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Pour obtenir le titre de

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Présentée par

Yahiaoui Leila

Rayon de stabilité des systèmes discrets avec des perturbations stochastiques et leur optimisation par un retour d'état

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Soutenue le 12 / 07 / 2023

Jury :

Président:	Rebiai Salah Eddine	Professeur	Université de Batna 2
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UNIVERSITY OF BATNA 2
Faculty of Mathematics and Computer Science
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THESIS

Presented for the degree of

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Yahiaoui Leila

Stability radius of discrete-time systems with stochastic perturbations and their optimization by state feedback

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Abstract

The principal objective of this dissertation is to examine the robustness stability and robustness stabilization problems for discrete-time systems on real Hilbert space. First, we consider invariant systems which are subjected to stochastic multi-perturbations. We research some features of the radius of stability through Lyapunov equations and the matching inequalities. These properties are utilized to compute the stability radius.

Afterwards, we treat the optimization of the stability radius based on state feedback. Necessary and sufficient conditions are proved for stabilizing the perturbed system by feedback with norm less than a given bound. These are in terms of a discrete-time Riccati equation. We show how we can acquire the expression of the supremal stability radii based on this equation.

Ultimately, we study the stability radius of discrete-time varying systems. We generalize some obtained consequences for invariant systems to the variant one. We derive bounds of the radius via the corresponding Lyapunov equation. in the end, we focus to apply the accomplished results to give conditions of the stability for periodic perturbed systems.

keywords: Discrete-time Lyapunov equation, Discrete-time Riccati equation, Robust stabilization, Stability radius, Stochastic perturbations.

Résumé

Le but fondamental de cette thèse est d'étudier les problèmes de stabilité robuste et de stabilisation robuste pour les systèmes à temps discret dans un espace de Hilbert réel. Tout d'abord, nous considérons des systèmes à temps invariant qui sont soumis à des multi-perturbations stochastiques. Nous établissons quelques caractérisations du rayon de stabilité en termes d'équations de Lyapunov et des inégalités correspondantes. Ces caractérisations sont utilisées pour donner une formule de calcul pour le rayon.

Ensuite, nous étudions l'optimisation du rayon de stabilité par retour d'état. Des conditions nécessaires et suffisantes sont établis pour l'existence d'un contrôleur stabilisateur qui réduit la norme de l'opérateur de perturbation en boucle fermée à un niveau inférieur à un seuil donné. Ces conditions sont en termes d'une équation discrète de Riccati. Nous montrons comment le suprême des rayons de stabilité peut être donné en fonction de cette équation.

Enfin, nous étudions le rayon de stabilité des systèmes discrets à temps variant. Nous avons montré comment on peut généraliser les résultats obtenus dans le cas invariant au cas variant. Nous établissons des bornes du rayon de stabilité en fonction de l'équation de Lyapunov. Les résultats sont appliqués pour obtenir des conditions de stabilité pour des systèmes périodiques.

Mots clés: Equations de Lyapunov à temps discret, Equation de Riccati à temps discret, Stabilisation robuste, Rayon de stabilité, Perturbations stochastiques.

الملخص

الهدف الرئيسي من هذه الاطروحة هو دراسة الإستقرار وصلابة الإستقرار لأنظمة الوقت المتقطع في فضاء هلبرت حقيقي.

أولا ، نعتبر الأنظمة ذات الوقت المتقطع حيث مؤثرالجملة لا يتغير بالزمن والتي تخضع لاضطرابات عشوائية متعددة. ندرس بعض خصائص نصف قطر الاستقرار باستعمال معادلات ومتراجحات ليابونوف، تستخدم هذه الخصائص لإعطاء صيغة حسابية لنصف القطر.

بعد ذلك ، ندرس تعظيم نصف قطر الإستقرار. نعطي شروط لازمة وكافية لوجود وحدة تحكم استقرار تقلل من نظيم مؤثر الحلقة المغلقة إلى مستوى أقل من عتبة معينة. هذه الشروط تعتمد على معادلة ريكاتي المتقطعة. نوضح كيف يمكن إعطاء نصف قطر الاستقرارالأعظمي وفقا لهذه المعادلة.

أخيرا ، ندرس نصف قطر استقرار الأنظمة ذات الوقت المتقطع و الزمن المتغير. لقد أظهرنا كيف يمكننا تعميم النتائج التي تم الحصول عليها في الحالة الثابتة الى الحالة المتغيرة. نستحدث حدود لنصف قطرالإستقرار بدلالة معادلة ليابونوف المتقطعة ذات الوقت المتغير. وفي الأخير يتم تطبيق النتائج للحصول على شروط الإستقرار لأنظمة الوقت المتقطع الدورية المضطربة.

الكلمات المفتاحية: معادلات ليابونوف في الوقت المتقطع، معادلات ريكاتي في الوقت المتقطع، صلابة الاستقرار، نصف قطر الاستقرار، الاضطرابات العشوائية.

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I am immensely grateful to all of you.

Contents

List of Symbols and Notations

$\mathcal{Z},\ U,\ V$	Real separable Hilbert spaces.
$L(U,\mathcal{Z})$	The space of bounded linear operator from U to $\mathcal{Z}$.
$L(\mathcal{Z})$	The space of bounded operators from $\mathcal{Z}$ to $\mathcal{Z}$.
$\langle .,. \rangle$	The inner product in $\mathcal{Z}$.
$\Vert . \Vert$	The norm in $\mathcal{Z}$.
$P \succ 0$	$P \in L(\mathcal{Z})$ is positive ($\langle Px, x\rangle > 0$ for all $x \in \mathcal{Z}$).
$P \succcurlyeq 0$	$P \in L(\mathcal{Z})$ non negative ($\langle Px, x\rangle \geqslant 0$ for all $x \in \mathcal{Z}$.
$P_1 \succcurlyeq P_2$	$P_1 - P_2 \succcurlyeq 0$.
B^*	The adjoint of the operator B.
$L^+(Z)$	The set of self-adjoint linear bounded operators $P \in L(Z)$ such that $P \succcurlyeq 0$.
$L^2_n(\mathcal{Z}) = L^2(\Omega, F_n, P, \mathcal{Z})$	The space of all equivalence classes of $\mathcal{Z}-$ valued random variables.
$l^2_\omega = l^2_\omega(\mathbb{N}, L^2(\Omega, Y)$	The space of all sequences $z(t) = (z(t))_{t\in\mathbb{N}}$ such that $z(t) \in L^2(\Omega, \mathcal{Z})$ is F_{t-1} measurable for all $t \in \mathbb{N}$.
$\mathbb{E}(x)$	The expectation of x.
$L^\infty(\mathbb{N}, L(U,\mathcal{Z}))$	The set of strongly measurable functions $h : \mathbb{N} \longrightarrow L(U,\mathcal{Z})$ such that $\Vert h(.)\Vert_\infty = \sup_{t\in\mathbb{N}} \Vert h(t)\Vert_{L(U,\mathcal{Z})} < +\infty$.

Introduction

Stochastic systems play a huge part in an immense field of applications containing mechanics, economics and finance, and have been vastly investigated over the last centuries (see [1], [39], [4]). Stochastic discrete-time systems are one of key themes in control theory which has enjoyed a considerable amount of scientific interest (see [5, 9, 33, 37, 40]).
In several cases, real-world events are influenced by random agents that practiced a crucial impact in the progressive process. Characteristics of discrete-time linear systems influenced by independent random perturbations and their utilization have attracted a great attention during the past century (see for e.g. [3], [16] and the references therein).

Robustness has been a basic topic in control theory. It indicates that a system must be ever capable to conserve stability and level of effectiveness with uncertainties in systems dynamics and in conditions of work. Robustness has extremely importance in control systems because real engineering systems are always affected by external disturbance and noise measure and there are always difference between models in mathematics and real models.
Problems of robustness stability and robustness stabilization have received great attention recently. One of the most suitable approach for those problems is based on the stability radius approach. This method was introduced in [12] for finite-dimensional systems and extended to other system classes and kinds of perturbations (see [14, 3, 10, 19, 20, 29]).

Pritchard and El Bouhtouri [6] presented and distinguished the stability radius for linear systems with respect to Lipschitzian structured stochastic perturbations. The relevant conclusions for discrete-time systems were treated by El Bouhtouri [7]. For infinite dimensional systems, Wirth [35] extended the concept of the stability radius for discrete-time systems discomfited to structured deterministic perturbations in infinite dimension. The great benefit of this research is that adopt notions and methods which can be extended to several classes of systems and several categories of perturbations.

For state space systems of the form $\dot{x}(t) = Ax(t)$ or $x(t+1) = Tx(t),\ T \in \mathcal{L}(\mathbb{K}^n),\ \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, linear perturbations of the form

1. $T + \Lambda,\ \Lambda \in \mathcal{L}(\mathbb{K}^n)$ unstructured perturbations.
2. $T + D\Lambda E,\ \Lambda \in \mathcal{L}(\mathbb{K}^q,\ \mathbb{K}^l)$ simple stuctured perturbations.
3. $T + \sum_{i=1}^{N} D_i \Lambda_i E_i,\ \Lambda \in \mathcal{L}(\mathbb{K}^q,\ \mathbb{K}^l)$ multi-perturbations.

have been considered, where $D,\ E,\ D_i,\ E_i$ are given operators defining the perturbations structure.
In this dissertation, we treat the concepts of robustness stability and stabilization

for discrete-time systems under stochastic structured multi-perturbations in Hilbert space. We follow the approach used in [17] to investigate these problems.
We consider discrete time invariant systems in infinite dimensions described by linear difference equation of the form $x(t+1) = Tx(t),\ t \in \mathbb{N}$, subjected to N structured stochastic perturbations. Our goal is to define and calculate the corresponding stability radius, which refers to the minimum complex or real perturbation Λ_i for which the perturbed system $x(t+1) = Tx(t) + \sum_{i=1}^{N} D_i\Lambda_i(E_i(x(t))w_i(t)$ is unstable. $D_i,\ E_i$ are given linear bounded operators defining the structure of the perturbations and $w_i(t),\ t \in \mathbb{N}$, are real independent random variables.

Our second objective is to discuss the optimization of the radius of stability by state feedback. We give the characteristics of the supremum of the stability radii that can be reached by state feedback, via parametrized Riccati equation.

Time-varying systems are one of the most difficult types of systems in mathematics that can be used to model more realistic systems. In practical applications, time-varying systems serve as valuable tools for modeling various dynamical systems that exhibit time-varying characteristics. These systems are capable of describing a wide range of phenomena, including periodic systems, sampled-data systems, switched systems, and more.

Recently, considerable attention has been addressed to time-varying discrete-time systems (cf. [2, 25, 28, 40, 41]). Our last objective in this thesis is to extend the robust stability conclusions to the time-varying case. Based on the results of time-varying Lyapunov equations, we establish some characterizations of the stability radius to infinite dimensional discrete-time varying systems with stochastic perturbations. In addition, we provide bounds for the stability radius of periodic systems.

This thesis is organized as follows.

Chapter 1: Necessary background and necessary mathematical preliminaries regarding the most important concepts used in this thesis are given in this chapter. We collect some conceptions of discrete-time systems in Hilbert space, exponential stability, stabilizability, random variables, stochastic discrete-time systems, and exponential stochastic stability.

Chapter 2: First, we introduce the description of the system and the stability radius definition. Robustness stability requirements for the uncertain system are examined through a discrete-time Lyapunov equation. The requirements are used to establish a computing expression for the stability radius. Two examples are presented to clarify the applicability of the suggested methodology.

Chapter 3: We examin controlled stochastic systems given by

$$\begin{aligned} x(t+1) &= Tx(t) + \sum_{i=1}^{N} D_i \Lambda_i (E_i(x(t)) w_i(t) + Bu(t), \ t \in \mathbb{N} \\ x(t_0) &= x_0 \end{aligned}$$

We generate results on suboptimal controllers. Under these results we give characterizations of the supreme reached stability radius through a Riccati equation in discrete-time. The chapter is ended with an example and a numerical application.

Chapter 4: This chapter is devoted to the robustness stability problem in the time variant case. We extend the stability radius definition for time-varying systems subjected to stochastic perturbations. We establish new results for the corresponding stability radii. We develop the obtained results for periodic systems. We end up with an illustrative example.

1 Fundamental concepts

Contents

1.1 Introduction

In this chapter, we give some foundation definitions and features, which will be employed throughout this dissertation. First, we recall some basic notions concerning discrete-time systems in Hilbert space and the corresponding notions of stability. Since our goal is to review the stability of stochastic discrete-time systems we collect some special concepts about stochastic processes and random variables. We end by recalling important results of stochastic discrete-time systems and we give the corresponding stability theorems.

1.2 Discrete-time invariant systems in Hilbert space

Let $\mathcal{Z}$ be a Hilbert space over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We consider time-invariant systems of the form

$$\begin{cases} x(t+1) = Tx(t),\ t \in \mathbb{N}, \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

where T is a bounded linear operator on $\mathcal{Z}$. System (1.1) is called a discrete-time invariant system in infinite dimension, its solution is written as

$$x(t) = T^{t-t_0}x_0,\ t \geqslant t_0.$$

Definition 1.1. *[27] We say that the system (1.1) is exponentially stable if and only if there exist $c \geqslant 1$ and $0 < a < 1$ such that*

$$\|T^{t-t_0}\| \leqslant ca^{t-t_0},\ t \in \mathbb{N},\ t \geqslant t_0.$$

For $T \in \mathcal{L}(\mathcal{Z})$ we define its spectrum by:

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}$$

and its spectral radius by $\rho(T) = \max\{|\lambda|, \lambda \in \sigma(T)\}$.

Lemma 1.1. *[23]*

The next statements are equivalent:

1. *(1.1) is exponentially stable;*

2. $\rho(T) < 1$*;*

3. *For every positive operator* $Q \in \mathcal{L}^+(\mathcal{Z})$*, there exists a unique positive operator* $X \in \mathcal{L}^+(\mathcal{Z})$ *which solves the Lyapunov equation*

$$T^*XT - X + Q = 0, \tag{1.2}$$

and it is provided by

$$X = \sum_{t=t_0}^{+\infty} (T^*)^{t-t_0} Q\mathcal{T}^{t-t_0};$$

4. *There exist* $\beta > 0$ *and* $a \in (0,1)$ *such that*

$$\left\|T^{t-t_0}\right\|^2 \leqslant \beta a^{t-t_0}, \ \ t \geqslant t_0.$$

Proposition 1.1. *[24] Let* $T \in \mathcal{L}(\mathcal{Z}, \mathbb{K})$*. We have the equivalently of succeeding assertions:*

1. $\rho(T) < 1$.

2. $\|T^n\| \leqslant \beta a^n$*, for every* $n \geqslant 0$*, for some* $\beta \geqslant 1$ *and* $\alpha \in (0,1)$*.*

3. $\sum\limits_{n=0}^{\infty} \|T^n\|^p < \infty, \ \forall p > 0.$

4. $\sum\limits_{n=0}^{\infty} \|T^n x\|^p < \infty, \ \forall x \in \mathcal{Z}, \ \forall p > 0.$

Consider the discrete-time system with input u and output y given by

$$\begin{cases} x_{n+1} = Tx_n + Bu_n, \ t \in \mathbb{N}, \\ y_n = Cx_n + Du_n, \\ x(0) = x_0, \end{cases} \tag{1.3}$$

where $T, B, \ C, \ D$ belongs to $\mathcal{L}(\mathcal{Z}), \ \mathcal{L}(U, \mathcal{Z}), \ \mathcal{L}(\mathcal{Z}, Y), \ \mathcal{L}(U, Y)$, respectively. The Riccati equation (DARE) related to the system (1.3) is

$$T^*XT - X + CC^* = (C^*D + T^*XB)(S + B^*XB)^{-1}(D^*C + B^*X\mathcal{T}) \tag{1.4}$$

where $S = I + D^*D$.

Definition 1.2. *[26] The system (1.3) is stabilizable if there exist* $F \in \mathcal{L}(\mathcal{Z}, U)$ *so that* $(T + BF)$ *be exponentially stable.*

Theorem 1.1. *[26] We have the equivalently of the expressions bellow :*

1. *The system (1.3)is stabilizable.*

2. *The DARE (1.4) has a non-negative self-adjoint solution.*

1.3 Discrete-time varying systems in infinite dimension

Let $\mathcal{Z}$ be a Hilbert space over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We consider time varying systems of the form

$$\begin{cases} x(t+1) = T(t)x(t), \ t \in \mathbb{N}, \\ x(t_0) = x_0, \end{cases} \tag{1.5}$$

where $T(.) = (T(t))_{t\in\mathbb{N}}$ is a sequence of bounded linear operators on $\mathcal{Z}$.
The evolution operator related to this system is defined by

$$\begin{cases} \phi(t,t) = I_{\mathcal{Z}}, \\ \phi(t,s) = T(t-1)T(t-2)...T(s), \end{cases} \quad s, t \in \mathbb{N}, \ t > s \tag{1.6}$$

System (1.5) is a discrete-time varying system in infinite dimension, its solution is given by

$$x(t) = \phi(t, t_0)x_0, \ t \in \mathbb{N}, \ t \geqslant t_0,$$

see [36].

Definition 1.3. *[36] The system (1.5) is called*

1. *Stable if for every* $\varepsilon > 0$ *and* $t_0 \in \mathbb{N}$, *there exists* $\sigma = \sigma(\varepsilon, t_0) > 0$ *such that*

$$\|x_0\| < \sigma \Longrightarrow \|\phi(t, t_0)x_0\| < \varepsilon \ \text{ for all } t \geqslant t_0.$$

2. *Exponentially stable if there exist constants* $a, \beta > 0$ *such that*

$$\|\phi(t,k)\|_{\mathcal{L}(X)} \leqslant ae^{-\beta(t-k)}, k, t \in \mathbb{N}, \ t \geqslant k.$$

Assume the system designated by

$$x(l+1) = T(s+l)x(l),\ x(0) = x \in X, \tag{1.7}$$

whose solution is

$$x(l) = \phi(s+l,k)x,\ \forall l \geqslant 0, \tag{1.8}$$

where $\phi(s+l,s),\ s,l \in \mathbb{N}$, is the evolution operator linked with the sequence of operators $(T(k):\ k \in \mathbb{N})$ in $\mathcal{L}(\mathcal{Z})$. Then we have the theorem bellow.

Theorem 1.2. *[22] The statements bellow are equivalent.*

T) $\lim\limits_{l\longrightarrow\infty} \sup\limits_{s>0} \|\phi(s+l,k)\| = 0.$

B) $\sup\limits_{s>0} \|\phi(s+l,s)\| < 1,\ \forall l \geqslant l_1$, *for some* $l_1 \geqslant 1$.

C) $\sup\limits_{s>0} \|T(s)\| < \infty$, *and* $\sup\limits_{s>0} \|\phi(s+l_0,s)\| < 1$, *for some* $l_0 \geqslant 1$.

D) $\limsup\limits_{n\longrightarrow\infty} \sup\limits_{s>0} \|\phi(s+l,s)\|^{\frac{1}{n}} < 1.$

E) There exist real constants $\beta \geqslant 1$, $\alpha \in (0,1)$ *such that*

$$\|\phi(s+l,s)\| \leqslant \beta\alpha^l,\ \forall k,l \geqslant 0.$$

F) For every $p > 0$, *there exists a positive number* σ_p *such that*

$$\sum_{l=0}^{\infty} \|\phi(s+l,s)\|^p \leqslant \sigma_p, \quad \forall k \geqslant 0.$$

G) For every $p > 0$, *there exists a positive number* σ_p *such that*

$$\sum_{l=0}^{\infty} \|\phi(s+l,s)x\|^p \leqslant \sigma_p \|x\|^p, \quad \forall k \geqslant 0,\ \forall x \in \mathcal{Z}.$$

H) For some $q > 0$, *there exists a positive number* σ_q *such that*

$$\sum_{l=0}^{\infty} \|\phi(s+l,s)\|^q \leqslant \sigma_q, \quad \forall k \geqslant 0.$$

I) For some $q > 0$, there exists a positive number σ_q such that

$$\sum_{l=0}^{\infty} \|\phi(s+l,s)x\|^q \leqslant \sigma_q \|x\|^q, \quad \forall k \geqslant 0,\ \forall x \in \mathcal{Z}.$$

J) For every $p > 0$, there exists a positive number μ_p such that

$$\|\phi(s+l,s)\|^p \leqslant (l+1)^{-1}\mu_p, \quad \forall s, l \geqslant 0.$$

K) For some $q > 0$, there exists a positive number μ_p such that

$$\|\phi(s+l,s)\|^p \leqslant (l+1)^{-1}\mu_p, \quad \forall k, l \geqslant 0.$$

L) For every $p > 0$, there exists a positive number ρ_p such that

$$\sum_{m=0}^{l} \|\phi(s+l,s+m)\|^p \leqslant \rho_p, \quad \forall s, l \geqslant 0.$$

M) For some $q > 0$, there exists a positive number ρ_p such that

$$\sum_{m=0}^{l} \|\phi(s+l,s+m)\|^q \leqslant \rho_p, \quad \forall s, l \geqslant 0$$

N) $\lim\limits_{n\longrightarrow\infty} \sup\limits_{s>0} \sup\limits_{v>0} \| \sum\limits_{l=n}^{n+v} \phi(s+l,s)\| = 0.$

As a consequence we get the next result.

Corollary 1.1. *The statements bellow are equivalent*

1. *There exist real constants $\beta \geqslant 1$, $\alpha \in (0,1)$ such that*

$$\|\phi(t,0)\| \leqslant \beta\alpha^t,\ \forall t \geqslant 0.$$

2. *For every $p > 0$, there exists a positive number σ_p such that*

$$\sum_{t=0}^{\infty} \|\phi(t,0)\|^p \leqslant \sigma_p.$$

3. *For every $p > 0$, there exists a positive number σ_p such that*

$$\sum_{t=0}^{\infty} \|\phi(t,0)x\|^p \leqslant \sigma_p \|x\|^p, \quad \forall x \in \mathcal{Z}.$$

1.3.1 Discrete-time periodic systems

An important subclass of time varying linear systems is given by systems that are periodic with period $p \in \mathbb{N}$.
We say that (1.5) is a periodic system with a period p if

$$T(t+p) = T(t).$$

Suppose the associated time invariant system

$$x(t+1) = \phi(p,0)x(t), \ t \in \mathbf{N}. \tag{1.9}$$

Suppose that (1.5) is a periodic system. Then we have the theorem bellow.

Theorem 1.3. *(i) (1.5) is exponential stable iff (1.9) is exponential stable.*

(ii) (1.5) is asymptotically stable iff (1.9) is asymptotically stable.

1.4 Stochastic discrete-time systems in Hilbert space

1.4.1 Stochastic processes

In this part, let Ω is a nonempty set and Σ a class of subsets of Ω.

Definition 1.4. *(σ-algebra)([1], [11]) A class of sets Σ is named a σ-algebra if it fulfills the following requirements :*

- $\Omega \in \Sigma$,
- *If A belongs to Σ is then the complement of A belongs to Σ*
- *If $(A_n)_{n\in\mathbb{N}} \in \Sigma$ then the countable union of A_n belongs to Σ .*

Definition 1.5. *[11]*
Let $(\Omega, \ \Sigma)$ is a measurable space. A function μ from Σ to $\mathbb{R}$ is named a measure if it satisfies the requirements :

- $\mu(A) \geqslant 0$, *for any $A \in \Sigma$.*
- $\mu(\phi) = 0$.

- *countable σ-additivity.*

Definition 1.6. *(Measure space)[11]*
A triplet $(\Omega,\ \Sigma,\ \mu)$ is a measure space if $(\Omega,\ \Sigma)$ is a measurable space and $\mu : \Sigma \longrightarrow [0, \infty)$ is a measure.

- *If $\mu(\Omega) = 1$, then μ is a probability measure, which we generally use notation $\mathbb{P}$, and the measure space is a probability space.*

- *If $(\Omega,\ \Sigma,\ \mathbb{P})$ is a probability space, we set*

$$\overline{\Sigma} = \{A \subset \Omega :\ \exists B,\ C \in \Sigma;\ B \subset A \subset C,\ \mathbb{P}(B) = \mathbb{P}(C)\}.$$

  $\overline{\Sigma}$ is a σ-filed, named the completion of Σ.

- *If $\overline{\Sigma} = \Sigma$ then the probability space $(\Omega,\ \Sigma,\ \mathbb{P})$ is told to be complete.*

Definition 1.7. *[1] If the map $x : \Omega \longrightarrow \mathcal{Z}$ which is measurable from $(\Omega,\ F)$ to $(\mathcal{Z},\ \mathcal{B}(\mathcal{Z}))$, we say that x is an $\mathcal{Z}$-valued random variable.*

Definition 1.8. *The sequence of $\mathcal{Z}$ valued random variables x_n is said to converge to a random variable x in mean square if*

$$\mathbb{E}\left(\|x_n - x\|^2\right) \longrightarrow 0 \text{ as } n \longrightarrow +\infty$$

1.4.2 Linear discrete-time stochastic systems

This section is mainly based on [31]. Suppose that $\mathcal{Z}$ is a real separable Hilbert space. Assume the following system:

$$\begin{cases} x_{n+1} = T_n x_n + \xi_n B_n x_n,\ n \in \mathbb{N}, \\ x_s = x,\ s \in \mathbb{N}. \end{cases} \tag{1.10}$$

where $T_n,\ B_n \in \mathcal{L}(\mathcal{Z})$ are bounded linear operator and ξ_n are real independent random variables such that $\mathbb{E}(\xi_n) = 0$ and $\mathbb{E}|\xi_n|^2 = \lambda_n < \infty$ for all $n \in \mathbb{N}$.
We symbolize by $L^2_n(\mathcal{Z}) = L^2(\Omega, F_n, \mathbb{P}, \mathcal{Z})$ the space of all equivalence classes of $\mathcal{Z}$-valued random variables $\eta : (\Omega, F_n) \longrightarrow (\mathcal{Z}, \mathcal{B}(\mathcal{Z}))$ such that $\mathbb{E}\|\eta\|^2 < \infty$.
We symbolize by $\phi(n, s),\ n \geqslant s \geqslant 0$, the random evolution operator given by the linear system (1.10) i.e $\phi(s, s) = I$ and

$$\phi(n, s) = (T_{n-1} + \xi_{n-1} B_{n-1})...(T_s + \xi_s B_k), \quad \text{for all } n > s$$

If $x_n = x_n(k, x)$ is the solution of the system (1.10) then it is unique and

$$x_n(s, x) = \phi(n, s)x.$$

Let $\mathcal{H}$ be the subspace of $\mathcal{L}(\mathcal{Z})$ includes all self-adjoint operators. If $S \in \mathcal{H}$ we design the operators $\mathcal{G}_n, \mathcal{T}(n,\ s) : \mathcal{H} \to \mathcal{H}$ as follows:

$$\mathcal{G}_n(S) = T_n^* S T_n + \lambda_n B_n^* S B_n. \tag{1.11}$$

$$\begin{cases} \mathcal{T}(n, s)(S) = \mathcal{G}_s(\mathcal{G}_{s+1}(...(\mathcal{G}_{n-1}(S)))), \text{ for all } \ n-1 \geqslant s \\ \mathcal{T}(s, s)(S) = S, \end{cases} \tag{1.12}$$

$\mathcal{G}_n$ and $\mathcal{T}(n, s)$ are linear bounded operators.

Definition 1.9. *[31] The system (1.10) is uniformly exponential stable if there exist $\beta \geqslant 1$, $a \in (0, 1)$ such that we have*

$$\mathbb{E}\|\phi(n, s)x\|^2 \leqslant \beta a^{n-k}\|x\|^2,$$

for all $x \in \mathcal{Z}$.

1.4.3 Stability and Lyapunov equation

Over the space $\mathcal{Z}$ we suppose the Lyapunov equation

$$X_n = T_n^* X_{n+1} T_n + \lambda_n B_n^* X_n B_n + J_n, \tag{1.13}$$

where J_n is a sequence in $\mathcal{Z}$ under the feature that there are $\alpha,\ \beta > 0$ such that:

$$\alpha\|x\|^2 \leqslant \langle J_n x,\ x\rangle \leqslant \beta\|x\|^2, \tag{1.14}$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{Z}$. We remark that if (1.14) holds, then $\|J_n\| \leqslant \beta$, for all $n \in \mathbb{N}$.

Theorem 1.4. *[31, Theorem16] The two expressions are equivalent*

1. *The system (1.10) is uniformly exponential stable.*

2. *The equation (1.13) has an only one solution $X = (X_n)_{n\in\mathbb{N}}$ with the feature that there exist $m,\ M > 0$ such that*

$$m\|x\|^2 \leqslant \langle X_n x,\ x\rangle \leqslant M\|x\|^2, \tag{1.15}$$

for all $n \in \mathbb{N}$ and $x \in \mathcal{Z}$.

Proposition 1.2. *[34]*

a) *System (1.10) is uniformly exponential stable if the evolution operator $T(n,k)$ is uniformly exponential stable.*

b) *If the sequence($J_n)_{n\geqslant 0} \subset \mathcal{L}(\mathcal{Z})$ is bounded on $\mathbb{N}$ and $T(n,k)$ is uniformly exponential stable, then the Lyapunov equation,*

$$X_n = T_n^* X_{n+1} T_n + \lambda_n B_n^* X_n B_n + J_n, \tag{1.16}$$

has an only one bounded solution in $\mathcal{L}(\mathcal{Z})$ given by

$$X_s = \sum_{n=s}^{\infty} \mathcal{T}(n,s)(W_n).$$

In addition, if there is a $p \geqslant 1$ such that $(T_n)_{n\in\mathbb{N}},\ (W_n)_{n\in\mathbb{N}}$ are $p-$periodic, then the unique solution $X = X_n$ of (1.16) is $p-$periodic.

In the case of $B_n = 0$ we get this result.

Corollary 1.2. *If the system (1.5) is uniformly exponentially stable, the Lyapunov equation*

$$X(n) = T^*(n)X(n+1)T(n) + Q(n),$$

has a unique solution given by

$$X_s = \sum_{n=s}^{\infty} \phi(n,s)^* Q(s)\phi(n,s).$$

.

1.5 Stochastic discrete-time invariant systems

In the time invariant case, satisfactory conditions for the exponential stability of the system (1.10) are given below

$$\begin{cases} x_{n+1} = Tx_n + \xi_n B x_n, \ n \in \mathbb{N}, \\ x_s = x, \ s \in \mathbb{N}. \end{cases} \tag{1.17}$$

Where $T,\ B$ are linear bounded operators and ξ is a real random variable which satisfies $\mathbb{E}(\xi) = 0$ and $\mathbb{E}|\xi|^2 = b < \infty$.

Consider the Lyapunov algebraic equation

$$X = T^*XT + \lambda B^*XB + Q \tag{1.18}$$

on the space $\mathcal{Z}$ where Q is a self adjoint and positive operator. In the time-invariant case Theorem (1.4) yields the following corollary.

Corollary 1.3. *[31] If $T_n = T,\ B_n = B$ and $\lambda_n = \lambda$ then the solution of (1.17) is uniformly exponential stable if and only if (1.18) has a unique positive solution.*

2 Stability radius: Definitions and characterizations

Contents

2.1 Introduction

In this chapter we investigate stable discrete-time systems subjucted to N perturbations of stochastic type. We give the definition of their radii of stability at first, and afterwards, by using the scaling method, we discuss their characterizations. The acquired characterizations are expressed via discrete time Lyapunov equation. Two Illustrative examples are given at the end of the chapter.

2.2 System representation

Letting us suppose the stochastic discrete-time system defined by the following equation over the real separable Hilbert space $\mathcal{Z}$

$$\begin{cases} x(t+1) = Tx(t) + \sum_{i=1}^{N} D_i\Lambda_i(E_i(x(t))w_i(t) \text{ for all } t \in \mathbb{N} \\ x(0) = x_0 \in \mathcal{Z}, \end{cases} \tag{2.1}$$

such that $T \in \mathcal{L}(\mathcal{Z})$ is stable, $D_i \in \mathcal{L}(U_i, \mathcal{Z})$ and $E_i \in \mathcal{L}(\mathcal{Z}, Y_i)$, $i \in \overline{N}$, where $\mathcal{Z}$; U_i and Y_i, $i \in \overline{N}$, are real separable Hilbert spaces, $\overline{N} := \{1, ..., N\}$ and $(w_i(t))_{t\in\mathbb{N}}$, $i \in \overline{N}$, are sequences of real independent random variables on a complete probability space $(\Omega, F, (F_n)_{n\in\mathbb{N}}, \mathbb{P})$.

Letting

$$\mathbb{N}_t := \{0, 1, ..., t\}, \quad t \in \mathbb{N}, \text{ and } F_t := \sigma\left(\{w_i(s);\ s \in \mathbb{N}_t,\ i \in \overline{N}\}\right).$$

We note that F_t is the $\sigma-$ algebra generated by $(w_i(s))_{s\in\mathbb{N}_t}$. Moreover, $(F_t)_{t\in\mathbb{N}}$ is an increasing sequence of σ-algebras $F_t \subset F$.

We will suppose that

$$\begin{cases} \mathbb{E}(w_i(t)) = 0, \\ \mathbb{E}(w_i(s)w_j(t)) = \lambda_i\delta_{ij}\delta_{st},\ s, t \in N,\ \ i, j \in \overline{N}, \end{cases}$$

where δ_{ij}, δ_{st} are the Kronecker symbols. For $i \in \overline{N}$, $\Lambda_i : Y_i \longrightarrow U_i$ is a measurable map such that

$$\Lambda_i(0) = 0, \quad \|\Lambda_i\| := \sup_{y\neq 0} \frac{\|\Lambda_i(y)\|}{\|y\|} < \infty.$$

The space $l^2_\omega\big(\mathbb{N}, l^2(\Omega, \mathcal{Z})\big)$ is equipped with the l^2-norm defined by

$$\|z(.)\|^2_{l^2_\omega} := \mathbb{E}\left(\sum_{t\in\mathbb{N}} \left(\|z(t)\|^2\right)\right) = \sum_{t\in\mathbb{N}} \mathbb{E}\left(\|z(t)\|^2\right).$$

According to these requirements, the system (2.1) has a unique solution x, for every initial state x_0, presented by

$$\begin{aligned} x(t) &= x(t, x_0) \\ &= T^t x_0 + \sum_{i=1}^{N}\sum_{k=0}^{t-1} T^{t-k-1} D_i \Lambda_i E_i(x(k)) w_i(k), \quad t \geq 0. \end{aligned} \tag{2.2}$$

2.3 Characteristics of the stochastic stability radius

During this part, we aim to establish which borders $\mathfrak{d}$ of the disturbance Λ_i guarantee the preservability of the stability of the nominal system $x(t+1) = Tx(t)$, with addition stochastic perturbations of the kind $\sum_{i=1}^{N} D_i\Lambda_i(E_i(x(t))w_i(t), \ t \in \mathbb{N}$. For this goal letting us set

$$Y := \bigoplus_1^N Y_i, \ \ U := \bigoplus_1^N U_i.$$

Let Λ be the perturbation operator such that $\Lambda = \bigoplus_{i=1}^{N} \Lambda_i$. We note that the norm of Λ is given by

$$\Lambda = \bigoplus_{i=1}^{N} \Lambda_i \text{ and } \|\Lambda\| = \max_{i\in\overline{N}} \|\Lambda_i\|.$$

We recall that the stability radius of (2.1) is the maximum $\mathfrak{d} > 0$ whereby the system (2.1) is exponentially stable.

Definition 2.1. *The stochastic stability radius of (2.1) according the perturbation structure* $(D_i, E_i)_{i\in\overline{N}}$ *is defined by*

$$\mathfrak{r}_\omega\big(T, (D_i, E_i)\big)_{i\in\overline{N}} := \inf \left\{ \begin{array}{c} \|\Lambda\| : \Lambda = \bigoplus_{i=1}^{N} \Lambda_i, \quad \textit{such that} \\ (2.1) \ \ \textit{is not exponentially stable} \end{array} \right\}.$$

For the purpose of characterizing the stochastic stability radius, $\mathfrak{r}_\omega(T,(D_i,E_i))_{i\in\bar{N}}$ of system (2.1), we follow the approach used in [16, 17] for continuous systems and we use some results established in [23, 32, 31, 35], for infinite-dimensional discrete-time systems. We start with this Lemma, which will be an important key in providing characterizations of the stochastic stability radius.

Lemma 2.1. *Let $E \in \mathcal{L}(\mathcal{Z}, Y)$. Assume that*

$$y(t) = ET^t x_0 + \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1}D_i v_i(k)w_i(k),$$

where $v_i \in l^2_\omega(\mathbb{N}, l^2(\Omega, U_i))$, $i \in \overline{N}$, *and* $t \geq 0$.
Then

$$\mathbb{E}\left(\|y(t)\|^2\right) = \left\|ET^t x_0\right\|^2 + \sum_{i=1}^{N}\lambda_i \sum_{k=0}^{t-1}\mathbb{E}\left\|ET^{t-k-1}D_i v_i(k)\right\|^2, \quad t \geq 0.$$

Moreover, $y(.) \in l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$ *and*

$$\|y\|^2_{l^2_\omega} = \sum_{t=0}^{\infty}\mathbb{E}\left(\|y(t)\|^2\right) = \sum_{t=0}^{\infty}\left\|ET^t x_0\right\|^2 + \sum_{i=1}^{N}\lambda_i\sum_{k=0}^{\infty}\mathbb{E}\left\langle D_i v_i(k), PD_i v_i(k)\right\rangle, \tag{2.3}$$

where $P \in \mathcal{L}(\mathcal{Z})$ is a self-adjoint non negative operator fulfilling

$$T^*PT - P + \sum_{i=1}^{N} E_i^* E_i = 0. \tag{2.4}$$

Proof. In fact,

$$y(t) = ET^t x_0 + \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1}D_i v_i(k)w_i(k), \quad t \geq 0.$$

Thus

$$\|y(t)\|^2 = \langle y(t), y(t)\rangle = \Big\langle ET^t\mathbf{x}_0 + \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1}D_i v_i(k)w_i(k), \\ ET^t\mathbf{x}_0 + \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1}D_i v_i(k)w_i(k)\Big\rangle$$

$$
\begin{aligned}
&= \langle ET^t x_0, ET^t x_0\rangle \\
&+ \left\langle ET^t x_0, \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k) \right\rangle + \langle \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k), ET^t x_0\rangle \\
&+ \left\langle \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k), \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k) \right\rangle \\
&= \left\|ET^t x_0\right\|^2 + \langle ET^t x_0, \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k)\rangle \\
&+ \langle \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k), ET^t \mathbf{x}_0\rangle + \left\| \sum_{i=1}^{N}\sum_{k=t_0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k) \right\|^2
\end{aligned}
$$

and then

$$
\begin{aligned}
\mathbb{E}\langle y(t), y(t)\rangle &= \left\langle ET^t x_0, ET^t x_0 \right\rangle \\
&+ \mathbb{E}\left\langle ET^t x_0, \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k) \right\rangle \\
&+ \mathbb{E}\left(\left\langle \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k), ET^t \mathbf{x}_0 \right\rangle\right) \\
&+ \mathbb{E}\left\| \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k) \right\|^2
\end{aligned}
$$

as in the proof of [31, Theorem 5], we obtain for $i = j$

$$
\mathbb{E}\left\langle \sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k), \sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k) \right\rangle = \lambda_i \sum_{k=0}^{t-1} \mathbb{E}\left\|ET^{t-k-1} D_i v_i(k)\right\|^2;
$$

and

$$
\mathbb{E}\left(\left\langle \sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k), \sum_{k=0}^{t-1} ET^{t-k-1} D_j v_j(k) w_j(k) \right\rangle\right) = 0, \quad \text{for} \quad i \neq j.
$$

Then

$$
\mathbb{E}\left(\left\| \sum_{i=1}^{N}\sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k) \right\|^2\right) = \sum_{i=1}^{N} \lambda_i \sum_{k=0}^{t-1} \mathbb{E}\left\|ET^{t-k-1} D_i v_i(k)\right\|^2.
$$

Note that since T is stable it follows that

$$
\exists\, \beta \geq 1 : \;\; a \in (0,1), \;\; \left\|T^t\right\|^2 \leq \beta a^t,
$$

which leads to

$$\begin{aligned}\sum_{t=0}^{\infty} \left\|ET^t x_0\right\|^2 &\leq \beta \left\|E\right\|^2 \left\|x_0\right\|^2 \sum_{t=0}^{\infty} a^t \\ &\leq \beta \left\|E\right\|^2 \left\|x_0\right\|^2 \frac{1}{1-a}.\end{aligned}$$

Hence

$$\sum_{t=0}^{\infty} \left\|ET^t x_0\right\|^2 < \infty, \tag{2.5}$$

and hence

$$\sum_{k=0}^{t-1} \left\|ET^{t-k-1} D_i v_i(k)\right\|^2 \leq \left\|D_i\right\|^2 \left\|E\right\|^2 \sum_{k=0}^{t-1} \beta a^{t-k-1} \left\|v_i(k)\right\|^2 .$$

Set $M_i = \beta \|D_i\|^2 \|E\|^2$, then we get

$$\sum_{t=0}^{\infty} \sum_{i=1}^{\infty} \lambda_i \sum_{k=0}^{t-1} \mathbb{E} \left\|ET^{t-k-1} D_i v_i(k)\right\|^2 \leqslant \left(\frac{1}{1-a}\right) \max_{i\in\{1,\ldots N\}} (\lambda_i M_i) \|v\|^2.$$

This implies that $y(.) \in l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$. Moreover, for $t \geqslant 0$, we have

$$\mathbb{E}\left(\|y(t)\|^2\right) = \left\|ET^t x_0\right\|^2 + \sum_{i=1}^{N} \lambda_i \sum_{k=0}^{t-1} \mathbb{E} \left\|ET^{t-k-1} D_i v_i(k)\right\|^2 .$$

So,

$$\sum_{t=0}^{\infty} \mathbb{E} \left\|y(t)\right\|^2 = \sum_{t=0}^{\infty} \left\|ET^t x_0\right\|^2 + \sum_{i=1}^{N} \lambda_i \sum_{k=0}^{\infty} \mathbb{E} \left\langle D_i v_i(k), P D_i v_i(k)\right\rangle ,$$

where P denotes the solution of (2.4) and it is provided by

$$P := \sum_{t=k+1}^{\infty} (T^{t-k-1})^* E^* E T^{t-k-1}.$$

□

Let us consider the input-output operator

$$\begin{aligned} L \quad &: \quad l^2_\omega(\mathbb{N}, l^2(\Omega, U)) \longrightarrow l^2_\omega(\mathbb{N}, l^2(\Omega, Y)), \\ (Lv)(t) \quad &:= \quad \left(L \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \right)(t) = \sum_{i=1}^{N} \sum_{k=0}^{t-1} ET^{t-k-1} D_i v_i(k) w_i(k). \end{aligned}$$

According to Lemma 2.1 it follows that $Lv \in l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$ for every $v \in l^2_\omega(\mathbb{N}, l^2(\Omega, U))$.

The following Lemma will be used to classify the norm of the above-mentioned linear map L.

Lemma 2.2. *We have*

$$\|L\| = \left(\max_{i\in\{1,\cdots,N\}} \Big(\lambda_i \|D_i^* P D_i\| \Big) \right)^{\frac{1}{2}},$$

where P satisfies (2.4).

Proof. Given $v \in l^2_\omega(\mathbb{N}, \mathcal{L}^2(\Omega, U))$, we can deduce from Lemma 2.1 that

$$\begin{aligned} \|Lv\|^2_{l^2_\omega} &= \sum_{i=1}^{N} \lambda_i \sum_{k=0}^{+\infty} \mathbb{E} \left\langle D_i v_i(k), \sum_{t=k+1}^{\infty} \left(T^{t-k-1}\right)^* E^* E T^{t-k-1} D_i v_i(k) \right\rangle \\ &= \sum_{i=1}^{N} \lambda_i \sum_{k=0}^{\infty} \mathbb{E} \left\langle v_i(k), D_i^* P D_i v_i(k) \right\rangle. \end{aligned}$$

Hence

$$\|Lv\|^2_{l^2_\omega} \leqslant \max_{i\in\{1,\cdots,N\}} \left(\lambda_i \|D_i^* P D_i\|\right) \|v\|^2_{L^2_\omega},$$

and hence

$$\|L\| \leqslant \left(\max_{i\in\{1,\ldots,N\}} \lambda_i \|D_i^* P D_i\| \right)^{\frac{1}{2}}.$$

Assume that

$$\max_{i\in\{1,\ldots,N\}} \Big(\lambda_i \|D_i^* P D_i\| \Big) = \lambda_{i_0} \left\langle v_{i_0}, D_{i_0}^* P D_{i_0} v_{i_0} \right\rangle, \quad \|v_{i_0}\| = 1.$$

Choose v^0 such that

$$\begin{cases} v_i^0(t) = 0, \ i \neq i_0, \\ v_{i_0}^0(t) = \beta(t) v_{i_0}, \quad t \in \mathbb{N}, \end{cases}$$

where $\beta(.) \in l^2(\mathbb{N}, \mathbb{R})$ is such that $\|\beta(.)\|_{l^2} = 1$. We get

$$\left\|v^0\right\|^2_{l^2} = \sum_{t=0}^{\infty} \|\beta(t) v_{i_0}\|^2 = \|v_{i_0}\|^2 \sum_{t=0}^{\infty} \|\beta(t)\|^2 = 1.$$

So,

$$\begin{aligned} \left\|Lv^0\right\|^2_{l^2} &= \lambda_{i_0} \sum_{k=0}^{\infty} \left\langle \beta(k) v_{i_0}, D_{i_0}^* P D_{i_0} \beta(k) v_{i_0} \right\rangle \\ &= \lambda_{i_0} \left\| D_{i_0}^* P D_{i_0} \right\|. \end{aligned}$$

We conclude that

$$\left\|Lv^0\right\|^2 = \max_{i\in\{1,\ldots,N\}} \left(\lambda_i \|D_i^* P D_i\|\right),$$

and finally that

$$\|L\| = \left(\max_{i\in\{1,\dots,N\}} (\lambda_i \|D_i^* P D_i\|) \right)^{\frac{1}{2}}.$$

□

Let $\alpha \in (0,+\infty)^N$, $\alpha = (\alpha_1, \alpha_2, ...\alpha_N)$ such that

$$D_i^{\alpha_i} = \alpha_i^{-1} D_i, E_i^{\alpha_i} = \alpha_i E_i, \Lambda_i^{\alpha_i} = \alpha_i \Lambda_i \alpha_i^{-1},\ i \in N \tag{2.6}$$

If we rewrite the perturbed system with $D_i^{\alpha_i}$, $E_i^{\alpha_i}$, $\Lambda_i^{\alpha_i}$, instead of D_i, E_i, Λ_i, both of the systems (2.1) and the scaled perturbed system

$$x(t) = T^t x_0 + \sum_{i=1}^{N} \sum_{k=0}^{t-1} T^{t-k-1} D_i^{\alpha_i} \Lambda_i^{\alpha_i} (E_i^{\alpha_i} x(k)) w_i(k),\ t \in \mathbb{N} \tag{2.7}$$

have the same solution. The input-output operator: $L^a : l^2_\omega(\mathbb{N}, l^2(\Omega, U)) \longrightarrow l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$ of the system $(T, (D_i^{\alpha_i}, E_i^{\alpha_i}))$ is presented by :

$$(L^\alpha v)(t) = \left(L \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} \right)(t) = \sum_{i=1}^{N} \sum_{k=0}^{t-1} \begin{pmatrix} E_1^{\alpha_1} \\ \vdots \\ E_n^{\alpha_n} \end{pmatrix} T^{t-k-1} D_i^{\alpha_i} v_i(k) w_i(k),\ t \geq 0.$$

We note that the solution of the scaled system does not depend on α but the input-output operator does change with α. We use this fact to examine a computing expression for the stability radius.

Theorem 2.1. *Assume there are $\alpha := (\alpha_i)_{i\in\overline{N}} \in (0,+\infty)^N$ and $P(\alpha) \in \mathcal{L}^+(\mathcal{Z})$ fulfilling*

$$T^* P(\alpha) T - P(\alpha) + \sum_{i\in\overline{N}} \alpha_i^2 E_i^* E_i = 0, \tag{2.8}$$

and

$$I - \left(\frac{\mathfrak{d}}{\alpha_j} \right)^2 \lambda_j D_j^* P(\alpha) D_j \succeq 0,\quad j \in \overline{N}. \tag{2.9}$$

Then $\mathfrak{r}_\omega(T, (D_i, E_i))_{i\in N} \geq \mathfrak{d}$.

Proof. Set $\Lambda = \bigoplus_{i=1}^{N} \Lambda_i$ such $\Lambda_i \in \mathcal{L}(Y_i, U_i)$, $\|\Lambda\| < \mathfrak{d}$ and assume that $\alpha \in (0,+\infty)^N$, $P(\alpha) \in \mathcal{L}^+(\mathcal{Z})$, are such that conditions (2.8) and (2.9) hold. The unique solution

of (2.1) satisfies the scaled system (2.7).
Let $v_i^{\alpha_i}(t) = \Lambda_i^{\alpha_i}(y_i^{\alpha_i}(t))$, $y_i^{\alpha_i}(t) = E_i^{\alpha_i}(x(t))$, $t \in \mathbb{N}^*$.

$$E(\alpha) = \begin{pmatrix} E_1^{\alpha_1} \\ \vdots \\ E_n^{\alpha_n} \end{pmatrix},\ y^a(t) = \begin{pmatrix} y_1^{\alpha_1}(t) \\ \vdots \\ y_n^{\alpha_n}(t) \end{pmatrix},\ v^\alpha(t) = \begin{pmatrix} v_1^{\alpha_1}(t) \\ \vdots \\ v_n^{\alpha_n}(t) \end{pmatrix},\ \Lambda^\alpha = \bigoplus_{i=1}^{N} \Lambda_i^{\alpha_i}, t \in \mathbb{N}^*$$

We have

$$y^\alpha(t) = E(\alpha)T^t x_0 + \sum_{i=1}^{N} \sum_{k=0}^{t-1} E(\alpha)T^{t-k-1} D_i^{\alpha_i} v_i^{\alpha_i}(k) w_i(k)$$

For $i \in \overline{N}$, define $v_{i,N_0}(.) \in l^2_\omega(\mathbb{N}, l^2(\Omega, U_i))$ by

$$v_{i,N_0}(t) = \begin{cases} v_i^{\alpha i} = \Lambda_i^{\alpha_i}(y_i^{\alpha_i}(t)),\ t \le N_0 \\ 0,\ t > N_0 \end{cases},\ v^\alpha_{N_0}(t) = \begin{pmatrix} v_{1,N_0}^{\alpha_1}(t) \\ \vdots \\ v_{n,N_0}^{\alpha_n}(t) \end{pmatrix}$$

then

$$\begin{aligned} \left\|v^\alpha_{N_0}\right\|^2_{L^2_\omega} &= \sum_{t=0}^{N_0} \mathbb{E}(\left\|v^\alpha_{N_0}(t)\right\|^2) \\ &= \sum_{t=0}^{N_0} \mathbb{E}\left(\sum_{i=1}^{N} \|\Lambda_i^{\alpha_i}(y_i^{\alpha_i}(t))\|^2\right) \\ &\le \max_{i=\{1,\dots,N\}} \|\Lambda_i^{\alpha_i}\|^2 \sum_{t=0}^{N_0}\left(\sum_{i=1}^{N} \mathbb{E}\left(\|(y_i^{\alpha_i}(t))\|^2\right)\right) \end{aligned}$$

Hence

$$\left\|v^\alpha_{N_0}\right\|^2_{l^2_\omega} \le \|\Lambda^\alpha\|^2 \sum_{t=0}^{+\infty} \mathbb{E}\left(\|(y^\alpha(t))\|^2\right) \tag{2.10}$$

Now suppose that $y^\alpha_{N_0}$ is the output of the system $(T, (D_i^{\alpha_i}, E_i^{\alpha_i}))_{i\in\overline{N}}$ engendred by the input $v^\alpha_{N_0}(t)$ with the initially state $x(0) = x_0$. Then

$$y^\alpha_{N_0}(t) = y^\alpha(t) = E(\alpha)T^t x_0 + L^\alpha v^\alpha_{N_0}(t),\ t \in \mathbb{N} \tag{2.11}$$

which yields

$$\left(\sum_{t=0}^{N_0} \mathbb{E}\left(\|(y^\alpha(t))\|^2\right)\right)^{\frac{1}{2}} \le \left\|y^\alpha_{N_0}\right\|_{L^2_\omega} \le \left\|E(\alpha)T^t x_0\right\| + \|L^\alpha\| \left\|v^\alpha_{N_0}\right\|$$

hence

$$\left(\sum_{t=0}^{N_0}\mathbb{E}\left(\|(y^\alpha(t))\|^2\right)\right)^{\frac{1}{2}} \leq \left\|E(\alpha)T^t x_0\right\| + \tag{2.12}$$

$$\|L^\alpha\|\,\|\Lambda^\alpha\|\left(\sum_{t=0}^{N_0}\mathbb{E}\left(\|(y^\alpha(t))\|^2\right)\right)^{\frac{1}{2}} \tag{2.13}$$

Condition (2.9) implies that

$$\max_{j\in\{1,\dots,N\}}\lambda_j\left\|\left(D_j^{\alpha_j}\right)^* P(\alpha)D_j^{\alpha_j}\right\| \leq \mathfrak{d}^{-2}$$

Using Lemma 2.2, we deduce that

$$\|L^\alpha\|^2 \leq \mathfrak{d}^{-2}$$

hence

$$s = \|L^\alpha\|\,\|\Lambda^\alpha\| < 1$$

(2.12) implies that

$$\left(\sum_{t=0}^{N_0}\varepsilon\left(\|(y^\alpha(t))\|^2\right)\right)^{\frac{1}{2}} \leq (1-s)^{-1}\left\|E(\alpha)T^t x_0\right\|,\ \text{for any } N_0\in\mathbb{N},$$

which require that $y^\alpha \in l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$ and $v^\alpha = \Lambda^\alpha(y^\alpha) \in l^2_\omega(\mathbb{N}, l^2(\Omega, U))$. Using Lemma 2.1, we conclude that the solution of (2.7) belongs to $l^2_\omega(\mathbb{N}, l^2(\Omega, \mathcal{Z}))$, hence $\mathfrak{r}_\omega(T, (D_i, E_i))_{i\in\overline{N}} \geq \mathfrak{d}$. □

Remark 2.1. *1. From the foregoing it comes that*

$$\mathfrak{r}_\omega(T, (D_i, E_i))_{i\in\overline{N}} \geq \sup_{\alpha\in(0,+\infty)^N}\left(\max_{j\in\overline{N}}\left\|\frac{\lambda_j}{\alpha_j^2}D_j^* P(\alpha)D_j\right\|\right)^{-\frac{1}{2}}$$

2. If the condition is satisfied using $\succ$ instead of $\succeq$ in (2.9), we get $\mathfrak{r}_\omega(T, (D_i, E_i))_{i\in\overline{N}} > \mathfrak{d}$.

If we change the Lyapunov equation (2.8) with the matching inequality, we achieve the same result.

Corollary 2.1. *Suppose that there exist $\alpha \in (0, +\infty)^N$ and $P \in \mathcal{L}^+(\mathcal{Z})$ holding*

$$T^*PT - P + \sum_{i\in\overline{N}}\alpha_i^2 E_i^* E_i \preceq 0, \tag{2.14}$$

and

$$I - \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 \lambda_j D_j^* P D_j \succeq 0, \quad \left(resp.\ I - \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 \lambda_j D_j^* P D_j \succ 0\right), \quad j \in \overline{N}. \quad (2.15)$$

Then

$$\mathfrak{r}_\omega(T, (D_i, E_i))_{i\in\bar{N}} \geq \mathfrak{d}, \quad \left(resp.\ \mathfrak{r}_\omega(T, (D_i, E_i))_{i\in\bar{N}} > \mathfrak{d}\right).$$

In this situation, the Lyapunov equation (2.8) admits a solution $P_0 \in \mathcal{L}^+(\mathcal{Z})$ with $P \succeq P_0$.

Proof. Due to T is stable, there is a solution P_0 to the Lyapunov equation (2.8). For $X := P - P_0$, we obtain

$$T^* X T - X \preceq 0.$$

This implies that $X \succeq 0$. Alternatively, $P \succeq P_0$. Thus, the requirements (2.8) and (2.15) are accomplished. Theorem 2.1 shows that

$$\mathfrak{r}_\omega(T, (D_i, E_i))_{i\in\bar{N}} \geq \mathfrak{d}.$$

□

For $i, j \in \overline{N}$, define the operator $\varphi_{i,j} \in \mathcal{L}(U_j)$ as follows

$$\varphi_{i,j} u_j = \lambda_j \sum_{t=0}^{+\infty} D_j^* \left(T^t\right)^* E_i^* E_i T^t D_j u_j$$

Assume that $J \subset \overline{N}$, $\alpha \in (0, +\infty)^J$, such that $(0, +\infty)^J = \{(\alpha_j)_{j\in J}, \alpha_j \in (0, +\infty)\}$. Thus the operator

$$P(\alpha^J) = \sum_{t=0}^{+\infty} \left(T^t\right)^* \left(\sum_{i\in J} \alpha_i^2 E_i^* E_i\right) T^t$$

is the only solution of the Lyapunov equation

$$T^* P T - P + \sum_{i\in J} \alpha_i^2 E_i^* E_i = 0 \quad (2.16)$$

We have

$$\begin{aligned} \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha^J) D_j u_j &= \frac{\lambda_j}{\alpha_j^2} D_j^* \sum_{t=0}^{+\infty} \left(T^t\right)^* \left(\sum_{i\in J} \alpha_i^2 E_i^* E_i\right) T^t D_j u \\ &= \sum_{i\in J} \left(\frac{\alpha_i}{\alpha_j}\right)^2 \varphi_{i,j} u_j \end{aligned}$$

Hence

$$\frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha^J) D_j u_j = \sum_{i\in J} \left(\frac{\alpha_i}{\alpha_j}\right)^2 \varphi_{i,j} u_j,\ j \in J$$

In order to examine a computing expression for the stability radius, the next Lemma established in [6] is needed.

Lemma 2.3. *[6] Set* $\overline{\mu} = \inf_{\alpha\in(0,+\infty)^N} \max_{j\in\overline{N}} \left\| \sum_{i=1}^{N} \left(\frac{\alpha_i}{\alpha_j}\right)^2 \varphi_{i,j} \right\|$, *there exist a sub-set* $J \subset \overline{N}$ *and for* $\delta > 0$, *a vector* $\alpha \in (0,+\infty)^N$ *so that*

$$\begin{cases} \left\| \sum_{i=1}^{N} \left(\frac{\alpha_i}{\alpha_j}\right)^2 \varphi_{i,j} \right\| \le \overline{\mu} + \delta,\ j \in \overline{N} \\ \left\| \sum_{i\in J} \left(\frac{\alpha_i}{\alpha_j}\right)^2 \varphi_{i,j} \right\| = \overline{\mu},\ j \in J \end{cases}$$

An important characterization of the stability radius is established in the next Theorem.

Theorem 2.2. *Let* $(T,(D_i,E_i))_{i\in\overline{N}}$ *and* $(w_i(t))_{i\in\overline{N}}$ *be as previously. Then the corresponding stability radius is represented by*

$$\mathfrak{r}_\omega(T,(D_i,E_i))_{i\in\overline{N}} = \sup_{\alpha\in(0,+\infty)^N} \left(\max_{j\in\overline{N}} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha) D_j \right\| \right)^{-\frac{1}{2}}, \tag{2.17}$$

such that $P(\alpha)$ *is the only solution of (2.8).*

Proof. Let $\overline{\mu} = \inf_{\alpha\in(0,+\infty)^N} \max_{j\in\overline{N}} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha) D_j \right\|$ then

1. If $\overline{\mu} = 0$, from Theorem 2.1 we get that $\mathfrak{r}_\omega(T,(D_i,E_i)) = +\infty$.
2. Assume that $\overline{\mu} > 0$. Let $\mathfrak{d}(\alpha)$ be the greatest $\mathfrak{d}$ for which (2.8) admits a solution $P(\alpha) \in \mathcal{L}^+(\mathcal{Z})$ holding the condition (2.9). Let $\mathfrak{d} < \mathfrak{d}(\alpha)$, for $j \in \overline{N}$, we have

   $$I - \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 \lambda_j D_j^* P(\alpha) D_j \ge 0,\ \forall j \in \overline{N} \Longrightarrow \max_{j\in\overline{N}} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha) D_j \right\| \le \mathfrak{d}^{-2}$$

   Thus $\mathfrak{d} \le \left(\max_{j\in\overline{N}} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha) D_j \right\| \right)^{-\frac{1}{2}}$. Set

   $$\mathfrak{d}(\alpha) = \left(\max_{j\in\overline{N}} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha) D_j \right\| \right)^{-\frac{1}{2}}$$

Hence

$$\sup_{\alpha\in(0,+\infty)^N} \mathfrak{d}^2(\alpha) = \sup_{\alpha\in(0,+\infty)^N} \left(\max_{j\in\overline{N}} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha) D_j \right\|\right)^{-1} = \overline{\mu}^{-1}$$

From Lemma 2.3, there exist $J \subset \overline{N}$ and a vector $\alpha^J = (\alpha_j)_{j\in J}$, $\alpha_j \in (0,+\infty)^J$, such that

$$\hat{\mu} = \left\| \sum_{i\in J} (\frac{\alpha_i}{\alpha_j})^2 \varphi_{i,j} \right\| = \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha^J) D_j \right\|, \; j \in J$$

where $P(\alpha) \in \mathcal{L}^+(\mathcal{Z})$ is the only solution of (2.16). Letting $v_j^0 \in U$, $\left\|v_j^0\right\|_U = 1$, $j \in J$ such that

$$\begin{aligned} \left\| \sum_{i\in J} (\frac{\alpha_i}{\alpha_j})^2 \varphi_{i,j} \right\| &= \left\langle v_j^0, \sum_{i\in J} \left(\frac{\alpha_i}{\alpha_j}\right)^2 \varphi_{i,j} v_j^0 \right\rangle \\ &= \left\langle v_j^0, \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha^J) D_j v_j^0 \right\rangle = \hat{\mu} \end{aligned}$$

Then

$$\left\langle v_j^0, \frac{\lambda_j}{\alpha_j^2} \hat{\mu}^{-1} D_j^* P(\alpha^J) D_j v_j^0 \right\rangle = 1$$

for $\hat{\mathfrak{d}} = \hat{\mu}^{-\frac{1}{2}}$ we get

$$\left\langle v_j^0, \frac{\lambda_j}{\alpha_j^2} \, \hat{\mathfrak{d}}^2 D_j^* P(\alpha^J) D_j v_j^0 \right\rangle = 1 \tag{2.18}$$

For $j \in J$, we define Λ_j by

$$\begin{cases} \Lambda_j(y_j) = \hat{\mathfrak{d}} \left\|y_j\right\| v_j^0 \,, & j \in J, \; y_j \in Y_j \\ \Lambda_j(y_j) = 0, & j \in \bar{N} \diagup J \end{cases}$$

hence $\|\Lambda\| = \hat{\mathfrak{d}}$ where $\Lambda = \bigoplus_{j=1}^{N} \Lambda_i$. We will provide that for this Λ the perturbed system can not be exponentially stable. Suppose that the system (2.1) is exponentially stable. Let $x_0 \in \mathcal{Z}$, the solution of the system (2.1) fulfills

$$x(t) = T^t x_0 + \sum_{j\in J} \sum_{k=0}^{t-1} T^{t-k-1} D_j^{\alpha j} \Lambda_j^{\alpha_j} (E_j^{\alpha_j} x(k)) w_j(k) \tag{2.19}$$

where $D_i^{\alpha_i}$, $E_i^{\alpha_i}$ and $\Lambda_i^{\alpha_i}$ are defined by (2.6) and set $y_j^{\alpha_j} = E_j^{\alpha_j}x$. Then $y_j^{\alpha_j} \in l^2(\mathbb{N}, l^2(\Omega, Y_j))$, $j \in J$. For $j \in J$, we have

$$\Lambda_j^{\alpha_j}(y_j) = \alpha_j \Lambda_j \alpha_j^{-1} y_j = \hat{\mathfrak{d}} \, \|y_j\| \, v_j^0, \quad y_j \in Y_j, \; j \in J$$

We define $y^{\alpha_J}$and E^{α_J} by

$$y^{\alpha_J} = (y_j^{\alpha_j})_{j \in J} \ , \quad E^{\alpha_j} x = \left(E_j^{\alpha_j} \mathbf{x}\right)_{j \in J}$$

Then

$$\begin{aligned} y^{\alpha_J}(t) &= E^{\alpha_J} T^t x_0 + \sum_{j \in J} \sum_{k=0}^{t-1} E^{\alpha_J} T^{t-k-1} D_j^{\alpha_j} \Lambda_j^{\alpha_j} y_j(k) w_j(k) \\ &= E^{\alpha_J} T^t x_0 + \hat{\mathfrak{d}} \sum_{j \in J} \sum_{k=0}^{t-1} E^{\alpha_J} T^{t-k-1} D_j^{\alpha_j} \left\| y_j^{a_j}(k) \right\| v_j^0 w_j(k) \end{aligned}$$

Using Lemma 2.1 we obtain

$$\sum_{t=0}^{+\infty} \mathbb{E} \left\| y^{\alpha_j}(t) \right\|^2 = \sum_{t=0}^{+\infty} \left\| E^{\alpha_J} T^t x_0 \right\|^2 + \hat{\mathfrak{d}}^2 \sum_{j \in J} \frac{\lambda_j}{\alpha_j^2} \left\langle v_j^0, D_j^* P(\alpha^J) D_j v_j^0 \right\rangle \sum_{k=0}^{+\infty} \mathbb{E} \left\| y_j^{\alpha_j}(k) \right\|^2$$

From (2.18) we obtain

$$\sum_{t=0}^{+\infty} \mathbb{E} \left\| y^{\alpha_j}(t) \right\|^2 = \sum_{t=0}^{+\infty} \left\| E^{\alpha_J} T^t x_0 \right\|^2 + \sum_{k=0}^{+\infty} \mathbb{E} \left\| y^{\alpha_j}(k) \right\|^2$$

From this equation we deduce that

$$\sum_{t=0}^{+\infty} \left\| E^{\alpha_J} T^t x_0 \right\|^2 = 0$$

which require that $P(\alpha^J) = 0$. This is a contradictory with the fact that $\hat{\mu} \neq 0$. Consequently the perturbed system (2.1) and the system (2.19) can not be exponentially stable. It infer that

$$\begin{aligned} \mathfrak{r}_\omega(T, (D_i, E_i))_{i \in \bar{N}} &\leq \mathfrak{r}_\omega(T, (D_j, E_j))_{j \in J} \\ &\leq \left(\max_{j \in J} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P(\alpha^J) D_j \right\| \right)^{-\frac{1}{2}} = \hat{\mu}^{-\frac{1}{2}} \end{aligned}$$

From Remark 2.1, we obtain that $\mathfrak{r}_\omega(T, (D_j, E_j))_{j \in J} \geq \hat{\mu}^{-\frac{1}{2}}$. Hence

$$\begin{aligned} \mathfrak{r}_\omega(T, (D_i, E_i))_{i \in \bar{N}} &\leq \mathfrak{r}_\omega(T, (D_j, E_j))_{j \in J} \\ &\leq \left(\max_{j \in J} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P(a^J) D_j \right\| \right)^{-\frac{1}{2}} = \hat{\mu}^{-\frac{1}{2}} \\ &\leq \mathfrak{r}_\omega(T, (D_i, E_i))_{i \in \bar{N}} \end{aligned}$$

□

Remark 2.2. *Using Lemma 2.2, we infer from (2.17) that*

$$\mathfrak{r}_\omega(T,(D_i,E_i))_{i\in\bar{N}} = \sup_{\alpha\in(0,+\infty)^N} \|L^\alpha\|^{-1} = \mathfrak{r}_\omega(T,(D_i^{\alpha_i},E_i^{\alpha_i}))_{i\in\overline{N}}$$

Theorem 2.2 yields the next result.

Corollary 2.2. *Let $\mathfrak{d} > 0$ such that $\mathfrak{r}_\omega(T,(D_i,E_i))_{i\in\bar{N}} > \mathfrak{d}$, then there exist $\alpha_i > 0$, $i \in \overline{N}$, and $P \in \mathcal{L}^+(\mathcal{Z})$ such that*

$$T^*PT - P + \sum_{i\in\overline{N}} \alpha_i^2 E_i^* E_i = 0,$$

and

$$I - \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 \lambda_j D_j^* P D_j \succ 0, \quad j \in \overline{N}. \tag{2.20}$$

Proof. Since $\mathfrak{r}_\omega(T,(D_i,E_i))_{i\in\bar{N}} > \mathfrak{d}$, there exists $\mathfrak{d}' \in \left]\mathfrak{d}, r^\omega(T,(D_i,E_i))_{i\in\bar{N}}\right[$.

Following Theorem 2.2, we infer that there exists $\alpha \in (0+\infty)^N$ such that

$$\mathfrak{d}' < \left(\max_{j\in\overline{N}} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P D_j \right\| \right)^{-\frac{1}{2}},$$

where P is the solution of the equation (2.8). Thus,

$$\mathfrak{d}^{-2} > \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* P D_j \right\|, \quad \text{for all} \quad j \in \overline{N}.$$

Therefore

$$\lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 \lambda_j D_j^* P D_j \prec I, \quad \text{for all} \quad j \in \overline{N}.$$

Consequently, P fulfills both the equation (2.8) and the requirement (2.20). □

The corollary bellow offers a characteristics of the stability radius via the strict Lyapunov inequality:

$$T^*PT - P + \sum_{i\in\overline{N}} \alpha_i^2 E_i^* E_i \prec 0. \tag{2.21}$$

Corollary 2.3. *Let $\mathfrak{d} > 0$ such that $\mathfrak{r}_\omega(T,(D_i,E_i))_{i\in\overline{N}} > \mathfrak{d}$. Then the inequality (2.21) has a solution $P \in \mathcal{L}^+(\mathcal{Z})$, with $P \succ 0$, for some $\alpha \in (0,+\infty)^N$, which satisfies*

$$I - \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 \lambda_j D_j^* P D_j \succ 0, \quad \textit{for all} \quad j \in \overline{N}.$$

Proof. Let $\mathfrak{d} > 0$ such that $\mathfrak{r}_\omega(T,(D_i,E_i))_{i\in\overline{N}} > \mathfrak{d}$. According to the previous proof, there exists $\alpha \in (0+\infty)^N$, in order that

$$\lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* P(\alpha) D_j \prec I, \quad \text{for all } \ j \in \overline{N},$$

where $P(\alpha)$ is the solution of the equation (2.8). Thus,

$$\lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* \sum_{t=0}^{+\infty} \left(T^t\right)^* \left(\sum_{i\in\overline{N}} \alpha_i^2 E_i^* E_i\right) T^t D_j \prec I, \quad \text{for all } \ j \in \overline{N}.$$

This means that there exist $\varepsilon > 0$ so as to

$$\lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* \sum_{t=0}^{+\infty} \left(T^t\right)^* \left(\sum_{i\in J} \alpha_i^2 E_i^* E_i + \varepsilon I\right) T^t D_j \prec I, \quad \text{for all } \ j \in \overline{N}.$$

By setting $P := \sum_{t=0}^{+\infty} (T^t)^* \left(\sum_{i\in J} \alpha_i^2 E_i^* E_i + \varepsilon I\right) T^t$, we obtain the desired result. □

2.4 Examples

We end this chapter with two interesting examples.

Example 2.1. *We will apply our results on the following system considered in [27]*

$$x(t+1) = Tx(t), \quad t \in \mathbb{N},$$

on the space of square summable sequences l_2, where the operator $T : l_2 \longrightarrow l_2$ is defined by the infinite matrix $(a_{nm})_{n,m\in\mathbb{N}^*}$ with

$$\begin{cases} a_{nm} = 0, & \text{if} \quad n \neq m, \\ a_{nn} = (\frac{1}{2})^n. \end{cases}$$

Note that $T \in \mathcal{L}(l_2)$ and $\mathfrak{d}(A) = \left\{\frac{1}{2^n}, n \in \mathbb{N}^*\right\}$ where $\mathfrak{d}(A)$ is the spectrum of T. Thus, the system is Schur stable. Now, let us consider the following perturbed system

$$x(t+1) = Tx(t) + \sum_{i=1}^{N} \delta_i A_i x(t) w_i(t), \quad N \in \mathbb{N}, \tag{2.22}$$

where $(\delta_i)_{i\in\overline{N}}$ *is a sequence of unknown scalar perturbations such that* $\sup_{i\in\overline{N}}|\delta_i| < \infty$, *and* $(w_i(t))_{i\in\overline{N}}$ *is a sequence of real independent random variables such that* $\varepsilon(w_i^2) = \lambda_i,\ \ i \in \overline{N}$. *Furthermore,* $A_i = (a^i_{nm})$ *with*

$$a^i_{nm} = \begin{cases} 1 & \textit{if } n = m = i, \\ 0 & \textit{otherwise}, \end{cases}$$

So,

$$T = \begin{pmatrix} \frac{1}{2} & 0 & \dots & 0 & \dots \\ 0 & \frac{1}{4} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{1}{2^n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},\quad D_i = A_i,\quad E_i = I,\quad i = 1, ..., N.$$

The appropriate Lyapunov equation for this problem is

$$T^*PT - P + \sum_{i=1}^{N} \alpha_i^2 I = 0. \tag{2.23}$$

Letting

$$P := \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} & \dots \\ p_{12} & p_{22} & \dots & p_{n2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{1n} & p_{n2} & \dots & p_{nn} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

The solution of the Lyapunov equation (2.23) is given by

$$p_{ij} = 0,\ \ \textit{for all}\ \ i \neq j\ \ \textit{and}\ \ p_{ii} = \frac{\alpha_1^2 + \alpha_2^2 + ...\alpha_N^2}{\left(\frac{1}{2}\right)^{2i} - 1}\ \ \textit{for all}\ \ i \geq 1.$$

Theorem 2.2 demonstrates that

$$\mathfrak{r}_\omega\Big(T, (D_i, E_i)\Big)_{i\in\overline{N}} = \sup_{\alpha\in(0,+\infty)^N} \left(\left(\max_{j\in\overline{N}} \left(\frac{\alpha_1^2 + \alpha_2^2 + ...\alpha_N^2}{\alpha_j^2} \right) \right) \left(\left| \frac{\lambda_j}{\left(\frac{1}{2}\right)^{2j} - 1} \right| \right) \right)^{\frac{-1}{2}}.$$

For example, for $N = 2$, $\lambda_1 = 1$ *and* $\lambda_2 = 0.5$, *we get*

$$\mathfrak{r}_\omega(T,(D_i,E_i))_{i\in\bar{N}} = \sup_{\alpha\in(0,+\infty)^2}\left(\max\left(\frac{4}{3}(1+\frac{\alpha_2^2}{\alpha_1^2}),\frac{8}{15}(\frac{\alpha_1^2}{\alpha_2^2}+1)\right)\right)^{-\frac{1}{2}}.$$

Therefore

$$\mathfrak{r}_\omega(T,(D_i,E_i))_{i\in\bar{N}} = \left(\max\left(\frac{4}{3},\frac{8}{15}\right)\right)^{-\frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{2}}$$

Thus, the system (2.22) is stable for $\delta = (\delta_1, \delta_2)$ *such that* $\sup(|\delta_1|, |\delta_2|) < \frac{\sqrt{3}}{\sqrt{2}}$.

Example 2.2. *Letting the stochastic heat equation of heat of one-dimensional represented by:*

$$\begin{cases} \frac{\partial z(x,t)}{\partial t} = \eta\frac{\partial^2 z(x,t)}{\partial \mathbf{x}^2} + cz(t,x)w(t), \quad c\in\mathbb{R}, \quad 0\le x\le 1, \quad t\ge 0, \\ z(0,t)=0, \ z(1,t)=0, \quad t\ge 0, \\ z(x,0)=z_0(x), \quad 0\le x\le 1, \end{cases} \tag{2.24}$$

where $z(x,t)$ *represents the temperature at time* t *and position* x, $z_0(x)$ *is the initial temperature profile,* $w(t)$ *is a scalar white noise with mean zero and variance* λ, *and* η *is the thermal diffusivity. Now, we will use the finite difference method to get a discrete-time approximation of the system (2.24). To accomplish this, define* x_i *and* t_{k+1} *as indicated bellow:*

$$\begin{aligned} x_i &= x_0 + i\Delta x, \ i = 1,2,,,,N, \ \Delta x = 1/N, \ x_0 = 0, \ x_N = 1, \\ t_{k+1} &= t_k + \Delta t, \ k = 0,...N_t, \ t_0 = 0. \end{aligned}$$

Hence

$$\frac{z(x_i,t_j+\Delta t)-z(x_i,t_j)}{\Delta t} = \eta\frac{z(x_i+\Delta x,t_j)-2z(x_i,t_j)+z(x_i-\Delta x,t_j)}{\Delta x^2} + cz(x_i,t_j)w(t_j),$$

such that $0\le j\le N_t$, *and hence*

$$z(x_i,t_j+\Delta t) = rz(x_i+\Delta x,t_j)+(1-2r)z(x_i,t_j)+rz(x_i-\Delta x,t_j)+c\Delta t z(x_i,t_j)w(t_j),$$

with $r = \frac{\eta\Delta t}{(\Delta x)^2}$.
Letting $y(t) = (z(x_1,t), z(x_2,t), ..., z(x_N,t))^T$, *we get*

$$y(t_{j+1}) = Ty(t_j) + c\Delta t y(t_j)w(t_j), \quad j\ge 0,$$

where

$$T := \begin{bmatrix} 1-2r & r & 0 & 0 & 0 \\ r & 1-2r & r & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & r & 1-2r & r \\ 0 & 0 & \dots & r & 1-2r \end{bmatrix}.$$

Assume that $2r < 1$*, and thus* T *is stable. The stability radius is obtained using the formula*

$$\mathfrak{r}_\omega(T,(D,E)) = \left(\left\|\lambda D^T P D\right\|\right)^{-\frac{1}{2}}, \tag{2.25}$$

where P *solves the following Lyapunov equation*

$$T^T P T - P + E^* E = 0. \tag{2.26}$$

with

$$E(y(t)) = y(t), \;\; \Lambda(y(t)) = c\Delta t y(t) \;\; \text{and} \;\; D(z(t)) = z(t).$$

For $\eta = 0.0001,\ \ r = 0.1,\ \ N = 120,\ \ \lambda = 10,$ *we solve the Lyapunov equation (2.26) and use the formula (2.25) to calculate the stability radius. We obtain* $\mathfrak{r}_\omega(T,(D_i,E_i) = 0.3672$*. Thus, the equation (2.24) is stable for* c *such that* $c\Delta t < 0.3672$*, and thus* $c < 5,3756$*.*

3 Optimization of the stability radius by state feedback

Contents

3.1 Introduction

Througout this part, we look on the problem of optimizing the radius of the stability with linear state feedback. First, we investigate some useful lemmas. Then we derive suboptimality conditions. We demonstrate that we can classify the supreme attainable stability radius by discrete-time infinite dimensional Riccati equations. Finally, we clarify the outcomes with two examples.

3.2 Suboptimality conditions

Suppose the controlled stochastic system

$$\begin{aligned} x(t+1) &= Tx(t) + \sum_{i=1}^{N} D_i \Lambda_i (E_i(x(t)) w_i(t) + Bu(t), \quad t \in \mathbb{N}, \quad t \geq 0, \\ x(0) &= x_0, \end{aligned}$$

where $u \in l^2_\omega(\mathbb{N}, l^2(\Omega, Z))$, Z is a separable Hilbert space, and $B \in \mathcal{L}(Z, \mathcal{Z})$ and the other operators are described as in the preceding section. By assuming that (T, B) is stabilizable. The goal is to obtain characteristics of the supremum of the stability radii that can be attained by state feedback $u = Fx$ such that $F \in \mathcal{L}(\mathcal{Z}, Z)$. Letting

$$\bar{\mathcal{S}} = \Big\{ F \in \mathcal{L}(\mathcal{Z}, Z), \quad T + BF \text{ is stable} \Big\},$$

and determine

$$\bar{\mathfrak{r}}_\omega \Big(T, (D_i, E_i)_{i \in \bar{N}} \Big) = \sup \Big\{ \mathfrak{r}_\omega (T + BF, (D_i, E_i)_{i \in \bar{N}}), \quad F \in \bar{\mathcal{S}} \Big\}.$$

To investigate the maximization problem, we adopt the approach established in [16].

For $F \in \bar{\mathcal{S}}$, $\alpha \in (0, +\infty)^N$ and $\varepsilon > 0$, suppose the following Lyapunov inequality

$$(T + BF)^* R (T + BF) - R + E^*(\alpha) E(\alpha) + \varepsilon^2 F^* F \preccurlyeq 0, \tag{3.1}$$

under the condition

$$I - \mathfrak{d}^2 \frac{\lambda_j}{\alpha_j^2} D_j^* R D_j \succcurlyeq 0, \quad j \in \overline{N}, \tag{3.2}$$

where $E^*(\alpha)E(\alpha) = \sum_{i=1}^{N} \alpha_i^2 E_i^* E_i$.
To investigate conditions for the existence of sub-optimal controllers $u(t) = Fx(t)$ such that $F \in \bar{\mathcal{S}}$ and $\mathfrak{d} \leqslant \mathfrak{r}_\omega(T+BF,(D_i,E_i)_{i\in\ \bar{N}})$, for $\mathfrak{d} > 0$, the Lemma bellow is required.

Lemma 3.1. *Let $\alpha \in (0,+\infty)^N$ and $\varepsilon > 0$. Assume that there exists $R \in \mathcal{L}^+(\mathcal{Z})$ such that*

$$\begin{gathered}\left(T - B\left(B^*RB+\varepsilon^2 I\right)^{-1} B^*RT\right)^* R\left(T - B\left(B^*RB+\varepsilon^2 I\right)^{-1} B^*RT\right) \\ -R + E^*(\alpha)E(\alpha) + \varepsilon^2\left(\left(B^*RB+\varepsilon^2 I\right)^{-1} B^*RT\right)^* \\ \left(\left(B^*RB+\varepsilon^2 I\right)^{-1} B^*RT\right) \preccurlyeq 0\end{gathered} \tag{3.3}$$

and

$$I - \lambda_j\left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R D_j \succcurlyeq 0,\ \ j \in \overline{N}. \tag{3.4}$$

Then

$$T_\varepsilon := T - B\left(B^*RB+\varepsilon^2 I\right)^{-1} B^*RT$$

is stable. Moreover,

$$\mathfrak{d} \leqslant \mathfrak{r}_\omega(T_\varepsilon,(D_i,E_i)_{i\in\bar{N}}).$$

Proof. Let

$$\begin{cases} x(s+1) = T_\varepsilon x(s), & s \in \mathbb{N}, \\ x(0) = x_0. \end{cases} \tag{3.5}$$

Set $V(x) := \langle x, Rx\rangle$, where x is the solution of (3.5). For $s \in \mathbb{N}$, we have

$$\begin{aligned}\Delta V(x(s)) &= V(x(s+1) - V(x(s)) \\ &= \langle x(s+1), Rx(s+1)\rangle - \langle x(s), Rx(s)\rangle \\ &= \langle\left(T_\varepsilon^* R T_\varepsilon - R\right)x(s), x(s)\rangle .\end{aligned}$$

From inequality (3.3), we get

$$\begin{aligned}\Delta V(x(s)) &\leqslant -\langle E(\alpha)x(s), E(\alpha)x(k)\rangle \\ &\quad -\varepsilon^2\left\langle\left(B^*RB+\varepsilon^2 I\right)^{-1} B^*RTx(s), \left(B^*RB+\varepsilon^2 I\right)^{-1} B^*RTx(s)\right\rangle \\ &\leqslant -\varepsilon^2\left\langle\left(B^*RB+\varepsilon^2 I\right)^{-1} B^*RTx(s), \left(B^*RB+\varepsilon^2 I\right)^{-1} B^*RTx(s)\right\rangle .\end{aligned}$$

Thus,

$$V(x(s+1)) - V(x(0)) \leqslant -\varepsilon^2 \sum_{j=0}^{s} \left\| \left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RTx(j) \right\|^2 .$$

Since $R \succcurlyeq 0$, we get

$$\varepsilon^2 \sum_{j=0}^{s} \left\| \left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RTx(j) \right\|^2 \leqslant V(x(0)), \text{ for } \quad s \geq 0,$$

which implies that

$$\sum_{j=0}^{+\infty} \left\| \left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RTx(j) \right\|^2 < +\infty.$$

Also, we have

$$\begin{aligned} x(s+1) &= T_\varepsilon x(s) \\ &= Tx(k) - B\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RTx(s). \end{aligned}$$

So, the solution $x(k)$ of system (3.5) is given by

$$x(s) = T^s x(0) - \sum_{j=0}^{s-1} T^{s-j-1} B \left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RTx(j),$$

and therefore

$$\|x(s)\| \leqslant \|T^s x(0)\| + \left\| \sum_{j=0}^{s-1} T^{s-j-1} B \left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RTx(j) \right\|.$$

Moreover, we get

$$\begin{aligned} \|x(s)\|^2 &\leqslant 2\left\|T^s x(0)\right\|^2 + 2\left\| \sum_{j=0}^{s-1} T^{k-j-1} B \left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RTx(j) \right\|^2 \\ &\leqslant 2\beta a^s \left\|x(0)\right\|^2 + 2\beta \|B\|^2 \sum_{j=0}^{s-1} a^{s-1-j} \left\| \left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RTx(j) \right\|^2 . \end{aligned}$$

Set $K_1 := 2\beta \left\|x(0)\right\|^2, \quad K_2 := 2\beta \|B\|^2$. Hence

$$\|x(s)\|^2 \leqslant K_1 a^k + K_2 \sum_{j=0}^{s-1} a^{k-1-j} \left\| \left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RTx(j) \right\|^2 ,$$

and hence

$$\sum_{k=0}^{\infty} \|x(k)\|^2 \leqslant K_1 \sum_{s=0}^{\infty} a^s + K_2 \sum_{s=0}^{\infty} \sum_{j=0}^{s-1} {a'}^{s-1-j} \left\| \left(B^* R B + \varepsilon^2 I\right)^{-1} B^* R T x(j) \right\|^2 .$$

Accordingly

$$\sum_{s=0}^{\infty} \|x(s)\|^2 \leqslant \frac{K_1}{1-a} + K_2 \sum_{j=0}^{\infty} a^{-1-j} \left\| \left(B^* R B + \varepsilon^2 I\right)^{-1} B^* R T x(j) \right\|^2 \frac{1}{1-a} a^{j+1},$$

which implies that

$$\sum_{s=0}^{\infty} \|x(k)\|^2 \leqslant K_1 \frac{1}{1-a} + K_2 \frac{1}{1-a} \sum_{j=0}^{\infty} \left\| \left(B^* R B + \varepsilon^2 I\right)^{-1} B^* R T x(j) \right\|^2 .$$

Therefore $x(.) \in \mathcal{L}^2(\mathbb{N}, \mathcal{Z})$, and as a consequence T_ε is stable.

According to the inequality (3.3), we obtain

$$T_\varepsilon^* R T_\varepsilon - R + E^*(\alpha) E(\alpha) \preccurlyeq -\varepsilon^2 \left(\left(B^* R B + \varepsilon^2 I\right)^{-1} B^* R T \right)^* \left(\left(B^* R B + \varepsilon^2 I\right)^{-1} B^* R T \right).$$

Consequently,

$$T_\varepsilon^* R T_\varepsilon - R + E^*(\alpha) E(\alpha) \preccurlyeq 0,$$

and

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j} \right)^2 D_j^* R D_j \succcurlyeq 0, \quad j \in \overline{N}.$$

Also,

$$F_\varepsilon = -\left(B^* R B + \varepsilon^2 I\right)^{-1} B^* R T \in \bar{\mathcal{S}}.$$

By the applicability of the Corollary 2.1 we obtain that $\mathfrak{d} \leqslant \mathfrak{r}_\omega(T_\varepsilon, (D_i, E_i)_{i \in \bar{N}})$, and if inequality (3.4) is strict, we get

$$\mathfrak{d} < \mathfrak{r}_\omega(T_\varepsilon, (D_i, E_i)_{i \in \bar{N}}).$$

□

The following Lemma plays an essential role in investigating the maximization problem.

Lemma 3.2. *Let $\alpha \in (0, +\infty)^N$, $\varepsilon > 0$ and $F \in \bar{\mathcal{S}}$. If the inequality (3.1) has a solution $R_0 \in \mathcal{L}^+(\mathcal{Z})$ holding requirement (3.2), then*

$$F_0 = -\left(B^* R_0 B + \varepsilon^2 I\right)^{-1} B^* R_0 T \in \bar{\mathcal{S}} \ \textit{and} \ \mathfrak{d} \leqslant \mathfrak{r}_\omega(T + B F_0, (D_i, E_i)_{i \in \bar{N}}).$$

In addition, there exists $R_1 \in \mathcal{L}^+(\mathcal{Z})$ *such that*

$$\begin{cases} (T+BF_0)^* R_1 (T+BF_0) - R_1 + E^*(\alpha)E(\alpha) \\ +\varepsilon^2 \left((B^*R_0B+\varepsilon^2 I)^{-1} B^*R_0T \right)^* \left((B^*R_0B+\varepsilon^2 I)^{-1} B^*R_0T \right) = 0, \\ I - \lambda_j \left(\dfrac{\mathfrak{d}}{\alpha_j} \right)^2 D_j^* R_1 D_j \succcurlyeq 0, \ j \in \overline{N}, \\ R_1 \preccurlyeq R_0. \end{cases}$$

Proof. We have

$$\begin{aligned} (T+BF)^* R_0 (T+BF) - R_0 &+ E^*(\alpha)E(\alpha) + \varepsilon^2 F^*F = T^*R_0T - R_0 + E^*(\alpha)E(\alpha) \\ &+ T^*R_0BF + F^*B^*R_0T + F^*(\varepsilon^2 I + B^*R_0B)F. \end{aligned}$$

Set $\grave{F} := (B^*R_0B+\varepsilon^2 I) F + B^*R_0T$, we get

$$\begin{aligned} \grave{F}^* \left(B^*R_0B+\varepsilon^2 I\right)^{-1} \grave{F} &= F^* \left(B^*R_0B+\varepsilon^2 I\right) F + F^*B^*R_0T \\ &+ T^*R_0BF + T^*R_0B \left(BR_0B^*+\varepsilon^2 I\right)^{-1} B^*R_0T. \end{aligned}$$

Hence

$$\begin{aligned} T^*R_0T - R_0 + E^*(\alpha)E(\alpha) &+ T^*R_0BF + F^*B^*R_0T + F^*(\varepsilon^2 I + B^*R_0B)F = T^*R_0T - R_0 \\ &+ E^*(\alpha)E(\alpha) + T^*R_0BF + F^*B^*R_0T + \grave{F}^* \left(B^*R_0B+\varepsilon^2 I\right)^{-1} \grave{F} \\ &- F^*B^*R_0T - T^*R_0BF - T^*R_0B \left(BR_0B^*+\varepsilon^2 I\right)^{-1} B^*R_0T. \end{aligned}$$

Since R_0 solves (3.1), it follows that

$$\begin{gathered} T^*R_0T - R_0 + E^*(\alpha)E(\alpha) - T^*R_0B (B^*R_0B+\varepsilon^2 I)^{-1} B^*R_0T \\ +\grave{F}^* (B^*R_0B+\varepsilon^2 I)^{-1} \grave{F} \preccurlyeq 0. \end{gathered} \tag{3.6}$$

So,

$$T^*R_0T - R_0 + E^*(\alpha)E(\alpha) - T^*R_0B \left(BR_0B^*+\varepsilon^2 I\right)^{-1} B^*R_0T \preccurlyeq -\grave{F}^* \left(B^*R_0B+\varepsilon^2 I\right)^{-1} \grave{F}.$$

Since $R_0 \succcurlyeq 0$ then $(BR_0B^*+\varepsilon^2 I)^{-1} \succcurlyeq 0$, so inequality (3.6) implies that

$$T^*R_0T - R_0 + E^*(\alpha)E(\alpha) - T^*R_0B \left(BR_0B^*+\varepsilon^2 I\right)^{-1} B^*R_0T \preccurlyeq 0.$$

Letting

$$S := \left(T - B\left(B^*R_0B+\varepsilon^2 I\right)^{-1} B^*R_0T\right)^* R_0 \left(T - B\left(B^*R_0B+\varepsilon^2 I\right)^{-1} B^*R_0T\right),$$

we obtain

$$\begin{aligned} S &= T^*R_0T - T^*R_0B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T - T^*R_0B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T \\ &+ T^*R_0B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T. \end{aligned}$$

Also, for $\grave{S} := T^*R_0T - T^*R_0B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T$, we get

$$\begin{aligned} \grave{S} &= S + T^*R_0B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T \\ &- T^*R_0B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T \\ &= S + T^*R_0B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}\left[B^*R_0B+\varepsilon^2 I - B^*R_0B\right]\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T. \end{aligned}$$

Thus,

$$\begin{aligned} T^*R_0T - T^*R_0BG_\varepsilon B^*R_0T &= \left(T - BG_\varepsilon B^*R_0T\right)^* R_0\left(T - BG_\varepsilon B^*R_0T\right) \\ &+ \varepsilon^2 T^*R_0BG_\varepsilon\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T \end{aligned}$$

where $G_\varepsilon := \left(B^*R_0B+\varepsilon^2 I\right)^{-1}$. Setting $F_0 := -\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T$ and $T_0 := T - B\left(B^*R_0B+\varepsilon^2 I\right)^{-1}B^*R_0T$, we get

$$T_0^*R_0T_0 - R_0 + E^*(\alpha)E(\alpha) + \varepsilon^2 F_0^*F_0 \preccurlyeq 0. \tag{3.7}$$

Applying Lemma 3.1, we infer that $F_0 \in \bar{F}$ and $\mathfrak{d} \leqslant \mathfrak{r}_\omega(T_0, (D_i, E_i)_{i\in\bar{N}})$.

Now, set $\tilde{E}(\alpha) := \begin{pmatrix} E(\alpha) \\ \varepsilon F_0 \end{pmatrix}$, we obtain

$$T_0^*R_0T_0 - R_0 + \tilde{E}^*(\alpha)\tilde{E}(\alpha) \preccurlyeq 0.$$

According to Corollary 2.1, there exists $R_1 \in \mathcal{L}^+(\mathcal{Z})$ such that

$$T_0^*R_1T_0 - R_1 + \tilde{E}^*(\alpha)\tilde{E}(\alpha) = 0, \quad \text{where} \quad R_1 \preccurlyeq R_0.$$

Consequently,

$$T_0^*R_1T_0 - R_1 + E^*(\alpha)E(\alpha) + \varepsilon^2 F_0^*F_0 = 0,$$

and

$$I - \lambda_j\left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^*R_1D_j \succcurlyeq 0, \quad j \in \overline{N}.$$

□

By the repetitively application of the lemma, we demonstrate in the next theorem that there exists $R \in \mathcal{L}^+(\mathcal{Z})$ such that

$$T^*RT - R + E^*(\alpha)E(\alpha) - T^*RB\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT = 0, \tag{3.8}$$

and

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R D_j \succcurlyeq 0, \quad j \in \overline{N}.$$

Theorem 3.1. *Let $F \in \bar{\mathcal{S}}$. Consider that there exist $\mathfrak{d} > 0$, $\alpha \in (0, +\infty)^N$ and $\varepsilon > 0$ such that the Lyapunov inequality (3.1) admits a solution $R_0 \in \mathcal{L}^+(\mathcal{Z})$ which holds the requirement (3.2). Then the Riccati equation (3.8) has a solution $R \in \mathcal{L}^+(\mathcal{Z})$ fulfilling*

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R D_j \succcurlyeq 0, \quad j \in \overline{N},$$

$$F_\varepsilon = -\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT \in \bar{\mathcal{S}}.$$

Moreover, $\mathfrak{d} \leqslant \mathfrak{r}_\omega(T + BF_\varepsilon, (D_i, E_i)_{i \in \bar{N}})$.

Proof. Applying Lemma 3.2 iteratively, we construct a sequence of linear operators $(R_k)_{k\in\mathbb{N}} \in \mathcal{L}^+(\mathcal{Z})$ which fulfills

$$\begin{aligned} & T_k^* R_{k+1} T_k - R_{k+1} + E^*(\alpha)E(\alpha) \\ + \quad & \varepsilon^2 \left(\left(B^*R_kB + \varepsilon^2 I\right)^{-1} B^*R_kT\right)^* \left(\left(B^*R_kB + \varepsilon^2 I\right)^{-1} B^*R_kT\right) = 0, \end{aligned}$$

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R_{k+1} D_j \succcurlyeq 0, \quad j \in \overline{N}, \quad R_{k+1} \preccurlyeq R_k,$$

where R_0 solves the inequality (3.1) and $T_k = T - B\left(B^*R_kB + \varepsilon^2 I\right)^{-1} B^*R_kT$. For $R := \lim\limits_{k\to+\infty} R_k$, we get

$$\begin{aligned} & T_\varepsilon^* R T_\varepsilon - R + E^*(\alpha)E(\alpha) \\ & + \varepsilon^2 \left(\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT\right)^* \left(\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT\right) = 0, \end{aligned}$$

and

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R D_j \succcurlyeq 0, \quad j \in \overline{N},$$

where $T_\varepsilon := T - B\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT$.

Lemma 3.1 gives

$$\begin{cases} F_\varepsilon = -\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT \in \bar{\mathcal{S}}, \\ \mathfrak{d} \le \mathfrak{r}_\omega(T + BF_\varepsilon, (D_i, E_i)_{i\in\bar{N}}). \end{cases}$$

Finally, from

$$\begin{aligned} T_\varepsilon^* R T_\varepsilon - R \quad &+ \quad E^*(\alpha)E(\alpha) + \varepsilon^2 \left(\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT\right)^* \left(\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT\right) \\ &= \quad T^*RT - R + E^*(\alpha)E(\alpha) - T^*RB\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT, \end{aligned}$$

we confirm that R satisfies the Riccati equation (3.8). □

We provide requirements for the existence of sub-optimal controllers based on the above results.

Proposition 3.1. *Let $\mathfrak{d} > 0$. Suppose that there exists $F \in \bar{F}$ such that $\mathfrak{d} < \mathfrak{r}_\omega(T + BF, (D_i, E_i)_{i\in\bar{N}})$. Then there exist $\alpha \in (0, +\infty)^N$, $\varepsilon > 0$, such that the Riccati equation (3.8) has a solution $R \in \mathcal{L}^+(\mathcal{Z})$ fulfilling*

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R D_j \succcurlyeq 0, \quad j \in \overline{N},$$

$$F_\varepsilon = -\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT \in \bar{\mathcal{S}}.$$

Proof. Since $\mathfrak{d} < \mathfrak{r}_\omega(T+BF, (D_i, E_i)_{i\in\bar{N}})$, there exists $\grave{\mathfrak{d}}$ such that $\mathfrak{d} < \grave{\mathfrak{d}} < \mathfrak{r}_\omega(T + BF, (D_i, E_i)_{i\in\bar{N}})$. Furthermore, there exists $\alpha \in (0, +\infty)^N$ such that

$$\grave{\mathfrak{d}} \leqslant \left(\max_{j\in\overline{N}} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* R(\alpha) D_j \right\|\right)^{-\frac{1}{2}},$$

where $R(\alpha)$ solves the equation

$$(T + BF)^* R (T + BF) - R + E^*(\alpha)E(\alpha) = 0.$$

Thus,

$$\mathfrak{d}^{-2} > \left(\max_{j\in\overline{N}} \left\| \frac{\lambda_j}{\alpha_j^2} D_j^* R(\alpha) D_j \right\|\right),$$

which leads to

$$I - \mathfrak{d}^2 \frac{\lambda_j}{\alpha_j^2} D_j^* R(\alpha) D_j \succ 0, \quad j \in \overline{N}.$$

Hence

$$\lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* \left[\sum_{t=0}^{+\infty} \left(T_F^t\right) E^*(\alpha)E(\alpha)T_F^t\right] D_j \prec I, \quad j \in \overline{N}, \quad \text{where } \; T_F = T + BF.$$

Let $\beta_j := \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2$. Assume that R_0 solves the Lyapunov equation

$$(T + BF)^* R (T + BF) - R + F^*F = 0.$$

Let $\varepsilon > 0$, such that $\beta_j M_0 \varepsilon^2 < 1 - \beta_j M$, with

$$M := \max_{j \in \overline{N}} \left\| D_j^* R(\alpha) D_j \right\|, \quad M_0 := \max_{j \in \overline{N}} \left\| D_j^* R_0 D_j \right\|.$$

Set $R_\varepsilon := R(\alpha) + \varepsilon^2 R_0$, then

$$I - \beta_j D_j^* R_\varepsilon D_j \succ 0, \quad \text{for all } \; u \neq 0.$$

We infer that there exists $\varepsilon > 0$ such that $R_\varepsilon \succcurlyeq 0$ and

$$(T + BF)^* R_\varepsilon (T + BF) - R_\varepsilon + E^*(\alpha)E(\alpha) + \varepsilon^2 F^*F = 0,$$

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R_\varepsilon D_j \succ 0, \quad j \in \overline{N}.$$

Applying Theorem 3.1 we infer that there exists $X \in \mathcal{L}^+(\mathcal{Z})$ holding

$$T^*XT - X + E^*(\alpha)E(\alpha) - T^*XB \left(B^*XB + \varepsilon^2 I\right)^{-1} B^*XT = 0,$$

and

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* X D_j \succ 0, \quad j \in \overline{N},$$

and for which

$$F_\varepsilon = -\left(B^*XB + \varepsilon^2 I\right)^{-1} B^*XT \in \bar{\mathcal{S}}.$$

□

Proposition 3.2. *Let* $\mathfrak{d}$, $\varepsilon > 0$. *Suppose that the Riccati equation (3.8) has a solution* R *in* $\mathcal{L}^+(\mathcal{Z})$ *such that*

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R_\varepsilon D_j \succcurlyeq 0, \quad j \in \bar{N}, \quad \textit{for some} \quad \alpha \in (0, +\infty)^N.$$

Then $F_\varepsilon = -\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT \in \bar{\mathcal{S}}$. *Moreover,* $\mathfrak{d} \leqslant \mathfrak{r}_\omega(T+BF_\varepsilon, (D_i, E_i)_{i \in \bar{N}})$.

Proof. Note that since R is a solution of the Riccati equation (3.8), we have

$$\left(T - B\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT\right)^* R \left(T - B\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT\right) - R$$
$$+E^*(\alpha)E(\alpha) + \varepsilon^2 \left(\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT\right)^* \left(\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT\right) = 0,$$

and

$$I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R D_j \succcurlyeq 0, \quad j \in \overline{N}.$$

Lemma 3.1 shows that

$$F_\varepsilon = -\left(B^*RB + \varepsilon^2 I\right)^{-1} B^*RT \in \bar{\mathcal{S}} \text{ and } \mathfrak{d} \leqslant \mathfrak{r}_\omega(T + BF_\varepsilon, (D_i, E_i)_{i\in\bar{N}}).$$

□

As a consequence of propositions 3.1 and 3.2, the supreme attainable stability radius is characterized via the Riccati equation (3.8) as indicated bellow.

Corollary 3.1. *We have*

$$\bar{r}^\omega(T, (D_i, E_i)_{i\in\bar{N}}) = \sup \left\{ \begin{array}{c} \mathfrak{d} > 0; \ \text{there exist } \ \alpha \in (0, +\infty)^N \quad \text{and} \quad \varepsilon > 0 \\ \text{such that (3.8) has a solution} \quad R \in \mathcal{L}^+(\mathcal{Z}) \\ \text{with} \quad I - \lambda_j \left(\frac{\mathfrak{d}}{\alpha_j}\right)^2 D_j^* R D_j \succcurlyeq 0 \text{ for all } j \in \overline{N} \end{array} \right\}.$$

3.3 Examples

Example 3.1. *We examine the controlled system corresponding to the system of Example 2.1*

$$x(t+1) = Tx(t) + \sum_{i=1}^{N} \delta_i T_i x(t) w_i(t) + Bu(t), \quad t \in , \quad N \in \mathbb{N},$$

where $Bu = u$ for all $u \in l^2$.

Suppose that R solves the Riccati equation (3.8) and that it takes the form bellow:

$$R = \begin{pmatrix} p_1 & 0 & \ldots & 0 & \ldots \\ 0 & p_2 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & p_i & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Then the corresponding Riccati equation is equivalent to

$$\begin{cases} p_{11}^2 + \left(\dfrac{3}{4}\varepsilon^2 - \mu\right) p_{11} - \varepsilon^2 \mu = 0 \\ p_{22}^2 + \left(\dfrac{15}{16}\varepsilon^2 - \mu\right) p_{22} - \varepsilon^2 \mu = 0 \\ \vdots \\ p_{nn}^2 + \left(-\left(12^{2n} - 1\right)\varepsilon^2 - \mu\right) p_{nn} - \varepsilon^2 \mu = 0 \end{cases} , \ \mu = \sum_{i=1}^{n} \alpha_i^2$$

Then the corresponding Riccati equation reads as

$$p_j^2 + \left(-\left(\left(\frac{1}{2}\right)^{2j} - 1\right)\varepsilon^2 - \mu\right) p_j - \varepsilon^2 \mu = 0, \quad \mu = \sum_{i=1}^{N} \alpha_i^2, \ \ j \geq 1,$$

which implies that

$$p_j = \frac{\left(\left(\frac{1}{2}\right)^{2j} - 1\right)\varepsilon^2 + \mu + \sqrt{\triangle_j}}{2}, \ \ \triangle_j = \left(-\left(\left(\frac{1}{2}\right)^{2j} - 1\right)\varepsilon^2 - \mu\right)^2 + 4\varepsilon^2\mu, \ \ j \geq 1.$$

since we have

$$D_1^* R(\alpha^J) D_1 = \begin{pmatrix} \dfrac{-\frac{3}{4}\varepsilon^2 + \mu + \sqrt{\triangle_1,}}{2} & 0 & ... & 0 & ... \\ 0 & 0 & ... & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

$$D_2^* R(\alpha^J) D_2 = \begin{pmatrix} 0 & 0 & ... & 0 & ... \\ 0 & \dfrac{-\frac{15}{16}\varepsilon^2 + \mu + \sqrt{\triangle_2}}{2} & ... & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

So,

$$D_i^* R D_i = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dfrac{\left(\left(\frac{1}{2}\right)^{2i} - 1\right)\varepsilon^2 + \mu + \sqrt{\triangle_i}}{2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \ i \geq 1.$$

The condition $I - \lambda_i \left(\dfrac{\mathfrak{d}}{\alpha_i}\right)^2 D_i^* R D_i \succeq 0, \ \ i \in \overline{N}$, *gives*

$$\mathfrak{d}^2 \leqslant \frac{2\alpha_i^2}{\left(\left(\left(\frac{1}{2}\right)^{2i} - 1\right)\varepsilon^2 + \mu + \sqrt{\left(-\left(\frac{1}{2}^{2i} - 1\right)\varepsilon^2 - \mu\right)^2 + 4\varepsilon^2\mu}\right)\lambda_i}, \quad 1 \leq i \leq N, \ \varepsilon > 0.$$

Thus,

$$\bar{\mathfrak{r}}_\omega\Big(T, (D_i, E_i)_{i\in\bar{N}}\Big) = \left(\min_{i=\{1,\cdots,N\}} \left\{\frac{1}{\lambda_i}\right\}\right)^{\frac{1}{2}}.$$

Inspired from [15], we consider the following example.

Example 3.2. *Suppose a one-dimensional rod with a length of 1 that can be heated along its length according to*

$$\begin{cases} \frac{\partial z(x,t)}{\partial t} = \eta \frac{\partial^2 z(x,t)}{\partial x^2} + b_1(x)u(t) + cb_2(x)z(t,x)w(t), \ \ 0 \leq x \leq 1, \ \ t \geq 0, \\ z(0,t) = 0, \ \ z(1,t) = 0, \ \ t \geq 0, \\ z(x,0) = z_0(x), \ \ 0 \leq x \leq 1, \end{cases} \tag{3.9}$$

where $z(x,t)$ *denotes the temperature of the rod at time* t *and position* x*, however,* $z_0(x)$ *is the initial temperature profile, and* $u(.) \in \mathcal{L}^2((0,\infty),\mathbb{R}))$ *is the addition of heat along the bar,*

$$b_1(x) = \begin{cases} 1 & \text{if } x \in [x_1, x_2], \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \ \ b_2(x) = \begin{cases} 1 & \text{if } x \in [x_3, x_4], \\ 0 & \text{otherwise.} \end{cases}$$

In this case, $(w(t))$ *is the scalar white noise with mean zero and variance* λ*, and* η *represents the thermal diffusivity. In addition,* $[x_1, x_2]$*,* $[x_3, x_4]$ *are the intervals where the input and noise are present and* $c \in \mathbb{R}$*.*

In the space $\mathcal{Z} = \mathcal{L}^2(0,1)$, the system (3.9) can be described in the following abstract form:

$$\begin{cases} \dot{y}(t) = (Ty(t) + B_1u(t)) + D\Lambda E(y(t))w(t), \quad t \geq 0, \\ y(0) = y_0, \end{cases} \tag{3.10}$$

where

$$T = \eta \frac{d^2 z(x,t)}{dx^2},$$

with domain

$$D(T) = \left\{ y \in \mathcal{Z}, \quad \frac{dy(x)}{dx} \in \mathcal{Z}, \quad \frac{d^2y(x)}{dx^2} \in \mathcal{Z}, \quad y(0) = y(1) = 0 \right\},$$

and

$$\begin{aligned} E(y(t)) &= y(t), \quad \Lambda(y(t)) = cy(t), \quad D(z(t)) = B_2 z(t), \\ B_1 u(t) &= b_1(x)u(t), \quad \textit{and} \ B_2 y(t) = b_2(x)y(t). \end{aligned}$$

The operator $B_1 \in \mathcal{L}(\mathbb{R}, Z)$ describes the input of the system and $B_2 \in \mathcal{L}(\mathbb{R}, Z)$ characterizes the noise on the system.

Furthermore, we will utilize the finite difference method to approximate the system in discrete time. To this goal, define

$$\begin{aligned} x_i &= x_0 + i\Delta x, \quad i = 1, 2, \cdots, N, \quad \Delta x = 1/N, \quad x_0 = 0, \quad x_N = 1, \\ t_{k+1} &= t_k + \Delta t, \quad k = 0, \cdots, N_t, \quad t_0 = 0. \end{aligned}$$

This shows that

$$\begin{aligned} \tfrac{z(x_i, t_j + \Delta t) - z(x_i, t_j)}{\Delta t} = \eta \tfrac{z(x_i + \Delta x, t_j) - 2z(x_i, t_j) + z(x_i - \Delta x, t_j)}{\Delta x^2} + B_1 u(t_j) \\ + B_2 z(x_i, t_j) w(t_j), \ 0 \leq j \leq N_t, \end{aligned}$$

and therefore

$$\begin{aligned} z(x_i, t_j + \Delta) &= rz(x_i + \Delta x, t_j) + (1 - 2r)z(x_i, t_j) + rz(x_i - \Delta x, t_j) \\ &+ \ \Delta t B_1 u(t_j) + \Delta t B_2 z(x_i, t_j) w(t_j), \end{aligned}$$

with $r = \frac{\eta \Delta t}{(\Delta x)^2}$. Letting $y(t) = (z(x_1, t), z(x_2, t), ..., z(x_N, t))^T$, we obtain

$$y(t_{j+1}) = Ty(t_j) + \Delta t B_1 u(t_j) + c\Delta t B_2 y(t_j) w(t_j), \quad j \geq 0,$$

where

$$T = \begin{bmatrix} 1-2r & r & 0 & 0 & 0 \\ r & 1-2r & r & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & r & 1-2r & r \\ 0 & 0 & \dots & r & 1-2r \end{bmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 0 \\ 1\vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Note that the non-zero elements of B_1 are located where the mesh points lie in $[x_1, x_2]$, however, the non-zero elements of B_2 are positioned where the mesh points lie in $[x_3, x_4]$.

Assume that $2r < 1$, and thus T is stable. To investigate the maximization problem, we must first solve the following Riccati equation:

$$T^T X T - X + E^T E - T^T X B_1 \left(B_1^T X B_1 + \varepsilon^2 I\right)^{-1} B_1^T X T = 0, \;\; \varepsilon > 0. \qquad (3.11)$$

In this case, we take $\eta = 0.001$, $\lambda = 10$ and $\epsilon = 0.01$. In fact,

$$I - \lambda \mathfrak{d}^2 D^T X D \succ 0 \Longleftrightarrow \mathfrak{d} < \left(\left\|\lambda D^T X D\right\|\right)^{-\frac{1}{2}}$$

Thus,

$$\bar{\mathfrak{r}}(T, D, E) = \sup_{\varepsilon > 0} \left(\left\|\lambda (\Delta t)^2 D^T X D\right\|\right)^{-\frac{1}{2}}.$$

The values of $\bar{\mathfrak{r}}(T, D, E)$ according to the length of the interval $[x_3, x_4]$ are shown in table (3.1).

We observe that $\bar{\mathfrak{r}}(T, D, E)$ decreases according to the length of the interval $[x_3, x_4]$. The larger stability margin is obtained for the smallest interval of noise.

The stabilizing feedback is obtained by the formula

$$F_\varepsilon = -(\Delta t)^{-1} \left(B_1^* X B_1 + \varepsilon^2 I\right)^{-1} B_1.$$

The following figures show F_ε for different intervals $[x_1, x_2]$.

We see that the largest gain is obtained in the smallest interval $[x_1, x_2]$.

Length $([x3, x4])$	$\bar{\mathfrak{r}}(T, D, E)$.
0.05	1.8719
0.15	1.0875
0.25	0.8435
0.35	0.7133
0.45	0.6293
0.55	0.5693
0.65	0.5238
0.75	0.4877
0.85	0.4581
0.95	0.4334

Table 3.1: *Values of* $\bar{\mathfrak{r}}(T, D, E)$

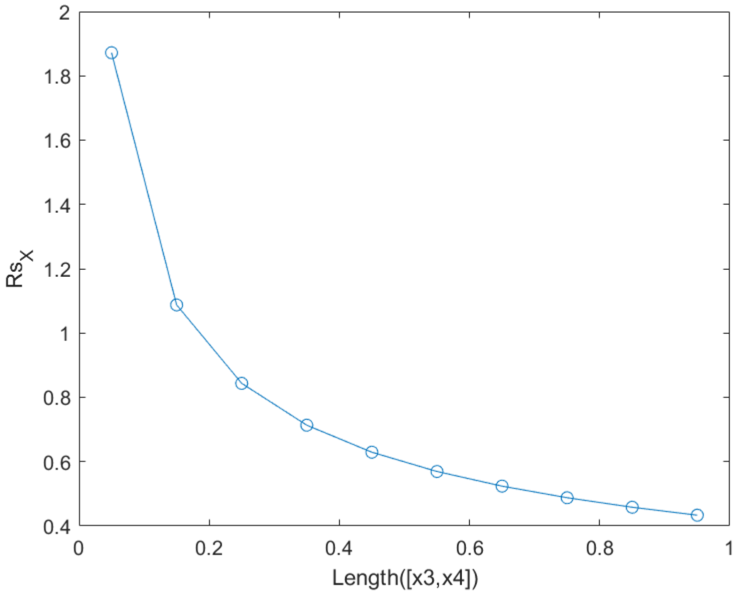

Figure 3.1: rmax the value of $\bar{r}(T, D, E)$

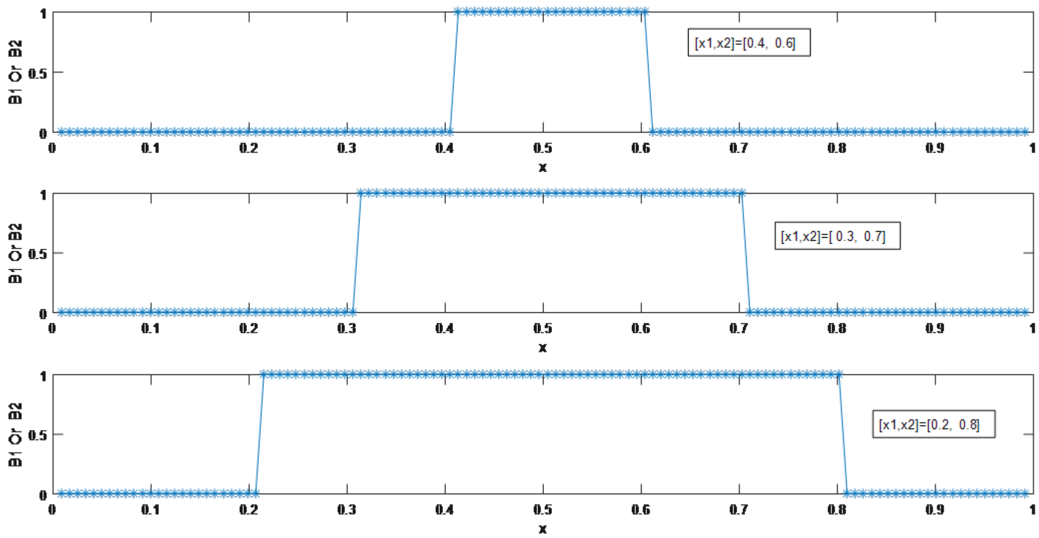

Figure 3.2: B_1 for $x \in [x_1, x_2]$ and B_2 for $x \in [x_3, x_4]$

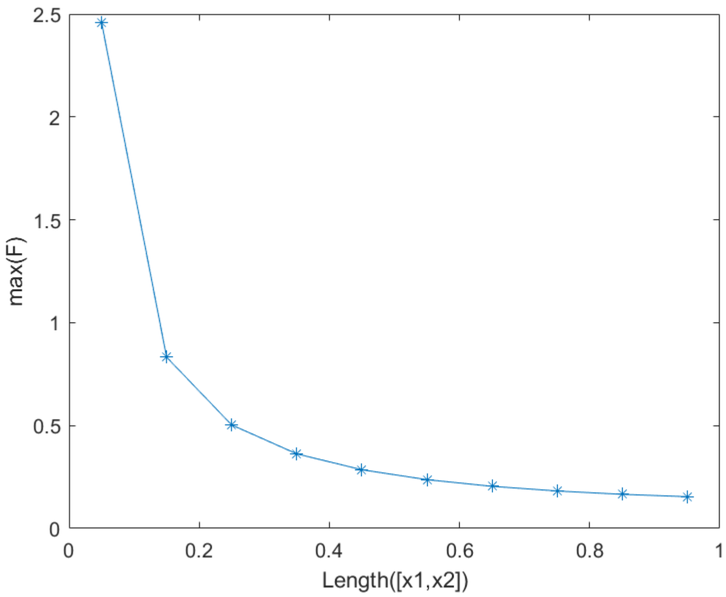

Figure 3.3: $\max |F_\varepsilon|$ for differentS intervales $[x_1, x_2]$

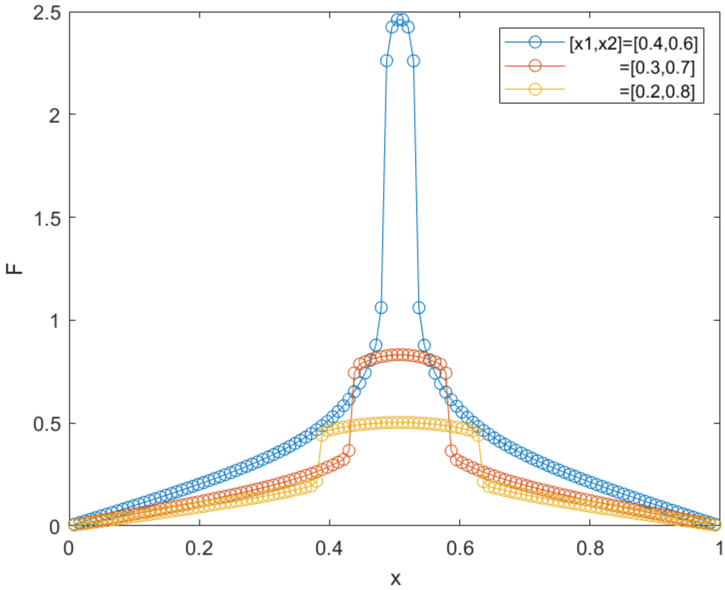

Figure 3.4: F_ε for different intervals $[x_1, x_2]$

4 | Stability radii of discete-time varying systems

Contents

4.1 Introduction

The aim of this chapter is to investigate the robust stability problem for discrete time-varying systems perturbed by stochastic perturbations. Following the approach developed in Chapter 3, we yield characteristics of the corresponding stability radius. Then, we examine the case of periodic systems.

4.2 Bounds of the stability radius

Let $\mathcal{Z}$ be a separable Hilbert space over $\mathbb{K}$ ($\mathbb{K}$=$\mathbb{C}$ or $\mathbb{R}$). Consider the linear time-varying system giving by

$$\begin{cases} x(t+1) = T(t)x(t) + D(t)\Lambda(E(x(t))w(t), \ t \in \mathbb{N}, \\ x(0) = x_0 \in \mathcal{Z}, \\ \| \triangle \|_\infty < \mathfrak{d}, \end{cases} \tag{4.1}$$

where

- $T(.) = (T(t))_{t\in\mathbb{N}}$ is a sequence of bounded linear operators on $\mathcal{Z}$.
- $D(.) \in \mathcal{L}^\infty(\mathbb{N}, \mathcal{L}(U, \mathcal{Z})), E(.) \in \mathcal{L}^\infty((\mathbb{N}, \mathcal{L}(\mathcal{Z}, Y))$ and $\Lambda(.) \in \mathcal{L}^\infty(\mathbb{N}, \mathcal{L}(Y, U)))$.
- $(w(t))_{t\in\mathbb{N}}$ is a real random variable on a complete probability space (Ω, F, P) satisfying that $\mathbb{E}(w(t)) = 0$ and $\mathbb{E}(|w(t)|^2) = \lambda(t)$, $t \geqslant 0$.

Using [32, Lemma7] and according to these hypothesis, the system (4.1) accepts a unique solution given by

$$x(t) = \phi(t,0)x_0 + \sum_{k=0}^{t-1} \phi(t,k+1)D(k)\Lambda(E(x(k))w(k), \ t \geqslant 0.$$

where $(\phi(t,s))$ is the evolution operator related to $(T(t))_{t\in\mathbb{N}}$ and it is defined as

$$\begin{cases} \phi(t,t) = I_{\mathcal{Z}}, \\ \phi(t,s) = T(t-1)T(t-2)\dots T(s), s,t \in \mathbb{N}, t > s, \end{cases}$$

such that $I_{\mathcal{Z}}$ is the identity operator on $\mathcal{Z}$.
Throughout this chapter we assume that the deterministic system

$$\begin{cases} x(t+1) = T(t)x(t), \ t \in \mathbb{N}, \\ x(0) = x_0, \end{cases} \tag{4.2}$$

is uniformly exponentially stable. Therefore the Lyapunov equation

$$T^*(t)R(t+1)T(t) - R(t) + E(t)E^*(t) = 0,$$

has a unique solution $R = (R(t))_{t\in\mathbb{N}}$ such that

$$m\|x\|^2 \leq \langle R(t)x, x\rangle \leq M\|x\|^2$$

for all $t \in \mathbb{N}$ and $x \in \mathcal{Z}$, where $m, \ M > 0$.

Definition 4.1. *The stochastic system (4.1) is uniformly exponentially stable iff there exist $\beta \geqslant 1$, $a \in (0,1)$ such that*

$$\mathbb{E}\|(x(t)\|^2 \leqslant \beta a^t \|x_0\|^2, \ \textit{for all } t \geqslant 0 \ \textit{and} \ x_0 \in \mathcal{Z}.$$

We are now in position to introduce the stability radius of (4.1).

Definition 4.2. *The stochastic stability radius of (4.1) according to the perturbation structure $(D(.), E(.))$ and $(w(t))_{t\in N}$ is defined by*

$$\mathfrak{r}_\omega\big(T(.), (D(.), E(.))\big) := \inf \left\{ \begin{array}{l} \|\Lambda\|_\infty : \ \Lambda(.) \in \mathcal{L}^\infty(\mathbb{N}, \mathcal{L}(Y,U))), \textit{such that} \\ (4.1) \ \textit{is not uniformly exponentially stable} \end{array} \right\}.$$

Lemma 4.1. *Assume that $E(.) \in \mathcal{L}^\infty((\mathbb{N}, \mathcal{L}(\mathcal{Z}, Y))$ and*

$$y(t) = E(t)\phi(t,0)x_0 + \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k), \ t \geqslant 0, \tag{4.3}$$

where $v \in l^2_\omega(\mathbb{N}, l^2(\Omega, U))$. Then

$$\mathbb{E}\left(\|y(t)\|^2\right) = \|E(t)\phi(t,0)x(0)\|^2 + \sum_{k=0}^{t-1} \lambda(k)\mathbb{E}\,\|E(t)\phi(t,k+1)D(k)v(k)\|^2, \ t \geqslant 0.$$

Moreover $y(.) \in l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$ *and*

$$\|y\|^2_{l^2_\omega} = \sum_{t=0}^{\infty} \mathbb{E}\left(\|y(t)\|^2\right) = \sum_{t=0}^{\infty} \|E(t)\phi(t,0)x(0)\|^2 + \sum_{k=0}^{\infty} \lambda(k)\mathbb{E}\left\langle D(k)v(k), R(k+1)D(k)v(k)\right\rangle.$$

where $(R(t))_{t\in\mathbb{N}}$ *is a sequence in* $\mathcal{Z}$ *satisfying*

$$T^*(t)R(t+1)T(t) - R(t) + E(t)E^*(t) = 0,\ t \geqslant 0. \tag{4.4}$$

Proof. We have for $t \geqslant 0$,

$$y(t) = E(t)\phi(t,0)x_0 + \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k),\ t \geqslant 0.$$

and

$$\|y(t)\|^2 = \langle y(t), y(t)\rangle = \Big\langle E(t)\phi(t,0)x_0 + \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k)), \\ E(t)\phi(t,0)x_0 + \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k)\Big\rangle$$

So

$$\begin{aligned}
\|y(t)\|^2 &= \langle E(t)\phi(t,0)x_0, E(t)\phi(t,0)x_0\rangle \\
&+ \left\langle E(t)\phi(t,0)x_0, \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k)\right\rangle \\
&+ \left\langle \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k), E(t)\phi(t,0)x_0\right\rangle \\
&+ \Big\langle \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k), \\
&\sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k)\Big\rangle \\
&= \|E(t)\phi(t,0)x_0\|^2 + \langle E(t)\phi(t,0)x_0, \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k)\rangle \\
&+ \langle\sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k), E(t)\phi(t,0)x_0\rangle \\
&+ \left\|\sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k)\right\|^2,
\end{aligned}$$

Then

$$
\begin{aligned}
\mathbb{E}\langle y(t), y(t)\rangle &= \mathbb{E}\left(\langle E(t)\phi(t,0)x_0, E\phi(t,0)x_0\rangle\right) \\
&+ \mathbb{E}\left\langle \langle E(t)\phi(t,0)x_0, \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k) \right\rangle \\
&+ \mathbb{E}\left\langle \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k), E(t)\phi(t,0)x_0 \right\rangle \\
&+ \mathbb{E}\left\| \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k) \right\|^2 .
\end{aligned}
$$

We have

$$
\mathbb{E}\left(\left\langle \langle \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k), \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k) \right\rangle\right) = 0, \text{for } i \neq j
$$

For $i = j$

$$
\mathbb{E}\left(\begin{array}{c} \left\langle \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k), \right. \\ \left. \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k) \right\rangle \end{array} \right) = \sum_{k=0}^{t-1} \lambda(k)\mathbb{E}\left(\|E(t)\phi(t,k+1)D(k)v(k)\|^2\right).
$$

Then

$$
\mathbb{E}\left(\left\| \sum_{i=1}^{N}\sum_{k=0}^{t-1} E(t)\phi(t,k+1)(k)v(k)w(k) \right\|^2\right) = \sum_{i=1}^{N}\sum_{k=0}^{t-1} \lambda(k)\mathbb{E}\left\|E(t)\phi(t,k+1)D(k)v(k)\right\|^2 .
$$

Since the nominal system (4.2) is uniformly exponentially stable it follows that

$$
\exists \beta \geqslant 1, a \in (0,1) : \|\phi(t,0)\|^2 \leqslant \beta a^t,\ \ t \geq 0.
$$

It yields

$$
\begin{aligned}
\sum_{t=0}^{\infty} \|E(t)\phi(t,0)x_0\|^2 &\leqslant \sum_{t=0}^{\infty} \|E(t)\| \, \|x_0\| \, \beta a^t \\
&\leqslant \beta \sup_{t \geqslant 0} \|E(t)\| \, \|x_0\| \sum_{t=0}^{\infty} a^t \\
&\leqslant \beta \sup_{t \geqslant 0} \|E(t)\| \, \|x_0\| \frac{1}{1-a} < \infty.
\end{aligned}
$$

It Holds that

$$\sum_{t=0}^{\infty} \|E(t)\phi(t,0)x_0\|^2 < \infty \tag{4.5}$$

and

$$\begin{aligned}\sum_{k=0}^{t-1} \|E(t)\phi(t,k+1)D(k)v(k)\|^2 &\leqslant \sum_{k=0}^{t-1} \|E(t)\|^2 \|\phi(t,k+1)\|^2 \|D(k)\|^2 \|v(k)\|^2 \\ &\leqslant \sup_{k\geqslant 0} \|D(k)\|^2 \sup_{t\geqslant 0} \|E(t)\|^2 \sum_{k=0}^{t-1} \beta a^{t-k-1} \|v(k)\|^2 .\end{aligned}$$

Set $M = \beta \sup\limits_{k\geqslant 0} \|D(k)\|^2 \sup\limits_{t\geqslant 0} \|E(t)\|^2$, then we get

$$\begin{aligned}\sum_{t=0}^{\infty}\sum_{k=0}^{t-1} \lambda(k)\mathbb{E}\|E(t)\phi(t,k+1)D(k)v(k)\|^2 &\leqslant \sup_{k\geqslant 0}(\lambda(k)) \sum_{t=0}^{\infty} M \sum_{k=t_0}^{t-1} a^{t-k-1}\mathbb{E}\|v(k)\|^2 \\ &\leqslant M \sup_{k\geqslant 0}(\lambda(k)) \sum_{t=0}^{\infty}\sum_{k=0}^{t-1} a^{t-k-1}\mathbb{E}\|v(k)\|^2 \\ &\leqslant M \sup_{k\geqslant 0}(\lambda(k)) \sum_{k=0}^{\infty} \mathbb{E}\|v(k)\|^2 (\sum_{t=k+1}^{\infty} a^t)a^{-k-1} \\ &\leqslant M \sup_{k\geqslant 0}(\lambda(k)) \sum_{k=0}^{\infty} (\mathbb{E}\|v(k)\|^2)\frac{1}{1-a}a^{k+1}a^{-k-1} \\ &\leqslant \left(\frac{M}{1-a}\right) \sup_{k\geqslant 0}(\lambda(k)) \sum_{k=0}^{\infty}(\mathbb{E}\|v(k)\|^2) \\ &= \left(\frac{M}{1-a}\right) \sup_{k\geqslant 0}(\lambda(k)) \|v\|_{L^2_\omega} .\end{aligned}$$

Hence

$$\sum_{t=0}^{\infty}\sum_{k=0}^{t-1} \lambda(k)\mathbb{E}\|E(t)\phi(t,0)Dv(k)\|^2 \leqslant \widehat{M}\|v\|_{L^2_\omega}, \ \widehat{M} > 0.$$

From this inequality and the inequality (4.5) it can be deduced that $y(.) \in l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$.

$$\mathbb{E}\left(\|y(t)\|^2\right) = \|E(t)\phi(t,0)x(0)\|^2 + \sum_{k=0}^{t-1} \lambda(k)\mathbb{E}\|E(t)\phi(t,k+1)D(k)v(k)\|^2 .$$

Then

$$\begin{aligned}\sum_{t=0}^{\infty}\mathbb{E}\left(\|y(t)\|^2\right) &= \sum_{t=0}^{\infty}\|E(t)\phi(t,0)x(0)\|^2 \\ &+ \sum_{k=0}^{\infty}\lambda(k)\mathbb{E}\left\langle D(k)v(k), \sum_{t=k+1}^{\infty}(\phi(t,k+1))^*E^*(t)E(t)\phi(t,k+1)D(k)v(k)\right\rangle. \\ &= \sum_{t=0}^{\infty}\|E(t)\phi(t,0)x(0)\|^2 \\ &+ \sum_{k=0}^{\infty}\lambda(k)\mathbb{E}\left\langle D(k)v(k), R(k+1)D(k)v(k)\right\rangle.\end{aligned}$$

where $R(t)$ is the solution of (4.4) and it is given by

$$R(t) = \sum_{t=k}^{\infty}(\phi(t,k))^*E^*(t)E(t)\phi(t,k).$$

□

Let the input-output operator

$$L : l^2_\omega(\mathbb{N}, l^2(\Omega, U)) \longrightarrow l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$$

$$(Lv)(t) = \left(L\begin{pmatrix} v_1 \\ \vdots \\ v_n\end{pmatrix}\right)(t) = \sum_{k=0}^{t-1}E(t)\phi(t,k+1)D(k)v(k)w(k).$$

From Lemma 4.1 we get that $Lv \in l^2_\omega(\mathbb{N}, \mathcal{L}^2(\Omega, Y))$ for every $v \in l^2_\omega(\mathbb{N}, \mathcal{L}^2(\Omega, U))$.

Lemma 4.2. *We have*

$$\|L\| \leqslant \left(\sup_{k\geqslant 0}\lambda(k)\|D^*(k)R(k+1)D(k)\|\right)^{\frac{1}{2}},$$

where $R(.)$ satisfies (4.4).

Proof. Let $v \in l^2_\omega(\mathbb{N}, l^2(\Omega, U))$. From Lemma 4.1 we have

$$\begin{aligned}
\|Lv\|^2_{l^2_\omega} &= \sum_{t=0}^{\infty} \mathbb{E} \left\langle \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k), \right. \\
&\quad \left. \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k) \right\rangle . \\
&= \sum_{k=0}^{+\infty} \lambda(k)\mathbb{E} \left\langle D(k)v(k), \sum_{t=k+1}^{\infty} \left(\phi(t,k+1)\right)^* E^*(t)E(t)\phi(t,k+1)D(k)v(k) \right\rangle \\
&= \sum_{k=0}^{\infty} \lambda(k)\mathbb{E} \left\langle D(k)v(k), R(k+1)D(k)v(k) \right\rangle \\
&= \sum_{k=0}^{\infty} \lambda(k)\mathbb{E} \left\langle v(k), D^*(k)R(k+1)D(k)v(k) \right\rangle .
\end{aligned}$$

we get that

$$\begin{aligned}
\|Lv\|^2_{L^2_\omega} &\leqslant \sup_{k\geqslant 0}(\lambda(k)) \sum_{k=0}^{\infty} \mathbb{E}\,\|v(k)\|^2 \,\|D^*(k)R(k+1)D(k)\| \\
&\leqslant \left(\sup_{k\geqslant 0} \lambda(k)\,\|D^*(k)R(k+1)D(k)\| \right) \sum_{k=0}^{\infty} \mathbb{E}\,\|v(k)\|^2 .
\end{aligned}$$

Then it holds that

$$\|Lv\|^2_{L^2_\omega} \leqslant \left(\sup_{k\geqslant 0} \lambda(k)\,\|D^*(k)R(k+1)D(k)\| \right) \|v\|^2_{L^2_\omega} .$$

We deduce that

$$\frac{\|Lv\|^2_{L^2_\omega}}{\|v\|^2_{L^2_\omega}} \leqslant \left(\sup_{k\geqslant 0}(\lambda(k))\,\|D^*(k)R(k+1)D(k)\| \right).$$

Thus

$$\|L\| \leqslant \left(\sup_{k\geqslant 0} \lambda(k)\,\|D^*(k)R(k+1)D(k)\| \right)^{\frac{1}{2}} .$$

which concludes the proof. □

Theorem 4.1. *Assume that $R(.) \in \mathcal{L}^\infty(\mathbb{N}, \mathcal{L}(\mathcal{Z}))$ is such that*

$$T^*(t)R(t+1)T(t) - R(t) + E^*(t)E(t) = 0, \; t \in \mathbb{N}, \tag{4.6}$$

$$I - \mathfrak{d}^2\lambda(t)D^*(t)R(t+1)D(t) \succeq 0, \; t \geqslant 0. \tag{4.7}$$

Then $\mathfrak{r}_\omega(T(.), (D(.), E(.))) \geqslant \mathfrak{d}$.

Proof. Let $\Lambda(.) \in \mathcal{L}^\infty(\mathbb{N}, \mathcal{L}(Y,U))$, $\|\Lambda(.)\|_\infty < \mathfrak{d}$. Let $R(.) \in \mathcal{L}^+(\mathcal{Z})$ such that conditions (4.6)and (4.7) hold. We have

$$x(t) = \phi(t,0)x_0 + \sum_{k=0}^{t-1} \phi(t,k+1)D(k)\Lambda(E(k)x(k))w(k), \ t \geqslant 0.$$

Set $y(t) = E(t)x(t)$ and $v(t) = \Lambda(y(t))$, then

$$y(t) = E(t)\phi(t,0)x_0 + \sum_{k=0}^{t-1} E(t)\phi(t,k+1)D(k)v(k)w(k), \ t \geqslant 0.$$

Let $N_0 \in \mathbb{N}^*$. Choose $v_{N_0}(t) \in L^2_\omega(\mathbb{N}, \mathcal{L}^2(\Omega, U))$ such that

$$v_{N_0}(t) = \begin{cases} v(t) = \Lambda(y(t)) & \text{if } t < N_0, \\ 0, \ \text{t} > N_0. \end{cases}$$

Then

$$\begin{aligned}
\|v_{N_0}\|^2_{L^2_\omega} &= \sum_{t=0}^{+\infty} \mathbb{E}(\|v_{N_0}(t)\|^2) \\
&= \sum_{t=0}^{N_0} \mathbb{E}(\|v(t)\|^2) \\
&= \sum_{t=0}^{N_0} \mathbb{E}\left(\|\Lambda(y(t))\|^2\right) \\
&\leqslant \sum_{t=0}^{N_0} \mathbb{E}\left(\|\Lambda\|^2_\infty \|(y(t))\|^2\right) \\
&\leqslant \sum_{t=0}^{N_0} \left(\|\Lambda\|^2_\infty \mathbb{E}\left(\|(y(t))\|^2\right)\right) \\
&\leqslant \|\Lambda\|^2_\infty \sum_{t=0}^{N_0} \left(\mathbb{E}\left(\|(y(t))\|^2\right)\right), \text{ for all } N_0 > 0.
\end{aligned}$$

Hence

$$\|v_{N_0}\|^2_{L^2_\omega} \leqslant \|\Lambda\|^2_\infty \sum_{t=0}^{+\infty} \left(\mathbb{E}\left(\|(y(t))\|^2\right)\right). \tag{4.8}$$

Now suppose that y_{N_0} is the output of the system $(T(.), (D(.), E(.)))$ engendered by the input $v_{N_0}(t)$ with the initial $x(0) = x_0$. Then

$$y_{N_0}(t) = E(t)\phi(t,0)x_0 + Lv_{N_0}(t),\ t \in \mathbb{N}. \tag{4.9}$$

Thus

$$\begin{aligned}\left(\sum_{t=0}^{N_0}\left(\mathbb{E}\left(\|(y(t))\|^2\right)\right)\right)^{\frac{1}{2}} &\leqslant \|y_{N_0}\|_{L^2_\omega} \\ &\leqslant \|E(t)\phi(t,0)x_0\| + \|L\|\,\|v_{N_0}\|.\end{aligned}$$

This inequality and inequality (4.8) yields to

$$\left(\sum_{t=0}^{N_0}\left(\mathbb{E}\left(\|(y(t))\|^2\right)\right)\right)^{\frac{1}{2}} \leqslant \|E(t)\phi(t,0)x_0\| + \|L\|\,\|\Lambda\|_\infty\left(\sum_{t=0}^{N_0}\left(\mathbb{E}\left(\|(y(t))\|^2\right)\right)\right)^{\frac{1}{2}}. \tag{4.10}$$

Condition (4.7) gives

$$\begin{aligned}I - \mathfrak{d}^2\lambda(t)D^*(t)R(t+1)D(t) \succeq 0 &\Rightarrow 1 - \mathfrak{d}^2\lambda(t)\,\|D(t)^*R((t+1))D(t)\| \geqslant 0 \\ &\Rightarrow \mathfrak{d}^2\lambda(t)\,\|D^*(t)R(t+1)D(t)\| \leqslant 1 \\ &\Rightarrow \lambda(t)\,\|D^*(t)R(t+1)D(t)\| \leqslant \mathfrak{d}^{-2}.\end{aligned}$$

for $t \in \mathbb{N}$. We conclude that

$$\sup_{t\geqslant 0}\lambda(t)\,\|D^*(t)R(t+1)D(t)\| \leqslant \mathfrak{d}^{-2}.$$

Using Lemma (4.2), we deduce that

$$\|L\|^2 \leqslant \mathfrak{d}^{-2}.$$

and because the operator $L\Lambda$ is a truncation on $l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$ and

$$c = \|L\|\,\|\Lambda\|_\infty < 1.$$

(4.10) implies that

$$\left(\sum_{t=0}^{N_0}\left(\mathbb{E}\left(\|(y(t))\|^2\right)\right)\right)^{\frac{1}{2}} \leqslant \|E(t)\phi(t,0)x_0\| + c\left(\sum_{t=0}^{N_0}\left(\mathbb{E}\left(\|(y(t))\|^2\right)\right)\right)^{\frac{1}{2}}.$$

Thus

$$(1-c)\left(\sum_{t=0}^{N_0}\left(\mathbb{E}\left(\|(y(t))\|^2\right)\right)\right)^{\frac{1}{2}} \leqslant \|E(t)\phi(t,0)x_0\|.$$

Therefore

$$\left(\sum_{t=0}^{N_0}\left(\mathbb{E}\left(\|(y(t))\|^2\right)\right)\right)^{\frac{1}{2}} \leqslant (1-c)^{-1}\|E(t)\phi(t,0)x_0\|, \text{ for } N_0 \in \mathbb{N}.$$

Then we infer that $y \in l^2_\omega(\mathbb{N}, l^2(\Omega, Y))$ and $v = \Lambda(y) \in l^2_\omega(\mathbb{N}, l^2(\Omega, U))$. Using Lemma 4.1 we conclude that the solution of (4.1) belongs to $l^2_\omega(\mathbb{N}, l^2(\Omega, \mathcal{Z}))$ which implies that $\mathfrak{r}_\omega(T(.), (D(.), E(.)) \geqslant \mathfrak{d}$. □

As a result we get the following bound for the stability radius.

Corollary 4.1. *Assume that there exist* $(R(t)) \in \mathcal{L}^\infty(\mathbb{N}, \mathcal{L}(\mathcal{Z}))$ *a sequence in* $\mathcal{Z}$*, solution of the Lyapunov equation (4.6) so*

$$\mathfrak{r}_\omega(T(.), (D(.), E(.)) \geqslant \left(\sup_{t\geqslant 0}\lambda(t)\|D^*(t)R(t+1)D(t)\|\right)^{\frac{-1}{2}}. \tag{4.11}$$

Proof. 1. Let $t \geq 0$. If $\|D^*(t)R(t+1)D(t)\| = 0$ then

$$I - \mathfrak{d}^2\lambda(t)D^*(t)R(t+1)D(t) > 0.$$

for all $\mathfrak{d} > 0$. According to Theorem 4.1, it comes that

$$\mathfrak{r}_\omega(T(.), (D(.), E(.)) \geqslant \mathfrak{d}.$$

Then we derive that

$$\mathfrak{r}_\omega(T(.), (D(.), E(.)) = +\infty.$$

2. Assume that $\sup_{t\geqslant 0}\|D^*(t)R(t+1)D(t)\| \neq 0$. Let $t \geq 0$. We have

$$\|D^*(t)R(t+1)D(t)\| = \sup_{v\neq 0}\frac{\langle D^*(t)R(t+1)D(t)v, v\rangle}{\|v\|^2}.$$

So

$$\|v\|^2 - \|D^*(t)R(t+1)D(t)\|^{-1}\langle D^*(t)R(t+1)D(t)v, v\rangle \geqslant 0.$$

hence

$$\|v\|^2 - \left(\left(\sup_{t\geqslant 0}(\lambda(t))\|D^*(t)R(t+1)D(t)\|\right)^{\frac{-1}{2}}\right)^2 \sup_{t\geqslant 0}(\lambda(t))\,\langle D^*(t)R(t+1)D(t)v, v\rangle \geqslant 0.$$

By Theorem 4.1, we conclude that

$$\mathfrak{r}_\omega(T(.),(D(.),E(.)) \geqslant \left(\left(\sup_{t\geqslant 0}(\lambda(t))\|D^*(t)R(t+1)D(t)\|\right)\right)^{\frac{-1}{2}}.$$

□

4.3 Periodic systems

Consider the following periodic system

$$x(t+1) = T(t)x(t) + D(t)\Lambda(E(x(t)))w(t), \quad t \geq 0, \tag{4.12}$$

such that $T(.), D(.), E(.)$ are p periodic bounded operators.
The solution of the Lyapunov equation (4.6) satisfies $R(t+p) = R(t)$. We get in this case

$$\|\lambda(t+p)D^*(t+p)R(t+p+1)D(t+p)\| = \|\lambda(t)D^*(t)R(t+1)D(t)\|, \tag{4.13}$$

which implies that

$$\sup_{t\geqslant 0}\|\lambda(t)D^*(t)R(t+1)D(t)\| = \sup_{t\in\{0\ldots p-1\}}\|\lambda(t)D^*(t)R(t+1)D(t)\|$$

Lemma 4.3. *The linear map L defined in Lemma 4.2 has the operator norm*

$$\begin{aligned}\|L\| &= \left(\sup_{k\geqslant 0}\lambda(k)\,\|D^*(k)R(k+1)D(k)\|\right)^{\frac{1}{2}}\\ &= \left(\sup_{k\in\{0,\ldots,p-1\}}\lambda(k)\,\|D^*(k)R(k+1)D(k)\|\right)^{\frac{1}{2}}\end{aligned}$$

where $R(.)$ satisfies (4.4).

Proof. According to the poof of Lemma 4.2 we can obtain the inequality

$$\|L\| \leqslant \left(\sup_{k\geqslant 0} \lambda(k)\, \|D^*(k)R(k+1)D(k)\|\right)^{\frac{1}{2}}$$

Now assume that

$$\begin{aligned}
\sup_{k\geqslant 0} \lambda(k)\|D^*(k)R(k+1)D(k)\| &= \sup_{k\in\{0,\ldots,p-1\}} \lambda(k)\|D^*(k)R(k+1)D(k)\| \\
&= \lambda(k_0)\|D(k_0)R(k_0+1)D(k_0)\| \\
&= \lambda(k_0)\langle\widehat{v}, D^*(k_0)R(k_0+1)D(k_0)\widehat{v}\rangle,
\end{aligned}$$

for $k_0 \in \{0, ...p-1\}$ and $\|\widehat{v}\| = 1$.
Choose v_0 such that

$$\begin{cases} v_0(t) = 0, & if\, t \neq k_0, \\ v_0(t) = \beta(k_0)\widehat{v}, & if\, t = k_0. \end{cases}$$

where $\beta(.) \in \mathcal{L}^2(\mathbb{N}, \mathbb{R})$ and

$$\begin{cases} \beta(t) = 0, & t \neq k_0, \\ \beta(t) = 1, & t = k_0. \end{cases}$$

Then

$$\begin{aligned}
\|v_0\|^2_{L^2} &= \sum_{t\geqslant 0} \mathbb{E}\|v_0(t)\|^2 \\
&= \sum_{t\geqslant 0} \|\beta(t)\widehat{v}\|^2 \\
&= \|\widehat{v}\|^2\|\beta(k_0)\|^2 \\
&= \|\widehat{v}\|^2 = 1
\end{aligned}$$

We have

$$\begin{aligned}
\|Lv_0\|^2 &= \sum_{k=0}^{k-1} \lambda(k)\mathbb{E}\left\langle v_0(k), D^*(k)R(k+1)D(k)v_0(k)\right\rangle \\
&= \sum_{k=0}^{\infty} \lambda(k)\left\langle \beta(k)\widehat{v}, D^*(k)R(k+1)D(k)\beta(k)\widehat{v}\right\rangle \\
&= \lambda(k_0)|\beta(k_0|^2\langle\widehat{v}, D^*(k_0)R(k_0+1)D(k_0)\widehat{v}\rangle \\
&= \lambda(k_0)\|D^*(k_0)R(k_0+1)D(k_0)\|.
\end{aligned}$$

Which concludes the proof. □

Thus for periodic systems, we get the following bound for the stability radius.

$$\mathfrak{r}_\omega(T(.),(D(.),E(.)) \geqslant \left(\left(\sup_{t\in\{0,\dots p-1\}} \lambda(t)\|D^*(t)R(t+1)D(t)\|\right)\right)^{\frac{-1}{2}}.$$

4.4 Example

Set

$$T(t)=\begin{cases} 1, & t \ even \\ a_0^2a_1^2, & t \ odd \end{cases}, \quad D(t)=\begin{cases} a_0^{-1}, & t \ even \\ a_0, & t \ odd \end{cases},$$

$$E(t)=\begin{cases} a_1^{-1}, & t \ even \\ a_1, & t \ odd \end{cases}.$$

such that $a_0, a_1 \in \mathbb{R}\setminus\{0\}$, $|a_0a_1|<1$. See [36]

$T(.), D(.)$ and $D(.)$ are operators of period tow. We solve the Lyapunov equation

$$R(t)=T(t)^*R(t+1)T(t)+E^*(t)E(t).$$

Since the system is periodic of period tow we just calculate $R(1),\ R(2)$. We have

$$\begin{cases} R(1)=a_0^2a_1^2R(2)a_0^2a_1^2+a_1^2 \\ R(2)=R(1)+1 \end{cases}$$

Hence

$$R(1)=a_0^4a_1^4R(1)+a_0^4a_1^4+a_1^2$$

then

$$R(1)=\frac{a_0^4a_1^4+a_1^2}{1-a_0^4a_1^4}$$

and

$$R(2)=\frac{a_1^2+1}{1-a_0^4a_1^4}$$

Now we have

$$\begin{cases} \|D^*(1)R(2)D(1)\|=|\dfrac{a_0^2a_1^2+a_0^2}{1-a_0^4a_1^4}| \\ \|D^*(2)R(3)D(2)\|=|\dfrac{a_1^4a_0^4+a_1^2}{a_0^2-a_1^4a_0^6}| \end{cases}.$$

Thus

$$\mathfrak{r}_\omega(T(.),(D(.),E(.)) \geqslant \left(\left(\lambda(1)|\frac{a_0^2a_1^2+a_0^2}{1-a_0^4a_1^4}|,\ \lambda(2)|\frac{a_1^4a_0^4+a_1^2}{a_0^2-a_1^4a_0^6}|\right)\right)^{\frac{-1}{2}}.$$

Conclusion

This dissertation concerns the robustness of stability and stabilization of linear discrete-time systems under stochastic perturbations. The stability radii approach is adopted. The research on this subject is uncomplished by this work and there are many open questions.

Summary of the obtained results

First, we considered discrete-time invariant systems with stochastic bounded structured perturbations. We proved new characterizations of the corresponding radius of stability extending the results of the continuous case. We design a computing expression for this radius by the use of the discrete Lyapunov equations and the corresponding inequalities.

The stability radius optimization according to state feedback was also studied. We discussed the existence of stabilizing feedback with a norm less than a given bound.

Since periodic systems have attracted important attention in the last decades, we have addressed in the last chapter robustness of stability of discrete-time variant systems with bounded stochastic perturbations. Some of the Chapter 3 results have been extended to the variant case. We expressed the stability radius via Lyapunov equation. But we have just established a lower bound for the stability radius.

All the results of the thesis were illustrated by theoretical or numerical examples.

Open problems

- It is important to note that the results reported in the time variant case provides only a lower bound for the stability radius. Our ongoing efforts focus on deriving necessary robustness conditions and hence upper bounds for the stability radius.

- Further research is required to study the maximization of the stability radius in the time variant case.

- In recent years, the class of jump linear systems has attracted the attention of a lot of researchers. Extention of our results to this class of systems is still an open problem.

- Another open problem is whether our results can be extended to bilinear discrete-time systems.

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