon^2 \langle Fx, Fx \rangle + 2\langle B^*P_0x, Fx \rangle.$$

Since P_0 solves (4.26), it follows that

$$2\langle P_0Ax, x \rangle + \langle Ex, Ex \rangle + 2\langle P_0BFx, x \rangle + \varepsilon^2 \langle Fx, Fx \rangle \leq 0,$$

so that

$$2 \langle P_0 A x, x \rangle + \langle E x, E x \rangle - \varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle + \langle F' x, F' x \rangle \leq 0. \quad (4.28)$$

Set $A_0 := A + B F_0$ where $F_0 := -\varepsilon^{-2} B^* P_0$, we obtain

$$2 \langle P_0 A_0 x, x \rangle = 2 \langle P_0 A x, x \rangle - 2\varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle.$$

From (4.28) it follows that

$$2 \langle P_0 A_0 x, x \rangle + \langle E x, E x \rangle + \varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle \leq - \langle F' x, F' x \rangle.$$

Therefore

$$2 \langle P_0 A_0 x, x \rangle + \langle E x, E x \rangle + \varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle \leq 0. \quad (4.29)$$

Lemma 4.3 shows that $F_0 \in \overline{\mathcal{F}}$ and $\sigma \leq r^w(A_0; (D, E))$.

Now, since P_0 solves (4.29), it follows that

$$2 \langle P_0 A_0 x, x \rangle + \langle \widehat{E} x, \widehat{E} x \rangle \leq 0,$$

where

$$\widehat{E} := \begin{pmatrix} E \\ \varepsilon^{-1} B^* P_0 \end{pmatrix}.$$

By Proposition 3.2, there exists $P_1 \in \mathcal{L}^+(\mathcal{H})$ such that

$$2 \langle P_1 A_0 x, x \rangle + \langle \widehat{E} x, \widehat{E} x \rangle = 0, \quad \text{with } P_1 \preceq P_0.$$

Consequently,

$$2 \langle P_1 A_0 x, x \rangle + \varepsilon^{-2} \langle B^* P_0 x, B^* P_0 x \rangle + \langle E x, E x \rangle = 0,$$

and

$$I - \theta \sigma^2 D^* P_1 D \succeq 0.$$

□

Theorem 4.2. *Let $F \in \overline{\mathcal{F}}$. Assume that there exist $\sigma > 0$ and $\varepsilon > 0$ such that the Lyapunov inequality (4.26) has a solution $P_0 \in \mathcal{L}^+(\mathcal{H})$ which satisfies the condition (4.27). Then the Riccati equation*

$$2 \langle A x, P x \rangle + \langle E x, E x \rangle - \varepsilon^{-2} \langle B^* P x, B^* P x \rangle = 0, \quad x \in \mathcal{D}((-A)^\delta) \quad (4.30)$$

has a solution P satisfying

$$I - \theta \sigma^2 D^* P D \succeq 0, \quad F_\varepsilon = -\varepsilon^2 B^* P \in \overline{\mathcal{F}} \quad \text{and} \quad r^w(A - \varepsilon^{-2} B B^* P; (D, E)) \geq \sigma.$$

Proof. The main basic idea of the proof is to iteratively apply the Lemma 4.4. To this end, we construct a sequence of linear operators $(P_k)_{k \in \mathbb{N}} \in \mathcal{L}^+(\mathcal{H})$ such that

$$2 \langle P_{k+1} A_k x, x \rangle + \langle E x, E x \rangle + \varepsilon^{-2} \langle B^* P_k x, B^* P_k x \rangle = 0, \quad x \in \mathcal{D}((-A)^\delta),$$

$$I - \theta \sigma^2 D^* P_{k+1} D \succeq 0, \quad \text{and} \quad P_k \succeq P_{k+1},$$

where P_0 is a solution of (4.26) and $A_k := A - \varepsilon^{-2} B B^* P_k$.

Letting $P := \lim_{k \rightarrow +\infty} P_k$. Hence, for all $x \in \mathcal{D}((-A)^\delta)$ we have

$$2 \langle P A_\varepsilon x, x \rangle + \langle E x, E x \rangle + \varepsilon^{-2} \langle B^* P x, B^* P x \rangle = 0, \quad \text{where} \quad A_\varepsilon := A - \varepsilon^{-2} B B^* P,$$

and

$$I - \theta \sigma^2 D^* P D \succeq 0.$$

Lemma 4.4 shows that

$$F_\varepsilon \in \overline{\mathcal{F}} \quad \text{and} \quad \sigma \leq r^w(A - \varepsilon^{-2} B B^* P; (D, E)).$$

Since

$$2 \langle P A_\varepsilon x, x \rangle + \langle E x, E x \rangle + \varepsilon^{-2} \langle x, P B B^* P x \rangle = 2 \langle P A x, x \rangle + \langle E x, E x \rangle - \varepsilon^{-2} \langle B^* P x, B^* P x \rangle,$$

it follows that P satisfies the Riccati equation (4.30). This ends the proof. \square

The following result is another consequence of the above Theorem. Here, we establish some conditions for the existence of suboptimal controllers.

Proposition 4.2. *Let $\sigma, \varepsilon > 0$. Assume that the Riccati equation (4.30) has a solution $P \in \mathcal{L}^+(\mathcal{H})$ such that $I - \theta \sigma^2 D^* P D \succeq 0$, then*

$$F_\varepsilon \in \overline{\mathcal{F}} \quad \text{and} \quad \sigma \leq r^w(A + B F_\varepsilon; (D, E)).$$

Proof. Since P solves the Riccati equation (4.30), we see that

$$2 \langle P(A - \varepsilon^{-2} B B^* P)x, x \rangle + \langle E x, E x \rangle + \varepsilon^{-2} \langle B^* P x, B^* P x \rangle = 0, \quad x \in \mathcal{D}((-A)^\delta).$$

$$I - \theta \sigma^2 D^* P D \succeq 0.$$

By Lemma 4.4, we obtain the desired result. \square

Let us now state a corollary of Proposition 4.2 for characterizing the supremal achievable stability radius via the Riccati equation (4.30). For this purpose, we set

$$\mathcal{S}' := \left\{ \begin{array}{l} \sigma > 0; \text{ there exist } \varepsilon > 0 \text{ such that (4.30) has a solution} \\ P \in \mathcal{L}^+(\mathcal{H}) \text{ with } I - \theta\sigma^2 D^* P D \succeq 0 \end{array} \right\}.$$

Corollary 4.2. *The following inequality holds*

$$\overline{r^w}(A; (D, E)) \geq \sup \mathcal{S}'.$$

Proof. Let $\sigma \in \mathcal{S}'$. Then there exist $\varepsilon > 0$ such that the Riccati equation (4.30) has a solution $P \in \mathcal{L}^+(\mathcal{H})$ with

$$I - \theta\sigma^2 D^* P D \succeq 0.$$

By proposition 4.2,

$$\sigma \leq r^w(A + BF_\varepsilon; (D, E)) \text{ with } F_\varepsilon \in \overline{\mathcal{F}}.$$

Therefore, we have the desired result. \square

4.3.3 Examples

In this section, we present two practical examples to illustrate the above theoretical results. Denoting by δ the Dirac Delta function at zero and by δ_ξ the Dirac Delta function at ξ .

Example 4.3. *Consider the following stochastic parabolic equation with Newman boundary conditions*

$$\left\{ \begin{array}{l} \frac{\partial y}{\partial t}(x, t) = \frac{\partial^2 y}{\partial x^2}(x, t) - y(x, t) + \delta_\xi(x)u(t), \quad 0 \leq x, \xi \leq 1, \quad 0 \leq t \leq T, \\ y(x, 0) = y_0(x), \\ \frac{\partial y}{\partial x}(0, t) = -c \langle f, y(0, t) \rangle \dot{w}(t), \quad c \in \mathbb{R}, \quad f \in L^2(0, 1), \\ \frac{\partial y}{\partial x}(1, t) = 0. \end{array} \right. \quad (4.31)$$

To put the problem (4.31) into the abstract setting, we introduce the self-adjoint operator $Ah = \frac{d^2h}{dx^2} - h$ defined on the Hilbert space $\mathcal{H} = L^2(0, 1)$ with

$$\mathcal{D}(A) = \{z \in L^2(0, 1), \frac{d^2z}{dx^2} \in L^2(0, 1), \frac{dz}{dx}(0) = \frac{dz}{dx}(1) = 0\}.$$

The operator A generates an analytic semigroup $S(t)$. According to [8], the eigenvalues of A are

$$\lambda_n = -(1 + n^2\pi^2).$$

However, the corresponding eigenfunctions are

$$\phi_0 = 1 \text{ and } \phi_n(x) = \sqrt{2} \cos n\pi x \text{ for all } n \geq 1.$$

Define the following operators:

$$\begin{aligned} Dy &= -\delta y, \text{ with } D \in \mathcal{L}(\mathbb{R}, H^{-\frac{1}{2}-\beta}(0, 1)), \\ Ez &= \langle f, z \rangle, \text{ with } E \in \mathcal{L}(\mathcal{H}, \mathbb{R}), \\ \Delta &= c \in \mathbb{R}, \\ Bu &= \delta_\varepsilon u, \text{ with } B \in \mathcal{L}(\mathbb{R}, H^{-\frac{1}{2}-\beta}(0, 1)), \end{aligned}$$

The problem (4.31) can be formulated in the abstract form as follows

$$\begin{cases} \mathcal{Y}(t) = A\mathcal{Y}(t)dt + D\Delta E(\mathcal{Y}(t))dw(t) + Bu(t)dt, \\ \mathcal{Y}(0) = \mathcal{Y}_0. \end{cases} \quad (4.32)$$

In this case, we have

$$\eta = \gamma = \frac{1}{4} + \frac{\beta}{2} \text{ such that } 0 < \beta < \frac{1}{2} \text{ and } \delta = 0.$$

The corresponding Riccati equation reads

$$2 \langle A\mathcal{Y}, P\mathcal{Y} \rangle + \langle E\mathcal{Y}, E\mathcal{Y} \rangle - \varepsilon^{-2} \langle B^*P\mathcal{Y}, B^*P\mathcal{Y} \rangle = 0. \quad (4.33)$$

Expressing the solution P of (4.33) as follows:

$$P\mathcal{Y} = \sum_{n,j=0}^{+\infty} P_{nj} \langle \mathcal{Y}, \phi_n \rangle \phi_j, \quad \mathcal{Y} \in \mathcal{H},$$

and using

$$A\mathcal{Y} = \sum_{n=0}^{+\infty} \lambda_n \langle \mathcal{Y}, \phi_n \rangle \phi_n, \quad \mathcal{Y} \in \mathcal{D}(A),$$

we get

$$\langle A\mathcal{Y}, P\mathcal{Y} \rangle = \sum_{n=0}^{+\infty} \lambda_n P_{nn} (\langle \mathcal{Y}, \phi_n \rangle)^2,$$

and

$$\langle E\mathcal{Y}, E\mathcal{Y} \rangle = |\langle f, \mathcal{Y} \rangle|^2 = \left(\sum_{n=0}^{+\infty} \langle \mathcal{Y}, \phi_n \rangle \langle f, \phi_n \rangle \right)^2.$$

Also, we have

$$B^*P\mathcal{Y} = B^* \left(\sum_{n,j=0}^{+\infty} P_{nj} \langle \mathcal{Y}, \phi_n \rangle \phi_j \right) = \sqrt{2} \sum_{n,j=0}^{+\infty} P_{nj} \cos j\pi\xi \langle \mathcal{Y}, \phi_n \rangle.$$

So,

$$\langle B^*P\mathcal{Y}, B^*P\mathcal{Y} \rangle = 2 \sum_{n,j=0}^{+\infty} \sum_{m,k=0}^{+\infty} P_{nj} P_{mk} \cos j\pi\xi \cos k\pi\xi \langle \mathcal{Y}, \phi_n \rangle \langle \mathcal{Y}, \phi_m \rangle.$$

Equation (4.33) becomes

$$\begin{aligned} & 2 \sum_{n=0}^{+\infty} \lambda_n P_{nn} (\langle \mathcal{Y}, \phi_n \rangle)^2 + \left(\sum_{n=0}^{+\infty} \langle \mathcal{Y}, \phi_n \rangle \langle f, \phi_n \rangle \right)^2 \\ & - 2\varepsilon^{-2} \sum_{n,j=0}^{+\infty} \sum_{m,k=0}^{+\infty} P_{nj} P_{mk} \cos j\pi\xi \cos k\pi\xi \langle \mathcal{Y}, \phi_n \rangle \langle \mathcal{Y}, \phi_m \rangle = 0. \end{aligned}$$

For $\mathcal{Y} = \phi_k$, $k \in \mathbb{N}$, we get

$$2\lambda_k P_{kk} + (\langle f, \phi_k \rangle)^2 - 2\varepsilon^{-2} P_{kk}^2 \cos^2 k\pi\xi = 0.$$

Hence

$$P_{kk} = \frac{\lambda_k + \sqrt{\Delta'}}{2\varepsilon^{-2} \cos^2 k\pi\xi}, \quad \text{where } \Delta' := \lambda_k^2 + 2\varepsilon^{-2} \langle f, \phi_k \rangle^2 \cos^2 k\pi\xi > 0,$$

and hence

$$P\mathcal{Y} = \sum_{n=0}^{+\infty} P_n \langle \mathcal{Y}, \phi_n \rangle \phi_n, \quad \text{where } P_n := \frac{\lambda_n + \sqrt{\Delta'}}{2\varepsilon^{-2} \cos^2 n\pi\xi}.$$

Now, we show that $I - \sigma^2(\theta)D^*PD \succeq 0$ for some $\sigma > 0$. In fact,

$$I - \sigma^2(\theta)D^*PD \succeq 0$$

is equivalent to

$$1 - 2\sigma^2\theta \sum_{k=0}^{+\infty} P_k \geq 0, \quad ,$$

or

$$\sum_{k=0}^{+\infty} \frac{\lambda_k + \sqrt{\Delta'}}{\varepsilon^{-2} \cos^2 k\pi\xi} \leq \frac{1}{\theta\sigma^2}.$$

which is equivalent to

$$\sum_{k=0}^{+\infty} \frac{2\langle f, \phi_k \rangle^2}{\sqrt{\lambda_k^2 + 2\langle f, \phi_k \rangle^2 \varepsilon^{-2} \cos^2 k\pi\xi} - \lambda_k} \leq \frac{1}{\theta\sigma^2},$$

But

$$\sum_{k=0}^{+\infty} \frac{2\langle f, \phi_k \rangle^2}{\sqrt{\lambda_k^2 + 2\langle f, \phi_k \rangle^2 \varepsilon^{-2} \cos^2 k\pi\xi} - \lambda_k} \leq \sum_{k=0}^{+\infty} \frac{2\langle f, \phi_k \rangle^2}{-2\lambda_k},$$

Assume $f(x)$ is such that $f(x) > 0$ for all $0 \leq x \leq 1$. Then

$$\sum_{k=0}^{+\infty} \frac{2\langle f, \phi_k \rangle^2}{-2\lambda_k} \leq \sum_{k=0}^{+\infty} \frac{4\langle f, \phi_0 \rangle^2}{-2\lambda_k}$$

We have

$$2 \sum_{k=0}^{+\infty} \frac{\langle f, \phi_0 \rangle^2}{-\lambda_k} = 2\langle f, \phi_0 \rangle^2 \sum_{k=0}^{+\infty} \frac{1}{(1 + n^2\pi^2)} = 2.313 \langle f, \phi_0 \rangle^2$$

Let σ such that

$$\sigma^2 \leq \frac{1}{2.313\theta \langle f, \phi_0 \rangle^2}.$$

then

$$\sum_{k=0}^{+\infty} \frac{2\langle f, \phi_k \rangle^2}{-2\lambda_k} \leq \frac{1}{\theta\sigma^2},$$

therefore

$$\sum_{k=0}^{+\infty} \frac{\lambda_k + \sqrt{\Delta'}}{\varepsilon^{-2} \cos^2 k\pi\xi} \leq \frac{1}{\theta\sigma^2}.$$

Thus, for $\frac{1}{1.5209\sqrt{\theta}\langle f, \phi_0 \rangle} \geq \sigma$ the Riccati equation (4.33) has a solution $P \in \mathcal{L}^+(\mathcal{H})$ with $I - \theta\sigma^2 D^* P D \succeq 0$.

According to Corollary 4.2, we get

$$\overline{r^w}(A; (D, E)) \geq \frac{1}{1.5209\sqrt{\theta}\langle f, \phi_0 \rangle}.$$

Example 4.4. Consider the following boundary controlled heat equation on the interval $(0, L)$, $L > 0$

$$\begin{cases} \partial_t z(x, t) = \partial_{xx} z(x, t) - z(x, t) + k\delta_\xi(x) \langle g, z(x, t) \rangle \dot{w}(t), & 0 \leq x, \xi \leq L, \quad t > 0, \\ g \in L^2(0, L), \\ z(x, 0) = z_0(x), \quad 0 \leq x \leq L, \\ \partial_x z(0, t) = u(t), \quad \partial_x z(L, t) = 0, \quad t > 0. \end{cases} \quad (4.34)$$

In this case, we consider the operator $Ah = \frac{d^2 h}{dx^2} - h$ defined on the Hilbert space $\mathcal{H} = L^2(0, L)$ with

$$\mathcal{D}(A) = \{z \in H^2(0, L), \frac{dz}{dx}(0) = \frac{dz}{dx}(L) = 0\}.$$

The operator A generates an analytic semigroup $S(t)$, the eigenvalues of A are

$$\lambda_n = \frac{-n^2 \pi^2}{L^2} - 1, \quad n \geq 0.$$

In addition, the corresponding eigenfunctions are

$$\phi_0 = \frac{1}{\sqrt{L}} \quad \text{and} \quad \phi_n(x) = \sqrt{\frac{2}{L}} \cos \frac{n\pi x}{L}, \quad \text{for all } n \geq 1.$$

Define the following operators:

$$\begin{aligned} Dz &= \delta_\xi z, \quad \text{with } D \in \mathcal{L}(\mathbb{R}, H^{-\frac{1}{2}-\beta}(0, L)), \\ Ez &= \langle g, z \rangle, \quad \text{with } E \in \mathcal{L}(\mathcal{H}, \mathbb{R}), \\ \Delta &= k \in \mathbb{R}, \\ B^* \psi(x) &= -\psi(0), \quad \text{with } \psi \in \mathcal{D}(B^*). \end{aligned}$$

The problem (4.34) takes the following abstract form

$$\begin{cases} dZ(t) = AZ(t)dt + D\Delta E(Z(t))dw(t) + Bu(t)dt, \\ Z(0) = Z_0. \end{cases} \quad (4.35)$$

For this system we have

$$\eta = \gamma = \frac{1}{4} + \frac{\beta}{2}, \quad 0 < \beta < \frac{1}{2} \quad \text{and} \quad \delta = 0.$$

Note that the corresponding Riccati equation is

$$2 \langle AZ, PZ \rangle + \langle EZ, EZ \rangle - \varepsilon^{-2} \langle B^* PZ, B^* PZ \rangle = 0, \quad Z \in \mathcal{D}(A) \quad (4.36)$$

Assume that we can express the solution P of (4.36) as

$$PZ = \sum_{n,j=0}^{+\infty} P_{nj} \langle Z, \phi_n \rangle \phi_j, \quad Z \in \mathcal{H}.$$

Hence

$$AZ = \sum_{n=0}^{+\infty} \lambda_n \langle Z, \phi_n \rangle \phi_n, \quad \text{for } Z \in \mathcal{D}(A),$$

and hence

$$\langle AZ, PZ \rangle = \sum_{n=0}^{+\infty} \lambda_n P_{nn} (\langle Z, \phi_n \rangle)^2.$$

Since

$$\begin{aligned} \langle EZ, EZ \rangle &= \langle g, Z \rangle^2, \\ B^* PZ &= B^* \left(\sum_{n,j=0}^{+\infty} P_{nj} \langle Z, \phi_n \rangle \phi_j \right) \\ &= -\sqrt{\frac{2}{L}} \left(\sum_{n,j=0}^{+\infty} P_{nj} \langle Z, \phi_n \rangle \right), \end{aligned}$$

Equation (4.36) takes the form

$$2 \sum_{n=0}^{+\infty} \lambda_n P_{nn} (\langle Z, \phi_n \rangle)^2 + \langle g, Z \rangle^2 - \frac{2\varepsilon^{-2}}{L} \sum_{n,k=0}^{+\infty} \sum_{m,j=0}^{+\infty} P_{nk} P_{mj} \langle Z, \phi_n \rangle \langle Z, \phi_m \rangle = 0.$$

For $Z = \phi_k$, $k \in \mathbb{N}$, we obtain

$$2\lambda_k P_{kk} + (\langle g, \phi_k \rangle)^2 - \frac{2\varepsilon^{-2}}{L} P_{kk}^2 = 0.$$

So,

$$P_{kk} = \frac{L(\lambda_k + \sqrt{\Delta'})}{2\varepsilon^{-2}}, \quad \text{where } \Delta' = \lambda_k^2 + \frac{2\varepsilon^{-2} \langle g, \phi_k \rangle^2}{L} > 0,$$

and so

$$PZ = \sum_{n=0}^{+\infty} P_n \langle Z, \phi_n \rangle \phi_n,$$

where

$$P_n = \frac{L(\lambda_n + \sqrt{\Delta'})}{2\varepsilon^{-2}} \quad \text{and} \quad \Delta' = \lambda_n^2 + \frac{2\varepsilon^{-2} \langle g, \phi_n \rangle^2}{L}.$$

Now, we show that $I - \sigma^2(\theta)D^*PD \succeq 0$ for some $\sigma > 0$.

We have

$$I - \sigma^2(\theta)D^*PD \succeq 0$$

is equivalent to,

$$1 - \frac{2}{L}\sigma^2\theta \sum_{n=0}^{+\infty} \cos^2\left(\frac{n\pi\xi}{L}\right)P_n \geq 0,$$

or,

$$\frac{L}{2\theta\sigma^2} \geq \sum_{n=0}^{+\infty} \frac{L(\lambda_n + \sqrt{\Delta'}) \cos^2\left(\frac{n\pi\xi}{L}\right)}{2\varepsilon^{-2}},$$

We have

$$\frac{2\varepsilon^{-2} \langle g, \phi_n \rangle^2}{L(\sqrt{\lambda_n^2 + \frac{2\varepsilon^{-2} \langle g, \phi_n \rangle^2}{L}} - \lambda_n)} = \lambda_n + \sqrt{\lambda_n^2 + \frac{2\varepsilon^{-2} \langle g, \phi_n \rangle^2}{L}}, \quad \text{for all } n \in \mathbb{N},$$

Since

$$\frac{\langle g, \phi_n \rangle^2}{\sqrt{\lambda_n^2 + \frac{2\varepsilon^{-2} \langle g, \phi_n \rangle^2}{L}} - \lambda_n} \leq \frac{\langle g, \phi_n \rangle^2}{-2\lambda_n}, \quad \text{for all } n \in \mathbb{N},$$

it follows that

$$\lambda_n + \sqrt{\lambda_n^2 + \frac{2\varepsilon^{-2} \langle g, \phi_n \rangle^2}{L}} \leq \frac{2\varepsilon^{-2} \langle g, \phi_n \rangle^2}{-2L\lambda_n},$$

thus

$$\sum_{n=0}^{+\infty} \frac{L(\lambda_n + \sqrt{\Delta'}) \cos^2\left(\frac{n\pi\xi}{L}\right)}{2\varepsilon^{-2}} \leq \sum_{n=0}^{+\infty} \frac{\langle g, \phi_n \rangle^2 \cos^2\left(\frac{n\pi\xi}{L}\right)}{-2\lambda_n}$$

Set

$$S = \sum_{n=0}^{+\infty} \frac{1}{-\lambda_n} = \sum_{n=0}^{+\infty} \frac{L^2}{n^2\pi^2 + L^2} = \frac{1}{2} + \frac{L \coth(L)}{2}$$

Let σ such that

$$\sigma \leq \sqrt{\frac{L}{2 \langle g, \phi_0 \rangle^2 \theta S}},$$

Assume $g(x)$ is such that $g(x) > 0$ for all $0 \leq x \leq L$. We have

$$\sum_{n=0}^{+\infty} \frac{\langle g, \phi_n \rangle^2 \theta \cos^2(\frac{n\pi\xi}{L})}{-2\lambda_n L} \leq \sum_{n=0}^{+\infty} \frac{2 \langle g, \phi_0 \rangle^2 \theta}{-2\lambda_n L} \leq \frac{1}{2\sigma^2},$$

and hence

$$\frac{1}{2\sigma^2} \geq \sum_{n=0}^{+\infty} \frac{\langle g, \phi_n \rangle^2 \theta \cos^2(\frac{n\pi\xi}{L})}{-2\lambda_n L},$$

Or equivalently,

$$\frac{L}{2\sigma^2\theta} \geq \sum_{n=0}^{+\infty} \frac{\langle g, \phi_n \rangle^2 \cos^2(\frac{n\pi\xi}{L})}{-2\lambda_n},$$

hence that

$$\frac{1}{2\sigma^2\theta} \geq \frac{1}{2\varepsilon^{-2}} \sum_{n=0}^{+\infty} \cos^2(\frac{n\pi\xi}{L})(\lambda_n + \sqrt{\Delta'}),$$

and finally that

$$1 - \frac{2}{L}\sigma^2\theta \sum_{n=0}^{+\infty} \cos^2(\frac{n\pi\xi}{L})P_n \geq 0, \quad .$$

Consequently, if

$$\sigma \leq \sqrt{\frac{L}{2 \langle g, \phi_0 \rangle^2 \theta S}},$$

then the Riccati equation (4.36) has a solution $P \in \mathcal{L}^+(\mathcal{H})$. Moreover, $I - \theta\sigma^2 D^* P D \succeq 0$.

According to Corollary 4.2, we get

$$\bar{r}^w(A; (D, E)) \geq \sqrt{\frac{L}{2 \langle g, \phi_0 \rangle^2 \theta S}}.$$

CONCLUSION

In this thesis, we have addressed the problem of robust stabilization for a class of continuous-time infinite dimensional linear systems subjected to stochastic perturbations. As a robustness metric, we used the concept of stability radius.

At first, an infinite-dimensional system subjected to stochastic structured bounded perturbations is explored with the goal of maximizing the stability radius using a dynamic compensator. Operator inequalities are used to express the necessary and sufficient conditions for the existence of a suboptimal compensator.

The second objective of the thesis is to study the robustness stabilization problem for systems with unbounded structured stochastic multi-perturbations. This case is more important in the applications because it covers the case of partial differential equations with boundary and pointwise noise. Under additional assumptions, we have extended the results of [41] to the unbounded case. We characterized the stability radius in terms of a Lyapunov equation and the supremal achievable stability radius in terms of a Riccati equation.

Finally, we have investigated the problem of maximizing the stability radius when the system is subjected to bounded or unbounded structured perturbations, with unbounded input operator. We established some conditions for the existence of suboptimal controllers. These conditions are obtained in terms of a Riccati equation which satisfies an operator inequality. The supremal achievable stability radius is characterized via the Riccati equation.

PERSPECTIVES

We recall here some possible directions for future investigation.

Most of the results in this thesis as well as in other papers on stabilization of stochastic equations have been proved for parabolic equations. Hence, it is quite restrictive for many applications.

Our future work will be focused on investigating robust stability problems for the class of stochastic well posed systems introduced in [48].

We are interested by the generalization of our results to a more general setting. More specifically, for partial differential equations with deterministic uncertainties and boundary noise.

Another possible direction for future work is to provide simulations to illustrate the effectiveness of the designed controllers.

While for time invariant systems, most of the robustness problems are well understood, many problems for time-varying systems are still open. These are interesting and challenging topics for future work.

Some real world problems in science and engineering can be modeled by stochastic differential equations driven by fractional Brownian motion and it will be interesting to investigate the stability radius problem for such systems.

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