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**DÉPARTEMENT DE MATHÉMATIQUES**
**LABORATOIRE DES TECHNIQUES MATHÉMATIQUES (LTM).**

# THÈSE

POUR OBTENIR LE DIPLÔME DE DOCTORAT EN MATHÉMATIQUES

**OPTION :** THÉORIE DE CONTRÔLE

**PRÈSENTÈE PAR :**

BEZAI ASSIA

## THÈME

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# Sur la résolution des équations de Lyapunov et Riccati

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**SOUTENUE LE : 24 /01/ 2024**

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# THESIS

FOR A DOCTORAL DEGREE IN MATHEMATICS

**SPECIALTY :** CONTROL THEORY

**DEFENDED ON :** 24 /01/ 2024

**PRESENTED BY :**
BEZAI ASSIA

## THEME

# On the solvability of the Lyapunov and Riccati equations

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# Dedication

*To my perents.*

*To my sister and my brothers*

*To all my family and friends .*

Assia.

# Acknowledgments

Above all, I thank Allah for giving me the opportunity and the ability to finish this research study.

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# List of publications

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# Abstract

The aim of this thesis is to study the solvability of the Lyapunov ($AX - XB = C$), the Sylvester ($AX - YB = C$) and the Riccati ($AX - XB + XDX = C$ ) operator equations in Hilbert spaces of infinite dimensions using generalized inverses. More precisely, we give new necessary and sufficient conditions for the solvability to the operator equations $AX - XB = C$ and $AX - YB = C$, where $A$ and $B$ are group invertible. In addition the general solutions to the equation $AX - YB = C$, are derived in terms of the group inverse of $A$ and $B$. As a consequence, new necessary and sufficient conditions for the solvability to the operator equation $AYB - Y = C$, are derived. Next by application of the generalized Drazin inverse, we give a new method for solving Riccati and Lyapunov operator equations in Hilbert space. Results are applied to Riccati and Lyapunov operator differential equations.


**Key words** : Hilbert spaces, Inner inverse, Drazin inverse, Group inverse, Generalized Drazin inverse, Lyapunov equation, Riccati equation, Sylvester equation, Pseudo-similarity, Pseudo-equivalence.
**AMS Classification** : Primary 47A62; Secondary 15A09.



# Résumé

Le but de cette thèse est d'étudier la solvabilité des équations opératorielles de Lyapunov ($AX - XB = C$), Sylvester ($AX - YB = C$) et Riccati ($AX - XB + XDX = C$ ) sur des espaces de Hilbert de dimension infinie en utilisant les inverses généralisés. Plus précisément, nous donnons de nouvelles conditions nécessaires et suffisantes d'existence des solutions des équations opératorielles $AX - XB = C$ et $AX - YB = C$, où $A$ et $B$ sont des opérateurs groupe inversibles. De plus les formes des solutions générales de l'équation $AX - YB = C$, sont données en terme des groupe inverses des opérateurs $A$ et $B$. Comme conséquence, nous déduisons de nouvelles conditions nécessaires et suffisantes pour la solvabilité de l'équation $AYB - Y = C$. Ensuite, par application de l'inverse de Drazin généralisé, nous donnons une nouvelle méthode pour résoudre les équations de Riccati et de Lyapunov. Les résultats obtenus sont appliqués aux problèmes de Cauchy pour les équations différentielles opératorielles de Riccati et de Lyapunov.


**Mots clés** : Espaces de Hilbert, Inverse intérieur, Inverse de Drazin, Groupe inverse, Inverse de Drazin généralisé, Équation de Lyapunov, Équation de Riccati, Équation de Sylvester, Pseudo-similarité, Pseudo-équivalence.

## الملخص

الهدف من هذه الاطروحة هو دراسة قابلية حل معادلات المؤثرات من نوع ليابونوف (AX - XB=C) و سيلفيستر (AX-YB=C) و ريكاتي (AX - XB + DXD = C) المعرفة في فضاءات هيلبرت ذات الابعاد اللانهائية باستخدام المعكوسات المعممة. بتعبير ادق نقوم بإيجاد شروط جديدة ضرورية وكافية لحل المعادلات AX - XB=C و AX - YB=C حيث المؤثران A و B يقبلان معكوس المجموعة, بالإضافة لذلك نعطي الشكل العام لحلول المعادلة AX - YB=C باستخدام معكوس المجموعة كنتيجة لذلك نستنتج شروط جديدة ضرورية وكافية لحل المعادلة AYB-Y=C. بعد ذلك, باستخدام معكوس درازن المعمم نطبق طريقة جديدة لحل معادلات ريكاتي و ليابونوف ثم نستخدم النتائج المتحصل عليها لإيجاد الحلول العامة لمسائل كوشي الخاصة بمعادلتي ريكاتي وليابونوف التفاضليتين.

الكلمات المفتاحية :فضاءات هلبرت, معكوس داخلي, معكوس درازين, معكوس المجموعة, معكوس درازين المعمم, معادلة ليابونوف, معادلة ريكاتي, تشابه زائف, تكافؤ زائف.

# Contents

# General introduction

Since 1955 a great number of papers on various aspects of generalized inverses and their applications have appeared. Generalized inverses pervade a wide range of mathematical areas, matrix theory, operator theory, differential equations, numerical analysis, Markov chains, $C^*$-algebras or rings. Numerous applications include areas such as statistics, cryptography, control theory, coding theory.

We recall that the concept of generalized inverses seems to have been first mentioned in print in 1903 by Ivar Fredholm [40], who formulated a pseudo inverse for a linear integral operator which is not invertible in the ordinary sense.

One year later, in 1904 Hilbert made implicit use of pseudo-inverses when considering the theory of linear ordinary differential equations. In fact, he introduced the notion of the generalized Green's function which was the integral kernel of the pseudo-inverse of the differential operator.

In 1913, Hurwitz [47] reconsidered the same problem of Fredholm and used the finite dimensionality of the null-space of Fredholm operators to give a simple algebraic construction.

Generalized inverses of differential and integral operators thus predated the generalized inverses of matrices whose existence was first noted in 1920 by Moore [66], who defined a unique inverse $A^\dagger$ called by him the "general reciprocal" for every finite matrix (square or rectangular). Moore established the existence and uniqueness of $A^\dagger$ for any $A$, and gave an explicit form for $A^\dagger$ in terms of the subdeterminants of $A$ and $A^*$. His work received practically no attention in the next 30 years, mostly because it used very complicated notation.

In 1951, Bjerhamar [9, 7, 8] recognized the least squares properties of certain generalized inverses and noted the relation between some generalized inverses and solutions to linear systems.

In 1955, Penrose [71] sharpened and extended Bjerhammar's results on linear systems, and showed that Moore's inverse, for a given matrix A, is the unique matrix X satisfying the following four equations

$$(1)\;\; AXA = A, \quad (2)\;\; XAX = X, \quad (3)\;\; (AX)^* = AX, \quad (4)\;\; (XA)^* = XA.$$

The latter discovery has been so important that this unique inverse is now commonly called the Moore Penrose inverse. Since 1955 the theory of generalized inverses and also the applications and computational methods have been developing rapidly. Throughout the years the Moore-Penrose inverse was intensively studied, one of the primary reasons for that being its usefulness in applications to dealing with diverse problems such as, for example, that of solving systems of linear equations, which constitutes one of the basic but at the same time the most important applications of this type of generalized inverse. As we will be working with a number of different subsets of the above mentioned set of four equations for operators defined on Hilbert spaces, we need some convenient notations for the generalized inverses satisfying those certain specified equations.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces of infinite dimension, and let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all linear bounded operators from $\mathcal{H}$ to $\mathcal{K}$. An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is generalized invertible if there is an operator $S \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that the equation $TST = T$ ((1) of Penrose ) is satisfied. The operator $S$ is not unique in general. In order to have its uniqueness, further conditions have to be imposed. The most convenient additional conditions are

$$(2)\ \ STS = S,\ \ (3)\ \ (TS)^* = TS,\ \ (4)\ \ (ST)^* = ST,\ \ (5)\ \ TS = ST.$$

One also considers the condition $(I_k)$ $T^kST = T^k$, where $k$ is a fixed positive integer. Clearly, $(1) = (I_1)$. Elements $S \in \mathcal{B}(\mathcal{H},\mathcal{K})$ satisfying (1) are called $\{1\}-$ inverses or inner inverses of $T$ and denoted by $S = T^-$. Similarly, $\{1,2,5\}$-inverse called group inverse and denoted by $S = T^\sharp$. $\{1,2,3,4\}$-inverse called Moore-Penrose inverse and denoted by $S = T^\dagger$. And $\{I_k,2,5\}$-inverse called Drazin inverse and denoted by $S = T^D$, where $k = ind(T)$ the Drazin index of $T$ is the smallest non-negative integer for which $\mathcal{R}(T^{k+1}) = \mathcal{R}(T^k)$ and $\mathcal{N}(T^{k+1}) = \mathcal{N}(T^k)$ (see [17]). In particular, when $ind(T) = 1$ the operator $T^D$ is the group inverse of $T$ denoted by $T^\sharp$.
In [53] Koliha introduced the concept of generalized Drazin inverse (GD-inverse), which is an element $T^d$ of $\mathcal{B}(\mathcal{K},\mathcal{H})$ satisfying the equations (2), (5) and $T - T^2T^d$ is quasi-nilpotent. If $T$ is GD-invertible, then the spectral idempotent corresponding to $\{0\}$ is given by $T^\pi = I - TT^d$.

There are many papers in which the basic aim is to find necessary and sufficient conditions for the existence of a solution to some matrix or operator equations. The reason for this is a large number of applications in physics, mechanics, control theory and many other fields.

In 1952 W.E. Roth [80] showed that the matrices equations of the form $AX - YB = C$ and $AX - XB = C$, over fields can be solved if and only if the block of matrices $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are equivalent or similar. In 1969, Rosenblum [79] showed that the result of Roth for the equation $AX - XB = C$, remains true, when $A$ and $B$ are selfadjoint operators on a complex separable Hilbert space. In 1982, Schweinsberg [82] extended the result to include finite rank operators and normal operators on Hilbert space.

The first aim of this thesis is to give new necessary and sufficient conditions for the existence of solutions to the operator equations $AX - YB = C$ and $AX - XB = C$, using generalized inverses and the concept of pseudo-similarity and pseudo-equivalence of the matrices of operators

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

In addition the general solutions to the equation $AX - YB = C$, are derived in terms of group inverse of $A$ and $B$. As a consequence, new necessary and sufficient conditions for the solvability to the operator equation $AYB - Y = C$, are derived. The problem of solving operator equations of the type

$$XDX + AX - XB - C = 0, \tag{R}$$

called the Ricatti equation appears in the study of certain control problems [85, 64, 59, 19] and it has been studied by several authors in different contexts [14, 27, 68]. The second aim of this thesis is to present a method using GD-inverse for solving equations (R), that permits to reduce this equation to a system of two linear operator equations. This reduction is available by means of the application of the GD-inverse of matrices of operators, we add some conditions in order to give the form of the general solution to the Riccati equation (R), then we deduce the same results for the operator equation $AX - XB = C$. The results are applied to Cauchy problems for Riccati and Lyapunov operator differential equations, because of the relation between algebraic and differential problems.

The thesis is organized in four chapters.

**Chapter 1**, is a reminder on essential notions and the main tools needed throughout this thesis. First we present some basic and important properties of projections. Then we introduce basic concepts of generalized inverses and we recall their algebraic and topological properties in Banach and Hilbert spaces, also we give some known methods about finding an explicit formulas for the GD-inverse and the group inverse of block operator matrix. Finally we recall some known results about matrix and operator equations.

**Chapter 2**, is divided into three sections. In Section 2.1, we generalize the notion of pseudo-similarity and pseudo-equivalence introduced by Hartwig and Hall [45] for matrices over a ring to the setting of bounded linear operators defined on Hilbert spaces of infinite dimension, in addition we give some properties of these two notions which are weaker than similarity and equivalence known in the literature. In Section 2.2, we prove that if $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $A$ and $B$ are group invertible, then the equation $AX - XB = C$ has a solution if and only if $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is pseudo-similar to $D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

In Section 2.3, we provide some equivalent conditions to the solvability of the operator equation $AX - YB = C$, the most important one is that the matrices of operators $M$ and $D$ are pseudo equivalent via $P$ and $Q$ and the operator $U = DQPDD^{\sharp} + I - DD^{\sharp}$ is invertible. As a consequence we deduce new necessary and sufficient conditions for the existence of solutions to the operator equation $AYB - Y = C$.

**Chapter 3**, is divided into two sections. In Section 3.1, we present a method using GD-invese for solving equations $AX - XB + XDX = C$ and $AX - XB = C$, that permits to reduce the equation $AX - XB + XDX = C$ to a system of two linear operator equations. This reduction is possible by means of the application of the GD-inverse of matrices of operators, we add some conditions in order to give the form of the general solution to the Riccati equation (R), then we deduce the same results for the Lyapunov operator equation $AX - XB = C$. In Section 3.2, we apply the results of Section 3.1 to Cauchy problems for Riccati and Lyapunov operator differential equations, because of the relation between algebraic and differential problems.

**Chapter 4**, is independent of the two previous ones. It is divided into three sections. Section 4.1 is a reminder on essential notions about the Kroneker product, the matrix hermite echelon form and some Matlab software functions.

In Section 4.2, we present a procedure for computing the Drazin inverse of matrix and we give its Matlab software code with numerical example.

In Section 4.3, we review some known results of [75] on the application of the Drazin inverse to study the positivity and stability of time-invariant descriptor systems. We give a Matlab software code to check the positivity and stability of the time-invariant descriptor systems with numerical examples.

## 0.1 Terminology

Here we present all the notations used in this thesis.

1. $\mathcal{H}$, $\mathcal{K}$ are two complex Banach or Hilbert spaces of infinite dimensions.
2. $\mathcal{B}(\mathcal{H},\mathcal{K})$ the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$.
3. $T^*$ the adjoint of $T$.
4. $T^{-1}$ the inverse of $T$.
5. $T^-$ the inner inverse of $T$.
6. $T^D$ the Drazin inverse of $T$.
7. $T^d$ the generalized Drazin inverse of $T$.
8. $T^\sharp$ the group inverse of $T$
9. $\mathcal{R}(T)$ the range of $T$.
10. $\mathcal{N}(T)$ the kernel of $T$.
11. $\sigma(T)$ the spectrum of $T$.
12. $asc(T)$ the ascent of $T$.
13. $des(T)$ the descent of $T$.
14. $ind(T)$ the Drazin index of $T$.
15. $\langle .,. \rangle$ the inner product.
16. $\otimes$ the kronecker product.
17. $\overline{\sim}$ pseudo-similarity.
18. $\mathbb{R}^{m\times n}$ the vector space of all $m$ by $n$ reel matrices.
19. $\mathbb{C}^{m\times n}$ the vector space of all $m$ by $n$ complex matrices.

Let $\mathcal{H}$, $\mathcal{K}$ be two Hilbert spaces of infinite dimensions, an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be

1. Projection or idempotent if $T^2 = T$.
2. An orthogonal projection if $T^2 = T$ and $T^* = T$.

3. Self-adjoint, $T^* = T$.
4. Isometry if $T^*T = I$.
5. Normal if $T^*T = TT^*$.
6. Unitary if $T^*T = TT^* = I$.
7. $n$-normal if $T^nT^* = T^*T^n$ where $n \in \mathbb{N}$.
8. Polynomially normal if there exists a non-trivial polynomial $p$ such that

$$p(T)T^* = T^*p(T).$$

9. Algebraic if there exists a non-trivial polynomial $p$ such that $p(T) = 0$.
10. Nilpotent if there exists $n \in \mathbb{N}^*$ such that $T^n = 0$.
11. Quasi-nilpotent if and only if $\|T^n\|^{\frac{1}{n}} \longrightarrow 0$.

Chapter 1

# Preliminaries

In this introductory chapter, we introduce some basic concepts and well-known results and mathematical backgrounds on generalized inverses their algebraic and topological properties. In particular we recall some basic notions and theorems of the Drazin inverse, also we give some known methods about finding an explicit formulas for the GD-inverse and the group inverse of block operator matrix. Finally we recall some known results about matrix and operator equations.

## 1.1 Projections

In this section, we give some definition and properties of projections, for more details and proofs see [86].

**Proposition 1.1.1.** *Each projection $P$ determines a direct sum decomposition of $\mathcal{H}$, namely*

$$\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P). \tag{1.1}$$

*Conversely every direct sum decomposition of $\mathcal{H}$ determines a projection.*

*Proof.* It is clear that $\mathcal{H} = \mathcal{R}(P) + \mathcal{N}(P)$, since each $x \in \mathcal{H}$ may be written in the form

$$x = Px + (x - Px).$$

Furthermore, forall $x \in \mathcal{R}(P)$ are characterized by the fact that $Px = x$. So, if $x \in \mathcal{R}(P) \cap \mathcal{N}(P)$, then $x = Px = 0$, that is $\mathcal{R}(P) \cap \mathcal{N}(P) = \{0\}$. This proves (1.1). Conversely let $\mathcal{H} = M \oplus N$, then $\forall x \in \mathcal{H}$ may be written uniquely in the form $x = x_1 + x_2$ with $x_1 \in M$ and $x_2 \in N$. If we define $P$ by $Px = x_1$, then it is clear that $P$ is a linear operator such $\mathcal{R}(P) = M$, $\mathcal{N}(P) = N$, and $P^2 = P$.
We call $P$ a projection of $\mathcal{H}$ onto $M$ along $N$. □

**Proposition 1.1.2.** *If $P$ is a projection of $\mathcal{H}$ onto $M$ along $N$, then the operator $(I - P)$ is also a projection of $\mathcal{H}$ onto $N$ along $M$.*

**Definition 1.1.1.** *A subspace $M$ of a Banach space $\mathcal{H}$ is said to have a complemented subspace if there exists a subspace $N$ such that $\mathcal{H} = M \oplus N$.*

**Theorem 1.1.1.** *Let $M$ and $N$ be closed subspaces such that $\mathcal{H} = M \oplus N$, then the projection $P$ of $\mathcal{H}$ onto $M$ along $N$ is continuous.*

*Proof.* Because of the closed graph Theorem it suffices to prove that $P$ is a closed operator. Suppose that

$$x_n \to x \quad \textit{and} \quad Px_n \to y.$$

Then

$$x_n - Px_n \to x - y.$$

Since $Px_n \in M$ and $x_n - Px_n \in N$, it follows that

$$y \in M \quad \textit{and} \quad x - y \in N = \mathcal{N}(P).$$

Then

$$Px - Py = 0 \quad \textit{and} \quad Px = Py = y.$$

Thus $P$ is closed. □

**Remark 1.1.1.** *The decomposition $\mathcal{H} = M \oplus N$ in Theorem 1.1.1 is a topological direct sum because $M$ and $N$ are both closed subspaces.*

**Lemma 1.1.1.**

1. *The range of a continuous projector $P$ on a Banach space $\mathcal{H}$ is closed.*

2. *A closed subspace of a Banach space $\mathcal{H}$ is complemented if and only if it is the range of some continuous projector in $\mathcal{H}$.*

*Proof.*

1. Since $\mathcal{R}(P) = \mathcal{N}(I - P)$, thus $\mathcal{R}(P)$ being the nullspace of a continuous linear operator is a closed subspace.

2. Let $M$ be a closed subspace of $\mathcal{H}$. If $M = \mathcal{R}(P)$ for some continuous projector $P$ on $\mathcal{H}$, then $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{N}(P)$ and $\mathcal{N}(P)$ is closed. Thus $M$ is complemented.
   Conversely, if $M$ is complemented, let $P$ be the continuous projector of $\mathcal{H}$ onto $M$, and the result follows.

□

## 1.2 Inner inverses

### 1.2.1 Inner inverses in Banach spaces

**Definition 1.2.1.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. An operator $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is said to be an inner inverse of $A$ if it satisfies the equation*

$$ASA = A. \tag{1.2}$$

*We denote the inner inverse by $A^-$. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called regular if $A^-$ exists.*

**Remark 1.2.1.**

1. *$A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has an inner inverse if and only if $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are closed and complemented subspaces of $\mathcal{H}$ and $\mathcal{K}$ respectively.*
2. *If $A$ has an inverse $A^{-1}$ in $\mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^{-1}$ is the only inner inverse of $A$.*

**Definition 1.2.2.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. An operator $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is said to be an outer inverse of $A$ if it satisfies the equation*

$$SAS = S. \tag{1.3}$$

**Example 1.2.1.** *Let $S_r$ be the unilateral right shift and $S_l$ the left shift defined on the Hilbert space $l_2(\mathbb{N})$by*

$$\begin{aligned} S_r(x_1, x_2, x_3, ....) &= (0, x_1, x_2, ......), \\ S_l(x_1, x_2, x_3, ....) &= (x_2, x_3, x_4......). \end{aligned}$$

*Since*

$$\begin{aligned} S_r S_l S_r(x_1, x_2, x_3, ....) &= S_r S_l(0, x_1, x_2, x_3, ....) \\ &= S_r(x_1, x_2, x_3, ....). \end{aligned}$$

$$\begin{aligned} S_l S_r S_l(x_1, x_2, x_3, ....) &= S_l S_r(x_2, x_3, ....) \\ &= S_l(0, x_2, x_3, ....) \\ &= (x_2, x_3, ......) \\ &= S_l(x_1, x_2, x_3, ....). \end{aligned}$$

*Then $S_l$ is an inner and outer inverse inverse of $S_r$.*

**Lemma 1.2.1.** *If $S$ is an inner inverse of $A$, then the operator $SAS$ satisfies both equations (1.2) and (1.3).*

*Proof.* The proof of this assertion is a simple verification. If $ASA = A$, then

$$A(SAS)A = (ASA)SA = ASA = A,$$

and

$$(SAS)A(SAS) = S(ASA)SAS = S(ASA)S = SAS.$$

□

**Theorem 1.2.1.** *[16] Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then*

1. *If $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is an inner and outer inverse of $A$, then $AS$ is a projection of $\mathcal{K}$ onto $\mathcal{R}(A)$ along $\mathcal{N}(S)$ and $SA$ is a projection of $\mathcal{H}$ onto $\mathcal{R}(S)$ along $\mathcal{N}(A)$.*
2. *If $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is an inner and outer inverse of $A$, then $\mathcal{R}(S)$ is a closed complemented subspace of $\mathcal{N}(A)$ and $\mathcal{N}(S)$ is a closed complemented subspace of $\mathcal{R}(A)$.*
3. *Suppose that $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are closed and complemented subspaces of $\mathcal{H}$ and $\mathcal{K}$ respectively, $P$ is a projection onto $\mathcal{R}(A)$ and $M$ is a complementary subspace to $\mathcal{N}(A)$. Then $S = A_1^{-1}P$ is an inner inverse of $A$, where $A_1$ is the restriction of $A$ to $M$.*

*Proof.*

1. Suppose that $A$ has an inner and outer inverse, so that there exists an operator $S$ which satisfies equation (1.2) and (1.3).

   Then, since
   $$(AS)^2 = ASAS = AS,$$
   and
   $$(SA)^2 = SASA = SA,$$
   it is clear that $AS$ and $SA$ are projection.

   Clearly, $\mathcal{R}(AS) \subseteq \mathcal{R}(A)$. Conversely, for each $y \in \mathcal{R}(A)$ there exists $x \in \mathcal{H}$ such that $y = Ax$, we can write
   $$Ax = ASAx,$$
   so that
   $$\mathcal{R}(A) \subseteq \mathcal{R}(AS).$$

In a similar way, we have $\mathcal{N}(S) \subseteq \mathcal{N}(AS)$ and if $ASx = 0$, then from equation (1.3) we know that

$$Sx = SASx = 0,$$

so that

$$\mathcal{N}(AS) \subseteq \mathcal{N}(S).$$

This means that, $AS$ is a projection onto $\mathcal{R}(A)$ along $\mathcal{N}(S)$.

On the other hand, clearly $\mathcal{R}(SA) \subseteq \mathcal{R}(S)$. Conversely, for each $y \in \mathcal{R}(S)$ such that $y = Sx$ we can write

$$Sx = SASx,$$

so that

$$\mathcal{R}(S) \subseteq \mathcal{R}(SA).$$

In a similar way, we have $\mathcal{N}(A) \subseteq \mathcal{N}(SA)$ and if $SAx = 0$, then from equation (1.2) we know that

$$Ax = ASAx = 0,$$

so that

$$\mathcal{N}(SA) \subseteq \mathcal{N}(A).$$

This means that $SA$ is a projection onto $\mathcal{R}(S)$ along $\mathcal{N}(A)$.

2. According to property 1, there exists a projections $SA \in \mathcal{B}(\mathcal{H}))$ and $AS \in \mathcal{B}(\mathcal{K})$ such that

$$\mathcal{N}(SA) = \mathcal{N}(A), \qquad \mathcal{R}(AS) = \mathcal{R}(A),$$

On the other hand, There exists a closed subspaces $\mathcal{R}(S)$ and $\mathcal{N}(S)$ such that

$$\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{R}(S), \qquad \mathcal{K} = \mathcal{N}(S) \oplus \mathcal{R}(A),$$

Thus, $\mathcal{R}(S)$ is a closed complemented subspace of $\mathcal{N}(A)$ and $\mathcal{N}(S)$ is a closed complemented subspace of $\mathcal{R}(A)$.

3. Suppose that $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented subspaces in $\mathcal{H}$ and $\mathcal{K}$ respectively

$$\mathcal{H} = \mathcal{N}(A) \oplus M, \qquad \mathcal{K} = N \oplus \mathcal{R}(A), \tag{1.4}$$

let $A_1 = A/M$, and let $P$ be the continuous projection of $\mathcal{K}$ onto $\mathcal{R}(A)$ along $N$. Since $A_1$ is a bijective map of $M$ onto $\mathcal{R}(A)$, which are both closed subspaces, it follows from the inverse mapping theorem that $A_1^{-1}$ is continuous, hence $A_1^{-1}P$ is also continuous. Clearly $A(A_1^{-1}P)A = A$. This proves that $S = A_1^{-1}P$ is a bounded inner inverse for $A$.

□

**Remark 1.2.2.** *Note that if $S$ is an inner inverse and not an outer inverse, then the arguments in the first and second part of Theorem 1.2.1, can be applied to the operator $SAS$ to show that the conclusions are true in general.*

**Theorem 1.2.2.** *[86] Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then the following conditions are equivalent:*

1. *There exists projections $P \in \mathcal{B}(\mathcal{H})$ and $Q \in \mathcal{B}(\mathcal{K})$ such that*

$$\mathcal{R}(P) = \mathcal{N}(A), \qquad \mathcal{R}(Q) = \mathcal{R}(A), \tag{1.5}$$

2. *There exists closed subspaces $M$ and $N$ such that*

$$\mathcal{H} = \mathcal{N}(A) \oplus M, \qquad \mathcal{K} = N \oplus \mathcal{R}(A), \tag{1.6}$$

3. *$A$ has an inner inverse.*

*Proof.*
(1) $\Rightarrow$ (2) Clear from Lemma 1.1.1.
(2) $\Rightarrow$ (3) Suppose that $M$ and $N$ are closed subspaces such that

$$\mathcal{H} = \mathcal{N}(A) \oplus M, \qquad \mathcal{K} = N \oplus \mathcal{R}(A),$$

from the third part of Theorem 1.2.1, $S = A_1^{-1}P$ is an inner inverse of $A$. Thus (2) $\Rightarrow$ (3).
(3) $\Rightarrow$ (1) Suppose that $S$ is an inner inverse of $A$. Then from Theorem 1.2.1, $SA$ is a projection and $\mathcal{N}(A) = \mathcal{N}(SA)$, also $AS$ is a projection and $\mathcal{R}(A) = \mathcal{R}(AS)$. So (3) $\Rightarrow$ (1), with $P = I - SA$ and $Q = AS$. □

**Remark 1.2.3.** *We have seen in Theorem 1.2.2 how to characterize the set of inner inverses in term of the formula $S = A_1^{-1}P$. However, in many situations, this is not so useful since we may not be able to describe all the projections onto $\mathcal{R}(A)$ and $\mathcal{N}(A)$. We are able to describe the set of inner inverses in another way; we first need a simple lemma.*

**Lemma 1.2.2.** *[16] Let $A, B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be regular operators and $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$, then the operator equation*

$$AXB = C,$$

*has a solution if and only if*

$$AA^{-}CB^{-}B = C.$$

*In which case, the general solution is*

$$X = A^{-}CB^{-} + U - A^{-}AUBB^{-},$$

*where $U \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is an arbitrary operator.*

**Corollary 1.2.1.** *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a regular operator and $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the operator equation*

$$AX = C,$$

*has a solution if and only if*

$$AA^{-}C = C.$$

*In which case, the general solution is*

$$X = A^{-}C + (I_{\mathcal{H}} - A^{-}A)U,$$

*where $U \in \mathcal{B}(\mathcal{H})$ is an arbitrary operator.*

**Corollary 1.2.2.** *Let $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a regular operator and $D \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the operator equation*

$$XB = D,$$

*has a solution if and only if*

$$DB^{-}B = D.$$

*In which case, the general solution is*

$$X = DB^{-} + U(I_{\mathcal{K}} - BB^{-}),$$

*where $U \in \mathcal{B}(\mathcal{K})$ is an arbitrary operator.*

**Theorem 1.2.3.** *If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has an inner inverse $A^{-} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying equations*

$$AA^{-}A = A \quad \text{and} \quad A^{-}AA^{-} = A^{-},$$

*then the set of inner inverses of $A$ consists of all operators of the form*

$$A^{-} + U - A^{-}AUAA^{-},$$

*where $U \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is an arbitrary operator.*

*Proof.* We know that $AA^{-}A = A$. From Lemma 1.2.2, we conclude that the other inner inverses are given by

$$A^{-}AA^{-} + U - A^{-}AUAA^{-} = A^{-} + U - A^{-}AUAA^{-},$$

where $U \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is an arbitrary operator. □

### 1.2.2 Inner inverses in Hilbert spaces

It is well-known that every closed subspace is complemented, therefore an operator $T$ in Hilbert space is regular if and only if it has closed range.

**Theorem 1.2.4.** *[69] Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The following statements are equivalent*

1. *$T$ has a bounded inner inverse.*
2. *$T^*$ has a bounded inner inverse.*
3. *$\mathcal{R}(T)$ is closed.*
4. *$\mathcal{R}(T^*)$ is closed.*
5. *The restriction of $T$ to $(\mathcal{N}(T))^{\perp}$ has a bounded inverse.*

**Remark 1.2.4.** *We recall that if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then the cartesian product space $\mathcal{H} \times \mathcal{K}$ itself is a Hilbert space and $\mathcal{H} \times \mathcal{K}$ will be denoted by $\mathcal{H} \oplus \mathcal{K}$.*

Now we give some well-known results which gives the inner inverses of block operators matrix.

**Theorem 1.2.5.** *[70] Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $A$, $B$ are regular operators, then*

$$M = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$$

*is regular if and only if $T = (I_{\mathcal{K}} - BB^-)C(I_{\mathcal{H}} - A^-A)$ is regular, in this case*

$$M^- = \begin{bmatrix} A^-AA^- - SCA^- & S \\ -B^-CA^-AA^- + B^-CSCA^- & -B^-CS + B^-BB^- \end{bmatrix},$$

*where $S = (I_{\mathcal{H}} - AA^-)T^-(I_{\mathcal{K}} - B^-B)$.*

**Lemma 1.2.3.** *[70] Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $A$ is regular, then $M = \begin{bmatrix} A & B \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}, \mathcal{K})$ is regular if and only if $S = (I_{\mathcal{K}} - AA^-)B$ is regular, in this case*

$$\begin{bmatrix} A & B \end{bmatrix}^- = \begin{bmatrix} A^-AA^- - A^-BS^-(I_{\mathcal{K}} - AA^-) \\ S^-(I_{\mathcal{K}} - AA^-) \end{bmatrix}.$$

**Lemma 1.2.4.** *[70] Let* $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *and* $B \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *such that* $A$ *is regular, then* $M = \begin{bmatrix} A \\ B \end{bmatrix} \in \mathcal{B}(\mathcal{H},\mathcal{K}\oplus\mathcal{K})$ *is regular if and only if* $S = B(I_{\mathcal{H}} - A^{-}A)$ *is regular, in this case*

$$\begin{bmatrix} A \\ B \end{bmatrix}^{-} = \begin{bmatrix} A^{-}AA^{-} - (I_{\mathcal{H}} - A^{-}A)S^{-}BA^{-} & (I_{\mathcal{H}} - A^{-}A)S^{-} \end{bmatrix}.$$

**Example 1.2.2.** *Consider the following block matrix* $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, *where*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

*It is easy to check that* $A^{-} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $B^{-} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ *are inner inverses of* $A$ *and* $B$ *respectively. According to Theorem 1.2.5,* $M^{-}$ *is given by*

$$M^{-} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

**Example 1.2.3.** *Consider the operator matrix* $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{B}(l_2(\mathbb{N}) \oplus l_2(\mathbb{N}))$, *where*

$$\begin{aligned} A(x_1, x_2, x_3, ....) &= (0, x_2, x_3, ....), \\ B(x_1, x_2, x_3, ....) &= (0, x_1, x_2, ....), \\ C(x_1, x_2, x_3, ....) &= (0, x_1, 0, ......). \end{aligned}$$

*We have* $A$ *is a projection, then* $A^{-} = A$, *and according to Exemple 1.2.1 we get*

$$B^{-}(x_1, x_2, x_3, ....) = S_l(x_1, x_2, x_3, ....) = (x_2, x_3, ....).$$

*Since* $T = (I_{\mathcal{H}} - AA^{-})C(I_{\mathcal{K}} - B^{-}B) = 0$, *then* $T$ *is regular which implies according to Theorem 1.2.5 that* $M$ *is regular and* $M^{-}$ *is given by*

$$M^{-} = \begin{bmatrix} A^{-} & -\tilde{C} \\ 0 & B^{-} \end{bmatrix},\ \text{where } \tilde{C}(x_1, x_2, x_3, ....) = (0, x_2, 0, ....).$$

**Lemma 1.2.5.** *[25] Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $D \in \mathcal{B}(\mathcal{K})$ such that $A$ is regular and $AA^-B = B$, $CA^-A = C$, then*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

*is regular if and only if $S = D - CA^-B$ is regular, in this case*

$$M^- = \begin{bmatrix} A^- + A^-BS^-CA^- & -A^-BS^- \\ -S^-CA^- & S^- \end{bmatrix}.$$

**Example 1.2.4.** *Consider the following block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where*

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 5 & 6 \end{bmatrix},\ C = \begin{bmatrix} 1 & 2 \end{bmatrix},\ D = \begin{bmatrix} 2 & 3 \end{bmatrix}.$$

*It is easy to check that $A^- = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is an inner inverse of $A$, we have $S = D - CA^-B = \begin{bmatrix} -4 & -5 \end{bmatrix}$ and $S^- = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Since $AA^-B = B$ and $CA^-A = C$, then according to Lemma 1.2.5 $M^-$ is given by the formula*

$$M^- = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}.$$

## 1.3 Drazin inverses and Generalized Drazin inverses

In this section we introduce basic concepts of Drazin and generalized Drazin inverses and we recall their properties in Hilbert spaces, also we give some known methods about finding an explicit formulas for the GD-inverse and the group inverse of block operator matrix.

### 1.3.1 Drazin inverses

First we recall the notions of ascent and descent of operators.

**Definition 1.3.1.** *[86] Let $T \in \mathcal{B}(\mathcal{H})$, then*

1. *The ascent of $T$ denoted by $asc(T)$ is the smallest $n \in \mathbb{N}$ such that*

$$\mathcal{N}(T^n) = \mathcal{N}(T^{n+1}),$$

*if such $n$ does not exists then $asc(T) = \infty$.*

2. *The descent of $T$ denoted by $des(T)$ is the smallest $n \in \mathbb{N}$ such that*

$$\mathcal{R}(T^n) = \mathcal{R}(T^{n+1}),$$

*if such $n$ does not exists then $des(T) = \infty$.*

**Theorem 1.3.1.** *[86] Let $T \in \mathcal{B}(\mathcal{H})$, then*

1. *$asc(T) = 0$ if and only if $T$ is injective.*
2. *$des(T) = 0$ if and only if $T$ is surjective.*
3. *If $asc(T) < \infty$ and $des(T) < \infty$, then $asc(T) = des(T) = k$ and*

$$\mathcal{H} = \mathcal{R}(T^k) \oplus \mathcal{N}(T^k).$$

**Definition 1.3.2.** *Let $T \in \mathcal{B}(\mathcal{H})$ such that $asc(T) < \infty$ and $des(T) < \infty$, then the non-negative integer $k$ such that*

$$k = asc(T) = des(T),$$

*is called the Drazin index of $T$ and denoted by $ind(T)$, in particular if $ind(T) = 0$, then $T$ is invertible.*

**Definition 1.3.3.** *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be Drazin invertible if there exists an operator $S \in \mathcal{B}(\mathcal{H})$ such that*

1. $TS = ST$
2. $STS = S$
3. $TST^k = T^k, \quad$ *where* $\quad k = ind(T)$.

*If such $S$ exists then it is called the Drazin inverse of $T$, denoted by $T^D$.*

**Theorem 1.3.2.** *[89] If the Drazin inverse of $T$ exists, then it is unique.*

In particular if $ind(T) = 1$, then $T^D$ is the group inverse of $T$, which is defined as follows

**Definition 1.3.4.** *The group inverse of $T \in \mathcal{B}(\mathcal{H})$ is the unique (if exists) operator $T^\sharp \in \mathcal{B}(\mathcal{H})$ such that*

1. $T^\sharp T T^\sharp = T^\sharp$,
2. $T T^\sharp T = T$,
3. $T T^\sharp = T^\sharp T$.

*If $T^\sharp$ exists, then the spectral idempotent $T^\pi$ is given by $T^\pi = I - TT^\sharp$.*

**Theorem 1.3.3.** *If $T \in \mathcal{B}(\mathcal{H})$ is Drazin invertible, $ind(T) = k$, then the following properties hold*

1. $\mathcal{R}(T^D) = \mathcal{R}(T^k)$ *and* $\mathcal{N}(T^D) = \mathcal{N}(T^k)$.
2. $T^\pi = I - TT^D$, *and so* $\mathcal{R}(T^\pi) = \mathcal{N}(T^k)$ *and* $\mathcal{N}(T^\pi) = \mathcal{R}(T^k)$. *In the particular case that* $ind(T) = 1$, *then* $TT^\pi = 0$.

**Lemma 1.3.1.** *[31, Lemma 2.2] Let $A,\ B \in \mathcal{B}(\mathcal{H})$. If $A$ is group invertible, then the following assertions hold*

1. $\mathcal{R}(A)$ *is closed.*
2. $\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A)$.
3. $\mathcal{R}(AA^\sharp) = \mathcal{R}(A)$ *and* $\mathcal{N}(AA^\sharp) = \mathcal{N}(A)$.
4. $(A^\sharp)^\sharp = A$.
5. $(A^*)^\sharp = (A^\sharp)^*, (A^k)^\sharp = (A^\sharp)^k$ *for each nonnegative integer* $k$.
6. *If* $AB = BA$, *then* $A^\sharp B = BA^\sharp$.
7. $A^\pi$ *is a projection of* $\mathcal{H}$ *onto* $\mathcal{N}(A)$ *along* $\mathcal{R}(A)$ *and* $AA^\sharp$ *is a projection of* $\mathcal{H}$ *onto* $\mathcal{R}(A)$ *along* $\mathcal{N}(A)$.

**Lemma 1.3.2.** *[31, Lemma 2.2] Let $A \in \mathcal{B}(\mathcal{H})$. The following assertions are equivalent*

1. *$A$ is group invertible.*
2. $\mathcal{R}(A) = \mathcal{R}(A^k)$, $\mathcal{N}(A) = \mathcal{N}(A^k)$, $k \geq 2$.
3. $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}$ *with respect to the space decomposition* $\mathcal{H} = \mathcal{R}(A) \oplus^\perp \mathcal{R}(A)^\perp$, *where $A_{11}$ is invertible.*

4. *$A = A_0 \oplus 0$, with respect to the space decomposition $\mathcal{H} = \mathcal{R}(A) \oplus \mathcal{N}(A)$, where $A_0$ is invertible.*

In the following theorem, we recall the group inverse of matrix of operators.

**Theorem 1.3.4.** *[31, Theorem 2.5] Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and let $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, then the following assertions hold*

1. *Assume that $B^\sharp$ exists (resp., $A^\sharp$ exists), then $M^\sharp$ exists if and only if $A^\sharp$ exists (resp., $B^\sharp$ exists) and $A^\pi C B^\pi = 0$.*
2. *Assume that $A^\sharp$, $B^\sharp$ exists, then $M^\sharp$ exists if and only if $A^\pi C B^\pi = 0$, in this case*
$$M^\sharp = \begin{bmatrix} A^\sharp & S \\ 0 & B^\sharp \end{bmatrix},$$
*where $S = (A^\sharp)^2 C B^\pi + A^\pi C (B^\sharp)^2 - A^\sharp C B^\sharp$.*
3. *Assume that $\mathcal{H}$ (resp., $\mathcal{K}$) is finite dimensional, then $M^\sharp$ exists if and only if $A^\sharp$, $B^\sharp$ exists and $A^\pi C B^\pi = 0$.*

**Example 1.3.1.** *Consider the following block triangular matrix $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, where*
$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$
*According to [76] we have $A^\sharp = \begin{bmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} \\ -\frac{3}{4} & 1 & 0 & -\frac{3}{4} \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$ and $A^\pi = \begin{bmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{3}{4} & 0 & 0 \\ \frac{1}{8} & -\frac{1}{4} & 1 & \frac{1}{8} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$.*

*Since $B^2 = B$, then $B$ is a projection and $B^\sharp = B$ so that $B^\pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.*

*We can check that* $A^{\pi}CB^{\pi} = 0$ *and* $S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, *thus we get*

$$M^{\sharp} = \begin{bmatrix} A^{\sharp} & S \\ 0 & B^{\sharp} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} & 2 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 & -\frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

### 1.3.2 Generalized Drazin inverses

In [53] Koliha introduced the concept of the generalized Drazin inverse (GD-inverse), which is defined as follows

**Definition 1.3.5.** *An operator* $T \in \mathcal{B}(\mathcal{H})$ *is said to be GD-invertible if there exists a unique operator* $S \in \mathcal{B}(\mathcal{H})$ *such that*

1. $TS = ST$,
2. $STS = S$,
3. $T - T^2S$ *is quasi-nilpotent.*

*If such* $S$ *exists, then it is called the GD-inverse of* $T$ *denoted by* $T^d$ *and its spectral idempotent is given by* $T^{\pi} = I - TT^d$.

**Remark 1.3.1.** *Note that since every nilpotent operator is quasi-nilpotent, then the Drazin inverse is the special case of GD-inverse.*

**Lemma 1.3.3.** *[53] An operator* $T \in \mathcal{B}(\mathcal{H})$ *is GD-invertible if and only if zero is not an accumulation point of* $\sigma(T)$.

**Lemma 1.3.4.** *[67] Let* $T \in \mathcal{B}(\mathcal{H})$ *be GD-invertible, then* $T$ *has the following operator matrix representaion*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(T^{\pi}) \\ \mathcal{R}(T^{\pi}) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(T^{\pi}) \\ \mathcal{R}(T^{\pi}) \end{bmatrix}, \tag{1.7}$$

*with respect to the decomposition* $\mathcal{H} = \mathcal{N}(T^{\pi}) \oplus \mathcal{R}(T^{\pi})$ *where* $T_1 : \mathcal{N}(T^{\pi}) \longrightarrow \mathcal{N}(T^{\pi})$ *is invertible and* $T_2 : \mathcal{R}(T^{\pi}) \longrightarrow \mathcal{R}(T^{\pi})$ *is quas-inilpotent,* *and the generalized Drazin inverse of $T$ is given by*

$$T^d = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{N}(T^\pi) \\ \mathcal{R}(T^\pi) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{N}(T^\pi) \\ \mathcal{R}(T^\pi) \end{bmatrix},$$

Now we give some well-know results concerning the GD-inverse of block triangular operator matrices .

**Lemma 1.3.5.** *[37, Lemma 2.2] Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$, $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $D \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then*

1. *If $A$ and $B$ are GD-invertible, then $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is GD-invertible and*

$$M^d = \begin{bmatrix} A^d & S \\ 0 & B^d \end{bmatrix},$$

*where $S = (A^d)^2 \left[\sum_{n=0}^{\infty} (A^d)^n C B^n\right] B^\pi + A^\pi \left[\sum_{n=0}^{\infty} A^n C (B^d)^n\right] (B^d)^2 - A^d C B^d$.*

2. *If $A$ and $B$ are GD-invertible, then $M = \begin{bmatrix} A & 0 \\ D & B \end{bmatrix}$ is GD-invertible and*

$$M^d = \begin{bmatrix} A^d & 0 \\ R & B^d \end{bmatrix},$$

*where $R = (B^d)^2 \left[\sum_{n=0}^{\infty} (B^d)^n D A^n\right] A^\pi + B^\pi \left[\sum_{n=0}^{\infty} B^n D (A^d)^n\right] (A^d)^2 - B^d D A^d$.*

**Example 1.3.2.** *Consider the following block triangular matrix $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, where*

$$A = \begin{bmatrix} -3 & -1 & -3 \\ 3 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

*According to [15, Algorithm 7.2.1] we have $A^d = \frac{1}{4} \begin{bmatrix} -9 & -1 & -9 \\ 6 & 2 & 6 \\ 5 & 1 & 5 \end{bmatrix}$ and since $B^2 = 0$, then $B$ is a nillpotent matrix which implies that $B^d = 0$ so that we get*

$S = \frac{1}{4}\begin{bmatrix} -3 & 0 & -3 \\ 6 & 0 & 6 \\ 3 & 0 & 3 \end{bmatrix}$, *thus*

$$M^d = \begin{bmatrix} A^d & S \\ 0 & B^d \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -9 & -1 & -9 & -3 & 0 & -3 \\ 6 & 2 & 6 & 6 & 0 & 6 \\ 5 & 1 & 5 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is still a hard problem to find an explicit formula for the Drazin inverse of the operator matrix $\begin{bmatrix} A & C \\ D & B \end{bmatrix}$ in terms of $A, B, C, D$ and related GD-inverses. GD-inverse formula for the operator matrix $\begin{bmatrix} A & C \\ D & B \end{bmatrix}$ under various conditions appears in literature, in the following theorem we give some of these results.

**Theorem 1.3.5.** *Consider the following block operator matrix* $M = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$, *where* $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$, $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ *and* $D \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

1. *under the condition that $A$ and $B$ are GD-invertible, D. S. Djordjevic et al. in [37] and C. Deng et al. in [32] proved that*

   a) *If $CD = 0$, $BD = 0$ and $CB = 0$, then $M$ is GD-invertible and*

   $$M^d = \begin{bmatrix} A^d & (A^d)^2C \\ D(A^d)^2 & B^d + D(A^d)^3C \end{bmatrix}. \tag{1.8}$$

   b) *If $CD = 0$, $CB = 0$, then $M$ is GD-invertible and*

   $$M^d = \begin{bmatrix} A^d & (A^d)^2C \\ S + T - B^dDA^d & B^d + (B^dT + SA^d)C - B^d(B^dD + DA^d)A^dC \end{bmatrix},$$

   c) *If $AC = 0$, $DC = 0$, then $M$ is GD-invertible and*

   $$M^d = \begin{bmatrix} A^d + C(B^dT + SA^d) - CB^d(B^dD + DA^d)A^d & C(B^d)^2C \\ S + T - B^dDA^d & B^d \end{bmatrix}, \tag{1.9}$$

   d) *If $DA = 0$, $DC = 0$, then $M$ is GD-invertible and*

   $$M^d = \begin{bmatrix} A^d + (T_0B^d + A^dS_0)D - A^d(CB^d + A^dC)B^dD & S_0 + T_0 - A^dCB^d \\ (B^d)^2D & B^d \end{bmatrix},$$

*e) If $CD = 0$, $BD = 0$, then $M$ is GD-invertible and*

$$M^d = \begin{bmatrix} A^d & S_0 + T_0 - A^dCB^d \\ D(A^d)^2 & B^d + D(T_0BB^d + AB^dS_0) - DAB^d(CB^d + A^dC)B^d \end{bmatrix},$$

*f) If $CD = 0$, $DC = 0$ and $BD = DA$, then $M$ is GD-invertible and*

$$M^d = \begin{bmatrix} A^d & S_0 + T_0 - A^dCB^d \\ -D(A^d)^2D & B^d + DA^d(A^dC + CB^d)B^d - D(A^dS_0 + T_0B^d) \end{bmatrix},$$

*g) If $CD = 0$, $DC = 0$ and $AC = CB$, then $M$ is GD-invertible and*

$$M^d = \begin{bmatrix} A^d + CB^d(DA^d + B^dD)A^d & -C(B^d)^2 \\ S + T - B^dDA^d & B^d \end{bmatrix},$$

*where*

$$S_0 = \Sigma_{n=0}^{\infty}(A^d)^{n+2}CB^nB^\pi, \quad T_0 = A^\pi\Sigma_{n=0}^{\infty}A^nC(B^d)^{n+2},$$

*and*

$$S = B^\pi\Sigma_{n=0}^{\infty}B^nD(A^d)^{n+2}, \quad T = \Sigma_{n=0}^{\infty}(B^d)^{n+2}DA^nA^\pi.$$

2. *Under the condition that $A$ is GD-invertible and the schur complement*

$$S = B - DA^dC,$$

*is invertible C. Deng et al. in [32] proved that*

*a) If $A^\pi CD = 0$, $DA^\pi C = 0$ and $A^\pi AC = A^\pi CB$, then $M$ is GD-invertible and*

$$M^d = \left[R - \begin{pmatrix} 0 & A^\pi C \\ 0 & 0 \end{pmatrix} R^2\right]\left[I + \Sigma_{n=0}^{\infty}R^{n+1}\begin{pmatrix} 0 & 0 \\ DA^\pi A^n & 0 \end{pmatrix}\right],$$

*b) If $CDA^\pi = 0$, $DA^\pi C = 0$ and $DAA^\pi = BDA^\pi$, then $M$ is GD-invertible and*

$$M^d = R + \Sigma_{n=0}^{\infty}\begin{pmatrix} 0 & A^nA^\pi C \\ 0 & 0 \end{pmatrix} R^{n+2} - \Sigma_{n=0}^{\infty}\begin{pmatrix} 0 & 0 \\ 0 & DA^nA^\pi C \end{pmatrix} R^{n+3},$$

*where* $R = \begin{bmatrix} A^d + A^dCS^{-1}DA^d & -A^dCS^{-1} \\ -S^{-1}DA^d & S^{-1} \end{bmatrix}.$

3. *Under the condition that $A$, $A\mathcal{G}$ are GD-invertible with $\mathcal{G} = AA^d + A^dCDA^d$ and the schur complement $S = B - DA^dC$ is equal to zero, Hartwing et al. in [44] and C. Deng et al. in [32] proved that*

   a) *If $DA^\pi = 0$, $A^\pi C = 0$, then $M$ is GD-invertible and*

$$M^d = \begin{bmatrix} I \\ DA^d \end{bmatrix} [(A\mathcal{G})^d]^2 A \begin{bmatrix} I & A^dC \end{bmatrix}.$$

   b) *If $A^\pi CD = 0$, $DA^\pi C = 0$ and $A^\pi AC = A^\pi CB$, then $M$ is GD-invertible and*

$$M^d = \left[R_0 - \begin{pmatrix} 0 & A^\pi C \\ 0 & 0 \end{pmatrix} R_0^2\right]\left[I + \Sigma_{n=0}^{\infty} R_0^{n+1} \begin{pmatrix} 0 & 0 \\ DA^\pi A^n & 0 \end{pmatrix}\right].$$

   c) *If $CDA^\pi = 0$, $DA^\pi C = 0$ and $DAA^\pi = BDA^\pi$, then $M$ is GD-invertible and*

$$M^d = R_0 - \Sigma_{n=0}^{\infty} \begin{pmatrix} 0 & A^nA^\pi C \\ 0 & 0 \end{pmatrix} R_0^{n+2} - \Sigma_{n=0}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & DA^nA^\pi B \end{pmatrix} R_0^{n+3}.$$

   *Where* $R_0 = \begin{bmatrix} I \\ DA^d \end{bmatrix} [(A\mathcal{G})^d]^2 A \begin{bmatrix} I & A^dC \end{bmatrix}.$

**Example 1.3.3.** *Consider the following block matrix* $M = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$, *where*
$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $B = \frac{1}{7}\begin{bmatrix} 1 & 3 & -19 & 11 \\ 0 & 0 & 7 & -7 \\ 1 & 3 & -12 & 11 \\ 0 & 0 & 7 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -3 & 0 & 0 & 6 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
*According to [44] we have $A$, $B$ are GD-invertible where*

$$A^d = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B^d = \frac{1}{3}\begin{bmatrix} 1 & 3 & -1 & 1 \\ \frac{10}{3} & 10 & \frac{11}{3} & \frac{13}{3} \\ 0 & 0 & 0 & 3 \\ -1 & -3 & 1 & 2 \end{bmatrix}.$$

*Since $A$ and $B$ are generalized Drazin invertible and $CD = 0, CB = 0$ and $BD =$*

0, *then* $M$ *is GD-invertible, and by using formula (1.8), we get*

$$M^d = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & \frac{1}{3} & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & -1 & \frac{10}{9} & \frac{10}{3} & \frac{11}{9} & \frac{13}{9} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -1 & \frac{1}{3} & \frac{2}{3} \end{bmatrix},$$

**Example 1.3.4.** *Consider the following block matrix* $M = \begin{bmatrix} A & C \\ D & B \end{bmatrix}$, *where*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

*Since* $A$ *and* $B$ *are GD-invertible and* $AC = 0, DC = 0$, *then* $M$ *is GD-invertible, and by using formula (1.9) we get*

$$M^d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

## 1.4 Matrix and operator equations

In this section we recall some known results about matrix and operator equations, first we need the following definitions.

**Definition 1.4.1.** *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $B \in \mathcal{B}(\mathcal{K})$. *We say that* $A$ *is similar to* $B$, *if there exists an invertible operator* $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ *such that*

$$A = TBT^{-1}.$$

**Definition 1.4.2.** *Let* $A \in \mathcal{B}(\mathcal{H})$ *and* $B \in \mathcal{B}(\mathcal{K})$. *We say that* $A$ *is equivalent to* $B$, *if there exists two invertible operators* $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ *and* $P \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *such that*

$$A = TBP.$$

The equation $AX - XB = C$ has first studied in the finite dimensional case where basic theorem was proved by Sylvester in [84] many authors find the extensions to operators as did Rosenblum [78] in the following Theorem

**Theorem 1.4.1.** *[6] If $A, B \in \mathcal{B}(\mathcal{H})$, such that $\sigma(A) \cap \sigma(B) = \emptyset$, then the operator equation $AX - XB = C$ has a unique solution $X$, for every $C \in \mathcal{B}(\mathcal{H})$.*

The Sylvester Rosenblum Theorem gives the following result.

**Theorem 1.4.2.** *Let $A, B \in \mathcal{B}(\mathcal{H})$. If $\sigma(A) \cap \sigma(B) = \emptyset$, then for every $C \in \mathcal{B}(\mathcal{H})$, the operators*

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ are similar.}$$

This result was first observed in the finite dimensional case by Roth [80] who gives an interesting necessary and sufficient condition for the equation

$$AX - XB = C,$$

to have a solution as follows.

**Theorem 1.4.3.** *[80] If A, B are operators on fnite-dimensional spaces, then*

1. *$AX - XB = C$ is solvable, if and only if $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are similar.*
2. *$AX - YB = C$ is solvable, if and only if $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are equivalent.*

A nice proof of Roth's Theorem was given by Flanders et al. in [39], Rosenblum in [79] showed that Roth's Theorem can be extend to infinite-dimensional Hilbert spaces in the special case when A and B are self-adjoint operators on a complex separable Hilbert space as follows.

**Theorem 1.4.4.** *[79] Suppose that A and B are bounded self-adjoint operators on complex separable Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ respectively, then the equation*

$$AX - XB = C,$$

*is solvable if and only if $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are similar.*

Also in [82] Schweinsberg proved that Roth's Theorem can be extend to include finite rank operators and normal operators on infinite-dimensional Hilbert spaces.

**Theorem 1.4.5.** *[82] Let A and B be bounded normal operators on complex Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ respectively, then $AX - XB = C$ has a solution $X$ if and only if*

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ are similar.}$$

*Proof.* If the equation $AX - XB = C$ has a solution , then

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

then $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are similar.

Suppose that $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ are similar, then there exists an invertible operator $\begin{bmatrix} Q & R \\ S & T \end{bmatrix}$ such that $\begin{bmatrix} Q & R \\ S & T \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} Q & R \\ S & T \end{bmatrix}$, which implies that

$$QA - AQ = CS, RB - AR = CT$$

$$SA = BS, TB = BT.$$

We apply the Fuglede-Putnam theorem, we obtain

$$AS^* = S^*B, T^*B = BT^*,$$

then $B$ commute with $SS^*$ and $TT^*$. We have also

$$C(SS^* + TT^*) = (QS^* + RT^*)B - A(QS^* + RT^*),$$

then there exists

$$X = -(QS^* + RT^*)(SS^* + TT^*)^{-1}, \text{ such that } \quad AX - XB = C.$$

□

**Theorem 1.4.6.** *[82] Suppose that $A$ and $B$ are finite rank operators on complex Hilbert spaces, then $AX - XB = C$ has a solution $X$ if and only if*

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \textit{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \textit{ are similar.}$$

Chapter 2

# Solvability of the operator equations $AX - XB = C$ and $AX - YB = C$.

This chapter is the subject of a submitted article [61] written in collaboration with Pr. Lombarkia Farida. In this chapter we generalize the notions of pseudo-similarity and pseudo-equivalence introduced by R.E. Hartwig and F. J. Hall [45] for matrices over a ring to the setting of bounded linear operators defined on Hilbert spaces of infinite dimension.
Also we provides new necessary and sufficient conditions for the solvability to the operators equations $AX - XB = C$ and $AX - YB = C$, where $A$ and $B$ are group invertible operators defined on an infinite dimensional Hilbert spaces.
In addition the general solutions to the equation $AX - YB = C$, are derived in terms of the group inverse of $A$ and $B$.
As a consequence, new necessary and sufficient conditions for the solvability to the operator equation $AYB - Y = C$, are derived.

## 2.1 Pseudo-similarity and pseudo-equivalence of operators

In this section we give the definitions and some properties of the notions of pseudo-similarity and pseudo-equivalence of bounded linear operators defined on Hilbert spaces of infinite dimension, which are weaker than similarity and equivalence known in the literature.

**Definition 2.1.1.** *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we say that $A$ is pseudo-similar to $B$, via $T$ and we write $A \overset{\approx}{\sim} B$, if there exists a regular operator $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that*

$$A = TBT^{=} \text{ and } B = T^{-}AT,$$

*where $T^{-}, T^{=}$ are two possibly different inner inverses of $T$.*

It is easy to prove that similarity implies pseudo similarity but in general pseudo-similarity does not imply similarity as seen from the following example.

**Example 2.1.1.** *Let $S_r$ the unilateral right shift and $S_l$ the left shift defined on the Hilbert space $l_2(\mathbb{N})$ by*

$$S_r(x_1, x_2, x_3, ....) = (0, x_1, x_2, ......),$$

*and*

$$S_l(x_1, x_2, x_3, ....) = (x_2, x_3, ......).$$

*Let $A = S_r S_l$, and $B = I$ is the identity operator on $l_2(\mathbb{N})$, since $S_l$ is the inner inverse of $S_r$, we have $S_l A S_r = B$ and $S_r B S_l = A$, so that $A$ is pseudo-similar to $B$, and $B$ is not similar to $A$.*

**Theorem 2.1.1.** *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, suppose that $A$ is pseudo-similar to $B$, with $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $T^-$ and $T^=$ are as in Definition 2.1.1, then the following are valid*

$$A = TT^- ATT^= = ATT^= = TT^- A,$$

*and*

$$B = TT^- BT^= T = TT^- B = BT^= T.$$

*Proof.* Since $A$ is pseudo-similar to $B$, with $T$, $T^-$ and $T^=$ are as in Definition 2.1.1, hence by substituting $B = T^- AT$ into $A = TBT^=$ yields $A = TT^- ATT^=$ which on post-multiplying by $TT^-$ and per-multiplying by $TT^=$ we get

$$\begin{aligned} TT^- A &= TT^- TT^- ATT^= \\ &= (TT^- T)T^- ATT^= \\ &= TT^- ATT^= \\ &= A, \end{aligned}$$

and

$$\begin{aligned} ATT^= &= TT^- ATT^= TT^= \\ &= TT^- A(TT^= T)T^= \\ &= TT^- ATT^= \\ &= A. \end{aligned}$$

Hence we get $A = TT^- ATT^= = ATT^= = TT^- A$ hold.
Similarly we proof that $B = TT^- BT^= T = TT^- B = BT^= T$. □

We deduce the following corollary.

**Corollary 2.1.1.** *Let $A \in \mathcal{B}(\mathcal{H})$ , $B \in \mathcal{B}(\mathcal{K})$ and let $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be a regular operator, then the following are equivalent*

i. $A \eqsim B$ *via* $T$,

ii. $AT = TB$, $A = ATT^{=}$ *and* $B = T^{-}TB$,

iii. $BT^{=} = T^{-}A$, $TT^{-}A = A$ *and* $BT^{=}T = B$.

*Proof.* $(i) \Rightarrow (ii)$ Since $A$ is pseudo-similar to $B$, then according to Theorem 2.1.1, we get

$$A = ATT^{=} = TT^{-}A \text{ and } B = TT^{-}B = BT^{=}T,$$

per-multiplying $B = T^{-}AT$ by $T$ yields

$$TB = TT^{-}AT = AT.$$

$(ii) \Rightarrow (iii)$ Subsisting $AT = TB$ in $A = ATT^{=}$ and $B = T^{-}TB$ yields $A = TBT^{=}$ and $B = T^{-}AT$, post-multiplying $B = T^{-}AT$ by $T^{=}$ we get

$$BT^{=} = T^{-}ATT^{=} = T^{-}A.$$

Which on post-multiplying and per-multiplying $BT^{=} = T^{-}A$ by $T$, yields

$$BT^{=}T = T^{-}AT = B \text{ and } TT^{-}A = TBT^{=} = A.$$

$(iii) \Rightarrow (i)$
Subsisting $BT^{=} = T^{-}A$ in $TT^{-}A = A$ and $BT^{=}T = B$ yields $A = TBT^{=}$ and $B = T^{-}AT$ which implies according to Definition 2.1.1 that $A \eqsim B$ via $T$. □

Now we give the definition of pseudo-equivalence for bounded operators defined on Hilbert spaces.

**Definition 2.1.2.** *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, we say that $A$ is pseudo-equivalent to $B$, if there exists two regular operators $P \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $Q \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that*

$$B = PAQ \text{ and } A = P^{-}BQ^{-},$$

*where $Q^{-}, P^{-}$ are the inner inverses of $Q$ and $P$ respectively.*

**Proposition 2.1.1.** *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, if $A$ is pseudo-equivalent to $B$ via $P$ and $Q$, then we have*

i. $P^{-}PA = A$ *and* $AQQ^{-} = A$,

ii. $PP^{-}B = B$ *and* $BQ^{-}Q = B$.

*Proof.* i. $A$ is pseudo-equivalent to $B$, then there exists two regular operators $P \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $Q \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that

$$B = PAQ \text{ and } A = P^{-}BQ^{-}.$$

From the equality $A = P^{-}BQ^{-}$, we obtain

$$\begin{aligned} P^{-}PA &= P^{-}PP^{-}BQ^{-} \\ &= P^{-}PP^{-}(PAQ)Q^{-} \\ &= P^{-}(PP^{-}P)AQQ^{-} \\ &= P^{-}(PAQ)Q^{-} \\ &= P^{-}BQ^{-} \\ &= A, \end{aligned}$$

and

$$\begin{aligned} AQQ^{-} &= P^{-}BQ^{-}QQ^{-} \\ &= P^{-}PA(QQ^{-}Q)Q^{-} \\ &= P^{-}(PAQ)Q^{-} \\ &= P^{-}BQ^{-} \\ &= A. \end{aligned}$$

ii. Similarly as the proof of $(i)$, we have $PP^{-}B = B$ and $BQ^{-}Q = B$.

□

## 2.2 Solvability of the operator equation $AX - XB = C$

In the following section , we give necessary and sufficient conditions for the existences of the solutions to the operator equation $AX - XB = C$, using the pseudo-similarity of the matrix of operators $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

**Theorem 2.2.1.** *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ such that $A$ and $B$ are group invertible. Then the following are equivalent.*

*i. There exists $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ solution to the equation*

$$AX - XB = C. \tag{2.1}$$

*ii. There exists $P \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ regular, such that $M$ is pseudo-similar to $D$ via $P$.*

*Proof.* $(i) \Rightarrow (ii)$ Suppose that $X$ is a solution of the equation (2.1) and let

$$P = \begin{bmatrix} AA^\sharp & -XBB^\sharp \\ 0 & BB^\sharp \end{bmatrix}.$$

Observe that $AA^\sharp$ and $BB^\sharp$ are group invertible and that $(AA^\sharp)^\pi = A^\pi$ and $(BB^\sharp)^\pi = B^\pi$. Since $A^\pi(-XBB^\sharp)B^\pi = 0$, then from [31, Theorem 2.5] $P$ is group invertible, hence $P$ is regular and

$$P^- = \begin{bmatrix} AA^\sharp & AA^\sharp XBB^\sharp \\ 0 & BB^\sharp \end{bmatrix},$$

is an inner inverse of $P$. Since from [16, Theorem 2] the set of inner inverses is given by $\{P^- + U - P^-PUPP^-\}$, where $U$ is arbitrary, If we choose $U = \begin{bmatrix} 0 & AA^\sharp X \\ 0 & 0 \end{bmatrix}$, we get that

$$P^= = \begin{bmatrix} AA^\sharp & AA^\sharp X \\ 0 & BB^\sharp \end{bmatrix}$$

is another inner inverse of $P$ and we have

$$\begin{aligned} PDP^= &= \begin{bmatrix} AA^\sharp & -XBB^\sharp \\ 0 & BB^\sharp \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} AA^\sharp & AA^\sharp X \\ 0 & BB^\sharp \end{bmatrix} \\ &= \begin{bmatrix} A & AX - XB \\ 0 & B \end{bmatrix} \\ &= \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \\ &= M, \end{aligned}$$

and

$$\begin{aligned} P^-MP &= \begin{bmatrix} AA^\sharp & AA^\sharp XBB^\sharp \\ 0 & BB^\sharp \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} AA^\sharp & -XBB^\sharp \\ 0 & BB^\sharp \end{bmatrix} \\ &= \begin{bmatrix} A & AA^\sharp C + AA^\sharp XB \\ 0 & B \end{bmatrix} \begin{bmatrix} AA^\sharp & -XBB^\sharp \\ 0 & BB^\sharp \end{bmatrix} \\ &= \begin{bmatrix} A & -AXBB^\sharp + AA^\sharp CBB^\sharp + AA^\sharp XB \\ 0 & B \end{bmatrix} \\ &= \begin{bmatrix} A & -AXBB^\sharp + AA^\sharp(AX - XB)BB^\sharp + AA^\sharp XB \\ 0 & B \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \\ &= D. \end{aligned}$$

Then $M$ is pseudo-similar to $D$, via $P$.
$(ii) \Rightarrow (i)$
Suppose that there exists $P = \begin{bmatrix} Q & R \\ S & T \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ regular such that $M$ is pseudo-similar to $D$, via $P$, it follows from Corollary 2.1.1 that

$$MP = PD, \; MPP^{=} = M \text{ and } P^{-}PD = D.$$

The equation $MP = PD$ implies that

$$\begin{bmatrix} AQ + CS & AR + CT \\ BS & BT \end{bmatrix} = \begin{bmatrix} QA & RB \\ SA & BT \end{bmatrix},$$

then

$$\begin{cases} CS &= -AQ + QA, \\ CT &= -AR + RB, \\ BS &= SA, \\ BT &= TB. \end{cases}$$

Hence $P$ is of the following form

$$P = \begin{bmatrix} AA^{\sharp} & R \\ 0 & BB^{\sharp} \end{bmatrix}.$$

Since $P$ is regular then $W = (I - AA^{\sharp})R(I - BB^{\sharp})$ is regular and it follows from [70, Theorem 1] that

$$P^{-} = \begin{bmatrix} AA^{\sharp} - AA^{\sharp}R\tilde{W} & -AA^{\sharp}RBB^{\sharp} + AA^{\sharp}R\tilde{W}RBB^{\sharp} \\ \tilde{W} & BB^{\sharp} - \tilde{W}RBB^{\sharp} \end{bmatrix},$$

is an inner inverse of $P$,where $\tilde{W} = (I - BB^{\sharp})W^{-}(I - AA^{\sharp})$.
Since the set of inner inverses of $P$ is given by $\{P^{-} + U - P^{-}PUPP^{-}\}$ where $U$ is arbitrary, then if we choose $U = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ we get that

$$P^{=} = \begin{bmatrix} I - AA^{\sharp}R\tilde{W} & -AA^{\sharp}RBB^{\sharp} + AA^{\sharp}R\tilde{W}RBB^{\sharp} \\ \tilde{W} & I - \tilde{W}RBB^{\sharp} \end{bmatrix},$$

is another inner inverse of $P$.
We have the equality $P^{-}PD = D$ is satisfied and the equality $MPP^{=} = M$ implies that $CBB^{\sharp} = C - AR + ARBB^{\sharp}$, substituting $CBB^{\sharp}$ in the equation $CBB^{\sharp} = -AR + RB$ we get

$$\begin{aligned} C &= -ARBB^{\sharp} + RB \\ &= A(-RBB^{\sharp}) - (-RBB^{\sharp})B, \end{aligned}$$

which means that $-RBB^{\sharp}$ is a solution of the equation $AX - XB = C$. □

Now we cite some of class of operators which are group invertible. In [34] Djordjevic et al. introduced the class of polynomially normal operator on a complex Hilbert space, extending the notion of $n$-normal and normal operators, recall that $A$ is polynomially normal if there exists a non-trivial polynomial $p$ such that $p(A)$ is normal.
In [21, Theorem 3.3] the authors proved that if $A \in \mathcal{B}(\mathcal{H})$ is polynomially normal such that $p(A)$ has closed range, where $p$ is non-trivial polynomial defined as follow $p(z) = \Sigma_{i=1}^{n} a_i z^i$ and $a_1 \neq 0$, then $A$ is group invertible.
Another class of operators $A \in \mathcal{B}(\mathcal{H})$, which are group invertible is the class of algebraic operators $p(A) = 0$, where $p(z) = \Sigma_{i=1}^{n} a_i z^i$ and $a_1 \neq 0$.

**Example 2.2.1.** *Consider the matrix equation*

$$AX - XB = C,$$

$$\text{such that } A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 1 & 1 & -2 & -5 \end{bmatrix},\ B = \frac{1}{4}\begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -1 & 4 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \text{ and}$$

$$C = \frac{1}{19}\begin{bmatrix} 43 & 39 & 43 & -125 \\ 24 & 20 & 24 & -68 \\ 0 & 0 & 0 & 0 \\ -24 & -20 & -24 & 68 \end{bmatrix}.$$

*According to [76] $A$ is group invertible and*

$$A^\sharp = \begin{bmatrix} -5 & 4 & 1 & -2 \\ -21 & 17 & 4 & -9 \\ 16 & -13 & -3 & 7 \\ -11 & 9 & 2 & -5 \end{bmatrix},\ AA^\sharp = \begin{bmatrix} -1 & 1 & 0 & -1 \\ -5 & 4 & 1 & -2 \\ 4 & -3 & -1 & 1 \\ -3 & 2 & 1 & 0 \end{bmatrix}.$$

*Similarly we find that $B$ is group invertible and*

$$B^\sharp = \frac{2}{1083}\begin{bmatrix} 265 & -61 & -96 & -108 \\ -96 & 300 & -96 & -108 \\ -115 & -137 & 246 & 6 \\ -210 & -156 & -210 & 576 \end{bmatrix},\ BB^\sharp = \frac{1}{19}\begin{bmatrix} 13 & -5 & -6 & -2 \\ -6 & 14 & -6 & -2 \\ -6 & -5 & 13 & -2 \\ -6 & -5 & -6 & 17 \end{bmatrix}.$$

*Then there exists*

$$P = \begin{bmatrix} AA^\sharp & R \\ 0 & -I \end{bmatrix} \text{ where } R = -\frac{4}{19}\begin{bmatrix} 6 & 5 & 6 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

*Since $P$ is regular then, it follows from [70, Theorem 1] that*

$$P^{-} = \begin{bmatrix} AA^\sharp & -AA^\sharp RBB^\sharp \\ 0 & BB^\sharp \end{bmatrix} \text{ and } P^{=} = \begin{bmatrix} AA^\sharp & R - AA^\sharp RBB^\sharp \\ 0 & BB^\sharp \end{bmatrix}$$

*are two inner inverses of $P$ and we have the equalities $MPP^{=} = M$, $P^{-}PD = D$ are satisfied, then $M$ and $D$ are pseudo-similar via $P$, hence according to Theorem 2.2.1, $AX - XB = C$ is solvable.*

## 2.3 Solvability of the operator equation $AX - YB = C$

In this section we give some equivalent conditions to the solvability of the operator equation $AX - YB = C$, the most important one is that the matrices of operators $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ and $D = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, are pseudo equivalent via $P$ and $Q$ and the operator $U = DQPDD^{\sharp} + I - DD^{\sharp}$ is invertible.

**Theorem 2.3.1.** *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $A$ and $B$ are group invertible. Then the following conditions are equivalent.*

1. *$M$ is group invertible,*
2. *$A^{\pi}CB^{\pi} = 0$,*
3. *There exists $X, Y \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ solutions of the equation $AX - YB = C$,*
4. *$M$ and $D$ are pseudo equivalent via $P$ and $Q$ and the operator $U = DQPDD^{\sharp} + I - DD^{\sharp}$ is invertible.*

*In this case the general solutions of $AX - YB = C$, are given by*

$$\left\{ \begin{array}{lcl} X & = & A^{\sharp}C + A^{\sharp}ZB + A^{\pi}W \\ Y & = & -A^{\pi}CB^{\sharp} + Z + AA^{\sharp}ZBB^{\sharp} - ZBB^{\sharp} \end{array} \right. \tag{2.2}$$

*where $W$, $Z \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ are arbitrary.*

*Proof.* (1) $\Leftrightarrow$ (2) Since $A$ and $B$ are group invertible it follows from [31, Theorem 2.5], that (1) and (2) are equivalent.
(2) $\Leftrightarrow$ (3) Since $A^{\pi}CB^{\pi} = 0$, then we get

$$A(A^{\sharp}C) - (AA^{\sharp}CB^{\sharp} - CB^{\sharp})B = C,$$

which implies that $AX_0 - Y_0B = C$, where

$$\left\{ \begin{array}{lcl} X_0 & = & A^{\sharp}C \\ Y_0 & = & (AA^{\sharp} - I)CB^{\sharp} \end{array} \right.$$

are particular solutions of the equation $AX - YB = C$.
Since there exists $X, Y \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ solutions to the equation $AX - YB = C$, then $A^{\pi}[AX - YB]B^{\pi} = A^{\pi}CB^{\pi}$, this implies that $A^{\pi}CB^{\pi} = 0$, thus (2) $\Leftrightarrow$ (3).
(3) $\Rightarrow$ (4) Suppose that $X$ and $Y$ are solutions to the equation $AX - YB = C$, which is equivalent to $A^{\pi}CB^{\pi} = 0$, then there exists

$$P = \begin{bmatrix} AA^{\sharp} & A^{\pi}CB^{\sharp} \\ 0 & BB^{\sharp} \end{bmatrix} \text{ and } Q = \begin{bmatrix} AA^{\sharp} & A^{\sharp}C \\ 0 & BB^{\sharp} \end{bmatrix}.$$

Since $AA^{\sharp}$, $BB^{\sharp}$ are idempotent and $A^{\pi}CB^{\sharp}B^{\pi} = 0$, $A^{\pi}A^{\sharp}CB^{\pi} = 0$, it follows from [31, Theorem 2.5] that $P$ and $Q$ are group invertible then they are regular, the inner inverses are given by

$$P^{-} = \begin{bmatrix} AA^{\sharp} & 0 \\ 0 & BB^{\sharp} \end{bmatrix} \text{ and } Q^{-} = \begin{bmatrix} AA^{\sharp} & -A^{\sharp}CBB^{\sharp} \\ 0 & BB^{\sharp} \end{bmatrix}.$$

We have

$$\begin{aligned} PDQ &= \begin{bmatrix} AA^{\sharp} & A^{\pi}CB^{\sharp} \\ 0 & BB^{\sharp} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} AA^{\sharp} & A^{\sharp}C \\ 0 & BB^{\sharp} \end{bmatrix} \\ &= \begin{bmatrix} A & -CBB^{\sharp} + AA^{\sharp}CBB^{\sharp} \\ 0 & B \end{bmatrix} \begin{bmatrix} AA^{\sharp} & A^{\sharp}C \\ 0 & BB^{\sharp} \end{bmatrix} \\ &= \begin{bmatrix} A & AA^{\sharp}C + CBB^{\sharp} - AA^{\sharp}CBB^{\sharp} \\ 0 & B \end{bmatrix} \\ &= \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \\ &= M. \end{aligned}$$

and

$$\begin{aligned} P^{-}MQ^{-} &= \begin{bmatrix} AA^{\sharp} & 0 \\ 0 & BB^{\sharp} \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} AA^{\sharp} & -A^{\sharp}CBB^{\sharp} \\ 0 & BB^{\sharp} \end{bmatrix} \\ &= \begin{bmatrix} A & AA^{\sharp}C \\ 0 & B \end{bmatrix} \begin{bmatrix} AA^{\sharp} & -A^{\sharp}CBB^{\sharp} \\ 0 & BB^{\sharp} \end{bmatrix} \\ &= \begin{bmatrix} A & -AA^{\sharp}CBB^{\sharp} + AA^{\sharp}CBB^{\sharp} \\ 0 & B \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \\ &= D. \end{aligned}$$

Then $M$ and $D$ are pseudo equivalent, in addition we have

$$U = DQPDD^{\sharp} + I - DD^{\sharp} = \begin{bmatrix} A + A^{\pi} & AA^{\sharp}CBB^{\sharp} \\ 0 & B + B^{\pi} \end{bmatrix}.$$

Since $A$ and $B$ are group invertible, $A^\pi$ and $B^\pi$ are projections, then from [77, Lemma 3], $A + A^\pi$ and $B + B^\pi$ are invertible, hence from [43, Lemma 1], $U$ is invertible.
(4) ⇒ (1) Suppose that $M$ and $D$ are pseudo equivalent via $P$ and $Q$, it follows from Proposition 2.1.1 that $P^-PD = D$ and $DQQ^- = D$, where $P^-, Q^-$ are the inner inverses of $P$ and $Q$ respectively, since $U$ is invertible, then from [31, Lemma 3.1], $M$ is group invertible. If one of the conditions (1) — (4) holds, substituting (2.2) into $AX - YB$ gives
$A[A^\sharp C + A^\sharp ZB + A^\pi W] - [(AA^\sharp - I)CB^\sharp + Z + AA^\sharp ZBB^\sharp - ZBB^\sharp]B$

$$\begin{aligned}
&= AA^\sharp C + AA^\sharp ZB - (AA^\sharp - I)CB^\sharp B - AA^\sharp ZB \\
&= AA^\sharp C + CB^\sharp B - AA^\sharp CB^\sharp B \\
&= C.
\end{aligned}$$

Hence (2.2) satisfies the equation $AX - YB = C$. Let $X_0$ and $Y_0$ are arbitrary solution to the equation $AX - YB = C$. Letting $W = X_0$ and $Z = Y_0$. Then

$$\begin{aligned}
X &= A^\sharp C + A^\sharp Y_0 B + A^\pi X_0 \\
&= A^\sharp C + A^\sharp Y_0 B + X_0 - AA^\sharp X_0 \\
&= X_0.
\end{aligned}$$

And

$$\begin{aligned}
Y &= (AA^\sharp - I)CB^\sharp + Y_0 + AA^\sharp Y_0 BB^\sharp - Y_0 BB^\sharp \\
&= -(AA^\sharp - I)Y_0 BB^\sharp + Y_0 + AA^\sharp Y_0 BB^\sharp - Y_0 BB^\sharp \\
&= Y_0.
\end{aligned}$$

This show that (2.2) are general solutions of the equation $AX - YB = C$ . □

**Example 2.3.1.** *Consider the matrix equation*

$$AX - YB = C,$$

$$\text{such that } A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 1 & 1 & -2 & -5 \end{bmatrix}, \; B = \frac{1}{4}\begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 4 & -2 & 0 \\ -2 & -1 & 4 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, \text{ and}$$

$$C = \begin{bmatrix} 3 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ -6 & 1 & 1 & -2 \end{bmatrix}.$$

*According to [76] we have $A$ is group invertible where*

$$A^{\sharp} = \begin{bmatrix} -5 & 4 & 1 & -2 \\ -21 & 17 & 4 & -9 \\ 16 & -13 & -3 & 7 \\ -11 & 9 & 2 & -5 \end{bmatrix} \text{ and } AA^{\sharp} = \begin{bmatrix} -1 & 1 & 0 & -1 \\ -5 & 4 & 1 & -2 \\ 4 & -3 & -1 & 1 \\ -3 & 2 & 1 & 0 \end{bmatrix}.$$

*Similarly we have $B$ is group invertible where*

$$B^{\sharp} = \frac{2}{1083} \begin{bmatrix} 265 & -61 & -96 & -108 \\ -96 & 300 & -96 & -108 \\ -115 & -137 & 246 & 6 \\ -210 & -156 & -210 & 576 \end{bmatrix} \text{ and } BB^{\sharp} = \frac{1}{19} \begin{bmatrix} 13 & -5 & -6 & -2 \\ -6 & 14 & -6 & -2 \\ -6 & -5 & 13 & -2 \\ -6 & -5 & -6 & 17 \end{bmatrix}.$$

*We obtain $A^{\pi}CB^{\pi} = 0$, then according to Theorem 2.3.1, the equation $AX - YB = C$ have a solutions.*

**Example 2.3.2.** *Consider the operators equation $AX - YB = C$, where A, $B$ and $C$ are defined on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ by*

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

*Then we have $A^{\pi}CB^{\pi} = 0$, consequently the equation $AX - YB = C$ have a solutions.*

Now we apply Theorem 2.3.1 to give new necessary and sufficient conditions for the existence of solution to the Stein equation $AYB - Y = C$.

**Corollary 2.3.1.** *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, such that $A + I$ and $B + I$ are group invertible, then*
*The equation $AYB - Y = C$ has a solution if and only if $M_1 = \begin{bmatrix} A+I & C \\ 0 & B+I \end{bmatrix}$ and $D_1 = \begin{bmatrix} A+I & 0 \\ 0 & B+I \end{bmatrix}$ are pseudo equivalent via $P$ and $Q$ and the operator $U_1 = D_1QPD_1D_1^{\sharp} + I - D_1D_1^{\sharp}$ is invertible.*

*Proof.* The equation $AYB - Y = C$ is equivalent to

$$\begin{cases} X & = & YB \\ AX - Y & = & C \end{cases} \tag{2.3}$$

(2.3) is equivalent to

$$(A + I)X - Y(B + I) = C. \tag{2.4}$$

Since $A + I$ and $B + I$ are group invertible, it follows from Theorem 2.3.1, that the equation (2.4) has a solution if and only if $M_1$ and $D_1$ are pseudo equivalent via $P$ and $Q$ and the operator $U_1 = D_1QPD_1D_1^{\sharp} + I - D_1D_1^{\sharp}$ is invertible. $\square$

# Chapter 3

# On the operator equation $AX - XB + XDX - C = 0$

This chapter is the subject of the following article [5] written in collaboration with Pr. Lombarkia Farida published in Rendiconti del Circolo Matematico di Palermo Series 2.
In this chapter we present a method using GD-invese to solve the following algebraic Riccati and Lyapunov equations in Hilbert spaces

$$XDX + AX - XB - C = 0, \tag{3.1}$$

$$AX - XB = C, \tag{3.2}$$

that permits to reduce the equation (3.1) to a system of two linear operator equations. This reduction is available by means of the application of the GD-inverse of matrices of operators, we add some conditions in order to give the form of the general solution to the Riccati equation (3.1), then we deduce the same results for the Lyapunov operator equation (3.2). This result is applied to the Cauchy problems for Riccati and Lyapunov operator differential equations, because of the relation between the algebraic and the differential problems.

## 3.1 Solvability of the Riccati equation

**Theorem 3.1.1.** *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$, such that*

$$W = \begin{bmatrix} A & C \\ D & B \end{bmatrix} \text{ is GD-invertible and } W^d = \begin{bmatrix} U & M \\ V & N \end{bmatrix}.$$

*i. If $X \in \mathcal{B}(\mathcal{H})$ is a solution of the Riccati equation (3.1) such that $A + XD$ is quasi-nilpotent, then $X$ satisfies the operator system*

$$\begin{cases} U + XV &= 0 \\ M + XN &= 0 \end{cases} \tag{3.3}$$

*ii. If $X \in \mathcal{B}(\mathcal{H})$ is a solution of the system (3.3) and $V^*$ or $N^*$ is injective, then $X$ satisfies the Riccati equation (3.1).*

*Proof.* i. Suppose that $X$ is a solution of equation (3.1), then we have

$$\begin{aligned}\left[I, X\right] W &= \left[I, X\right] \begin{bmatrix} A & C \\ D & B \end{bmatrix} \\ &= \left[A + XD, C + XB\right] \\ &= \left[A + XD, AX + XDX\right] \\ &= (A + XD) \left[I, X\right],\end{aligned}$$

hence $X$ satisfies

$$\left[I, X\right].W = (A + XD)\left[I, X\right].$$

Thus, we have from [53, Theorem 4.4]

$$\left[I, X\right].W^d = (A + XD)^d \left[I, X\right].$$

Since $(A + XD)^d = 0$, we obtain $\left[I, X\right].W^d = 0$, which implies that

$$\begin{cases} U + XV &= 0 \\ M + XN &= 0 \end{cases}$$

Thus $X$ satisfies the operator system (3.3).

ii. Suppose that $X$ satisfies the system (3.3), by the equality $W^d W = W W^d$, we obtain

$$\begin{cases} UA + MD &= AU + CV \\ VA + ND &= DU + BV \end{cases}, \tag{3.4}$$

and

$$\begin{cases} VC + NB &= DM + BN \\ UC + MB &= AM + CN \end{cases} \tag{3.5}$$

Let $X$ be the solution of the system (3.3), then using (3.4), we have the following equation

$$\begin{aligned}(XN + M)D &= X(DU + BV - VA) + (AU + CV - UA) \\ &= 0,\end{aligned} \tag{3.6}$$

by substituting $U = -XV$ in (3.6), we obtain

$$(-C + AX - XB + XDX)V = 0.$$

Since $V^*$ is injective, then $-C + AX - XB + XDX = 0$, thus $X$ satisfies the Riccati equation.
Let $X$ is the solution of the system (3.3), then using (3.5) we have the following equation

$$\begin{aligned}(XV + U)C &= X(DM + BN - NB) + (AM + CN - MB) \\ &= 0,\end{aligned} \tag{3.7}$$

by substituting $M = -XN$ in (3.7), we obtain

$$(-C + AX - XB + XDX)N = 0.$$

Since $N^*$ is injective, then $-C + AX - XB + XDX = 0$, thus $X$ satisfies the Riccati equation (3.1).

□

It is still a hard problem to find an explicit formula for $W^d$ in terms of $A, B, C, D$ and related GD-inverses. GD-inverse formula for the operator matrix $W$ under various conditions appears in literature, see [18, 20, 22, 28, 36, 32] and the references therein.
In the case when $C = 0$ or $D = 0$, the authors in [37] got the form of the GD-inverse of $W$, so if we put $C = 0$ in the Riccati equation (3.1) we get the following Corollary.

**Corollary 3.1.1.** *Let $A, B, D \in \mathcal{B}(\mathcal{H})$, such that $A$ and $B$ are GD-invertible, then*

*i. If $X \in \mathcal{B}(\mathcal{H})$ is a solution of the equation $XDX + AX - XB = 0$, such that $A + XD$ is quasi-nilpotent, then $X$ satisfies the operator system*

$$\begin{cases} A^d + XV &= 0 \\ XB^d &= 0 \end{cases} \tag{3.8}$$

*where*

$$V = (B^d)^2 \left[\sum_{n=0}^{n=\infty} (B^d)^n D A^n\right] A^\pi + B^\pi \left[\sum_{n=0}^{n=\infty} B^n D (A^d)^n\right] (A^d)^2 - B^d D A^d. \tag{3.9}$$

*ii. If $X \in \mathcal{B}(\mathcal{H})$ is a solution of the system (3.8) and $V^*$ or $(B^d)^*$ is injective, then $X$ satisfies the equation $XDX + AX - XB = 0$.*

*Proof.* If $A$ and $B$ are GD-invertible, it follows from [37, Lemma 2.2] that

$$W^d = \begin{bmatrix} A^d & 0 \\ (B^d)^2 \left[\sum_{n=0}^{n=\infty} (B^d)^n D A^n\right] A^\pi + B^\pi \left[\sum_{n=0}^{n=\infty} B^n D (A^d)^n\right] (A^d)^2 - B^d D A^d & B^d \end{bmatrix}.$$

□

In the following theorem we add some conditions to obtain the form of the solutions to the Riccati operator equation (3.1).

**Theorem 3.1.2.** *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$, such that*

$$W = \begin{bmatrix} A & C \\ D & B \end{bmatrix} \text{ is GD-invertible and } W^d = \begin{bmatrix} U & M \\ V & N \end{bmatrix}.$$

*Suppose that $A + XD$ is quasi-nilpotent, where $X$ is the solution of (3.1).*
*If $V$ is surjective, $E = [V \quad N]$ is regular and $FE^-E = F$ with $F = [U \quad M]$, then the general solution $X$ of the equation (3.1) is given by*

$$\begin{aligned} X &= -FE^- + Z(I - EE^-), \\ &= -UV^-VV^- + (UV^-N - M)S^-(I - VV^-) + Z(I - SS^-)(I - VV^-), \end{aligned} \tag{3.10}$$

*where $Z \in \mathcal{B}(\mathcal{H})$ is arbitrary and $S = (I - VV^-)N$.*

*Proof.* If $X$ is a solution of the equation (3.1), then from Theorem 3.1.1, $X$ is the solution of the system (3.3) which is equivalent to the equation $XE + F = 0$, we have $V$ is surjective, it follows that $V$ has closed range, then $V$ is regular and since $FE^-E = F$ is satisfied, we deduce from [16] that the equation $XE + F = 0$ is solvable and the solutions is given by

$$X = -FE^- + Z(I - EE^-),$$

where $Z \in \mathcal{B}(\mathcal{H})$ is arbitrary.
Since $E$ is regular, it follows from [70, Theorem 3] that $S = (I - VV^-)N$ is regular and

$$E^- = \begin{bmatrix} V^-VV^- - V^-NS^-(I - VV^-) \\ S^-(I - VV^-) \end{bmatrix},$$

hence by substituting $E^-$ in the equation $X = -FE^- + Z - ZEE^-$, we obtain

$$X = -UV^-VV^- + (UV^-N - M)S^-(I - VV^-) + Z(I - SS^-)(I - VV^-),$$

where $Z \in \mathcal{B}(\mathcal{H})$ is arbitrary. Since $V$ is surjective, it follows that $V^*$ is injective, then from Theorem 3.1.1, we have $X$ is the solution of the Riccati equation (3.1). □

**Corollary 3.1.2.** *Let $A, B, D \in \mathcal{B}(\mathcal{H})$, such that $A$ and $B$ are GD-invertible. Suppose that $A + XD$ is quasi-nilpotent, where $X$ is the solution of the equation*

$$XDX + AX - XB = 0.$$

*If $V$ defined in equation (3.9) is surjective, $E = [V \;\; B^d]$ is regular and $FE^-E = F$ with $F = [A^d \;\; 0]$, then the general solution $X$ of the equation $XDX + AX - XB = 0$ is given by*

$$X = -A^dV^-VV^- + A^dV^-B^dS^-(I - VV^-) + Z(I - SS^-)(I - VV^-),$$

*where $Z \in \mathcal{B}(\mathcal{H})$ is arbitrary, $S = (I - VV^-)B^d$ and $V$ is defined in (3.9)*

**Example 3.1.1.** *Consider the matrix equation $-C + AX - XB + XDX = 0$, where*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

*Since $A$ is a projection, then $A$ is GD-invertible and $A^d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since $S = B - DA^dC = 0$, $DA^\pi C = 0 = A^\pi CD$, $A^\pi AC = A^\pi CB$ and $A\mathcal{G} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is GD-invertible where $\mathcal{G} = AA^d + A^dCDA^d = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Then from [32, Theorem 11], $W$ is GD-invertible and*

$$W^d = \left[R_0 - \begin{pmatrix} 0 & A^\pi C \\ 0 & 0 \end{pmatrix} R_0^2\right]\left[I + \Sigma_{n=0}^{\infty} R^{n+1} \begin{pmatrix} 0 & 0 \\ DA^\pi A^n & 0 \end{pmatrix}\right] = \frac{1}{9}\begin{bmatrix} -4 & 3 & 0 & -1 \\ 5 & -4 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ -4 & 3 & 0 & -1 \end{bmatrix},$$

*where $R_0 = \begin{bmatrix} I \\ DA^d \end{bmatrix} [(A\mathcal{G})^d]^2 A \begin{bmatrix} I & A^dC \end{bmatrix}$ and $(A\mathcal{G})^d = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.*
*Let $E = \begin{bmatrix} V & N \end{bmatrix}$, $F = \begin{bmatrix} U & M \end{bmatrix}$, where $U = \frac{1}{9}\begin{bmatrix} -4 & 3 \\ 5 & -4 \end{bmatrix}$, $V = \frac{1}{9}\begin{bmatrix} 1 & -1 \\ -4 & 3 \end{bmatrix}$, $N = \frac{1}{9}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ and $M = \frac{1}{9}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$.*

*We have $E^- = \begin{bmatrix} -27 & -9 \\ -36 & -9 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, since the condition $FE^-E = F$ holds, then according to Theorem 3.1.2*

$$X = -FE^- + T(I - EE^-) = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix},$$

*is a solution of the matrix equation $C = AX - XB + XDX$, we can chek that $A + XD$ is quasi-nilpotent matrix.*

**Example 3.1.2.** *Consider the operator equation* $XDX + AX - XB = 0$, *where* $A, B, D \in \mathcal{B}(l_2(\mathbb{N}))$ *are defined by :*

$$A(x_1, x_2, x_3, ....) = (0, x_1, 0, x_3, 0, x_5, ....),$$
$$B(x_1, x_2, x_3, ......) = (0, x_2, x_3, x_4, ......),$$

*and we choose* $D \in \mathcal{B}(l_2(\mathbb{N}))$ *a surjective operator such that* $A + XD$ *is quasi-nilpotent, where* $X$ *is the solution of the equation* $XDX + AX - XB = 0$.
*We have* $A^2 = 0$, *then* $A$ *is quasi-nilpotent and* $A^d = 0$, *the operator* $B$ *is a projection, hence* $B^d = B$, *and*

$$(B^d)^2 \left[ \sum_{n=0}^{n=\infty} (B^d)^n D A^n \right] A^\pi + B^\pi \left[ \sum_{n=0}^{n=\infty} B^n D (A^d)^n \right] (A^d)^2 - B^d D A^d = BD + BDA.$$

*Since the operators* $A$, $B$ *and* $D$ *are surjective, then* $V$ *is surjective and the condition* $FE^- E = F$ *is satisfied, it follows from Corollary 3.1.2 that the operator equation* $XDX + AX - XB = 0$ *is solvable and the general solution is given by*

$$X = Z(I - SS^-)(I - VV^-),$$

*where* $Z \in \mathcal{B}(l_2(\mathbb{N}))$ *is arbitrary.*

If we put $D = 0$ in Theorem 3.1.1, we obtain the following corollary which gives information on the solutions of the Lyapunov operator equation (3.2).

**Corollary 3.1.3.** *Let* $A, B, C \in \mathcal{B}(\mathcal{H})$, *such that* $A$ *is quasi-nilpotent and* $B$ *is GD-invertible (*$B$ *is not quasi-nilpotent and* $B$ *is not invertible).*

*i. If* $X \in \mathcal{B}(\mathcal{H})$ *is a solution of the equation (3.2), then* $X$ *satisfies the operator equation*

$$\left[ \sum_{n=0}^{n=\infty} (A^n) C (B^d)^n \right] (B^d)^2 + XB^d = 0. \tag{3.11}$$

*ii. If* $X \in \mathcal{B}(\mathcal{H})$ *is a solution of the equation (3.11) and* $(B^d)^*$ *is injective, then* $X$ *satisfies the Lyapunov equation (3.2).*

*Proof.* Since $A$ is quasi-nilpotent, then $A^d = 0$ and it follows from [37, Lemma 2.2] that

$$W^d = \begin{bmatrix} 0 & \left[\sum_{n=0}^{n=\infty} (A^n) C (B^d)^n\right] (B^d)^2 \\ 0 & B^d \end{bmatrix}.$$

If we suppose that $B$ is also quasi-nilpotent, we obtain $W^d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so we have nothing to prove.

If we suppose that $B$ is invertible, we obtain that the spectrum of $A$ and $B$ are disjoint, this condition implies that the Lyapunov equation (3.2) has a unique solution which is a known result in the literature. □

In the following proposition we add some conditions to obtain the form of the solution to the equation (3.2)

**Proposition 3.1.1.** *Let $A, B, C \in \mathcal{B}(\mathcal{H})$, such that $A$ is quasi-nilpotent and $B$ is GD-invertible ($B$ is not quasi-nilpotent and $B$ is not invertible).*
*Suppose that $B^d$ is surjective and*

$$\left[\sum_{n=0}^{n=\infty}(A^n)C(B^d)^n\right](B^d)^2(B^d)^- B^d = \left[\sum_{n=0}^{n=\infty}(A^n)C(B^d)^n\right](B^d)^2. \tag{3.12}$$

*If $X \in \mathcal{B}(\mathcal{H})$ is a solution of the Lyapunov equation (3.2), then*

$$X = \left[\sum_{n=0}^{n=\infty}(A^n)C(B^d)^n\right](B^d)^2(B^d)^- + T(I - B^d(B^d)^-), \tag{3.13}$$

*where $T \in \mathcal{B}(\mathcal{H})$ is arbitrary.*

*Proof.* If $X$ is a solution of the Lyapunov equation (3.2), then from Corollary 3.1.3 $X$ is the solution of the equation (3.11), since $B^d$ is surjective, it follows that $B^d$ has closed range, then $B^d$ is regular and since the condition (3.12) is satisfied, we deduce from [16] that the equation (3.11) is solvable and the solutions is given by

$$X = \left[\sum_{n=0}^{n=\infty}(A^n)C(B^d)^n\right](B^d)^2(B^d)^- + T(I - B^d(B^d)^-),$$

where $T \in \mathcal{B}(\mathcal{H})$ is arbitrary.
Since $B^d$ is surjective, it follows that $(B^d)^*$ is injective, then from Corollary 3.1.3, we have $X$ is the solution of the equation (3.2). □

**Example 3.1.3.** *Consider the Lyapunov operator equation $AX - XB = C$, where $A, B, C \in \mathcal{B}(l_2(\mathbb{N}))$ are defined by :*

$$A(x_1, x_2, x_3, ....) = (0, x_1, 0, x_3, 0, x_5, ...),$$
$$B(x_1, x_2, x_3, ....) = (0, x_2, x_3, x_4, ........),$$

*and we can choose $C$ an arbitrary operator in $\mathcal{B}(l_2(\mathbb{N}))$.*
*We have $A^2 = 0$, then $A$ is quasi-nilpotent and $A^d = 0$, the operator $B$ is a projection, hence $B^d = B^- = B$, and*

$$\left[\sum_{n=0}^{n=\infty}(A^n)C(B^d)^n\right](B^d)^2 = CB + ACB.$$

*Since $B$ is surjective and $(CB + ACB)B^{-}B = (CB + ACB)B^2 = CB + ACB$, it follows from Proposition 3.1.1 that the Lyapunov operator equation (3.2) is solvable and the general solution is given by*

$$X = CB + ACB + T(I - B),$$

*where $T \in \mathcal{B}(l_2(\mathbb{N}))$ is arbitrary.*

## 3.2 Application to solve differentiel problems

Cauchy problems for Riccati operator equations of the type

$$\begin{cases} \frac{d}{dt}X(t) &= -C + AX(t) - X(t)B + X(t)DX(t) \\ X(0) &= P_0 \end{cases} \tag{3.14}$$

where $A, B, C, P_0 \in \mathcal{B}(\mathcal{H})$, and $t$ belongs to the interval $J$ that contains zero, arise in control theory and transport theory.
The infinite dimensionel case has ben studied in [14, 27, 57, 68, 93], the author in [49] showed that from the existence of a solution for the algebraic Riccati equation (3.1), the problem (3.14) may be transformed into others for which explicit expressions for solutions are available.
In this section we show that from the existence of a solution for the Riccati equation (3.1) given in Theorem 3.1.2, the problem (3.14) may be transformed into other for which the expressions for general solutions are given, finally we deduce the general solution for the Cauchy problem for Lyapunov equation.
Let us suppose that the conditions of Theorem 3.1.2 are satisfied, then there exists a solution $X_0$ of the algebraic equation (3.1) given in (3.10) considering the change $U(t) = X(t) - X_0$, the problem (3.14) is equivalent to the problem

$$\begin{cases} \frac{d}{dt}U(t) &= A_0U(t) - U(t)B_0 + U(t)DU(t) \\ U(0) &= U_0, \end{cases} \tag{3.15}$$

where $A_0 = A + X_0D$, $B_0 = B - DX_0$ and $U_0 = P_0 - X_0$.
Similarly as in [49], we get the following theorem

**Theorem 3.2.1.** *Let us suppose that the conditions of Theorem 3.1.2 are satisfied and $X_0$ is a solution of the operator equation (3.1), and let us consider the Cauchy problem (3.14).*
*Then there exists an interval $J_1$ containing the origin such that the general solution of (3.14) is given by*

$$X(t) = X_0 + exp(tA_0)U_0\left(I - \int_0^t exp(-vB_0)Dexp(-vA_0)dvU_0\right)^{-1} exp(-tB_0), \ \ \forall t \in J_1,$$

*where $A_0 = A + X_0D$, $B_0 = B - DX_0$ and $U_0 = P_0 - X_0$.*

If we put $D = 0$ in Theorem 3.2.1, we obtain the following corollary which gives the form of the solutions of the following Cauchy problem

$$\begin{cases} \frac{d}{dt}X(t) &= -C + AX(t) - X(t)B \\ X(0) &= P_0 \end{cases} \tag{3.16}$$

where $A, B, C, P_0 \in \mathcal{B}(\mathcal{H})$, and $t$ belongs to the interval $J$ that contain zero.

**Corollary 3.2.1.** *Let us suppose that the conditions of Corollary 3.1.1 are satisfied and $X_0$ is a solution of the operator equation (3.2), given in 3.13 and let us consider the Cauchy problem (3.16).*
*Then there exists an interval $J_1$ containing the origin such that the general solution of (3.16) is given by*

$$X(t) = X_0 + exp(tA_0)U_0 exp(-tB_0), \quad \forall t \in J_1.$$

# Chapter 4

# Drazin inverse and applications

This chapter is written in collaboration with Dr. Maissa Kada.
In this chapter we present a procedure for computing the Drazin inverse of matrix and we review some known result of [75] on the application of the Drazin inverse to study the positivity and stability of the time-invariant descriptor systems. We give a Matlab software code to check the positivity and the stability of the time-invariant descriptor systems with numerical examples.

## 4.1 Preliminaries

### 4.1.1 Kronecker Product

**Definition 4.1.1.** *[83] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$, then the Kronecker product of $A$ and $B$ is the $\mathbb{C}^{(mp)\times(nq)}$ matrix defined by*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \dots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix}.$$

*Sometimes the Kronecker product is also called direct product or tensor product.*

**Example 4.1.1.** *Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ then*

$$A \otimes B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 4 & 0 & 2 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix}, \quad B \otimes A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}.$$

*Note that $A \otimes B \neq B \otimes A$.*

The following lemma list some basic properties of the Kronecker product.

**Lemma 4.1.1.** *[83] Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, $C \in \mathbb{C}^{s\times t}$ and $c \in \mathbb{C}$. Then the following properties hold.*

1. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$.
2. $(cA \otimes B) = (cA) \otimes B = A \otimes (cB)$.
3. $(A + B) \otimes (C) = A \otimes C + B \otimes C$.
4. $A \otimes (B + C) = A \otimes B + A \otimes C$.
5. $(A \otimes B)^T = A^T \otimes B^T$.

**Definition 4.1.2.** *[83] Let $A \in \mathbb{C}^{m\times n}$, the vec operator of $A$ denoted by $vec(A)$ is the operation of stacking the columns of $A$ under an other matrix to form a single column.*

**Example 4.1.2.** *Let* $A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$, *then* $vec(A) = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

The following Theorem investigate the connection of the vec operator with the Kronecker product.

**Theorem 4.1.1.** *[83] Let $A, B, C$ be matrices such that the matrix product $ABC$ exists, then*

$$vec(ABC) = (C^T \otimes A)vec(B).$$

**Corollary 4.1.1.** *Let $A, B \in \mathbb{C}^{n\times n}$, then*

1. $vec(AB) = (I_n \otimes A)vec(B)$.
2. $vec(AB) = (B^T \otimes A)vec(I_n)$.

**Remark 4.1.1.** *In order to compute the Kronecker product and the vec operator of a given matrices, we make use of the kron() and reshape() Matlab functions.*

**Example 4.1.3.** *Let* $A = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$, *by using the kron() and reshape() Matlab functions we will compute* $A \otimes B$ *and* $vec(A)$.

```
>> A=[1 -1;0 -2 ];
B = [2 0 1; 3 0 0];
AB_Kroneckerproduct = kron(A, B)
vec_A = reshape(A,[],1)

```

```
AB_Kroneckerproduct =

     2     0     1    -2     0    -1
     3     0     0    -3     0     0
     0     0     0    -4     0    -2
     0     0     0    -6     0     0


vec_A =

     1
     0
    -1
    -2
```

### 4.1.2 Matrix Hermite echelon form

**Definition 4.1.3.** *[15] A matrix $H \in \mathbb{C}^{n\times n}$ is said to be in the Hermite echelon form, if its elements $h_{ij}$ satisfies the following conditions.*

1. *$H$ is upper triangular (i.e. $h_{ij} = 0$ if $i > j$).*
2. *$h_{ii}$ is either $0$ or $1$.*
3. *If $h_{ii} = 0$ , then $h_{ik} = 0$ for every $k$, $1 \leq k \leq n$.*
4. *If $h_{ii} = 1$ , then $h_{ki} = 0$ for every $k \neq i$.*

**Remark 4.1.2.** *In order to compute the Hermite echelon form of a given matrix we make use of the $rref()$ (i.e., Reduced row echelon form) Matlab function which returns the reduced row echelon form.*

**Example 4.1.4.** *Let $A = \begin{bmatrix} 7 & \frac{3}{4} & 2 & 0 \\ 0 & -1 & 3 & 0 \\ -4 & 1 & 3 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}$, by using the $rref()$ Matlab function we will compute the Hermite echelon form of the matrix $A$.*

```
>> A=[7 0.75 2 0; 0 -1 3 0; -4 1 3 0; 0 3 0 0]
H = rref(A)

A =

    7.0000    0.7500    2.0000         0
         0   -1.0000    3.0000         0
   -4.0000    1.0000    3.0000         0
```

```
          0    3.0000         0         0


H =

     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     0
```

## 4.2 Procedure for computing the Drazin inverse of matrix

In this section we present a procedure for computing the Drazin inverse of matrix. The following procedure for computing the Drazin inverse is obtained from [15, Algorithm 7.2.1 ].

**Algorithm 4.2.1.**
***Input*** *The matrix* $A \in \mathbb{C}^{n\times n}$ *such that* $ind(A) = k$.
***Output*** *The Drazin inverse of* $A$, $A^D$.
***Step 1 :*** *Let* $p$ *be an integer such that* $p \geq k$ *(*$p$ *can be always taken to equal* $n$ *if no smaller value can be determined), if* $A^P = 0$, *then* $A^D = 0$ *thus we assume that* $A^P \neq 0$.
***Step 2 :*** *Rew reduced* $A^P$ *to it's hermite echelon form* $H_{A^P}$ *(the sequence of reducing matrices need not to be saved).*
***Step 3 :*** *By noting the position of the non-zero diagonal elements in* $H_{A^P}$, *select the distinguished columns from* $A^P$ *and call them* $v_1$, $v_2$ ,..., $v_r$, *this is a basis for* $\mathcal{R}(A^k)$.
***Step 4 :*** *From the matrix* $I - H_{A^P}$ *save it's non-zero columns and call them* $v_{r+1}$, $v_{r+2}$,..., $v_n$ *this is a basis for* $\mathcal{N}(A^k)$.
***Step 5 :*** *Construct the non-singular matrix* $P = \begin{bmatrix} v_1 & \dots & v_r & v_{r+1} & \dots & v_n \end{bmatrix}$.
***Step 6 :*** *Compute* $P^{-1}$.
***Step 7 :*** *From the product* $P^{-1}AP$, *this matrix will be of the form*

$$P^{-1}AP = \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix},$$

*where* $C$ *is non-singular matrix and* $N$ *is a nilpotent matrix.*
***Step 8 :*** *Compute* $C^{-1}$.
***Step 9 :*** *Compute* $A^D$ *by forming the product* $A^D = P \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$.

This is the Matlab software code of Algorithm 4.2.1 .

**Code 4.2.1.**

```
%---------- Compute the Drazin inverse of Matrix -------
%------------------ Step 1 ----------------
X = input('Please enter your matrix X :')
[n, n] = size(X);
G = X^n;
%----------------- Steps 2, 3, 4, 5 -----------
H = rref(G);
H_1 = eye(n)-H;
for(i = 1:1:n)
     for (k = 1:1:n)
     if ( H(i,i)~= 0)
       P(k,i) = G(k,i);
     elseif ( H_1(i,i)~= 0)
          P(k,i) = H_1(k,i);
     end
     end
end
%----------------- Steps 6, 7 ---------------
K_2 =  mtimes(X,P);
K = mtimes(inv(P),K_2)
%----------------- Step 8 -----------------
C = input('Please enter the non-singular matrix C:')
C_inverse = inv(C);
%----------------- Step 9 ----------------
[r,r] = size(C);
for(i = 1:1:n)
     for (j = 1:1:n)
     if ( i <= r && j<= r)
      Q(i,j) = C_inverse(i,j);
     else
      Q(i,j) = 0;
     end
     end
end
X_Drazin = mtimes(mtimes(P,Q),inv(P))
```

**Example 4.2.1.** *Let* $A = \begin{bmatrix} -3 & -1 & -3 \\ 3 & 2 & 3 \\ 2 & 1 & 2 \end{bmatrix}$, *by using the Matlab code 4.2.1 we will compute the Drazin inverse* $A^D$ *of* $A$.

```
>> Drazinfinal
Please enter your matrix X :[-3 -1 -3;3 2 3;2 1 2]

```

```
X =

    -3     -1     -3
     3      2      3
     2      1      2


K =

   -1.0000    -0.0000         0
    3.0000     2.0000         0
         0          0         0

Please enter the non-singular matrix C:[ -1.0000   -0.0000; 3.0000
     2.0000]

C =

    -1      0
     3      2


X_Drazin =

   -2.2500    -0.2500    -2.2500
    1.5000     0.5000     1.5000
    1.2500     0.2500     1.2500
```

## 4.3 Application of the Drazin inverse to the analysis of positivity and stability of descriptor systems

Consider the time-invariant descriptor system described by the following abstract differential equation

$$A\dot{x}(t) = Bx(t), \tag{4.1}$$

where $A, B \in \mathbb{R}^{n\times n}$ are constant coefficient matrices, $x(t) \in \mathbb{R}^n$ is the state vector. If $det(A) \neq 0$, then the system (4.1) turns into

$$\dot{x}(t) = A^{-1}Bx(t),$$

which is called a standard time-invariant descriptor system, if $det(A) = 0$, then the system (4.1) is known in the literature as descriptor systems [87, 60, 65, 94], singular systems [11, 12], differential-algebraic equations [10, 73], generalized state

space systems [51] and implicit linear systems [?, 1], these systems have many applications in different areas such as electrical circuits [50, 42, 41], economic systems [63], engineering [56] and many others.
In this section we review some of the results given of [75] on the application of Drazin inverse to study the positivity and stability of the time-invariant descriptor system.

**Definition 4.3.1.** *Let $A, B \in \mathbb{R}^{n\times n}$ such that $det(A) = 0$, then the pencil $(A,\ B)$ of system (4.1) is called regular pencil, if there exists $c \in \mathbb{C}$, such that*

$$det(cA - B) \neq 0.$$

Assuming that $det(A) = 0$ and that for a chosen $c \in \mathbb{C}$ the pencil $(A,\ B)$ is regular, hence by per-multiplying system (4.1) by $(cA - B)^{-1}$ we obtain

$$\tilde{A}\dot{x}(t) = \tilde{B}x(t)$$

where

$$\tilde{A} = (cA - B)^{-1}A, \quad \tilde{B} = (cA - B)^{-1}B. \tag{4.2}$$

In [?] Campbell et al. proved that the matrices $\tilde{A}$ and $\tilde{B}$ commute, which implies that the following properties hold

$$\tilde{A}\tilde{B}^D = \tilde{B}^D\tilde{A},\ \tilde{A}^D\tilde{B} = \tilde{B}\tilde{A}^D,\ \tilde{A}^D\tilde{B}^D = \tilde{B}^D\tilde{A}^D.$$

**Definition 4.3.2.** *The set of intial conditions for which system (4.1) has at least one solution is called the set of admissible initial conditions and it will be denoted by $\mathcal{X}_0$.*

By using the Drazin inverse Campbell in [13, Theorem 3.1.1, Theorem 3.1.3] obtained the following results on the solvabilty of system (4.1).

**Theorem 4.3.1.** *[13] The descriptor system (4.1) has a unique solution for each admissible initial condition if and only if the pencil $(A,\ B)$ is regular, in this case the set of admissible initial conditions is given by $\mathcal{X}_0 = \mathcal{R}(\tilde{A}^D\tilde{A})$ and the solution of (4.1) is given by*

$$x(t) = \mathrm{e}^{\tilde{A}^D\tilde{B}t}\,\tilde{A}^D\tilde{A}v, \tag{4.3}$$

*where $v \in \mathbb{R}^n$ is arbitrary and $\tilde{A},\ \tilde{B}$ are defined as in (4.2).*

### 4.3.1 Positivity analysis of the time-invariant descriptor system

It is obvious that when $A$ is invertible, system (4.1) reduce to the standard sytem $\dot{x}(t) = A^{-1}Bx(t)$ which is positive if and only if $A^{-1}B$ is a Metzler matrix (i.e. its off diagonal entries are non-negative ). In this sub-section we summarize the notion of positivity given by Rami et al. in [75] to analyze the positivity of system (4.1) in the general case (i.e. when $A$ is not necessarily invertible).

**Definition 4.3.3.** *[75] The time-invariant descriptor system (4.1) is called positive if for any nonnegative admissible initial condition $x_0 \geq 0$ we have $x(t) \geq 0$ for all $t \geq 0$.*

We can show that $x(t)$ given by (4.3) satisfies the following system

$$\begin{aligned} \dot{x}(t) &= \overline{B}x(t) \\ x_0 &\in \mathcal{R}(\overline{A}), \end{aligned} \tag{4.4}$$

where $\overline{B} = \tilde{A}^D\tilde{B}$, $\overline{A} = \tilde{A}^D\tilde{A}$, hence systems (4.1) and (4.4) are equivalent in the sense that they have the same solution.
Before we present the principal Theorem to study the positivity of system (4.4) we need the following Lemmas

**Lemma 4.3.1.** *[75] Let $\overline{B} = \tilde{A}^D\tilde{B}$, $\overline{A} = \tilde{A}^D\tilde{A}$, then the following hold.*

*i.* $\overline{A}\ \overline{B} = \overline{B}\ \overline{A} = \overline{B}$.

*ii. For any solution $x(t)$ of system (4.1) or equivalently to sytem (4.4) we have*

$$\overline{A}x(t) = x(t).$$

*Proof.* According to (4.2) since $\tilde{A}\tilde{B} = \tilde{B}\tilde{A}$, then $\tilde{A}^D\tilde{B} = \tilde{B}\tilde{A}^D$ thus we get

$$\begin{aligned} \overline{A}\ \overline{B} &= \tilde{A}^D\tilde{A}\tilde{A}^D\tilde{B} = \tilde{A}^D\tilde{B} = \overline{B}. \\ \overline{B}\ \overline{A} &= \tilde{A}^D\tilde{B}\tilde{A}^D\tilde{A} = \tilde{B}\tilde{A}^D\tilde{A}\tilde{A}^D = \tilde{B}\tilde{A}^D = \overline{B}. \end{aligned}$$

Using $\overline{A}\ \overline{B} = \overline{B}\ \overline{A}$ we get

$$\overline{A}x(t) = \overline{A}\,\mathrm{e}^{\overline{B}t}\,\overline{A}v = (\overline{A} + t\overline{A}\ \overline{B} + \frac{t^2}{2}\overline{A}\ \overline{B^2} + ...)\overline{A}v = \mathrm{e}^{\overline{B}t}\,\overline{A}\ \overline{A}v = \mathrm{e}^{\overline{B}t}\,\overline{A}v = x(t).$$

□

**Lemma 4.3.2.** *[17] Let $F \in \mathbb{R}^{p\times n}$, $M \in \mathbb{R}^{n\times n}$ and consider the linear system $\dot{z}(t) = Mz(t)$, then the following implication hold*

$$[Fz(0) \geq 0] \Rightarrow [Fz(t) \geq 0, \forall t \geq 0],$$

*if and only if there exists a Metzler matrix $H$ such that $FM = HF$.*

**Theorem 4.3.2.** *[75] The following statements are equivalent*

1. *System (4.4) is positive.*
2. *There exists a Metzler matrix $H$ such that $\overline{B} = H\overline{A}$.*
3. *There exists a matrix $D$ such that $H = \overline{B} + D(I - \overline{A})$ is Metzler.*

*Proof.* $1 \Leftrightarrow 2$ According to Lemma 4.3.1 since $\overline{A}\,\overline{B} = \overline{B}\,\overline{A} = \overline{B}$, then any solution of (4.4) can be written as follow

$$x(t) = \mathrm{e}^{\overline{B}t}\,\overline{A}^{D}\,\overline{A}v_0 = (\overline{A} + t\overline{B}\,\overline{A} + \frac{t^2}{2}\overline{B}^2\overline{A} + ....)v_0 = \overline{A}\,\mathrm{e}^{\overline{B}t}\,v_0 = \overline{A}z(t),$$

where $z(t) = \mathrm{e}^{\overline{B}t}v_0$ is a solution to $\dot{z}(t) = \overline{B}z(t), \quad z(0) = v_0 \in \mathbb{R}^n$, thus the positivity of (4.4) is equivalent to $[x(0) = \overline{A}v_0 = \overline{A}z(0) \geq 0] \Rightarrow [x(t) = \overline{A}z(t) \geq 0]$, therefore according to Lemma 4.3.2 there existe a Metzler matrix $H$ such that $\overline{A}\,\overline{B} = H\overline{A}$ since $\overline{A}\,\overline{B} = \overline{B}$ we get $\overline{B} = H\overline{A}$.

$2 \Rightarrow 3$

Suppose that there exists a Metzler matrix $H$ such that $\overline{B} = H\overline{A}$ since $\overline{A}$ is idempotent then $(\overline{A})^{-} = \overline{A}$ and since $\overline{B}\,(\overline{A})^{-}\,\overline{A} = \overline{B}$, then according to [16] $H$ is given by $H = \overline{B}\,(\overline{A})^{-} + T(I - \overline{A}\,(\overline{A})^{-})$ which implies that

$$H = \overline{B} + D(I - \overline{A}),$$

where $D$ is arbitrary matrix.

$3 \Rightarrow 2$

Suppose that there exists a matrix $D$ such that $H = \overline{B} + D(I - \overline{A})$ is Metzler, postmultipling $H$ by $\overline{A}$ we get

$$H\overline{A} = \overline{B}\,\overline{A} + D(I - \overline{A})\overline{A} = \overline{B}\,\overline{A} = \overline{B}.$$

□

**Example 4.3.1.** *Let system (4.1) be given by the following matrices*

$$A = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

*Since* $det(A) = 0$, *and* $det(A-B) = \begin{vmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ -1 & \frac{3}{4} & 0 \\ 0 & 0 & -\frac{1}{2} \end{vmatrix} = 0.1875$, *then* $(A-B)^{-1}$ *always exists, for* $c = 1$ *we get* $(A-B)^{-1} = \begin{bmatrix} -2 & 0 & -2 \\ -2.67 & 1.33 & -2.67 \\ 0 & 0 & -2 \end{bmatrix}$, *therefore* $\tilde{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1.33 & 0.67 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1.33 & -0.33 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, *by using the Matlab code 4.2.1* *we get* $\tilde{A}^D = \begin{bmatrix} 1.0000 & 0 & 0 \\ -1.9997 & 1.499 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *which implies that*

$$\overline{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

*Since* $\overline{B} = \overline{B}\,\overline{A}$ *and all the off diagonal entries of* $\overline{B}$ *are non-negative, then there exists a Metzler matrix* $H = \overline{B}$ *such that* $\overline{B} = H\overline{A}$ *which implies that system (4.1) is positive.*

### 4.3.2 Stability analysis of the time-invariant descriptor system

In this sub-section we summarize the notion of stability given by Rami et al. in [75] to analyze the stability of system (4.1) under the positivity constraint.

**Definition 4.3.4.** *[75] The time-invariant descriptor system (4.1) is said to be stable if* $x(t) \longrightarrow 0$ *as* $t \longrightarrow \infty$ *for all* $x(0) \in \mathcal{R}(\overline{A}) \cap \mathbb{R}^n_+$.

Before we present the principal Theorem to study the stability of system (4.4) we need the following Lemma

**Lemma 4.3.3.** *[75] Let* $B$ *be a Metzler matrix. Then the following are equivalent.*

1. $B$ *is Hurwitz (i.e. every eigenvalue* $\lambda$ *of* $B$ *has strictly negative real part).*
2. *There exists* $w \in \mathbb{R}^n$ *such that* $w > 0$ *and* $Bw < 0$.
3. *There exists* $u \in \mathbb{R}^n$ *such that* $u > 0$ *and* $u^T B < 0$.

**Theorem 4.3.3.** *[75] Assume there exists* $v_0 \in \mathbb{R}^n$ *such that* $\overline{A}v_0 > 0$ *(i.e.* $\mathcal{R}(\overline{A}) \cap \mathbb{R}^n_+ \neq \varnothing$*). Then the following statements are equivalent.*

1. *System (4.4) is positive and stable for all* $x(0) \in \mathcal{R}(\overline{A}) \cap \mathbb{R}^n_+$.

2. *There exists a matrix* $D$ *such that* $H = \overline{B} + D(I - \overline{A})$ *is Metzler and Hurwitz.*

*Proof.* $(1 \Rightarrow 2)$
Since (4.4) is stable we get

$$x(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty.$$

By integrating (4.4) we get

$$\overline{B} \int_0^\tau x(t)dt = x(\tau) - x(0).$$

Since $x(0) = \overline{A}v_0 > 0$ and $x(t)$ is continuous and positive then $w = \int_0^T x(t)dt > 0$ and we can show that for $\tau$ sufficiently large we get

$$\overline{B} \int_0^\tau x(t)dt = \overline{B}w = x(\tau) - x(0) < 0.$$

According to Lemma 4.3.1 since $\overline{A}x(t) = x(t)$ we get $\overline{A}w = \int_0^\tau \overline{A}x(t)dt = \int_0^\tau x = w$ thus $(I - \overline{A})w = 0$.
And since (4.4) is positive, then according to Theorem 4.3.2 there exists a matrix $D$ such that $H = \overline{B} + D(I - \overline{A})$ is a Metzler matrix, post-multiplying $H$ by $w$ we get $Hw = \overline{B}w < 0$ which implies according to Lemma 4.3.3 that $H$ is Hurwitz.
$(2 \Rightarrow 1)$
Suppose that there exists a matrix $D$ such that $H = \overline{B} + D(I - \overline{A})$ is a Metzler and Hurwitz then according to Theorem 4.3.2, $\overline{B} = H\overline{A}$ thus

$$\dot{x}(t) = \overline{B}x(t) = H\overline{A}x(t) = Hx(t),$$

which implies that (4.4) is positive and stable since $H$ is a Metzler and Hurwitz. □

### 4.3.3 Numerical example

In this sub-section we present a Matlab software code to check the positivity conditions given in Theorem 4.3.2.
In [75] Rami et al prouved that the positivity condition $\exists D \in \mathbb{R}^{n\times n}$ such that $H = \overline{B} + D(I - \overline{A})$ is Metzler matrix is equivalent to the following inequality

$$\overline{B} + D(I - \overline{A}) + \alpha I \geq 0,$$

which can be expressed as follow

$$\begin{bmatrix} (\overline{A} - I)^T \otimes I & -vec(I) \end{bmatrix} \begin{bmatrix} vec(D) \\ \alpha \end{bmatrix} \leq vec(\overline{B}) \tag{4.5}$$

where $\alpha \in \mathbb{R}$, $D \in \mathbb{R}^{n\times n}$ is the unknown matrix. Thus we can check the positivity conditions of system (4.1) given in Theorem 4.3.2 by solving the standard linear programming problem (4.5).

**Code 4.3.1.**

```
A = input('Please enter your system matrix A:')
B = input('Please enter your system matrix B:')
[q,q] = size(A);
A_0 = A-B;
%---------- Compute cE-A ---------------------
if(det(A_0) ~= 0)
    A_1 =  mtimes(inv(A_0),A);
    B_1 =  mtimes(inv(A_0),B);
else
    c = input('Please chose a value for c:')
    A_0 = c*A-B;
     A_1 =  mtimes(inv(A_0),A);

    B_1 =  mtimes(inv(A_0),B);
end
%---------- Compute the Drazin inverse of E1 ---------
%---------------- Step 1 ----------------
[n, n] = size(A_1);
G = A_1^n;
%---------------- Steps 2, 3, 4, 5 -----------
H = rref(G);
H_1 = eye(n)-H;
for(i = 1:1:n)
    for (k = 1:1:n)
    if ( H(i,i)~= 0)
      P(k,i) = G(k,i);
    elseif ( H_1(i,i)~= 0)
        P(k,i) = H_1(k,i);
    end
    end
end
%---------------- Steps 6, 7 ----------------
K_2 =  mtimes(A_1,P);
K = mtimes(inv(P),K_2)
%---------------- Step 8 ------------------
C = input('Please enter the non-singular matrix C:')
C_inverse = inv(C);
%---------------- Step 9 ----------------
[r,r] = size(C);
for(i = 1:1:n)
    for (j = 1:1:n)
    if ( i <= r && j<= r)
```

```
      Q(i,j) = C_inverse(i,j);
     else
      Q(i,j) = 0;
     end
     end
end
Drazin = mtimes(mtimes(P,Q),inv(P))
% - - - - - - - - Compute the dynamic and the projector matrices - -
A_projectormatrix = mtimes(Drazin,A_1)
B_dynamicmatrix = mtimes(Drazin,B_1)
% - - - - - - - - Solve the linear inequality of our system - - - -
L_0 = A_projectormatrix'-eye(q);
I_0 =  reshape(eye(q),[],1);
I_1 = mtimes(I_0,-1);
K_0 = kron(L_0,eye(q));
[z,z] = size(K_0);
for (i = 1:1:z)
     for (j = 1:1:z+1)
          if( j<=z)
            L_1(i,j) = K_0(i,j) ;
          else
            L_1(i,z+1) = I_1(i,1);
          end
     end
end
disp(L_1)
[l,w]=size(L_1);
B_3 = reshape(B_dynamicmatrix,[],1);
f = zeros(1,w);
x = linprog(f,L_1,B_3 );
x([w])=[];
D = reshape(x, q, q);
% - - - - - - - - Compute the Metzler matrix H - - - - - - - - - - - -
A_3=eye(q)-A_projectormatrix;
H = B_dynamicmatrix + mtimes(D,A_3)
H_eigenvalues = eig(H)
```

**Example 4.3.2.** *Let system (4.1) be given by the matrices*

$$A = \begin{bmatrix} \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{5}{4} & 0 & -\frac{7}{4} \\ -5 & -3 & 4 \\ -5 & 0 & 0 \end{bmatrix},$$

*by using the Matlab code 4.3.1 we will check the positivity and the stability of system (4.1)*

```
>> programmefinal
Please enter your system matrix A:[0.75 0.75 0.75;0.75 0 0.75;0.25
    0.75 0.25]

A =

    0.7500    0.7500    0.7500
    0.7500         0    0.7500
    0.2500    0.7500    0.2500

Please enter your system matrix B:[-1.25 0 -1.75; -5 -3 4; -5 0 0]

B =

   -1.2500         0   -1.7500
   -5.0000   -3.0000    4.0000
   -5.0000         0         0


K =

    0.1429    0.3429         0
    0.5022   -0.0494         0
   -0.0000   -0.0000         0

Please enter the non-singular matrix C:[0.1429 0.3429;0.5022
    -0.0494]

C =

    0.1429    0.3429
    0.5022   -0.0494


Drazin =

   -0.3815    0.8732   -0.3815
    2.8016   -0.7968    2.8016
    0.6567    1.0397    0.6567


A_projectormatrix =

    0.3840   -0.1739    0.3840
    0.0001    0.9999    0.0001
    0.6159    0.1739    0.6159
```

```
B_dynamicmatrix =

    0.7655   -1.0471    0.7655
   -2.8014    1.7966   -2.8014
   -0.0408   -0.8658   -0.0408


Optimal solution found.


H =

   -0.4694         0    0.0000
         0   -1.5466    0.0000
    3.0264         0   -1.9537


H_eigenvalues =

   -1.5466
   -0.4694
   -1.9537
```

*Since all the off diagonal elements of $H$ are nonnegative, then $H$ is a Metzler matrix and for $v_0 = [-4, -2, -6.5]$ we get $A_{projectormatrix} v_0 > 0$ and since*

$$\sigma(H) = \{-1.5466, -0.4694, -1.9537\},$$

*then $H$ is Hurwitz, which implies that system (4.1) is positive stable for the given matrices $A$ and $B$.*

**Example 4.3.3.** *Let system 4.1 be given by the matrices*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix},$$

*by using the Matlab code 4.3.1 we will check the positivity and the stability of system (4.1)*

```
>> programmefinal
Please enter your system matrix A:[1 0 0;-1 0 0;0 0 0]

A =

     1     0     0
```

```
    -1         0         0
     0         0         0

Please enter your system matrix B:[2 2 2; 1 2 2; 2 1 2]

B =

     2         2         2
     1         2         2
     2         1         2


K =

     2         0         0
     0         0         0
     0         0         0

Please enter the non-singular matrix C:2

C =

     2


Drazin =

    0.5000              0              0
    0.2500              0              0
   -0.6250              0              0


A_projectormatrix =

    1.0000              0              0
    0.5000              0              0
   -1.2500              0              0


B_dynamicmatrix =

    0.5000              0              0
    0.2500              0              0
   -0.6250              0              0


Optimal solution found.
```

```
H =

    0.5000         0         0
    0.2500         0         0
         0         0    0.5000


H_eigenvalues =

         0
    0.5000
    0.5000
```

*Since all the off diagonal elements of* $H$ *are nonnegative, then* $H$ *is a Metzler matrix and for* $v_0 = [1, -2, 3]$ *we get* $A_{projectormatrix}v_0 > 0$ *and since*

$$\sigma(H) = \{0, 0.5000, 0.5000\},$$

*then* $H$ *is not Hurwitz, which implies that system (4.1) is positive and not stable for the given matrices* $A$ *and* $B$*.*

# Conclusion

In this thesis we have studied the solvability of the Riccati, Sylvester and Lyapunov operator equations in Hilbert spaces of infinite dimensions using generalized inverses under some new conditions.

1. We presented new necessary and sufficient conditions for the solvability to the Lyapunov, Sylvester and Stien operator equations.
2. We presented new method for solving Riccati and Lyapunov operator equations in Hilbert space, the results are applied to the problem of finding general solutions to Cauchy problems for Riccati and Lyapunov operator differential equations.
3. We got some Matlab software codes to compute the Drazin inverse of a matrix and to check the positivity and stability of the time-invariant descriptor systems with numerical examples.

## Perspectives

In the future, we will try to extend our results to the case of unbounded operators in order to find some applications in control theory.

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