RÉPUBLIQUE ALGERIENNE DÉMOCRATIQUE ET POPULAIRE
MINISTÉRE DE L'ENSEIGNEMENT SUPÉRIEURE
ET DE LA RECHERCHE SCIENTIFIQUE

UNIVERSITÉ MOSTAPHA BEN BOULAID BATNA 2
FACULTÉ DES MATHÉMATIQUES ET D'INFORMATIQUE
DÉPARTEMENT DE MATHÉMATIQUES
LABORATOIRE DES ÉQUATIONS AUX DÉRIVÉES PARTIELLES ET APPLICATIONS, LEDPA

THÈSE

PRÉSENTÉE EN VUE D'OBTENIR LE DIPLÔME DE
DOCTORAT EN SCIENCES
SPÉCIALITÉ MATHÉMATIQUES

Par
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THÈME

QUELQUES ASPECTS DE LA DYNAMIQUE DES TOURBILLONS EN MÉCANIQUE DES FLUIDES

Soutenue le: 15-07-2021
Devant le jury composé de:

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Année Universitaire 2020-2021

Acknowledgments

First of all, I am extremely grateful to my supervisor Prof. Amar Youkana, for his excellent guidance, constant encouragement, patience and care during preparation of this research. I fell very fortunate to have an opportunity work under his patient supervisor.

It gives me an immense debt of gratitude to express my sincere thanks to Dr. Taoufik Hmidi, MC, HDR, I can say he is my co-supervisor with high respect and deep sense of appreciation for guiding me well throughout this research work from title's selection to finding the results. His immense knowledge, motivation and patience have given me more power and spirit to excel in this research writing. I really appreciate his willingness to meet me every time and going through several drafts of my thesis. I remain amazed that despite his busy schedule, he was able to go through this work by giving his comments and suggestions on almost every page. He is an inspiration.

I wish to express my sincere appreciation to my graduate professor Dr. Ahmed Zerrouk Mokrane, MCA, he honored me by chairing this Jury. I am very grateful for the knowledge and virtuous morals that he taught us throughout many years of academic study and fellowship.

I would like to thank Prof. Khaled Melkemi, Prof. Aissa Aibeche and Prof. Abbes Benaissa for their acceptance to be part of this Jury. I send them my deepest gratitude.

I am very much thankful to all the team of the department of mathematics from university of Batna 2.

I have a special gratitude to all my teachers and professors who have taught me during the primary, preparatory, secondary and university education. I address them with finest expressions of gratitude.

I would like to express my heartfelt gratitude to my dear parents for their continuous and unparalleled love, help and support. Thank you for educating me, for praying, for hoping that achieving this work is arduous but not impossible. I could never thank you enough. God bless you.

My sincere gratitude is reserved for my brothers and sisters for their much needed support, understanding, and encouragement in every possible way.

I also wish to thank my children. Your presence in my life helped me a lot on this path. Words cannot express the love I have for you.

Finally, I would like to acknowledge my husband. He has been a constant source of strength and inspiration. There were times during these years when everything seemed hopeless and I didn't have any hope. I can honestly say that it was only his determination and ceaseless encouragement, that ultimately made it possible for me to see this work through to the end.

So, thank you all for being by my side to dispel doubts and share joys.

Halima

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Nothing in life is to be feared, it's only to be understood.
Now is the time to understand more, so that we may fear less.

Marie Curie

Contents

Notations

$\mathbb{R}^d$:	space of variables $x=(x_1,\cdots,x_d)$.
$\vec{e}_2=(1,0)$:	unit vector of $\mathbb{R}^2$.
(e_r,e_θ,e_z) :	cylindrical coordinates of $\mathbb{R}^3$.
$\partial_t, \frac{\partial}{\partial t}$:	partial derivative with respect to time.
$\partial_j, \frac{\partial}{\partial x_j}$:	partial derivative with respect to space.
$\frac{D}{Dt}$:	material derivative $(\partial_t + v\cdot\nabla)$.
$\cdot$:	scalar product.
$\times$:	vector cross product.
$x^\perp$:	perpendicular vector $x=(-x_2,x_1)$ (rotation of x with angle $\frac{\pi}{2}$).
∇ :	gradient operator.
$\nabla^\perp$:	gradient perpendicular $\nabla^\perp=(-\partial_2,\partial_1)$.
div :	divergence operator.
curl :	curl operator.
Δ :	Laplacien operator.
$\mathscr{F}(u),\widehat{u}$:	Fourier transform of u.
L^p :	Lebesgue space.
$\mathscr{S}$:	Schwartz space.
$\mathscr{S}'$:	tempered distributions space.
H^s :	Sobolev space.
Lip :	space of Lipschitzian functions.
LL :	space of log-Lipschitz functions.
Δ_q, S_q :	inhomogeneous cut-off operators.
$\dot{\Delta}_q, \dot{S}_q$:	homogeneous cut-off operators.
$T_u v$:	paraproduct of u with respect to v.
$R(u,v)$:	remainder operator of u and v.
$B^s_{p,r}$:	inhomogeneous Besov space.
$\dot{B}^s_{p,r}$:	homogeneous Besov space.
BV :	space of bounded variations functions.
C^s :	Hölder space.
$L^\beta_T B^s_{p,r}$, $\widetilde{L}^\beta_T B^s_{p,r}$:	mixed space-time spaces.
$X_{t,\lambda}$:	admissible family of vector fields.
$\partial_{X_t} u$:	directional derivative of u along X_t.
$[\Delta, X_t]$:	commutator between Δ and X_t.
$C^\varepsilon(X_{t,\lambda})$:	anisotropic H'older space.
C^ε_Σ :	co-normal space associated to closed curve Σ with $C^{1+\varepsilon}$ regularity.

1 Introduction and overview of the main results

The current thesis treats essentially the local/global persistence of geometric structures for the planar Boussinesq system in different cases in the context of a smooth patch. Specifically, we show that if the initial vorticity is a smooth patch, that is to say, a characteristic function of a bounded domain with an adequate regularity, then the transported patch keeps its initial regularity through the time. Afterwards, we study the inviscid limit when the viscosity parameter (resp. diffusion parameter) goes to zero and quantify the rate of convergence.

This dissertation comprises four chapters and an appendix.

In the first Chapter, we state the main results for each chapter and a brief outline of the proof.

In the second Chapter, we carefully collect some general materials and gather specific tools used along the next three chapters of the thesis.

The third Chapter cares with the optimal rate of convergence in stratified Boussinesq system. First, we handle with the global well-posedness issue for the viscous Boussinesq system in space dimension two, where the boundary of the initial vorticity is a Jordan curve with $C^{1+\varepsilon}$ Hölder regularity, with $0 < \varepsilon < 1$. We prove, globally in time, that the velocity vector field is a Lipschitz function by means of logarithmic estimate due to J.-Y. Chemin [19]. Second, we discuss the inviscid limit problem when the viscosity goes to zero and we evaluate the rate of convergence in Lebesgue spaces. In particular, when the patch vortex is of Rankine's type, we establish that this rate of convergence is optimal.

The last Chapter is devoted to the study of the local persistence of a smooth vortex patch for the Boussinesq system with zero viscosity. We prove that if the initial vorticity is a smooth vortex patch, then the regularity of the transported patch persists locally in time and the corresponding velocity vector field is a Lipschitz function. Thereafter, our second task is to explore the rate of convergence among velocities, densities and involved flows.

We end this dissertation with an appendix chapter, where we describe the Boussinesq model and its origin. In addition, we explain the relationship between the 3d-axisymmetric Euler equations with swirl flows and the inviscid 2d- Boussinesq equations.

1.0.1 A brief concise on vortex patches problem

The objective in this dissertation is to study the dynamic of vortices for some evolution equations resulting from fluid mechanics. In particular, we focus our attention on models like Euler and Navier-Stokes equations incompressible stratified where the stratification occurs in the buoyancy term. The dynamic of vortices is a very old subject but always useful for the study in many aspects. It goes back to the works of Helmholtz [40], Kelvin [56], Kirchhoff [57] and other references. Here, we will give some important results that fit with our subject. To be precise, let us recall that the Euler equations for an ideal incompressible fluid defined in the whole space $\mathbb{R}^d$ are given by

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0, & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^d, \\ \text{div} v = 0, \\ v_{|t=0} = v^0. \end{cases} \tag{E}$$

where $v=(v^1,v^2,\cdots,v^d)$ represents the velocity vector field, p is a scalar function designates the pressure. In (E), the first equation is the momentum conservation equation and the second one means the mass conservation.
In (1933), Wolibner [73] proved that (E) is locally well-posed for smooth initial data. Later on Kato and Ponce [55] established the local well-posedness in the framework of Sobolev spaces H^s for $s>\frac{d}{2}+1$ and the maximal solution belonging to $\mathcal{C}([0,T^\star];H^s)$, satisfied the blow-up criterion

$$T^\star<+\infty \Rightarrow \int_0^{T^\star}\|\nabla v(\tau)\|_{L^\infty}d\tau=+\infty.$$

This criterion is very limited in applications and other criterion, equivalent to the first one presented above, was proven later by Beale, Kato and Majda [6]. It is related with the vorticity ω, which is defined in any dimension by the anti-symmetric matrix of the velocity gradient. This criterion reads as follows

$$T^\star < +\infty \Rightarrow \int_0^{T^\star} \|\omega(\tau)\|_{L^\infty} d\tau = +\infty.$$

We point out that further extensions of [55] have been implemented in various function spaces for example of type Besov, Hölder, Tribel, $\cdots$, for instance see [14, 19, 71, 79]. We mention that the global well-posedness problem of classical solution is still now an open problem except the space dimension two which is proved by Wolibner [73], and by Ukhovskii and Yudovich [70] in the three dimensional case with axisymmetric initial data. The global existence of solutions for Euler equations in dimension two of space, derives from the special structure of vorticity ω which satisfies a transport equation named Helmholtz equation

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \\ \omega_{|t=0} = \omega^0. \end{cases} \tag{1.1}$$

By the characteristic method, we can find infinite conservation laws, that is to say, for all $p \in [1, \infty]$

$$\|\omega(t)\|_{L^p} = \|\omega^0\|_{L^p}. \tag{1.2}$$

These laws are the key point to prove the global existence of Kato's classical solutions. We mention that the vorticity theory for incompressible fluids was founded by Helmholtz [40] who has formulated the principal laws that it governs. Here, we recall the dynamic of vortices in the case of dimension two of space. For this subject, we mention that the incompressibility of the fluid $\mathrm{div} v = 0$ implies the existence of a stream function Ψ such that $v = \nabla^\perp \Psi$, and so $\Delta \Psi = \omega$.

We observe that the connection between the vorticity and the stream function is of Poisson's kind, so

$$\Psi(\cdot, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \omega(\cdot, y) dy.$$

It follows that the velocity can be recovered from the vorticity by the well-known *Biot-Savart law*

$$v(\cdot, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(\cdot, y) dy.$$

1.1 Vortex patches: general presentation

We embark on this paragraph with a brief concise about Yudovich's solutions. Let us denote that the vorticity-velocity formulation associated with Euler equations (1.1) and the conservation laws (1.2) give a more opportunity to Yudovich [75] to relax the hyperbolic regularity following Kato and formulate a new concept of weak solutions global in time for Euler equations in the following sense.

Definition 1.1.1. *Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. We say that (v, ω) is a weak solution for (E) if and only if*

(i) $\omega \in L^\infty\big(\mathbb{R}_+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\big)$,

(ii) $v = \nabla^\perp \Delta^{-1}\omega$,

(iii) *for all $\phi \in C^1\big(\mathbb{R}_+; C_0^1(\mathbb{R}^2)\big)$, we have*

$$\int_{\mathbb{R}_+\times\mathbb{R}^2} \big(\partial_t\phi + v\cdot\nabla\phi\big)\omega dt dx = -\int_{\mathbb{R}^2} \phi(0,x)\omega^0(x)dx.$$

From the above definition, Yudovich succeed to recover the Euler equations globally in time as soon as $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. More precisely, he demonstrated the following theorem.

Theorem 1.1.2. *Let $\omega^0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Then (E) admits a unique solution globally in time*

$$\omega \in L^\infty\big(\mathbb{R}_+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\big)$$

in the sense of definition 1.1.1. As well, the velocity v admits a unique flow $\psi \in C(\mathbb{R}_+ \times \mathbb{R}^2)$ expressed by the integral equation

$$\psi(t,x) = x + \int_0^t v(\tau, \psi(\tau,x))d\tau.$$

Furthermore, this flow is an homeomorphism preserves the Lebesgue measure and satisfies the following degenerate regularity

$$\psi(t) - \mathbb{I} \in C^{e^{-t\|\omega^0\|_{L^1\cap L^\infty}}}.$$

This theorem is remarkable in so far as it is viable to define uniquely a flow in no Lipschitzian context. We also explicit the vorticity at each time in accordance of characteristic method $\omega(t,x) = \omega^0(\psi^{-1}(t,x))$. In this case, the vorticity is transported by the flow. An important sub-class of Yudovich's class under study encompasses the so-called vortex patches, meaning that if the initial vorticity is the characteristic function of a bounded planar domain: $\omega^0 = \mathbf{1}_{\Omega_0}$, then the transported vorticity by the flow ψ, being a vortex patch of the domain $\Omega_t \triangleq \psi(t,\Omega_0)$ and the dynamic is reduced to the evolution of the boundary of the patch that moves with the flow. However, if γ_t is a parametrization of this boundary, that it to say

$$\partial_t\gamma(t,s) = v(t,\gamma(t,s)).$$

The Biot-Savart law and the vorticity is a patch imply that the velocity is given by

$$v(t,x) = \frac{1}{2\pi}\int_0^{2\pi} \log|x-\gamma(t,s)|\partial_s\gamma(t,s)ds,$$

so, the equation of γ becomes

$$\partial_t\gamma(t,s) = \frac{1}{2\pi}\int_0^{2\pi} \log|\gamma(t,s)-\gamma(t,\tau)|\partial_\tau\gamma(t,\tau)d\tau. \tag{1.3}$$

The evolution of the regularity of the boundary can't be ensured from Yudovich's theory, which in turn does not give helpful information about the boundary's regularity of the transported patch. This is due to the lack of flow's regularity, that is ψ degenerates in time. For the optimality of such result and for more details we refer to [5] and [19].

The regularity persistence of the boundary was solved for the first time by Chemin in his famous work [19]. He proved that, when the boundary of the initial patch is of Hölderian class of type $C^{1+\varepsilon}$ where $0<\varepsilon<1$, then the transported patch that moves with the flow keeps it's initial regularity through the time. Chemin's proof is essentially based on a logarithmic estimate witch relies the Lipschitz norm of velocity with the co-normal regularity of vorticity transported by the flow. More accurately, Chemin's result is read as follows

Theorem 1.1.3. *Let $0<\varepsilon<1$, X_0 be a family of admissible vector fields of the plan and v^0 be a free-divergence vector field over $\mathbb{R}^2$ belongs to $C^1_\star$ such that its vorticity $\omega^0 \in L^a \cap C^\varepsilon(X_0)$ with $a>1$. Then (E) admits a unique global solution $v \in L^\infty_{loc}(\mathbb{R}; Lip)$ and $\nabla v \in L^a$.*
Furthermore, if ψ is the flow associated to the velocity v, then we have $\partial_{X_0}\psi \in L^\infty_{loc}(\mathbb{R}; C^\varepsilon)$. In addition, X_t the push-forward of X_0 by the flow ψ is also an admissible family and we have $\|\omega(t)\|_{C^\varepsilon(X_t)} \in L^\infty_{\rm loc}(\mathbb{R})$.

We mention here that the vortex patch problem was posed by Yudovich [75] in some smooth vessel, and this problem was renewed interest by Majda [7] in the whole plane with vorticity bounded and compactly supported. Numerical results was improved by Zabusky and al. [76]. Furthermore, Alinhac [4] has proved that the last equation develops instabilities close to the blow-up in finite time. Chemin's paradigm is different and it consists to propagate the tangential regularity of Euler equations for arbitrary time. The notion of tangential regularity with respect to an admissible family of vector fields was introduced by Bony [12] in the study of hyperbolic equations and it was refined to study semi linear PDE's by Alinhac [4] and Chemin [19]. We recall also that Bertozzi and Constantin [10] have established similar result to the Chemin's one with a shorter proof but their approach works well only in the context of dimension two of space, however the Chemin's method can be used for higher dimension and in the viscous case that, for example, Gamblin and Saint-Raymond in [34] proved the well-posedness of three dimensional vortex patches for Euler equations but locally in time. In the same period, Serfati [67] proved a version of persistence of striated regularity in slightly less generality than Chemin's result [19].
For the viscous case especially Navier-Stokes equations

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = 0, \quad \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \operatorname{div} v = 0,\\ v_{|t=0} = v^0, \end{cases} \tag{NS}$$

the situation is different and contributes some hardness. Among them, the dissipation term $\mu\Delta\omega$ breaks instantly the patch structure and does not commute with the admissible family. Consequently, the tangential derivative $\partial_{X_t}\omega$ evolves the following transport-diffusion equation

$$(\partial_t + v\cdot\nabla - \mu\Delta)\partial_{X_t}\omega = -\mu[\Delta, X_t]\omega,$$

where the vorticity formulation associated to (NS) is given by

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega - \mu\Delta\omega = 0,\\ \omega_{|t=0} = \omega^0. \end{cases}$$

The successful attempt in this way dates back to Danchin [22], where he claimed that if the initial patch belongs to the Hölderian class $C^{1+\varepsilon}$ with $0<\varepsilon<1$, then the velocity is lipschitzian uniformly to the viscosity term and the transported vorticity by the viscous flow is of class $C^{1+\varepsilon'}, \varepsilon'<\varepsilon$. Even though, to treat the term $\mu[\Delta, X_t]$

he required to propagate the Besov regularity $B^{\varepsilon}_{p,\infty}$ for an equation of type

$$\begin{cases} (\partial_t + v\cdot\nabla - \mu\Delta)a = f + \mu g, \\ a_{|t=0} = a_0. \end{cases}$$

Let us denote that this regularity is limited to $p < \infty$ which explain the artificial lose for the boundary regularity. Later, Hmidi in [41] has validated this smoothing effect in the Hölderian case corresponding to $p = \infty$. For the proof, he explored some smoothing effect of an equation of type transport-diffusion by means of Lagrangian coordinates.
The smooth vortex patch problem for 2d-Boussinesq equations was studied firstly by Hmidi and Zerguine [52]. The authors established a global well-posedness result in the frame work of vortex patch issue. Furthermore, they proved that the transported domain that moves with the flow preserves its initial Hölderian regularity. In [78], Zerguine studied the same equations by replacing the usual Laplacian by the fractional one $(-\Delta)^{\frac{1}{2}}$ and deduced a sharper result compared to the Euler equations. Other results in the same context can be found in [13, 15, 26, 25, 32, 43, 49, 48, 50] and the references therein.

1.2 Main results and headlines of the proof

This section comprises two main topics. The first one gives an affirmative answer to the question of global well-posedness for stratified Navier-Stokes equations or full viscous Boussinesq equations. Next, we study the inviscid limit whenever the viscosity goes to zero and give an optimal rate convergence in L^p spaces which extends the result of [1] for Navier-Stokes equations for $p = 2$. We reach the optimal rate when the vorticity is of Rankine type. The second topic deals with the persistence of geometric structures for partial viscous Boussinesq equations. Indeed, we have proved the local well-posedness of solutions in the framework of regular vortex patches and investigated the convergence of solutions of the equations towards the inviscid Boussinesq equations when the diffusivity tends to zero.

1.2.1 First Topic

Let us recall that stratified Navier-Stokes system couples a Navier-Stokes perturbed equation with a transport diffusion type equation for density. It reads as follows:

$$\begin{cases} \partial_t v_\mu + v_\mu\cdot\nabla v_\mu - \mu\Delta v_\mu + \nabla p_\mu = \rho_\mu \vec{e}_2 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \partial_t \rho_\mu + v_\mu\cdot\nabla\rho_\mu - \Delta\rho_\mu = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \text{div} v_\mu = 0, \\ (v_\mu,\rho_\mu)_{|t=0} = (v_\mu^0,\rho_\mu^0). \end{cases} \tag{B$_\mu$}$$

Where $v_\mu(t,x)\in\mathbb{R}^2$ denotes the velocity vector field which assumed to be incompressible, $\rho_\mu(t,x)\in\mathbb{R}$ the density and $p_\mu(t,x)\in\mathbb{R}$ the pressure. The coefficient μ designates the kinematic viscosity of the fluid and the vector unity $\vec{e}_2=(0,1)$. Applying the *curl* operator to the first equation in (B$_\mu$), we deduce a vorticity- density formulation given by

$$\begin{cases} \partial_t \omega_\mu + v_\mu\cdot\nabla\omega_\mu - \mu\Delta\omega_\mu = \partial_1\rho_\mu & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \partial_t \rho_\mu + v_\mu\cdot\nabla\rho_\mu - \Delta\rho_\mu = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ v_\mu = \nabla^\perp\Delta^{-1}\omega_\mu, \\ (\rho_\mu,\omega_\mu)_{|t=0} = (\rho_\mu^0,\omega_\mu^0). \end{cases} \tag{VD$_{\mu,\kappa}$}$$

We have established three main results for the system (B$_\mu$) in Chapter three. The first one deals with the global in time existence and uniqueness of solution with vortex patch initial data. The second one treats the inviscid limit of the problem towards the stratified Euler equations when the viscosity vanishes. In the last result we give an optimal rate of convergence for vortices if the initial patches are of Rankine type. Here we shall give our main results and explore the ideas of the proofs.
Our first main result is presented in the theorem bellow

Theorem 1.2.1. *Let Ω_0 be a simply connected bounded domain such that its boundary $\partial\Omega_0$ is $C^{1+\varepsilon}$ with $0<\varepsilon<1$. Let $\omega_\mu^0=\mathbf{1}_{\Omega_0}$ and $\rho_\mu^0\in L^1\cap L^\infty$, then the following assertions hold.*

(i) *The system (B$_\mu$) admits a unique global solution (v_μ,ρ_μ) such that*

$$(v_\mu,\rho_\mu)\in L^\infty_{loc}(\mathbb{R}_+;\text{Lip})\times L^\infty_{loc}(\mathbb{R}_+;L^1\cap L^\infty).$$

More precisely, there exists $C_0 \triangleq C(\varepsilon, \Omega_0) > 0$ *independent of the viscosity such that, for all* $\mu \in]0,1[$ *and for all* $t \in \mathbb{R}_+$ *we have*

$$\|\nabla v_\mu(t)\|_{L^\infty} \leq C_0 e^{C_0 t \log^2(1+t)}. \tag{1.4}$$

(ii) *The boundary of the transported domain* $\Omega_\mu(t) \triangleq \Psi_\mu(t, \Omega_0)$ *is* $C^{1+\varepsilon}$ *for every* $t \geq 0$ *uniformly on* μ*, where* Ψ_μ *denotes the viscous flow associated to* v_μ*.*

The proof is essentially based on the Chemin's work [19] to obtain a lipschitz norm for the velocity by means of logarithmic estimates which requires a co-normal regularity of vorticity $\partial_{X_{t,\lambda}}\omega$ in $C^{\varepsilon-1}, 0 < \varepsilon < 1$, where $X_{t,\lambda}$ is an admissible family of vector fields which is tangential to the boundary and verifies a transport equation of type

$$(\partial_t + v_\mu \cdot \nabla)\partial_{X_{t,\lambda}} = X_{t,\lambda} \cdot \nabla v_\mu.$$

The difference between the inviscid case and the viscous one is that this family does not commute with the Laplacian operator. In contrast, we have

$$(\partial_t + v_\mu \cdot \nabla - \mu\Delta)\partial_{X t,\lambda}\omega_\mu = -\mu[\Delta, X_{t,\lambda}]\omega_\mu + \partial_{X t,\lambda}\partial_1\rho_\mu, \tag{1.5}$$

where $[\Delta, X_{t,\lambda}]$ is the commutator between Δ and $X_{t,\lambda}$. Thus the difficulties reduce to bound the terms $[\Delta, X_{t,\lambda}]\omega_\mu$ and $\partial_{X t,\lambda}\partial_1\rho_\mu$ which apparently need more regularity to be well-defined than what is initially prescribed. In [63], we have treated the first term by using the formalism developed in [22, 41] for $2d-$incompressible Navier-Stokes equations. To bound the second term, we diagonalize the vorticity-density formulation and introduce a new unknown named *coupled function* defined by

$$\Gamma_\mu \triangleq (1-\mu)\omega_\mu - \partial_1\Delta^{-1}\rho_\mu.$$

We recall tha the notion of coupled function was introduced in [50]. This function satisfies an equation of a transport-diffusion type

$$\partial_t\Gamma_\mu + v_\mu \cdot \nabla\Gamma_\mu - \mu\Delta\Gamma_\mu = [\partial_1\Delta^{-1}, v_\mu \cdot \nabla]\rho_\mu \triangleq H_\mu.$$

Then $\partial_{X t,\lambda}\Gamma$ satisfies the following equation

$$(\partial_t + v_\mu \cdot \nabla - \mu\Delta)\partial_{X_{t,\lambda}}\Gamma_\mu = -\mu[\Delta, X_{t,\lambda}]\Gamma_\mu + \partial_{X t,\lambda}H_\mu.$$

Thus, the heart work in this case is to deal with the two quantities in the right hand side of the last equation.

Our second outcome concerning the rate of convergence of velocities, densities and vortices of Boussinesq equations (B_μ) towards the inviscid one, when the viscosity μ tends to zero in the frame work of Lebesgue spaces. More precisely, we have

Theorem 1.2.2. *Let (v_μ, ρ_μ), (v, ρ), (ω_μ, ρ_μ) and (ω, ρ) be the solutions of* (B_μ), (B_0), (VD_μ) *and* (VD_0) *respectively with the same initial data such that*

$$\omega_\mu^0 = \omega^0 = \mathbf{1}_{\Omega_0},$$

where Ω_0 is a $C^{1+\varepsilon}$ simply connected bounded domain. Then for all $t \geq 0, \mu \in]0,1[$ and $p \in [2, +\infty[$ we have

(i) $\|v_\mu(t) - v(t)\|_{L^p} + \|\rho_\mu(t) - \rho(t)\|_{L^p} \leq C_0 e^{e^{C_0 t \log^2(2+t)}} (\mu t)^{\frac{1}{2}+\frac{1}{2p}}$.

(ii) $\|\omega_\mu(t) - \omega(t)\|_{L^p} \leq C_0 e^{e^{C_0 t \log^2(1+t)}} (\mu t)^{\frac{1}{2p}}$.

We recall that the inviscid limit problem for Navier-Stokes equations towards Euler equations was studied by Constantin [20] for initial velocity $v^0 \in H^s$, with $s > 4$ and the rate of convergence in the Lebesque space L^2 is bounded by μt. However, Chemin proved in [18] a rate of convergence equal to $(\mu t)^{\frac{1}{2}e^{-Ct}}$ for initial data of Yudovich's type. In [21], Constantin and Wu obtained a rate of convergence controlled by $(\mu t)^{\frac{1}{2}}$ for vortex patch's initial data. Later, Abidi and Danchin [1] enhanced the last result, that the rate of convergence will be controlled by $(\mu t)^{\frac{3}{4}}$ in L^2 space. Furthermore, the obtained result is optimal for initial vortices with Rankine vortex patch. In [63], we have succeed to extend this result for the Boussinesq system. We mention here that we have obtained a rate of convergence in the Lebesgue spaces L^p for $p \in [2, +\infty[$ controlled by $(\mu t)^{\frac{1}{2}+\frac{1}{2p}}$ for velocities and densities, whilst the vortices are bounded by $(\mu t)^{\frac{1}{2p}}$.
Let us notice that the difference between the rates of convergence between velocities and their vortices is equal to $\frac{1}{2}$. In fact, really there is a gain of one derivative between them.
Finally, we present our interesting result in this way, witch gives an affirmative answer for the optimality of the rate of convergence between vortices for initial data of Rankine type vortex patch.

Theorem 1.2.3. *We assume that ρ_μ^0 and ρ^0 being constants and $\omega_\mu^0 = \omega^0 = \mathbf{1}_{\mathbb{D}}$ with $\mathbb{D}$ the unit disc. Then there exist two positive constants C_1 and C_2 independent on μ and t, such that for $\mu t \leq 1$, and $p \in [2, +\infty[$ we have*

$$C_1 (\mu t)^{\frac{1}{2p}} \leq \|\omega_\mu(t) - \omega(t)\|_{L^p} \leq C_2 (\mu t)^{\frac{1}{2p}},$$

with C_1 and C_2 depending on p.

If $p = 2$, we find the same result obtained in [1]. Our proof presented here is completely different from that given by [1] which is specific to the case $p = 2$ by using the explicit form of Fourier transform of the Rankine vortex patch. Our approach is based essentially on the computation in physical variables using the explicit form of the heat kernel.

1.2.2 Second topic

The second subject of the present thesis concerns the local persistence of geometric structures for partial viscous Boussinesq system under vortex patch initial data. The system in question couples a perturbed 2d-Euler equations with a transport-diffusion equation for density. More accurately, we have

$$\begin{cases} \partial_t v_\kappa + v_\kappa \cdot \nabla v_\kappa + \nabla p_\kappa = \rho_\kappa \vec{e}_2, \quad t \geq 0,\ x \in \mathbb{R}^2 \\ \partial_t \rho_\kappa + v_\kappa \cdot \nabla \rho_\kappa - \kappa \Delta \rho_\kappa = 0, \\ \operatorname{div} v_\kappa = 0, \\ v_{\kappa|t=0} = v_\kappa^0, \quad \rho_{\kappa|t=0} = \rho_\kappa^0. \end{cases} \tag{B$_\kappa$}$$

Where κ designates the molecular diffusivity of the fluid.
As we have seen above, the Boussinesq equations (B$_\kappa$) are a perturbation of (E), it will be of interest to ask whether the known results for Euler equations can be extended to the Boussinesq system as well. The topic of local/global posedness for (B$_\kappa$) for $\kappa > 0$ is of great interest. We recall here, that Chae in [15] succeed to prove that (B$_\kappa$) is globally well-posed whenever $(v^0, \rho^0) \in H^s \times H^s$, with $s > 2$. This result was improved later by Hmidi and Keraani in [45], where they imposed that $(v^0, \rho^0) \in B_{p,1}^{1+\frac{2}{p}} \times B_{p,1}^{-1+\frac{2}{p}} \cap L^r$, with $r > 2$. In this direction, Hmidi and Zerguine [51] established similar result in the setting of fractional laplacian $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in]1, 2]$. In [26], Danchin and Paicu extended weak solutions of Yudovich's type to the system (B$_\kappa$). Another results concerning this subject can be found in [1, 9, 16, 17, 25, 33] and the references therein.
In [62], we are interested by the study of the vortex patch problem for the system (B$_\kappa$) and to investigate the convergence towards the inviscid system when the diffusivity parameter κ goes to zero. Note that the limit system is simply obtained by

taking $\kappa = 0$, so we get

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = \rho\vec{e}_2, \quad t\geq 0, x\in\mathbb{R}^2\\ \partial_t\rho + v\cdot\nabla\rho = 0,\\ \operatorname{div} v = 0. \end{cases} \tag{B$_0$}$$

We point out that for the latter system, local well-posedness can be implemented in various function spaces similarly to Euler equations. For instance, Chae and Nam showed in [16] that (B$_0$) is locally well-posed in Sobolev spaces H^s with $s > 2$. This result was extended to critical Besov spaces $B^{1+\frac{2}{p}}_{p,1}$, $p\in]1,\infty[$ by Liu, Wang and Zhang in [54]. The global existence of classical solutions is an outstanding open problem.

In [39], Hassainia and Hmidi proved that the system (B$_0$) is locally well-posed with initial patch has a regular/singular structure. Our goal in [62] was to extend this result for the system (B$_\kappa$). The first main result of Chapter four is summarized in the following Theorem where we deal with local theory for the vortex patch problem uniformly with respect to the parameter κ. More precisely, we have

Theorem 1.2.4. *Let $\kappa\in[0,1]$ and consider a bounded domain Ω_0 in $\mathbb{R}^2$ whose boundary $\partial\Omega_0$ is a Jordan curve of $C^{1+\varepsilon}-$regularity, with $0<\varepsilon<1$. Let v^0_κ be a divergence-free vector field such that its vorticity $\omega^0_\kappa = \mathbf{1}_{\Omega_0}$ and the initial density $\rho^0_\kappa\in L^2\cap C^{1+\varepsilon}$ with $\nabla\rho^0_\kappa\in L^2$.*
Then there exists $T>0$ independent of κ such that the system (B$_\kappa$) admits a unique local solution

$$(v_\kappa,\rho_\kappa)\in\Big(L^\infty\big([0,T];\mathrm{Lip}(\mathbb{R}^2)\big)\Big)^2.$$

Furthermore, for all $t\in[0,T]$ the boundary $\partial\Omega_t$ is a Jordan curve of class $C^{1+\varepsilon}$, with $\Omega_t = \Psi_t(\Omega_0)$.

Here, an interesting remark is that the initial condition $\rho^0_\kappa\in C^{1+\varepsilon}$ doesn't persist in time. Indeed, $\rho_\kappa(t)\in C^{1+\varepsilon}$ requires more regularity on the velocity sharper than the lipschitzian one.

The proof of this Theorem is to derive the Lipschitz norm of the velocity locally in time uniformly on κ. For this aim, we walk in the footsteps of Chemin's approach [19]. Thus we shall control $\|\nabla v_\kappa(t)\|_{L^\infty}$ with respect to the co normal regularity of the vorticity $\partial_{X_{t,\lambda}}\omega_\kappa$ in $C^{\varepsilon-1}$, with $0<\varepsilon<1$ by means of logarithmic estimate. The family of vector fields $X_t = (X_{t,\lambda})_{\lambda\in\Lambda}$ obeys to the equation (4.2). The tangential derivative of the vorticity $\partial_{X_t}\omega_\kappa$ satisfies similarly to (4.3)

$$(\partial_t + v_\kappa\cdot\nabla)\partial_{X_{t,\lambda}}\omega_\kappa = \partial_{X_{t,\lambda}}\partial_1\rho_\kappa$$

This follows from the fact that the vorticity-density formulation of (B_κ) is given by

$$\begin{cases} \partial_t\omega_\kappa + v_\kappa\cdot\nabla\omega_\kappa = \partial_1\rho_\kappa, \quad t\geq 0,\ x\in\mathbb{R}^2\\ \partial_t\rho_\kappa + v_\kappa\cdot\nabla\rho_\kappa - \kappa\Delta\rho_\kappa = 0,\\ \operatorname{div} v_\kappa = 0, \end{cases} \tag{VD$_\kappa$}$$

We can write the term in the right-hand side of the first equation of (VD$_\kappa$) as follows

$$\partial_{X_t}\partial_1\rho_\kappa = \partial_1(\partial_{X_t}\rho_\kappa) + [\partial_{X_t},\partial_1]\rho_\kappa$$

and keeping in mind that the commutator $[\partial_{X_t},\partial_1]\rho_\kappa$ behaves well, then the problem reduces to follow the regularity of $\partial_{X_t}\rho_\kappa$ in C^ε. It is straightforward that the quantity $\partial_{X_t}\rho_\kappa$ satisfies the following evolution equation

$$(\partial_t + v_\kappa\cdot\nabla - \kappa\Delta)X_{t,\lambda}\rho_\kappa = -\kappa[\Delta, X_{t,\lambda}]\rho_\kappa. \tag{1.6}$$

Thus, the commutator term contributes with additional obstructs. The heart work is to treat carefully the commutator by means of the maximal smoothing effect of the transport -diffusion equation as in [22, 41]. The last step in the proof is doing by a classical argument introduced in [19], by building an initial admissible family X_0 such that the initial vortex patch $\mathbf{1}_{\Omega_0}$ belongs to the anisotropic space $C^\varepsilon(X_0)$ and check the regularity of the transported boundary by the flow associated to the velocity vector field.
Our second main result of Chapter four is dedicated to study the inviscid limit problem of the system (B$_\kappa$) towards (B$_0$) when the diffusivity κ vanishes, in the Lebesgue spaces L^p. We have established a rate of convergence for velocities, densities and the associated flows. More accurately, we have

Theorem 1.2.5. *Let (v_κ,ρ_κ) and (v,ρ) be the solutions of* (B$_\kappa$) *and* (B$_0$) *respectively with the same initial data given by Theorem 4.1.1. Then the following assertions hold true.*

(i) *For every $p\in[2,\infty]$*

$$\sup_{t\in[0,T]}\big(\|v_\kappa(t)-v(t)\|_{L^p} + \|\rho_\kappa(t)-\rho(t)\|_{L^p}\big) \leq C_0\kappa^{1/4+1/2p}.$$

(ii) *If Ψ_κ and Ψ denote the flow associated to v_κ and v respectively. Then we have*

$$\sup_{t\in[0,T]}\|\Psi_\kappa(t)-\Psi(t)\|_{L^\infty} \leq C_0\kappa^{1/4},$$

where $C_0 = C(\|\nabla\rho^0\|_{L^2\cap L^\infty}, T)$.

The proof of the rate of convergence for velocities and densities is done by using some classical L^p-estimates, the classical complex interpolation between Lebesgue spaces and the so-called Gagliardo-Nirenberg inequality. For the rate of convergence for flows, we use the integral equation of the flow associated to the velocity vector field v:

$$\Psi(t,x) = x + \int_0^t v(\tau, \Psi(\tau,x))d\tau$$

and some general estimates associated to the flow.

2 Preliminaries

In this Chapter, we have gathered all the ingredients that are usefull during this dissertion. We start by introducing the Littlewood-Paley theory, in particular, the decomposition of the unity, Bernstein's lemma, pardifferential calculus and state a few properties related to this subject. Afterward, we define some functional spaces like Besov, Hölder and bounded variations spaces and characterize them in terms of dyadic blocks and announce the main embedding theorems. Next, we shall discuss some properties concerning some commutator estimates by means of the so-called Bony's decomposition. Moreover, we furnish some useful technical lemmas, in particular two smoothing effects estimates for transport and transport-diffusion equations governing respectively the time evolution of the density and the vorticity.

2.1 An outline about Littlewood-Paley theory

We embark this section by a brief concise about Littlewood-Paley theory. We state by so-called dyadic decomposition of the unity and some of their properties, alike Bernstein's lemma and paradifferential calculus.
So, let χ be a smooth radial function supported in a ball $\mathcal{B}(0, \frac{4}{3})$ such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq \frac{3}{4} \\ 0 & \text{if } |\xi| \geq 1, \end{cases}$$

and define the function φ as follows

$$\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$$

We can easily check that φ is supported in the annulus $\mathcal{C}(0, \frac{3}{4}, \frac{8}{3})$.

A nice properties of the functions χ and φ are listed in the following proposition.

Proposition 2.1.1. *Let* $\mathcal{C}(0,\frac{3}{4},\frac{8}{3})$ *and* $\widetilde{\mathcal{C}}=\mathcal{B}(0,\frac{2}{3})+\mathcal{C}(0,\frac{3}{4},\frac{8}{3})$ *be two annulus. The following assertions hold*

$$q\geq 1\Longrightarrow 2^q\widetilde{\mathcal{C}}\cap 2^p\mathcal{C}=\emptyset, \tag{2.1}$$

$$|q-p|\geq 2\Longrightarrow \text{supp }\varphi(2^{-q}\cdot)\cap\text{supp }\varphi(2^{-p}\cdot)=\emptyset, \tag{2.2}$$

$$q\geq 1\Longrightarrow \text{supp }\chi\cap\text{supp }\varphi(2^{-q}\cdot)=\emptyset, \tag{2.3}$$

$$\chi(\xi)+\Sigma_{q\geq 0}\varphi(2^{-q}\xi)=1. \tag{2.4}$$

Proof. To prove (2.1) we proceed by absurd by assuming that $2^q\widetilde{\mathcal{C}}\cap 2^p\mathcal{C}\neq\emptyset$ and that $p\geq q$. This provides us

$$|p-q|\geq 2\Longrightarrow 2^q\widetilde{\mathcal{C}}\cap 2^p\mathcal{C}=\emptyset, \tag{2.5}$$

which implies that $2^p\frac{3}{4}\leq 2^{q+1}\frac{4}{3}$, hence $p-q\leq 1$. This contradicts the fact $|p-q|\geq 2$. For (2.2), let $\mathcal{C}'$ be the annulus centered at origin with two radius $\frac{1}{\alpha}$ and 2α with $0<\alpha<1$, and choosing a radial function $\theta\in\mathcal{C}_0^\infty$ with values in $[0,1]$, supported in $\mathcal{C}$ that equal to 1 in a neighborhood of $\mathcal{C}'$. Set

$$S(\xi)=\Sigma_{q\in\mathbb{Z}}\theta(2^{-q}\xi),$$

then, in view of (2.1), we can deduce that this sum is locally finite in $\mathbb{R}^d\diagdown\{0\}$. So, we get $S\in\mathcal{C}^\infty$ and $\cup_{q\in\mathbb{Z}}2^q\mathcal{C}'=\mathbb{R}^d\diagdown\{0\}$. From the properties of θ, one can said that S is strictly positive. Let us prove that the function defined by $\varphi=\frac{\theta}{S}$ is suitable. Clearly we have $\varphi\in\mathcal{C}^\infty(\mathcal{C})$.
On the other hand, we observe that the function $1-\Sigma_{q\geq 0}\varphi(2^{-q}\xi)$ is in $\mathcal{C}^\infty$ class. From supp $\theta\subset\mathcal{C}$, we obtain that

$$|\xi|\geq\frac{4}{3}\Longrightarrow\Sigma_{q\geq 0}\varphi(2^{-q}\xi)=1. \tag{2.6}$$

Hence, by assuming

$$\chi(\xi)=1-\Sigma_{q\geq 0}\varphi(2^{-q}\xi),$$

we get (2.4) and (2.2). The estimate (2.3) is a consequence of (2.6) and (2.5). □

The following *Bernstein's* lemma describes a bound on the derivatives of a function in the L^b-norm in terms of the value of the function in the L^a-norm, under the assumption that the Fourier transform of the function is compactly supported. For more details we refer to [5, 19].

Lemma 2.1.2. *Let $R_1 < R_2$. There exists a constant $C > 0$ such that for $1 \leq a \leq b \leq \infty$, for every function $u \in L^a$ and every $k \in \mathbb{N}$, we have*

(i) *(Direct Bernstein's inequality) If* $\operatorname{supp} \widehat{u} \subset \mathcal{B}(0, R_1\lambda)$*, then*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{q\left(k+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|u\|_{L^a},$$

(ii) *(Reverse Bernstein's inequality) If* $\operatorname{supp} \widehat{u} \subset \mathcal{C}(0, R_1\lambda, R_2\lambda)$*, then*

$$C^{-k}\lambda^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^k \lambda^k \|u\|_{L^a}.$$

Proof. By changing variables by setting $v(x) = u(\frac{1}{\lambda}x)$, the proof reduces to the case $\lambda = 1$. Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ be equal to 1 in neighborhood of the ball $\mathcal{B}(0, R_1)$, and let g being its Fourier inverse transform. One can observe that $\widehat{u}(\xi) = \phi(\xi)\widehat{u}(\xi)$. We deduce that

$$u(x) = \int_{\mathbb{R}^d} g(y)u(x-y)dy.$$

By derivation, we have

$$\partial^\alpha u(x) = \int_{\mathbb{R}^d} \partial^\alpha g(y)u(x-y)dy.$$

Applying Young's inequality, we get

$$\|\partial^\alpha u\|_{L^b} \leq \|\partial^\alpha g\|_{L^c} \|u\|_{L^a}, \qquad \frac{1}{c} = \frac{1}{b} - \frac{1}{a} + 1.$$

Now, we bound the term $\|\partial^\alpha g\|_{L^c}$ in the following way

$$\begin{aligned}
\|\partial^\alpha g\|_{L^c} &\leq \|\partial^\alpha g\|_{L^\infty} + \|\partial^\alpha g\|_{L^1} \\
&\leq \|(1 + |\cdot|^2)^d \partial^\alpha g\|_{L^\infty} \\
&\leq \|(Id - \Delta)^d((\cdot)^\alpha \phi)\|_{L^1} \\
&\leq C^k.
\end{aligned}$$

To prove the second point of Bernstein's lemma, we now assume that ϕ is compactly supported away from the origin and takes value 1 over the annulus $\mathcal{C}(0, R_1, R_2)$. Then in view of $|\xi|^{2k} = \Sigma_{|\alpha|=k}(i\xi)^\alpha(-i\xi)^\alpha$, we claim that

$$\widehat{u}(\xi) = \Sigma_{|\alpha|=k}\Big(\frac{(i\xi)^\alpha(-i\xi)^\alpha}{|\xi|^{2k}}\phi(\xi)\Big)\widehat{u}(\xi),$$

and

$$\widehat{u}(\xi) = \Sigma_{|\alpha|=k} \frac{(i\xi)^\alpha}{|\xi|^{2k}} \phi(\xi) \partial^\alpha \widehat{u}(\xi).$$

Setting $\widehat{g_\alpha}(\xi) = \frac{(i\xi)^\alpha}{|\xi|^{2k}} \phi(\xi)$, so we deduce that

$$\widehat{u}(\xi) = \mathcal{F}\big(\Sigma_{|\alpha|=k}(g_{\alpha)} * \partial^\alpha u)\big)(\xi).$$

Young's inequality leads to

$$C^{-k-1} \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a}.$$

□

2.1.1 Cut-off operators

Through the functions χ and φ, we state the so-called Littlewood-Paley or cut-off operators, as well as the low frequency cut-off operators. To be precise, we have

Definition 2.1.3. *Let $u \in \mathscr{S}'$. Then the cut-off operators are defined as follows*

$$\Delta_{-1} u \triangleq \chi(\mathrm{D}) u, \quad \Delta_q u \triangleq \varphi(2^{-q}\mathrm{D}) u, \quad \textit{if } q \in \mathbb{N}.$$

while the low frequency cut-off operators are given by

$$S_q u \triangleq \sum_{j \leq q-1} \Delta_j u \quad \textit{for } q \geq 0$$

Where in general case, $f(\mathcal{D})$ can be seen as the pseudo-differential operator $u \longmapsto \mathcal{F}^{-1}(f\widehat{u})$ with constant symbol.

The cut-off operators defined above are characterized by means of Fourier transform. Especially, we have

Proposition 2.1.4. *Let $u, v \in \mathscr{S}'$. We have*

(i) $\mathrm{supp}\, \mathcal{F}(\Delta_q u) \subset \mathcal{C}(0, \frac{3}{4}2^q, \frac{8}{3}2^q)$, $\quad \mathrm{supp}\, \mathcal{F}(\Delta_{-1} u) \subset \mathcal{B}(0, \frac{3}{4})$, $\quad \mathrm{supp}\, \mathcal{F}(\mathcal{S}_q u) \subset 2^q \mathcal{B}(0, \frac{4}{3})$, $\quad \mathrm{supp}\, \mathcal{F}(\mathcal{S}_{q-1} u \Delta_q v) \subset \mathcal{C}(0, \frac{1}{12}2^q, \frac{10}{3}2^q)$,

(ii) $\Delta_q u \equiv 2^{qd} h(2^q \cdot) * u$, $\quad \Delta_{-1} u \equiv \widetilde{h} * u$, $\quad \mathcal{F}(\mathcal{S}_q u) = \chi(2^{-q} \cdot) * u$, *where $h = \mathcal{F}^{-1}(\varphi)$ and $\widetilde{h} = \mathcal{F}^{-1}(\chi)$,*

(iii) $|p-q| \geq 2 \Longrightarrow \Delta_p\Delta_q u \equiv 0,\ |p-q| \geq 4 \Longrightarrow \Delta_q(\mathcal{S}_{p-1}u\Delta_p v) \equiv 0,$

Proof. **(i)** By definition we have $\mathcal{F}(\Delta_q u)(\xi) = \phi(2^{-q}\xi)u(\xi)$ and the fact that $\operatorname{supp}\phi \subset \mathcal{C}(0, \frac{3}{4}, \frac{8}{3})$, we deduce that

$$\operatorname{supp}\mathcal{F}(\Delta_q u) \subset \mathcal{C}\big(0, \frac{3}{4}2^q, \frac{8}{3}2^q\big).$$

(ii) For $u \in \mathscr{S}'$ we have $\mathcal{F}(\Delta_q u)(\xi) = \phi(2^{-q}\xi)u(\xi) = \mathcal{F}\big(\mathcal{F}^{-1}(\phi(2^{-q}\cdot))\widehat{u}\big)$. Since $\mathcal{F}^{-1}$ is an isomorphism from $\mathcal{S}'$ into itself, we get that

$$\Delta_q u \equiv \mathcal{F}^{-1}(\phi(2^{-q}\cdot)) * u.$$

Using the change of variables $\eta = 2^{-q}\xi$ to obtain

$$\mathcal{F}^{-1}(\phi(2^{-q}\cdot))(x) = \int_{\mathbb{R}^d} e^{i2^q x\cdot\eta}\phi(\eta)d\eta = 2^{qd}h(2^q x).$$

(iii) In view of $\Delta_q u = \mathcal{F}^{-1}\phi(2^{-q}\cdot) * u$, Young's inequality leads to

$$\|\Delta_q u\|_{L^p} \leq \|2^{qd}h(2^q\cdot)\|_{L^1}\|u\|_{L^p}.$$

The fact that $\|2^{qd}h(2^q\cdot)\|_{L^1} = 2^{qd}\int_{\mathbb{R}^d} h(2^q x)dx$. Hence, by a change of variable, it follows that

$$\|2^{qd}h(2^q\cdot)\|_{L^1} = \|h\|_{L^1}.$$

For $q = -1$, the proof can be done in a similar way.

$\square$

Likewise, we can define the homogeneous cut-off operators $\dot{\Delta}_q$ and $\dot{S}_q$ as follows

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q = \varphi(2^q D)u, \quad \dot{S}_q = \sum_{j\leq q-1}\dot{\Delta}_j v.$$

We mention that $\dot{\Delta}_q$ and $\dot{S}_q$ satisfy the same assertions as in the previous proposition.

Remark 2.1.5. *Let us mention that*

(i) *the cut-off operators Δ_q (resp. $\dot{\Delta}_q$) map L^p into itself with norms independent of q and p , it suffices to apply* **(ii)** *in Proposition 2.1.4 and Young's inequality,*

(ii) *we have (formally) the following Littlewood-Paley decomposition of unity,*

$$\mathbb{I} = \sum_{q\geq -1} \Delta_q, \quad \mathbb{I} = \sum_{q\in\mathbb{Z}} \dot{\Delta}_q.$$

Indeed, in the inhomogeneous Besov spaces the above decomposition makes a sense in $\mathscr{S}'$ in the following way: for $u \in \mathscr{S}'$, then $u = \lim_{q\to\infty} S_q u$. For $f \in \mathscr{S}$ we have $\langle u - S_q u, f\rangle = \langle u, f - S_q f\rangle$. Since the Fourier transform is an isomorphism of $\mathscr{S}$ into itself, so it suffices to establish our claim in $\mathscr{S}$, that is $\lim_{q\to\infty} \chi(2^q \hat{f}) = \hat{f}$.

From the properties above, the Bernstein Lemma 4.2.7 for Δ_d (resp. $\dot{\Delta}_d$) and S_q (resp. $\dot{S}_q$) can be viewed as follows.

Corollary 2.1.6. *There exists a constant $C > 0$ such that for $1 \leq a \leq b \leq \infty$, for every function u and every $q \in \mathbb{N} \cup \{-1\}$, we have*

(i)

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q\left(k+d\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|S_q u\|_{L^a},$$

(ii)

$$C^{-k} 2^{qk} \|\Delta_q u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \leq C^k 2^{qk} \|\Delta_q u\|_{L^a}.$$

2.1.2 Besov spaces

Definition 2.1.7. *For $(s,p,r) \in \mathbb{R} \times [1,\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ (resp. the homogeneous Besov space $\dot{B}^s_{p,r}$) is the set of all temepred distributions $u \in \mathscr{S}$ (resp. $u \in \mathscr{S}'_{|\mathbf{P}}$) such that*

$$\|u\|_{B^s_{p,r}} \triangleq \left(2^{qs}\|\Delta_q u\|_{L^p}\right)_{\ell^r} < \infty.$$

$$\left(\textit{resp. } \|u\|_{\dot{B}^s_{p,r}} \triangleq \left(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\right)_{\ell^r(\mathbb{Z})} < \infty\right).$$

With $\mathbf{P}$ denotes the set of polynomials of $\mathbb{R}^d$.

Proposition 2.1.8. *The spaces $B^s_{p,r}$ (resp. the homogeneous Besov spaces $\dot{B}^s_{p,r}$) equipped with $\|u\|_{B^s_{p,r}}$ (resp. $\|u\|_{\dot{B}^s_{p,r}}$) are normed spaces.*

Some remarks are in order.

Remark 2.1.9. *From the definition above, we can verify immediately that*

(i) *The definition of the $B^s_{p,r}$ (resp. the homogeneous Besov space $\dot{B}^s_{p,r}$) are independent of the choice of χ and φ employed for defining the cut-off operators Δ_q (resp. $\dot{\Delta}_q$), and changing χ and φ give an equivalent norm.*

(ii) *For all $s \in \mathbb{R}$, $B^s_{2,2}$ coincides with the nonhomogeneous Sobolev space H^s. More precisely, there exists a constant C such that for all $s \in \mathbb{R}$ we have*

$$C^{-|s|-1}\|u\|_s^2 \leq \Sigma_q 2^{2qs}\|\Delta_q u\|_{L^2}^2 \leq C^{|s|+1}\|u\|_s^2,$$

where $\|u\|_s^2 = \int_{\mathbb{R}^d}\left(1+|\xi|^2\right)^s|\widehat{u}(\xi)|^2 d\xi$.

(iii) *As a consequence of the corollary 2.1.6, the following embedding holds true.*

$$B^s_{p,r} \hookrightarrow {B^{s\prime}}_{p,r'} \text{ if } s' < s \text{ or } s = s', r' \geq r$$

and

$$B^s_{p,r} \hookrightarrow B^{s-d(\frac{1}{p}-\frac{1}{p'})}_{p',r} \text{ if } p \geq p'.$$

2.1.2.1 Paradifferential calculus

The algorithm of paraproduct was introduced by Bony [12]. It consists to write the product of two tempered distributions u and v into three pieces. Specifically, we have the following definition for paraproduct and remainder terms.

Definition 2.1.10. *For a given $u, v \in \mathscr{S}'$ we have*

$$uv = T_u v + T_v u + \mathscr{R}(u,v),$$

with

$$T_u v = \sum_q S_{q-1}u\Delta_q v, \quad \mathscr{R}(u,v) = \sum_q \Delta_q u\widetilde{\Delta}_q v \quad \text{and} \quad \widetilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

In the product uv, the interactions between the frequencies of u and v are mixed, the paraproduct enables us to separate the different types of interactions: $T_u v$ and $T_v u$ which collect the interactions between low frequencies and high frequencies of u and v, and $\mathscr{R}$ the interaction between similar frequencies of u and v. The way how these bilinear operators act on Besov spaces is described bellow.

Proposition 2.1.11. *For all $s \in \mathbb{R}$, there exists a constant C such that for any $(p, r) \in [1, \infty]^2$ we have*

$$\|T_u v\|_{B^s_{p,r}} \leq C\|u\|_{L^\infty}\|v\|_{B^s_{p,r}},$$

for all $(u, v) \in L^\infty \times B^s_{p,r}$.

Proof. From proposition 2.1.4, $S_{q-1}u\Delta_q v$ is spectrally supported in an annulus $2^q\widetilde{C}$. Thus, by definition of cut-off operators, we infer that

$$\|S_{q-1}u\Delta_q v\|_{L^p} \leq \|u\|_{L^\infty}\|\Delta_q v\|_{L^p}.$$

Remark 2.1.9 achieves the proof. □

The continuity properties of the remainder term are listed in the following proposition.

Proposition 2.1.12. *For all $(s_1, s_2) \in \mathbb{R}^2$ with $s_1 + s_2 > 0$, there exists a constant C such that for all $(p_1, p_2, r_1, r_2) \in [1, \infty]^4$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$, we have*

$$\|\mathscr{R}(u, v)\|_{B^{s_1+s_2}_{p,r}} \leq C\|u\|_{B^{s_1}_{p_1,r_1}}\|v\|_{B^{s_2}_{p_2,r_2}},$$

for all $u \in B^{s_1}_{p_1,r_1}$ and $v \in B^{s_2}_{p_2,r_2}$.

To claim such estimate, we need the following technical lemma

Lemma 2.1.13. *Let B be a ball of $\mathbb{R}^d$, $s \in \mathbb{R}_+$ and $(p, r) \in [1, \infty]^2$. Let $(u_q)_{q\in\mathbb{N}}$ be a sequence of smooth functions such that*

$$\text{supp } \widehat{u_q} \subset 2^q B, \quad \text{and} \quad \|(2^{qs}\|u_q\|_{L^p})_{q\in\mathbb{N}}\|_{l^r} < \infty.$$

Then, we have

$$u = \sum_{q\in} u_q \in B^s_{p,r}, \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq C\|(2^{qs}\|u_q\|_{L^p})_{q\in\mathbb{N}}\|_{\ell^r}.$$

Proof. The fact that $\|u_q\|_{L^p} \leq C2^{-qs}$, for any $q \in \mathbb{N}$ and s is positive, imply that $(u_q)_{q\in}$ converges in L^p. Thus, we will study $\Delta_j u_q$. Proposition 2.1.4, ensures that there exists an integer N such that

$$j \geq q + N \Longrightarrow 2^j C \cap 2^q B = \emptyset,$$

where B and C are a ball and an annulus as defined in proposition 2.1.4. This, implies that

$$j \geq q + N \Longrightarrow \mathcal{F}(\Delta_j u_q) = 0 \Longrightarrow \Delta_j u_q = 0.$$

Besides, we have

$$\begin{aligned}
\|\Delta_j u_q\|_{L^p} &\leqslant \|\sum_{q\geq j-N} \Delta_j u_q\|_{L^p} \\
&\leqslant \sum_{q\geq j-N} \|\Delta_j u_q\|_{L^p} \\
&\leqslant \sum_{q\geq j-N} \|u_q\|_{L^p}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
2^{js}\|\Delta_j u_q\|_{L^p} &\leqslant \sum_{q\geq j-N} \|2^{js}\Delta_j u_q\|_{L^p} \\
&\leqslant \sum_{q\geq j-N} 2^{(j-q)s}2^{qs}\|u_q\|_{L^p} \\
&= (c_k) * (d_l),
\end{aligned}$$

where $c_k = \mathbf{1}_{[-N,\infty[}(k)e^{-ks}$ and $d_l = 2^{ls}\|u_l\|_{L^p}$. This completes the proof.

□

Proof. of proposition 2.1.12. By definition, the remainder term spectrally supported in a ball. Thus, Hölder's inequality gives

$$2^{q(s_1+s_2)}\|\Delta_{q-i}u\Delta_q v\|_{L^p} \leq \sum_{i=-1}^{1} \|\Delta_{q-i}u\|_{L^{p_1}}\|\Delta_q v\|_{L^{p_2}}.$$

The last estimate is nothing but the sum of three series which are the product of two series of type ℓ^{r_1} and ℓ^{r_2} respectively. This gives the desired estimate. □

From the previous results, we bound the product uv in the framework of Besov spaces. More precisely, we have

Proposition 2.1.14. *For any real positive number s and any $(p,r) \in [1,\infty]^2$, $L^\infty \cap B^s_{p,r}$ is an algebra space. Furthermore, there exists a constant C such that*

$$\|uv\|_{B^s_{p,r}} \leq C\big(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}}\big)\|v\|_{L^\infty}.$$

The proof will be done by exploring once again Bony's decomposition definition and the two previous propositions of paraproduct and remainder terms.
We end up this part by studying the action of smooth functions on Besov spaces. More accurately, we have the following theorem

Theorem 2.1.15. *Let f be a smooth functions, s a positive real number and $(p,r) \in [1,\infty]^2$. If u belongs to $L^\infty \cap B^s_{p,r}$, then $f \circ u$ belongs to $B^s_{p,r}$ and*

$$\|f \circ u\|_{B^s_{p,r}} \leq C(s, f, \|u\|_{L^\infty})\|u\|_{B^s_{p,r}}.$$

We observe that if $s > \frac{d}{p}$ or if $s = \frac{d}{p}$ and $r = 1$, then $B^{p,r}$ is included in L^∞. This implies that $B^{p,r}$ is stable under the action of f by composition.

Proof. Let us introduce the telescopic series

$$\sum_q f_q \quad \text{with} \quad f_q \triangleq f(\mathcal{S}_{q+1}u) - f(\mathcal{S}_q u).$$

As $(\mathcal{S}_q u)_{q\in}$ converges to u in L^p, and $f(0) = 0$, we obtain

$$f(u) = \sum_q f_q.$$

Applying Taylor formula of order 1, one gets

$$f_q = m_q \Delta_q u \quad \text{with} \quad m_q \triangleq \int_0^1 f'(\mathcal{S}_q u + t\Delta_q u)dt.$$

To estimate the terms m_q, let us recall the following formula, in general case

$$\partial_\alpha v(a) = \sum_{\sum_{j=1}^p \alpha_j = |\alpha|, |\alpha_j| \geq 1} \Big(\Pi_{j=1}^p \partial^{\alpha_j} a\Big) v^{(p)}(a).$$

Applying this formula on m_q, one gets

$$\partial_\alpha m_q = \sum_{\sum_{j=1}^p \alpha_j=|\alpha|,|\alpha_j|\geq 1} \int_0^1 \Big(\Pi_{j=1}^p \partial^{\alpha_j}(\mathcal{S}_q u + t\Delta_q u)\Big) f^{(p+1)}(\mathcal{S}_q u + t\Delta_q u)dt.$$

To conclude the proof, we need the following lemma, which can be found in [77].

Lemma 2.1.16. *Let $s \in \mathbb{R}_+$ and $(p,r) \in [1,\infty]^2$. There exists a constant C_s such that if $(u_q)_{q\in\mathbb{N}}$ is a sequence of smooth functions that satisfies*

$$\Big(\sup_{|\alpha|\leq[s]+1} 2^{q(s-[\alpha])}\|\partial^\alpha u_q\|_{L^p}\Big)_q \in \ell^r,$$

then, we have

$$u = \sum_{q\in} u_q \in B^s_{p,r} \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq C_s \Big(\sup_{|\alpha|\leq[s]+1} 2^{q(s-[\alpha])}\|\partial^\alpha u_q\|_{L^p}\Big)_q \|_{\ell^r}.$$

Thus, one can write

$$\begin{aligned}\|\partial^\alpha m_q\|_{L^\infty} &\leq C_\alpha(f) \sum_{\sum_{j=1^p}\alpha_j=|\alpha|,|\alpha_j|\geq 1} \int_0^1 \left(\Pi_{j=1}^p 2^{q|\alpha_p|}\|u\|_{L^\infty}\right)\\ &\leq C_\alpha(f,\|u\|_{L^\infty})2^{q|\alpha|}.\end{aligned}$$

Using Leibniz formula and the previous lemma once again, we obtain

$$\|\partial^\alpha f_q\|_{L^p} \leq \sum_{\beta\leq\alpha} C^\alpha_\beta 2^{q|\beta|} C_\beta(f,\|u\|_{L^\infty}) 2^{q|\alpha|-|\beta|)}\|\Delta_q u\|_{L^p}$$

This yields to

$$\begin{aligned}\|\partial^\alpha f_q\|_{L^p} &\leq_\alpha (f,\|u\|_{L^\infty})2^{q|\alpha|}\|\Delta_q u\|_{L^p}\\ &\leq c_{q\alpha}(f,\|u\|_{L^\infty})2^{-q(s-|\alpha|}\|u\|_{B^{p,r}_s}.\end{aligned}$$

This achieves the proof. $\square$

2.1.2.2 Hölder spaces

Now, we intend to recall the notion of Hölder spaces, witch we used repeatedly during this thesis. For more details, we refer to [5, 19]. Firstly, let us remember the classical definition of Hölder spaces.

Definition 2.1.17. *Let $r \in \mathbb{R}_+ \backslash \mathbb{N}$*

(i) *When $0 < r < 1$, the Hölder space denoted by C^r is the set of all bounded functions u in $\mathbb{R}^d$, such that there exists a constant C, for all x and y in $\mathbb{R}^d$ we have*

$$|u(x) - u(y)| \leq C|x-y|^r.$$

(ii) *When $r > 1$, the Hölder space C^r is the set of all functions u such that for all multi-index α with $|\alpha| \leq [r]$, we have*

$$\partial^\alpha u \in C^{r-[r]}$$

equipped with the norm

$$\widetilde{\|}u\|_r = \Sigma_{|\alpha|\leq[r]}\left(\|\partial^\alpha u\|_{L^\infty} + \sup_{x\neq y}\frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{r-[r]}}\right).$$

The first thing to do is to characterize the Hölder spaces already invoked for $r \in \mathbb{R}_+ \backslash \mathbb{N}$ by means of cut-off operators. Specifically, we have

Proposition 2.1.18. **(i)** *There exists a constant C, such that for all $r \in \mathbb{R}_+ \backslash \mathbb{N}$ and for all function u in C^r, we have*

$$\|u\|_{C^r} \triangleq \sup_q 2^{qr}\|\Delta_q u\|_{L^\infty} \leq \frac{C^{r+1}}{[r]!}\widetilde{\|}u\|_{C^r}$$

(ii) *Let B a ball in $\mathbb{R}^d$ and $u \in \mathscr{S}'$ with $u = \sum_q u_q$, $\quad \operatorname{supp}\, \widehat{u_q} \subset 2^q B$. There exists a constant C such that, if the sequence $(2^{qr}\|u_q\|_{L^\infty})_{q\in\mathbb{N}}$ is bounded, then*

$$\widetilde{\|}u\|_{C^r} \leq C\left(\frac{1}{r-[r]} + \frac{1}{[r]+1-r}\right)\|u_q\|_{C^r}.$$

Proof. To prove **(i)**, we use the integral definition of Δ_q

$$\Delta_q u(x) = 2^{qd} \int_{\mathbb{R}^d} h\big(2^q(x-y)\big)u(y)dy.$$

Since $h = \mathcal{F}^{-1}(\phi)$ with ϕ identically vanishes in a neighborhood of the origin, thus we have

$$\int_{\mathbb{R}^d} x^\alpha h(x)dx = 0.$$

It follows that

$$\Delta_q u(x) = 2^{qd} \int_{\mathbb{R}^d} h\big(2^q(x-y)\big)\big(u(y) - \sum_{k=1}^{[r]} \frac{1}{k!} D^k u(x)(y-x)^{(k)}\big)dy. \tag{2.7}$$

By a Taylor formula of order $[r]$, one gets

$$u(y) - \sum_{k=1}^{[r]} \frac{1}{k!} D^k u(x)(y-x)^{(k)} = \int_0^1 \frac{(1-t)^{[r]-1}}{[r-1]!}\big(D^{[r]}u(x+t(y-x)) - D^{[r]}u(x)\big)\cdot(y-x)^{([r])}dt.$$

The fact that $\partial^\alpha u$ belongs to $C^{r-[r]}$, gives

$$|u(y) - \sum_{k=1}^{[r]} \frac{1}{k!} D^k u(x)(y-x)^{(k)}| \le \frac{C}{[r]!}|y-x|^r \widetilde{\|u\|}_{C^r}.$$

This implies that

$$|\Delta_q u(x)| \le \frac{C}{[r]!} 2^{qd} \widetilde{\|u\|}_{C^r} \int_{\mathbb{R}^d} |x-y|^r h(2^q(x-y))|dy.$$

The proof of **(ii)** is deeply based on the remark that $\|u_q\|_{L^\infty} \le C2^{-qr}$. This yields, by means of Bernstein's lemma, that the sequence $(\partial^\alpha u_q)_{q\in\mathbb{N}}$ converges in L^∞, so for any $\alpha \in \mathbb{N}^d$ such that $|\alpha| \le [r]$, we get

$$\partial^\alpha u \in L^\infty, \quad \|\partial^\alpha u\|_{L^\infty} \le C \sup_{q\in\mathbb{N}} 2^{qr}\|u_q\|_{L^\infty}. \tag{2.8}$$

To reach the proof, we must study the case of partial derivatives of order $[r]$. To do this, we explore the following frequential decomposition.

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \le \underbrace{\sum_{q<N} |\partial^\alpha u_q(x) - \partial^\alpha u_q(y)|}_{\text{(I)}} + \underbrace{\sum_{q\ge N} |\partial^\alpha u_q(x) - \partial^\alpha u_q(y)|}_{\text{(II)}},$$

where N is a parameter that will be chosen later. To get bound term **(I)** in the right-hand side, we apply the mean value theorem, thus, we deduce that

$$|\partial^\alpha u_q(x) - \partial^\alpha u_q(y)| \leq C|x-y| \sup_{|\beta|=[r]+1} \|\partial^\beta u_q\|_{L^\infty}.$$

Using again Bernstein's lemma, it holds

$$|\partial^\alpha u_q(x) - \partial^\alpha u_q(y)| \leq C|x-y|2^{-q(r-[r]-1)} \sup_{q\in\mathbb{N}} 2^{qr}\|u_q\|_{L^\infty}. \tag{2.9}$$

For the second term **(II)**, we write

$$|\partial^\alpha u_q(x) - \partial^\alpha u_q(y)| \leq 2^{1-qr} \sup_{q\in\mathbb{N}} 2^{qr}\|u_q\|_{L^\infty}.$$

By virtue of (2.9), we achieve that

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq C(\sup_{q\in\mathbb{N}} 2^{qr}\|u_q\|_{L^\infty})\big(\sum_{q=0}^{N} 2^{-q(r-[r]-1)}|x-y| + \sum_{q\geq N} 2^{-q(r-[r])}\big).$$

Exploiting (2.8), we can suppose that $|x-y| \leq 1$. By choosing

$$N = [-log_2|x-y|] + 1,$$

this completes the proof. □

In general case, where $r \in \mathbb{R}$, the Hölder spaces are defined as follows

Definition 2.1.19. *Let $r \in \mathbb{R}$. The Hölder space C^r is the set of all $u \in \mathscr{S}'$ such that*

$$\|u\|_{C^r} \triangleq \sup_q 2^{qr}\|\Delta_q u\|_{L^\infty} < +\infty.$$

Let us denote that these spaces are Banach spaces equipped with the norm $\|\cdot\|_{C^r}$.

Proposition 2.1.20. *Let $\widetilde{\mathcal{C}}$ be an annulus in $\mathbb{R}^d$. There exists a constant C, such that for a given real r and a sequence $(u_q)_{q\in\mathbb{N}}$ converges towards u in $\mathscr{S}'$ with* $\operatorname{supp}\, \widehat{u}_q \subset 2^q\widetilde{\mathcal{C}}$, *we have*

$$\sup_q 2^{qr}\|u_q\|_{L^\infty} < +\infty \Leftrightarrow \quad u \in C^r, \|u\|_{C^r} \leq C^{|r|+1} \sup_q 2^{qr}\|u_q\|_{L^\infty}.$$

Proof. First we bound $\|u_q\|_{L^\infty}$. From the hypothesis on the support of $\widehat{u}_q$, there exists an integer N such that

$$\Delta_q u = \sum_{q/|p-q|\leq N} \Delta_p u_q.$$

Since $\Delta_p : L^\infty \to L^\infty$ is continuous, it follows that

$$\|\Delta_p u_q\|_{L^\infty} \leq \sum_{q/|p-q|\leq N} \|u_q\|_{L^\infty}.$$

Even more, $2^q \leq C2^p$ because $|p-q| \leq N$. This gives the desired estimation. $\square$

Remark 2.1.21. *Since r is an integer, the obtained space is denoted sometimes by $C^r_\star$ and called Hölder-Zygmund space.*

We end this subsection by the relationship between Besov spaces $B^s_{p,r}$ and the so-called bounded variations spaces still denoted BV. For this purpose, we embark with the following definition.

Definition 2.1.22. *The space of functions of bounded variations denoted $BV(\mathbb{R}^d)$ (or simply BV), is the space of all functions $u \in L^1$ whose first order partial derivatives in the distributional sense are finite signed Random measures, that is to say, for all $i = 1, \cdots, d$, there exists a finite signed measure $\lambda_i : \mathscr{B}(\mathbb{R}^d) \to \mathbb{R}$ such that for all $\phi \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} u \frac{\partial \phi}{\partial x_i} dx = -\int_{\mathbb{R}^d} \phi d\lambda_i.$$

The measure λ_i is called the weak, or distributional, partial derivative of u with respect to x_i and is denoted $D_i u$. Let us setting

$$Du = (D_1 u, \cdots, D_d u), \quad \forall u \in BV.$$

Thus, if $u \in BV$, then $Du \in \mathcal{M}_b(\mathbb{R}^d; \mathbb{R}^d)$, then the total variation measure of Du, defined by

$$|Du|(E) = \sup\{\sum_{j\geq 1} |Du(E_j)|\}, \quad E \in \mathscr{B}(\mathbb{R}^d),$$

where the supremum is taken over all partitions $\{E_j\} \subset \mathscr{B}(\mathbb{R}^d)$ of E, is a finite Radon measure with $\mathscr{B}(\mathbb{R}^d)$ denotes a Borel $\sigma-$algebra. Furthermore, we have

$$|Du|(\mathbb{R}^d) = \|Du\|_{\mathcal{M}_b(\mathbb{R}^d;\mathbb{R}^d)} = \sup\Big\{\sum_{i=1}^d \int_{\mathbb{R}^d} \phi_i dD_i u : \phi \in C_0(\mathbb{R}^d;\mathbb{R}^d), \|\phi\|_{C_0(\mathbb{R}^d;\mathbb{R}^d)} \leq 1\Big\} < \infty.$$

Definition 2.1.23. *Let $u \in L^1_{loc}(\mathbb{R}^d)$. Define the variation of u in $\mathbb{R}^d$ as follows*

$$V(u,\mathbb{R}^d) = \sup\Big\{ \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial \phi_i}{\partial x_i} u dx : \quad \phi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d), \|\phi\|_{C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)} \leq 1 \Big\}.$$

The motivation to state the spaces BV is for the purpose to uphold the statement of the following axle result used in the next chapters. Especially we have

Proposition 2.1.24. *Let Ω be a bounded domain which its boundary is a Jordan curve of $C^{1+\varepsilon}-$regularity, with $0 < \varepsilon < 1$. Then $\mathbf{1}_\Omega \in L^\infty \cap BV$.*

Before giving the proof, let us firstly characterize Besov spaces by means of bounded variations spaces. For that, given a function $u : \mathbb{R}^d \to \mathbb{R}$, for every $h \in \mathbb{R}$, $i = 1, \cdots, d$ and $x \in \mathbb{R}^d$, we define

$$\mathbf{\Delta}_i^h u(x) = u(x + he_i) - u(x),$$

where e_i is the ith vector of canonical basis in $\mathbb{R}^d$. If $d = 1$, we write $\Delta^h u = \Delta_1^h u$.

Definition 2.1.25. *Let $0 < s < 1$ and $(p,r) \in [1,\infty]^2$. A function $u \in L^1_{loc}(\mathbb{R}^d)$ belongs to the Besov space $B^s_{p,r}$ if and only if the following norm is finite.*

$$\|u\|_{B^s_{p,r}} = \|u\|_{L^p} + |u|_{B^s_{p,r}} < \infty,$$

with

$$|u|_{B^s_{p,r}} \triangleq \begin{cases} \sum_{i=1}^d \Big(\int_0^{+\infty} \|\mathbf{\Delta}_i^h u\|_{L^p}^r \frac{dh}{h^{1+sr}} \Big)^{\frac{1}{r}} & \text{if } r < \infty, \\ \sum_{i=1}^d \sup_{h>0} \frac{1}{h^s} \|\mathbf{\Delta}_i^h u\|_{L^p} & \text{if } r = \infty. \end{cases}$$

We mention that $B^s_{p,r}$ is a Banach space. For a proof see [59] page 416.
Worthwhile results relates Besov spaces with bounded variations spaces are announced bellow.

Theorem 2.1.26. *Let $0 < t < s < 1$ and $(p,r) \in [1,\infty]^2$. Then, there exists a positive constant $C = C(t,r)$ such that for all $u \in B^s_{p,r}$, we have*

$$|u|_{B^t_{p,r}} \leq |u|_{B^s_{p,r}} + C\|u\|_{L^p}.$$

In particular $B^s_{p,r} \subset B^t_{p,r}$.

Proof. We distinguish two cases.

• *First case:* $1 \leq r < +\infty$. For $0 < h < 1$ we have $h^s < h^t$. Then, we can write

$$\begin{aligned}\Big(\int_0^\infty \|\Delta_i^h u\|_{L^p}^r \frac{dh}{h^{1+tr}}\Big)^{\frac{1}{r}} &\leq \Big(\int_0^1 \|\Delta_i^h u\|_{L^p}^r \frac{dh}{h^{1+tr}}\Big)^{\frac{1}{r}} + \Big(\int_1^\infty \|\Delta_i^h u\|_{L^p}^r \frac{dh}{h^{1+tr}}\Big)^{\frac{1}{r}}\\ &\leq \Big(\int_0^1 \|\Delta_i^h u\|_{L^p}^r \frac{dh}{h^{1+sr}}\Big)^{\frac{1}{r}} + 2\|u\|_{L^p}\Big(\int_0^\infty \frac{dh}{h^{1+tr}}\Big)^{\frac{1}{r}},\end{aligned}$$

where in the last term of the right-hand side we have used the definition of Δ_i^h and the convergence of the integral $\int_0^\infty \frac{dh}{h^{1+tr}}$. Thus, we get the desired estimate for this case.

• *Second case:* $r = +\infty$. For the proof, we must replace the integral by supremum. That is to say

$$\sup_{h>0} \frac{1}{h^t}\|\Delta_i^h u\|_{L^p} \leq \sup_{0<h<1} \frac{1}{h^t}\|\Delta_i^h u\|_{L^p} + \sup_{h\geq 1} \frac{1}{h^t}\|\Delta_i^h u\|_{L^p} \leq \sup_{0<h<1} \frac{1}{h^s}\|\Delta_i^h u\|_{L^p} + 2\|u\|_{L^p}.$$

This accomplishes the proof. □

Furthermore, we have the following embedding

Theorem 2.1.27. *Let $0 < s < 1$ and$(p,r) \in [1,\infty]^2$. Then, there exists a constant $C = C(s,r)$ such that for all $u \in W^{1,p}$, we have*

$$|u|_{B^s_{p,r}} \leq C\|u\|_{W^{1,p}}.$$

In particular $W^{1,p} \subset B^s_{p,r}$.

Proof. For $1 \leq r < \infty$, we remark that

$$\begin{aligned}\Big(\int_0^\infty \|\Delta_i^h u\|_{L^p}^r \frac{dh}{h^{1+sr}}\Big)^{\frac{1}{r}} &\leqslant \|\frac{\partial u}{\partial x_i}\|_{L^p}\Big(\int_0^1 \frac{dh}{h^{1+(s-1)r}}\Big)^{\frac{1}{r}} + 2\|u\|_{L^p}\Big(\int_1^\infty \frac{dh}{h^{1+sr}}\Big)^{\frac{1}{r}}\\ &\leqslant \Big(\frac{1}{(1-s)r}\Big)^{\frac{1}{r}}\|\frac{\partial u}{\partial x_i}\|_{L^p} + 2\Big(\frac{1}{sr}\Big)^{\frac{1}{r}}\|u\|_{L^p}.\end{aligned}$$

If $r = \infty$, we infer that

$$\begin{aligned}\sup_{h>0} \frac{1}{h^s}\|\Delta_i^h u\|_{L^p} &\leqslant \sup_{0<h<1} \frac{1}{h^{s-1}}\|\frac{\partial u}{\partial x_i}\|_{L^p} + \sup_{h\geq 1} \frac{1}{h^s}\|\Delta_i^h u\|_{L^p}\\ &= \|\frac{\partial u}{\partial x_i}\|_{L^p} + 2\|u\|_{L^p}.\end{aligned}$$

□

For $p = 1$, we have the following strong result, that we are going to use in Chapter three (see proposition 3.5.2).

Theorem 2.1.28. *Let $0 < s < 1$ and $1 \leq r \leq +\infty$. Then, there exists a positive constant $C = C(s, r)$ such that for all $u \in BV$, we have*

$$\|u\|_{B^s_{1,r}} \leq C\|u\|_{BV}.$$

In particular $BV \subset B^s_{1,r}$.

Proof. The proof can be done in the same way as the previous theorem, with the following difference

$$\|\Delta^h_i u\|_{L^1} \leq h|D_i u|(\mathbb{R}^d),$$

where the last estimate is deduced from the following result giving in [59] page 413:

$$\int_{\mathbb{R}^d} |u(x + he_i) - u(x)dx| \leq h|D_i u|(\mathbb{R}^d)$$

and

$$\lim_{h\to 0^+} \int_{\mathbb{R}^d} \frac{u(x + he_i) - u(x)}{h} dx = |D_i u|(\mathbb{R}^d),$$

for all $u \in BV$.
Conversely, if $u \in L^1$, such that

$$\lim \inf_{h\to 0^+} \int_{\mathbb{R}^d} \frac{u(x + he_i) - u(x)}{h} dx < \infty,$$

then $u \in BV$. □

2.2 Regularity for transport-diffusion and transport equations

This section gives details on the transport-diffusion and transport equations, in particular, the regularity persistence and the maximal regularity in Besov spaces. Firstly, we embark on the following proposition which deals with the persistence of Besov regularities in a transport-diffusion regime.

Proposition 2.2.1. *Let $(s, r, p) \in]-1, 1[\times[1, \infty]^2$, $f_0 \in B^s_{p,r}$ and $g \in L^1_{loc}(\mathbb{R}_+; B^s_{p,r})$. Let v be a smooth free-divergence vector field and f be a smooth solution of the* *transport-diffusion equation*

$$\begin{cases} \partial_t f + v\cdot\nabla f - \mu\Delta f = g \\ f_{|t=0} = f_0. \end{cases} \tag{TD$_\mu$}$$

Then for every $t \geq 0$ we have

$$\|f(t)\|_{B^s_{p,r}} \leq Ce^{CV(t)}\Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|g(\tau)\|_{B^s_{p,r}}d\tau\Big),$$

with

$$V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau$$

and C a constant which depends only on s and not on the viscosity. For the limit case

$$s = -1, r = \infty \text{ and } p \in [1,\infty] \quad \text{or} \quad s = 1, r = 1 \text{ and } p \in [1,\infty]$$

the above estimate remains true despite, we change $V(t)$ by $Z(t) \overset{def}{=} \|v\|_{L^1_t B^1_{\infty,1}}$. In addition if $f = \mathrm{curl}\, v$, then the above estimate holds true for all $s \in [1, +\infty[$.

Proof. We will only restrict ourselves to the proof of the limiting cases $s = \mp 1$. The rest cases are done for example in [5]. To begin with, we localize the equation (TD$_\mu$) via the operator Δ_q. Let $f_q \triangleq \Delta_q$, $g_q \triangleq \Delta_q g$. Then, they satisfy

$$\begin{aligned} \partial_t f_q + (v\cdot\nabla)f_q - \nu\Delta f_q &= g_q + (v\cdot\nabla)f_q - (v\cdot\nabla f_q) \\ &= {g_q}^2 - [\Delta_q, v\cdot\nabla]f. \end{aligned}$$

Multiply the previous equation by $|f_q|^{p-2}f_q$, so Hölder's inequality yields

$$\|f_q(t)\|_{L^p} \leq \|f_q^0\|_{L^p} + \int_0^t \|g_q(\tau)\|_{L^p}d\tau + \int_0^t \Big\|[\Delta_q, v(\tau)\cdot\nabla]f(\tau)\Big\|_{L^p}d\tau.$$

The famous Bony's decomposition, see definition 2.1.10 allows us to write

$$\begin{aligned} [\Delta_q, v\cdot\nabla]f &= \Delta_q\mathscr{R}(v^j, \partial_j f) + \Delta_q T_{\partial_j f}v^j - T'_{\Delta_q\partial_j f}u^j + [\Delta_q, T_{u^j}]\partial_j f \\ &:= \sum_{i=1}^4 \mathscr{M}_q^i, \end{aligned}$$

where $T'_v u$ stands for $T_v u + \mathscr{R}(v,u)$. Here, we have used Enstein's convention for the summation over the repeated indices. We treat the first quantity $\mathscr{M}_q^1$ by exploring the definition 2.1.10 to state

$$\mathscr{M}_q^1 = \sum_{k\geq q-3} \Delta_q\partial_j(\Delta_k f\widetilde{\Delta}_k v^j).$$

By means of Bernstein lemma 4.2.7, it follows for $s=-1$ that

$$\sup_{q\geq -1} 2^{-q}\|\mathscr{M}_q^1\|_{L^p} \lesssim \|f\|_{B^{-1}_{p,\infty}}\|v\|_{B^1_{p,\infty}} \tag{2.10}$$

To estimate $\mathscr{M}_q^2$, we write

$$\mathscr{M}_q^2 = \Delta T_{\partial_j f}v^j = \sum_{|q-k|\leq 4} \Delta_q(S_{k-1}\partial_j f\Delta_k v^j).$$

The Bernstein and Young inequalities lead to

$$\begin{aligned}
\sup_q 2^{-q}\|\mathscr{M}_q^2\|_{L^p} &\lesssim \sup_q 2^{-q}\|S_{q-1}f\|_{L^p}2^q\|\Delta_q v^j\|_{L^p} \\
&\lesssim \|v\|_{B^1_{\infty,\infty}}\sup_q \sum_{-1\leq m\leq q-2} 2^{m-q}2^{-m}\|\Delta_m f\|_{L^p} \\
&\lesssim \|f\|_{B^{-1}_{p,\infty}}\|v\|_{B^1_{\infty,\infty}}.
\end{aligned} \tag{2.11}$$

Concerning the member $\mathscr{M}_q^3$, we obviously check that it can be rewritten as follows

$$\mathscr{M}_q^3 = T_{\Delta_q\partial_j f}v^j = \sum_{k\geq q-2} S_{k+2}\Delta_q\partial_j\Delta_k v^j.$$

We apply once again the Bernstein inequality, we shall have

$$2^{-q}\|\mathscr{M}_q^3\|_{L^\infty} \lesssim 2^{-q}\|\Delta_q\|_{L^p}\sum_{k\geq q-2} 2^{q-k}2^k\|\Delta_k v\|_{L^\infty},$$

Therefore the convolution inequality yields

$$\sup_{q\geq -1} 2^{-q}\|\mathscr{M}_q^3\|_{L^p} \lesssim \|f\|_{B^{-1}_{p,\infty}}\|v\|_{B^1_{p,\infty}}. \tag{2.12}$$

For the last member we write

$$\mathscr{M}_q^4 = [\Delta_q, T_{v^j}]\partial_j f = \sum_{|k-q|\leq 4} [\Delta_q, S_{k-1}v^j]\Delta_k\partial_j f.$$

The following estimate is classical (see for example [19]),

$$\begin{aligned}\|[\Delta_q, S_{k-1}v^j]\Delta_k\partial_j f\|_{L^\infty} &\lesssim 2^{-q}\|\nabla S_{k-1}v\|_{L^\infty}\|\partial_j\Delta_k f\|_{L^p}\\ &\lesssim 2^{k-q}\|\nabla v\|_{L^\infty}\|\Delta_k f\|_{L^p}.\end{aligned}$$

This implies

$$\sup_{q\geq -1} 2^{-q}\|\mathscr{M}_q^4\|_{L^\infty} \lesssim \|f\|_{B^s_{p,\infty}}\|\nabla v\|_{L^\infty}. \tag{2.13}$$

Gathering the estimates (2.10), (2.11), (2.12) and (2.13) to obtain

$$\sup_{q\geq -1} 2^{-q}\Big\|[\Delta_q, v\cdot\nabla]f\Big\|_{L^p} \lesssim \|f\|_{B^{-1}_{p,\infty}}\|v\|_{B^1_{p,\infty}}.$$

This implies

$$\|f(t)\|_{\mathscr{B}^{-1}_{\infty,\infty}} \lesssim \|f_0\|_{\mathscr{B}^s_{\infty,\infty}} + \int_0^t \|g(\tau)\|_{\mathscr{B}^{-1}_{p,\infty}}d\tau + \int_0^t \|f(\tau)\|_{\mathscr{B}^{-1}_{p,\infty}}\|v(\tau)\|_{\mathscr{B}^1_{p,1}}d\tau.$$

To conclude the desired result it suffices to apply Gronwall's inequality.
Let us now move to the case $s=1$ which will briefly explained. We estimate $\mathscr{M}_q^1$ as follows

$$\begin{aligned}\sum_q 2^q\|\mathscr{M}_q^1\|_{L^p} &\lesssim \sum_{k\geq q-3} 2^{q-k}2^k\|\Delta_k f\|_{L^p}2^k\|\widetilde{\Delta}_k v^j\|_{L^\infty}\\ &\lesssim \|f\|_{B^1_{p,1}}\|v\|_{B^1_{\infty,\infty}}.\end{aligned}$$

For the second member we have

$$\begin{aligned}\sum_q 2^q\|\mathscr{M}_q^2\|_{L^p} &\lesssim \sum_q 2^q\|S_{q-1}\partial_j f\|_{L^p}\|\Delta_q v^j\|_{L^\infty}\\ &\lesssim \|\nabla f\|_{L^p}\|v\|_{\mathscr{B}^1_{\infty,1}}\\ &\lesssim \|f\|_{B^1_{p,1}}\|v\|_{B^1_{\infty,1}}.\end{aligned}$$

Similarly, the third and last members will done like the first one. □

Another important result was founded an enormous applications in two last chapters cares with the maximal smoothing effect regularity for equation (TD_μ) in mixed space-time spaces $\widetilde{L}^r_t B^s_{p_1,p_2}$ and reads as follows.

Theorem 2.2.2. *Let $(p, p_1, r, \theta_1) \in [1, +\infty]^4$ with $p \leq p_1$. Let $s \in \mathbb{R}$ satisfy*

$$\begin{cases} s < 1 + \frac{d}{p_1} \quad or \quad s \leq 1 + \frac{d}{p_1}, \quad if \quad r = 1, \\ s > -d \min\left\{\frac{1}{p_1}, \frac{1}{p'}\right\} \quad or \quad s > -1 - d\left\{\frac{1}{p_1}, \frac{1}{p'}\right\}, \quad if \quad \operatorname{div} v = 0. \end{cases} \tag{2.14}$$

There exists a constant C, depending only on d, r, s, and $s - 1 - \frac{d}{p_1}$, such that for any smooth solution f of (TD$_\mu$) with $\mu \geq 0$, and $\theta \in [\theta_1, \infty]$, the following a priori estimate holds true.

$$\mu^{\frac{1}{\theta}} \|f\|_{\widetilde{L}^\theta_t \dot{B}^{s+\frac{2}{\theta}}_{p,r}} \leq C e^{CV_{p_1}(t)} \Big(\|f^0\|_{\dot{B}^s_{p,r}} + \mu^{\frac{1}{\theta_1}-1} \|g\|_{\widetilde{L}^{\theta_1}_t \dot{B}^{s-2+\frac{2}{\theta_1}}_{p,r}} \Big), \quad \forall t \in \mathbb{R}_+, \tag{2.15}$$

with

$$\begin{cases} V_{p_1}(t) \triangleq \int_0^t \|\nabla v(\tau)\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,\infty} \cap L^\infty} d\tau \quad if\ s < \frac{d}{p_1} + 1, \\ V_{p_1}(t) \triangleq \int_0^t \|\nabla v(\tau)\|_{\dot{B}^{\frac{d}{p_1}}_{p_1,1} \cap L^\infty} d\tau \quad if\ s = \frac{d}{p_1} + 1, \end{cases}$$

Proof. The proof will be done in the spirit of [5, 46]. Roughly speaking, it consists first in localizing in frequency the evolution equation and second in rewriting the equation in Lagrangian coordinates. This will lead to some technical difficulties, especially, when we have to treat a commutator term coming from the commutation between the Laplacian and the regularized flows. To start with, let $f_q = \dot{\Delta}_q f$ and $g_q = \dot{\Delta}_q g$, thereafter we localize (TD$_\mu$) by applying $\dot{\Delta}_q$ to get

$$(\partial_t + \dot{S}_{q-1} v \cdot \nabla - \mu\Delta) f_q = g_q + (\dot{S}_{q-1} v - v) \cdot \nabla f_q - [\dot{\Delta}_q, v \cdot \nabla] f = A_q.$$

Let Ψ_q be the flow of the regularized velocity vector field $\dot{S}_{q-1} v$ defined by

$$\Psi_q(t, x) = x + \int_0^t \dot{S}_{q-1} v(\tau, \Psi_q(\tau, x)) d\tau. \tag{2.16}$$

Set $\bar{f}_q(t, x) = f_q(t, \Psi_q(t, x)), \bar{g}_q(t, x) = g_q(t, \Psi_q(t, x))$ and $\bar{A}_q(t, x) = A_q(t, \Psi(t, x))$. An elementary calculus gives

$$(\partial_t - \mu\Delta) \bar{f}_q = \bar{g}_q + \bar{A}_q + \mu B_q, \quad B_q = ((\Delta f_q)(t, \Psi(t, x)) - \Delta \bar{f}_q). \tag{2.17}$$

At this stage of the proof one can remark that the function $\bar{f}_q$ is not necessarily localized in frequency. Thus in order to quantify the smoothing effects we need once again to localize (2.17). Now, let $j \in \mathbb{N}$ then applying the cut-off operator $\dot{\Delta}_j$ to the equation (2.17) gives

$$(\partial_t - \mu\Delta) \dot{\Delta}_j \bar{f}_q = \dot{\Delta}_j \bar{g}_q + \dot{\Delta}_j A_q + \mu \dot{\Delta}_j B_q = \dot{\Delta}_j \bar{B}_{q,j}. \tag{2.18}$$

Let us denote that the previous equation can be seen as a particular heat equation, so, in view of

$$\|\dot{\Delta}_j e^{-\mu t\Delta} h\|_{L^p} \leq e^{-c\mu t 2^{2j}} \|\dot{\Delta}_j h\|_{L^p} \tag{2.19}$$

we end up with

$$\|\dot{\Delta}_j \bar{f}_q\|_{L^p} \leq C e^{-c\mu t 2^{2j}} \|\dot{\Delta}_j \bar{f}_q\|_{L^p} + C\int_0^t e^{-c\mu(t-\tau)2^{2j}} \|\bar{B}_{q,j}(\tau)\|_{L^p} d\tau.$$

To treat the quantity $\|\bar{B}_{q,j}(\tau)\|_{L^p}$, we write

$$\|\bar{B}_{q,j}(\tau)\|_{L^p} \leq \|\dot{\Delta}_j \bar{g}_q(\tau)\|_{L^p} + \|\dot{\Delta}_j \bar{A}_q(\tau)\|_{L^p} + \mu\|\dot{\Delta}_j B_q\|_{L^p}. \tag{2.20}$$

We first start on the term $\|\dot{\Delta}_j \bar{A}_q(\tau)\|_{L^p}$. Based on the formula (2.16) and a serious calculus lead to

$$\begin{aligned} \bar{A}_q &= \Delta f_q - \mathrm{Tr}\big(\nabla\Psi_q \cdot D^2 f_q \circ \Psi_q \cdot D\Psi_q\big) - Df_q \circ \Psi_q \cdot \Delta\Psi_q \\ &= \mathrm{Tr}\big((\mathbb{I} - \nabla\Psi_q) \cdot D^2 f_q \circ \Psi_q \cdot D\Psi_q\big) - \mathrm{Tr}\big(D^2 f_q \circ \Psi_q \cdot (\mathbb{I} - D\Psi_q)\big) - Df_q \circ \Psi_q \cdot \Delta\Psi_q, \end{aligned} \tag{2.21}$$

with in general case DF represents the Jacobian matrix of F, and ∇F denotes the transposed matrix of DF. If F has a d components, we thus set $J_F \triangleq \det DF$. Consequently,

$$\|\dot{\Delta}_j \bar{A}_q\|_{L^p} \leq C\big(\|D\Psi_q\| + 1\big)\|\mathbb{I} - D\Psi_q\|_{L^\infty}\|D^2 f_q \circ \Psi_q\|_{L^p} + C\|\Delta\Psi_q\|_{L^\infty}\|Df_q \circ \Psi_q\|_{L^p}.$$

In accordance with Bernstein's inequality combined with an adequate change of variable when computing the L^p-norm guided to

$$\begin{aligned} \|Df_q \circ \Psi_q\|_{L^p} &\leq C2^q \big\|J_{\Psi_q^{-1}}\big\|_{L^\infty}^{\frac{1}{p}} \|f_q\|_{L^p}, \\ \|D^2 f_q \circ \Psi_q\|_{L^p} &\leq C2^{2q} \big\|J_{\Psi_q^{-1}}\big\|_{L^\infty}^{\frac{1}{p}} \|f_q\|_{L^p}. \end{aligned}$$

Since the flows Ψ_q and Ψ_q^{-1} satisfy the classical estimates

$$\begin{aligned} \|D\Psi_q^{\pm 1}(t)\|_{L^\infty} &\leq e^{V(t)}, \\ \|D\Psi_q^{\pm 1}(t) - \mathbb{I}\|_{L^\infty} &\leq e^{V(t)} - 1, \\ \|D^2\Psi_q^{\pm 1}(t)\|_{L^\infty} &\leq e^{V(t)} \int_0^t \|D^2 v(\tau)\|_{L^\infty} e^{V(\tau)} d\tau, \end{aligned} \tag{2.22}$$

combined with Bernstein's inequality

$$\|\dot{\Delta}_j \bar{A}_q\|_{L^p} \leq C e^{2q}\Big(e^{CV(t)} - 1\Big)\|f_q\|_{L^p}. \tag{2.23}$$

For the term $\|\dot{\Delta}_j\bar{g}_q(\tau)\|_{L^p}$, we employ once again Bernstein's inequality to find

$$\|\dot{\Delta}_j\bar{g}_q(\tau)\|_{L^p}\approx 2^{-j}\|\dot{\Delta}_j D\bar{g}_q(\tau)\|_{L^p}.$$

We also have $D\bar{g}_q = Dg_q\circ\Psi_q\cdot D$. Again Bernstein's inequality and (2.22) give

$$\|\dot{\Delta}_j\bar{g}_q\|_{L^p}\le Ce^{CV(t)}2^{q-j}\|\bar{g}_q\|_{L^p}. \tag{2.24}$$

Step by step, we also get

$$\|\dot{\Delta}_j B_q\|_{L^p}\le Ce^{CV(t)}2^{q-j}\|B_q\|_{L^p}. \tag{2.25}$$

On the other hand, always Bernstein's inequality implies

$$\begin{aligned}\|(\dot{S}_{q-1}v-v)\cdot\nabla f_q\|_{L^p} &\le C2^q\|\dot{S}_{q-1}v-v\|_{L^\infty}\|f_q\|_{L^p}\\ &\le C\sum_{q'\ge q}2^{q-q'}\|\nabla\Delta_{q'}v\|_{L^\infty}\|f_q\|_{L^p}\\ &\le C\|\nabla v\|_{B^0_{\infty,\infty}}\|f_q\|_{L^p},\end{aligned}$$

combined with (2.25), we eventually get

$$\|\dot{\Delta}_j B_q\|_{L^p}\le Ce^{CV(t)}2^{q-j}c_q(t)2^{-qs}V_{p_1}(t)\|f(t)\|_{B^s_{p,r}}, \tag{2.26}$$

with $\|c_j(t)\|_{\ell^r}=1$ and V_{p_1} as defined in the statement of Theorem 2.2.2.

Putting together (2.23), (2.24) and (2.26) and plug them in (2.18), taking the $L^\theta-$norm over $[0,t]$, and multiplying by $\mu^{\frac{1}{\theta}}2^{2\frac{j}{\theta}}$ to obtain

$$\begin{aligned}\mu^{\frac{1}{\theta}}2^{2\frac{j}{\theta}}\|\dot{\Delta}_j\bar{f}_q\|_{L^\theta_tL^p} \le\ & C\Big(\|\dot{\Delta}_jf^0_q\|_{L^p}+2^{q-j}\mu^{-\frac{1}{\theta'_1}}2^{-2\frac{j}{\theta_1}}e^{CV(t)}\|g_q\|_{L^{\theta_1}_tL^p}\\ &+2^{2(q-q')}\mu^{\frac{1}{\theta}}e^{2\frac{j}{\theta}}\big(e^{CV(t)}-1\big)\|f_q\|_{L^\theta_tL^p}+2^{q-j}\int_0^t c_q(\tau)2^{-qs}V_{p_1}(\tau)e^{CV(\tau)}\|f\|_{B^s_{p,r}}d\tau\Big)\end{aligned}$$

with θ'_1 represents for the conjugate exponent of θ_1.

Multiplying the both sides of the last estimate by $2^{qs}2^{\frac{2}{\theta}(q-j)}$ to conclude that

$$\begin{aligned}\mu^{\frac{1}{\theta}}2^{q(s+\frac{2}{\theta})}\|\dot{\Delta}_j\bar{f}_q\|_{L^\theta_tL^p} \le\ & C\Big(2^{\frac{2}{\theta}(q-j)}2^{qs}\|\dot{\Delta}_jf^0_q\|_{L^p}+\mu^{-\frac{1}{\theta'_1}}2^{(1+\frac{2}{\theta}+\frac{2}{\theta'_1})(q-j)}e^{CV(t)}2^{q(s-\frac{2}{\theta'_1})}\|g_q\|_{L^{\theta_1}_tL^p}\\ &+2^{2(q-j)}\mu^{\frac{1}{\theta}}2^{q(s+\frac{2}{\theta})}\big(e^{CV(t)}-1\big)\|f_q\|_{L^\theta_tL^p}\\ &+2^{(1+\frac{2}{\theta})(q-j)}\int_0^t c_q(\tau)V_{p_1}(\tau)e^{CV(\tau)}\|f\|_{B^s_{p,r}}d\tau\Big).\end{aligned} \tag{2.27}$$

Now, let $N \in \mathbb{N}$ be a fixed number that will be chosen later. Since

$$f_q = \sum_{j \leq q-N} \dot{\Delta}_j \bar{f}_q \circ \Psi_q^{-1} + \sum_{j \leq q-N} \dot{\Delta}_j \bar{f}_q \circ \Psi_q^{-1}.$$

Because Ψ_q^{-1} preserves Lebesgue's measure then for $t \in [0,T]$ we get

$$\|f_q\|_{L_t^\theta L^p} \leq e^{CV(t)} \Big(\sum_{j \leq q-N} \|\dot{\Delta}_j \bar{f}_q\|_{L_t^\theta L^p} + \sum_{j \geq q-N} \|\dot{\Delta}_j \bar{f}_q\|_{L_t^\theta L^p} \Big).$$

To treat the term $\sum_{j\leq q-N} \|\dot{\Delta}_j \bar{f}_q\|_{L_t^\theta L^p}$, a Lemma 2.4 page 56 in [5] allows us for $t \in [0,T]$ to write

$$\sum_{j \leq q-N} \|\dot{\Delta}_j \bar{f}_q\|_{L^p} \leq C2^{-q} \big\|J_{\Psi_q^{-1}}\big\|_{L^\infty} \|f_q(t)\|_{L^p} \Big(\big\|DJ_{\Psi_q^{-1}}\big\|_{L^\infty} \big\|J_{\Psi_q}\big\|_{L^\infty} + 2^{q-N} \big\|D\Psi_q\big\|_{L^\infty} \Big).$$

In accordance with (2.22), the two quantities $J_{\Psi_q^{-1}}$ and J_{Ψ_q} are bounded by $e^{CV(t)}$. On the other hand

$$DJ_{\Psi_q^{-1}} \cdot h = D\big(\det D\Psi_q^{-1}\big) \cdot h = \sum_{m=1}^{d} \det\big(D\Psi_{1,q}^{-1}, \ldots, D^2\Psi_{m,q}^{-1} \cdot h, \ldots, D\Psi_{d,q}^{-1}\big).$$

Again (2.22) furnishes

$$\begin{aligned}
\big\|DJ_{\Psi_q^{-1}}\big\|_{L^\infty} &\leq e^{CV(t)} \int_0^t \big\|D^2 S_{q-1} v(\tau)\big\|_{L^\infty} e^{CV(\tau)} d\tau \\
&\leq Ce^{CV(t)} 2^q \int_0^t \big\|\nabla v(\tau)\big\|_{L^\infty} e^{CV(\tau)} d\tau \\
&\leq Ce^{CV(t)} 2^q \big(e^{CV(t)} - 1\big).
\end{aligned}$$

Then it happens

$$\sum_{j \leq q-N} \|\dot{\Delta}_j \bar{f}_q\|_{L_t^\theta L^p} \leq Ce^{CV(t)} \big(2^{-N} + e^{CV(t)} - 1\big) \|f_q\|_{L_t^\theta L^p}. \tag{2.28}$$

Now, let us move to bound $\sum_{j\geq q-N} \|\dot{\Delta}_j \bar{f}_q\|_{L_t^\theta L^p}$. For this purpose, the fact $\Delta_j f_q^0 = 0$ for $|j-q| > 1$ and summing (2.27) over $j \geq q-N$, it holds

$$\begin{aligned}
\sum_{j \geq q-N} \mu^{\frac{1}{\theta}} 2^{q(s+\frac{2}{\theta})} \|\dot{\Delta}_j \bar{f}_q\|_{L_t^\theta L^p} \leq\;& C\Big(2^{qs} \|f_q^0\|_{L^p} + e^{CV(t)} 2^{3N} \mu^{-\frac{1}{\theta_1'}} 2^{q(s-\frac{2}{\theta_1'})} \|g_q\|_{L_t^{\theta_1} L^p} \\
&+ 2^{2N} \big(e^{CV(t)} - 1\big) \mu^{\frac{1}{\theta}} 2^{q(s+\frac{2}{\theta})} \|f_q\|_{L_t^\theta L^p} \\
&+ 2^{3N} \int_0^t c_q(\tau) V_{p_1}(\tau) e^{CV(\tau)} \|f\|_{B_{p,r}^s} d\tau \Big).
\end{aligned}$$

Plugging this and (2.28) into (2.27), we conclude that

$$\begin{aligned}\mu^{\frac{1}{\theta}}2^{q(s+\frac{2}{\theta})}\|f_q\|_{L^\theta_tL^p} &\leq Ce^{CV(t)}\Big(2^{qs}\|f^0_q\|_{L^p}+2^{3N}\mu^{-\frac{1}{\theta'_1}}2^{q(s-\frac{2}{\theta'_1})}\|g_q\|_{L^{\theta_1}_tL^p}\\ &\quad+\big(2^{-N}+2^{2N}\big(e^{CV(t)}-1\big)\big)\mu^{\frac{1}{\theta}}2^{q(s+\frac{2}{\theta})}\|f_q\|_{L^\theta_tL^p}\\ &\quad+2^{3N}\int_0^t c_q(\tau)V_{p_1}(\tau)e^{CV(\tau)}\|f(\tau)\|_{B^s_{p,r}}d\tau\Big).\end{aligned}$$

It is easy to check that there exists two absolute constants $N\in\mathbb{N}$ and $C_1>0$ such that

$$V(t)\leq C_1\Rightarrow\big(2^{-N}+2^{2N}\big(e^{CV(t)}-1\big)\big)\mu^{\frac{1}{\theta}}2^{q(s+\frac{2}{\theta})}\leq\frac{1}{2}.$$

Indeed, we start with taking t such that $V(t)\leq 1$, which is possible since $\lim_{t\to 0}V(t)=0$. Next, we choose N in order to have $2C2^{-N}\leq\frac{1}{4}$. Now, we take $V(t)$ sufficiently small such that $2^{2N}\big(e^{CV(t)}-1\big)\mu^{\frac{1}{\theta}}2^{q(s+\frac{2}{\theta})}\leq\frac{1}{4}$. This proves the above assertion. Under this assumption $V(t)\leq C_1$, one has

$$\begin{aligned}\mu^{\frac{1}{\theta}}2^{q(s+\frac{2}{\theta})}\|f_q\|_{L^\theta_tL^p} &\leq Ce^{CV(t)}\Big(2^{qs}\|f^0_q\|_{L^p}+\mu^{-\frac{1}{\theta'_1}}2^{q(s-\frac{2}{\theta'_1})}\|g_q\|_{L^{\theta_1}_tL^p}\\ &\quad+\int_0^t c_q(\tau)V_{p_1}(\tau)\|f(\tau)\|_{B^s_{p,r}}d\tau\Big).\end{aligned}$$

To close our claim, we perform an ℓ^r summation, we get for $t\in[0,T]$ and $\theta\in[\theta_1,\infty]$,

$$\mu^{\frac{1}{\theta}}\|f\|_{\widetilde{L}^\theta_t\dot{B}^{s+\frac{2}{\theta}}}\leq C\Big(\|f^0\|_{\dot{B}^s_{p,r}}+\mu^{-\frac{1}{\theta'_1}}\|g\|_{L^{\theta_1}_t\dot{B}^{s-\frac{2}{\theta'_1}}}+\int_0^t V_{p_1}(\tau)\|f(\tau)\|_{B^s_{p,r}}d\tau\Big)\tag{2.29}$$

This gives the desired result for small time. In order to get the estimate for arbitrary time $t>$, we consider a partition $(t_i)_{1\leq i\leq L}$ of $[0,t]$ such that

$$\int_{t_i}^{t_{i+1}}\|\nabla v(\tau)\|_{L^\infty}\approx C_1.$$

Then reproducing the same calculation to obtain the aimed result. This completes the proof of the Theorem. □

One can wonder if the estimates stated in Theorem 2.2.3 remain true in nonhomogeneous Besov spaces. On one hand, the block $\Delta_{-1}f$ corresponding to the low frequencies of f cannot be handled by mean of (2.19). On the other hand, if we take $f\cdot\nabla v=0$ in (TD_μ), one obtains the following simple heat equation

$$\begin{cases}\partial_t f-\mu\Delta f=g,\\ f_{|t=0}=f^0.\end{cases}$$

Thus, the solution of this equation is given by

$$f(t,x)=e^{\mu t\Delta}f^0(x)+\int_0^t e^{\mu(t-\tau)\Delta}g(\tau,x)d\tau.$$

It is not difficult to verify that

$$\|\Delta_{-1}f(t)\|_{L^p}\leq\|\Delta_{-1}f^0\|_{L^p}+\int_0^t\|\Delta_{-1}g(\tau)\|_{L^p}d\tau,$$

consequently, if $1\leq\theta\leq\theta_1\leq\infty$,

$$\|\Delta_{-1}f(t)\|_{L_t^{\theta_1}L^p}\leq t^{\frac{1}{\theta_1}}\|\Delta_{-1}f^0\|_{L^p}+t^{1+\frac{1}{\theta_1}-\frac{1}{\theta}}\|\Delta_{-1}g\|_{L_t^\theta L^p}.$$

Of course the other cut-off operators may be treated as in the homogeneous case, see Theorem 2.2.2.
We end up with the following statement.

Theorem 2.2.3. *Let $(p_1,p,r)\in[1,+\infty]^3$, with $p_1\leq p$ and $s\in\mathbb{R}$ fulfills (2.14) and V_{p_1} be defined as in Theorem 2.2.2. Then there exists a constant C which depends only on d,r,s, and $s-1-\frac{d}{p_1}$, so for any smooth solution f of (TD_μ) and $1\leq\theta_1\leq\theta\leq\infty$, we have*

$$\mu^{\frac{1}{\theta}}\|f\|_{\widetilde{L}_t^\theta B_{p,r}^{s+\frac{2}{\theta}}}\leq Ce^{C(1+\mu t)^{\frac{1}{\theta}}V_{p_1}(t)}\Big((1+\mu t)^{\frac{1}{\theta}}\|f^0\|_{B_{p,r}^s}+(1+\mu t)^{1+\frac{1}{\theta}-\frac{1}{\theta_1}}\mu^{\frac{1}{\theta_1}-1}\|g\|_{\widetilde{L}_t^{\theta_1}B_{p,r}^{s-2+\frac{2}{\theta_1}}}\Big).\tag{2.30}$$

Remark 2.2.4. *If $r=\infty$, then both Theorems 2.2.2 and 2.2.3 hold true with*

$$V_{p_1}(t)\triangleq\int_0^t\|\nabla v(\tau)\|_{\dot B_{p_1,1}^{\frac{d}{p_1}}},\quad V_{p_1}(t)\triangleq\int_0^t\|\nabla v(\tau)\|_{B_{p_1,1}^{\frac{d}{p_1}}},$$

respectively, in the borderline case

$$s=-d\min\Big\{\frac{1}{p_1},\frac{1}{p'}\Big\}\quad\text{or}\quad s=-1-d\Big\{\frac{1}{p_1},\frac{1}{p'}\Big\},\quad\text{if}\quad \mathrm{div}v=0.$$

2.3 Striated regularity of the vorticity

We start this section by stating a brief concise about the origin of the vortex patch topic for Euler's equations. These equations are given by the following Cauchy problem

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0, \\ \mathrm{div} v = 0; \\ v_{|t=0} = v_0. \end{cases} \tag{E}$$

Let us denote that the set equations (E) model the motion of inviscid fluids. For better understanding it is very convenient to use the vorticity formulation for that equations. This quantity ω is very efficient in the analysis of fluid dynamics, in particular, it measures how fast the fluid rotates and can be identified as a scalar function $\omega = \partial_1 v^2 - \partial_2 v^1$ in dimension two of space. To derive an evolution equation of ω, taking the *curl* operator to the momentum equation in (E) one obtains

$$\begin{cases} \partial_t \omega + v\cdot\nabla\omega = 0, \\ v = \nabla^{\perp}\Delta^{-1}\omega, \\ \omega_{|t=0} = \omega_0. \end{cases} \tag{2.31}$$

Obviously (2.31) is a nonlinear transport equation, so, the characteristic method ensures that $\omega(t,\Psi(t,x)) = \omega_0(x)$, with Ψ refers to the flow of velocity vector field. The fact that Ψ preserves Lebesgue's measure, then it follows for $t\geq 0$ that $\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}$ for $p\in[1,\infty[$, the case where $p=\infty$ can be done by a maximum principle. By summarizing for $t\geq 0$ we obtain the following infinite conservation laws, that is

$$\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}, \quad p\in[1,\infty]. \tag{2.32}$$

These infinite conservation laws (2.32) enable to Yudovich to relax the hyperbolic regularity and to formulate a weak solution to (2.31) in the following way.

Definition 2.3.1. *Let $\omega_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. The couple (v,ω) is said a weak solution for (2.31) with initial date ω_0 if and only if*

(i) $\omega \in L^\infty\big(\mathbb{R}_+; L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)\big)$,

(ii) $v = \mathscr{N}_2 \star \omega$ *with* $\mathscr{N}_2 = \frac{1}{2\pi}|x|^2$,

(iii) *For every* $\varphi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}^2)$,

$$\int_{\mathbb{R}_+ \times \mathbb{R}^2} \omega(\partial_t \varphi + v \cdot \nabla \varphi) dt dx = -\int_{\mathbb{R}^2} \varphi(0,x)\omega_0(x) dx.$$

As a consequence from the previous definition, Yudovich [75] was able to work slightly below the Lipschitzian regularity. More precisely, he demonstrates the following theorem.

Theorem 2.3.2. *Let* $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *then the system* (E) *admits a unique solution* $\omega \in L^\infty\big(\mathbb{R}_+; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\big)$ *in the sense of the Definition 2.3.1. Moreover, the corresponding velocity possesses a unique flow* Ψ *defined via the integral equation;*

$$\Psi(t,x) = x + \int_0^t v(\tau, \Psi(\tau,x)) d\tau.$$

Even though, Ψ *is an isomorphism preserves Lebesgue's measure of regularity*

$$\Psi - \mathbb{I} \in C^{e^{-\alpha t}}.$$

This theorem is prominent in so far as one can uniquely define the flow in a non-Lipschitzian framework. This allows us to explicit the vorticity by $\omega(t, \Psi(t,x)) = \omega_0(x)$. We said in this situation that the vorticity is transported by the flow Ψ (mappings trajectories). One of the consequences of the explicit vorticity's form is the global persistence of geometric structures like "vortex" (a characteristic function of a bounded domain). In other words, if the initial vorticity $\omega_0 = \mathbf{1}_{\Omega_0}$ with Ω_0 is a bounded domain, so for $t \geq 0$ we have $\omega(t) = \mathbf{1}_{\Omega_t}$ with $\Omega_t = \Psi(t, \Omega_0)$ the patch that moves through the time. Nevertheless, Yudovich's Theorem 2.3.2 does not allow us to forecast what happens for the regularity of the boundary of Ω_t since the flow admits a regularity that degenerates over time, which is moreover an optimal result according to an example of [5].

2.4 On the vortex patches topic

This section addresses the main ingredients about the smooth vortex in general case. We start with the push-forward of a family of vector field, thereafter we treat the particular case, where the vector field is in free-divergence which satisfies a package of properties. Among them that it evolves an inhomogeneous transport equation and a commutation with the material derivative $\partial_t + v\cdot\nabla$. Afterwards, we define the anisotropic Hölder spaces and their connection with the famous so-called *stationary logarithmic estimate*. Finally, we give some interpretation geometric of striated regularity of the vorticity.

2.4.1 Push-forward: definitions and properties

Given a smooth family of a vector fields $X : \mathbb{R}^d \to \mathbb{R}^d$ and a smooth function $f : \mathbb{R}^d \to \mathbb{R}$. The derivative's concept of f in the direction X is denoted generally by $\partial_X f$ and defined by

$$X(f) = \partial_X f = \sum_{i=1}^{d} X^i \partial_i f = X\cdot\nabla f.$$

This is the Lie derivative of the function f with respect to the vector field X, denoted usually by $\mathcal{L}_X f$ and in the previous formula we adopt different notations for this object.

Definition 2.4.1. *Let $X, Y : \mathbb{R}^d \to \mathbb{R}^d$ be a two family of vector fields. Their commutator is defined as the Lie bracket $[X, Y]$ which is given in the coordinates system by*

$$\begin{aligned}[X,Y]^i &= \sum_{j=1}^{d}(X^j\partial_j Y^i - Y^j\partial_j X^i)\\ &= \partial_X Y^i - \partial_Y X^i.\end{aligned}$$

We observe that the previous identity can also be written in the following form

$$\partial_X\partial_Y - \partial_Y\partial_X = \partial_{\partial_X Y - \partial_Y X}.$$

Let us denote that if f is not sufficiently smooth, for example $f \in L^\infty$ and X a family of vector field we define $\partial_X f$ in a weak sense as

$$\partial_X f = \mathrm{div}(Xf) - f\mathrm{div}X. \tag{2.33}$$

Next, we define the *push-forward* of a vector field X by a diffeomorphism Φ on $\mathbb{R}^d$ by the following statement.

Definition 2.4.2. *Let $X : \mathbb{R}^d \to \mathbb{R}^d$ be a vector field and Φ be a diffeomorphism of $\mathbb{R}^N$. The push-forward of X by Φ, denoted by $\Phi_\star X$ or $\Phi \uparrow X$ is defined by*

$$(\phi_\star X)(x) = \big(X \cdot \nabla\Phi\big)(\Phi^{-1}(x)).$$

Let us mention that the important factor in the push forward process of a vector field by a diffeomorphism changes the base point and is accompanied by a linear transformation which describes the modification of the tangent space due to the action of the diffeomorphism.
In the particular case, where X is replaced by a smooth time-dependent vector field $v(t,\cdot)$, we recall that this vector generates a flow map which is considered as a solution of the following differential equation

$$\partial_t\Psi(t,x) = v(t,\Psi(t,x)), \quad \Psi(0,x) = x.$$

When $v(t,\cdot)$ belongs to the Lipschitz class, so, the flow map is a diffeomorphism from $\mathbb{R}^d$ into itself. Consequently, the push-forward for a given family of vector fields X_0 by the flow $\Psi(t,\cdot)$ is the time-dependent family of vector fields (X_t) that can be written in the local coordinates as follows:

$$X_t(x) = \big(X_0 \cdot \nabla\Psi(t,x)\big)\big(\Psi^{-1}(t,x)\big). \tag{2.34}$$

The first important property of such family is that it evolves the following inhomogeneous transport equation

$$\partial_t X_t + v \cdot \nabla X_t = X_t \cdot \nabla v. \tag{2.35}$$

Another main feature of the family (X_t) given by the equation (2.35) reflects in its commutation with the material derivative $D_t = \partial_t + v\cdot\nabla$. This implies an important consequence about the dynamics of the tangential regularity of the vorticity subject to the system (5.5). Actually, one obtains easily the following result.

Proposition 2.4.3. *Let X be the push-forward of a smooth family of vector fields X_0 defined by (2.34). Then X commutes with the transport operator $D_t = \partial_t + v\cdot\nabla$ in the sense*

$$[X, D_t] = \partial_X D_t - D_t\partial_X = 0.$$

Proof. By definition we have:

$$
\begin{aligned}
[X, D_t] &= XD_t - D_tX \\
&= X \cdot \nabla(\partial_t + v \cdot \nabla) - (\partial_t + v \cdot \nabla)X \cdot \nabla \\
&= X \cdot \nabla\partial_t + X \cdot \nabla v \cdot \nabla + v \cdot X \cdot \nabla^2 - \partial_t X \cdot \nabla - X \cdot \partial_t \nabla - v \cdot \nabla X \cdot \nabla - X \cdot v\nabla^2 \\
&= X \cdot \nabla v \cdot \nabla - \partial_t X \cdot \nabla - v \cdot \nabla X \cdot \nabla.
\end{aligned}
$$

By virtue of the formula (2.35), we find the desired identity. □

The *vortex patch topic* have been handled in its early stages with the bidimensional incompressible Euler equations. A vortex patch, meaning that the initial vorticity ω_0 is the characteristic function of some bounded domain Ω_0. According to Yudovich's theorem the system (E) admits a global solution with bounded vorticity. Furthermore, that solution generates a flow map $\Psi(t,\cdot)$, and ω satisfies

$$\partial_t\omega + v \cdot \nabla\omega = 0, \quad v(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\omega(t,y)dy.$$

We may conclude that $\omega(t,\cdot)$ is the characteristic function of the domain transported by the flow:

$$\omega(t,\cdot) = \mathbf{1}_{\Omega_t}(\cdot), \quad \Omega_t = \Psi(t,\Omega_0).$$

We remark that $\omega(t,\cdot) \in L^1 \cap L^\infty$ does not imply that the velocity v is Lipschitz, see, Theorm 2.3.2, so $\Psi(t,\cdot)$ need not be Lipschitz either. Consequently, the previous relation does not enough to propagate the initial smoothness of the transported patch. However, we shall require more smoothness, for example, if $\partial\Omega_0$ is a $C^{1+\varepsilon}$ Jordan curve for some $\varepsilon \in]0,1[$, then $\partial\Omega_t$ preserves this regularity through the time.

2.4.2 Results related to striated regularity

Let us present that if $\omega = \mathbf{1}_\Omega$ where Ω is a $C^{1+\varepsilon}$ Jordan curve of $\mathbb{R}^2$, then ω is "more regular" in the direction which is tangent to $\partial\Omega$. In fact, for any smooth vector filed X which is tangent to $\partial\Omega$, we have

$$\partial_X\omega \triangleq X^1\partial_1\omega + X^2\partial_2\omega = 0.$$

Due to the fact that $\mathrm{div}(X\omega)-\partial_X\omega=\omega\mathrm{div}X$, we can derive that if X is sufficiently smooth and has a bounded divergence then $\mathrm{div}(Xv)$ is in L^∞ in lieu of being a linear combination of derivatives of L^∞ function if ω merely bounded. This motivates the following definition.

Definition 2.4.4. *A family $X=(X_\lambda)_{\lambda\in\Lambda}$ of vector fields over $\mathbb{R}^2$ is said to be nondegenerate whenever*

$$I(X)\triangleq\inf_{x\in\mathbb{R}^2}\sup_{\lambda\in\Lambda}\mid X_\lambda(x)\mid>0.$$

Let $\epsilon\in]0,1[$ and $X=(X_\lambda)_{\lambda\in\Lambda}$ be a nondegenerate family of C^ε vector fields over $\mathbb{R}^2$ with $\mathrm{div}X_\lambda\in C^\epsilon$. A bounded function ω is said to be in the normed Hölder anisotropic space $C^\epsilon(X)$ if it satisfies

$$\|\omega\|_{C^\epsilon(X)}\triangleq\frac{1}{I(X)}\left(\|\omega\|_{L^\infty}\sup_{\lambda\in\Lambda}\widetilde{\|X_\lambda\|}_{C^\epsilon}+\sup_{\lambda\in\Lambda}\|\partial_{X_\lambda}\omega\|_{C^{\epsilon-1}}\right),$$

with $\widetilde{\|X_\lambda\|}_{C^\epsilon}=\|X_\lambda\|_{C^\epsilon}+\|\mathrm{div}X_\lambda\|_{C^\epsilon}$.

Next, we state some regularity properties of the family $X=(X_t)$ in L^p-norms with $p\in[2,\infty]$. To be precise, we will prove:

Proposition 2.4.5. *Let $p\in[2,\infty]$ and $X_0\in L^p(\mathbb{R}^2)$. Then $X_t\in L^p(\mathbb{R}^2)$ for any $t>0$ and*

$$\|X_0\|_{L^p}e^{-V(t)}\le\|X_t\|_{L^p}\le\|X_0\|_{L^p}e^{-V(t)}.\tag{2.36}$$

Proof. We distinguish two cases.

•*First case $p\in[2,\infty[$.* Multiplying (2.35) by $|X_t|^{p-2}X_t$ and integrate by parts over $\mathbb{R}^2$, so incompressibility condition and Hölder inequality leading

$$\frac{1}{p}\frac{d}{dt}\|X_t\|_{L^p}^p\le\|\nabla v(t)\|_{L^\infty}\|X_t\|_{L^p}^p.$$

However, Gronwall's inequality gives for any $T>0$

$$\|X_t\|_{L^p}\le\|X_0\|_{L^p}e^{V(t)}.$$

For the first inequality of (2.36), apply the time derivative to $\partial_{X_0}\Psi(t,x)$, invoking (2.35) and (2.34) to obtain

$$\partial_t\partial_{X_0}\Psi(t,x)=\nabla v\big(t,\Psi(t,x)\big)\cdot\partial_{X_0}\Psi(t,x),\ \partial_{X_0}\Psi(0,x)=X_0.$$

The time reversibility of this equation gives the desired estimate.
•*Second case* $p = \infty$. Employ the continuity with repect to p it suffices to take $p \to \infty$. The proposition is then proved. □

One of the fundamental tools to derive the Lipschitzian norm of the velocity is the following estimate, which states that any velocity vector field with striated vorticity is Lipschitz and may be bounded in terms of $\|\omega\|_{L^\infty}$ and logarithmic of $\|\omega\|_{C^\varepsilon(X)}$. This logarithmic estimate dates back to J.Y. Chemin [19] to treat the Euler equations.

Theorem 2.4.6. *Let $\epsilon \in]0,1[$ and $X = (X_{t,\lambda})_{\lambda\in\Lambda}$ be a family of vector fields as in Definition 2.4.4. Let v be a free-divergence vector field such that its vorticity ω belongs to $L^2 \cap C^\epsilon(X)$. Then there exists a constant C depending only on ϵ, such that*

$$\|\nabla v\|_{L^\infty} \leq C\left(\|\omega\|_{L^2} + \|\omega\|_{L^\infty}\log\left(e + \frac{\|\omega\|_{C^\epsilon(X)}}{\|\omega\|_{L^\infty}}\right)\right). \tag{2.37}$$

Proof. The proof is very hard and requires the Biot-Savart law and an intense paradifferential calculus. For more details, we refer to [5, 19].

□

We recall that it was shown in [19] that the striated regularity is transported for all time by the Eulerian flow. As a consequence, one can get that a regular vortex patch of $C^{1+\epsilon}$ regularity remains so for all time. We mention that other proofs have been provided by Bertozzi and Constantin [10] and by Serfati [67]. The situation of singular vortex patches for Euler equations was treated by Danchin [23] and for Navier-Stokes equations by Hmidi [42]. The case of vortex patches in bounded domain was studied by Depauw [29] in dimension two of space and by Dutrifoy [30] in dimension three. More singular solutions as the so called *vortex sheet* was treated by Delort [28].
We end this paragraph with a precise interpretation of the boundary regularity and the tangent space which will be explored in the proof of the principal theorems of the next chapters.

Definition 2.4.7. *Let $0 < \varepsilon < 1$, then we have the following definitions.*

(1) *A closed curve Σ is said to be $C^{1+\varepsilon}-$regular if there exists $f \in C^{1+\varepsilon}(\mathbb{R}^2)$ such that Σ is a locally zero sets of f, that is to say, there exists a neighborhood V of Σ such that*

$$\Sigma = f^{-1}\{0\} \cap V, \quad \nabla f(x) \neq 0 \quad \forall x \in V. \tag{2.38}$$

(2) *A vector field X with $C^{\varepsilon}-$regularity is said to be tangent to Σ if $X \cdot \nabla f_{|\Sigma} = 0$. The set of such vector field is denoted by $\mathcal{T}^{\varepsilon}_{\Sigma}$.*

Given a compact curve Σ of the regularity $C^{1+\varepsilon}$, with $0 < \varepsilon < 1$. The striated or co-normal space C^{ε}_{Σ} associated to Σ is defined by

$$C^{\varepsilon}_{\Sigma} \triangleq \big\{u \in L^{\infty}(\mathbb{R}^2); \forall X \in \mathcal{T}^{\varepsilon}_{\Sigma},\ (\text{div}X = 0) \Rightarrow \text{div}(Xu) \in C^{\varepsilon-1}\big\}.$$

According Danchin's result [22], the class C^{ε}_{Σ} doesn't covers only the vortex patch of the type $\omega_0 = \mathbf{1}_{\Omega_0}$, but also encompass the so-called general vortex patch. Specifically, we have the following result proved in [22].

Proposition 2.4.8. *Let Ω_0 be a $C^{1+\varepsilon}-$bounded domain, with $0 < \varepsilon < 1$. Then for every function $f \in C^{\varepsilon}$, we have*

$$f\mathbf{1}_{\Omega_0} \in C^{\varepsilon}_{\Sigma}.$$

3 Optimal rate of convergence in stratified Boussinesq system

This Chapter is the subject of the following publication:

H. Meddour and M. Zerguine: *Optimal rate of convergence in Stratified Boussinesq system*, Dynamics of PDE, Vol.15, No.4, (2018), 235–263.

3.1 Introduction

This chapter is mainly motivated by the analysis of the initial value problem for the stratified Navier-Stokes system. This system of partial differential equations governs the evolution of a viscous incompressible fluid like the atmosphere and the ocean where one should take into account the friction forces and the stratification under the Boussinesq approximation, see [65]. The state of the fluid is described by a triplet (v_μ, p_μ, ρ_μ) where $v_\mu(t,x)$ denotes the velocity field which is assumed to be incompressible and the thermodynamical variables $p_\mu(t,x)$ and $\rho_\mu(t,x)$ which are two scalar functions representing respectively the pressure and the density. The equations being solved take the form

$$\begin{cases} \partial_t v_\mu + v_\mu\cdot\nabla v_\mu - \mu\Delta v_\mu + \nabla p_\mu = \rho_\mu \vec{e}_2 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \partial_t\rho_\mu + v_\mu\cdot\nabla\rho_\mu - \kappa\Delta\rho_\mu = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \mathrm{div} v_\mu = 0, & \\ (v_\mu,\rho_\mu)_{|t=0} = (v_\mu^0,\rho_\mu^0). & \end{cases} \tag{$\mathrm{B}_{\mu,\kappa}$}$$

The two coefficients μ,κ stand respectively for the kinematic viscosity and molecular diffusivity and $\vec{e}_2 = (0,1)$. For a better understanding of the system $(\mathrm{B}_{\mu,\kappa})$ it is more convenient to write it using the vorticity-density formulation. Thus the vorticity

$\omega \triangleq \partial_1 v^2 - \partial_2 v^1$ and the density satisfy the equivalent system,

$$\begin{cases} \partial_t \omega_\mu + v_\mu \cdot \nabla \omega_\mu - \mu \Delta \omega_\mu = \partial_1 \rho_\mu & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \rho_\mu + v_\mu \cdot \nabla \rho_\mu - \kappa \Delta \rho_\mu = 0 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ v_\mu = \nabla^\perp \Delta^{-1} \omega_\mu, \\ (\rho_\mu, \omega_\mu)_{|t=0} = (\rho_\mu^0, \omega_\mu^0). \end{cases} \tag{VD$_{\mu,\kappa}$}$$

It is clear that ($\mathrm{B}_{\mu,\kappa}$) coincides with the classical incompressible Navier-Stokes system when the initial density ρ_μ^0 is identically constant. For a general review on the mathematical theory of the Navier-Stokes system we refer for instance to [5, 60]. We notice that the system ($\mathrm{VD}_{\mu,\kappa}$) is the subject of intensive research activities especially in the last decades. A lot of results have been obtained and we shall restrict the discussion to some of them. When the coefficients μ and κ are strictly positive, it was proved in [13, 37] that the system ($\mathrm{B}_{\mu,\kappa}$) admits a unique global solution for arbitrarily large data. For $\mu > 0, \kappa = 0$ the global well-posedness problem was solved independently by Chae [15] and Hou and Li [53] for smooth initial data in Sobolev spaces $H^s, s > 2$. Those results were improved by Abidi and Hmidi in [2] for $(v^0, \rho^0) \in B^{-1}_{\infty,1} \cap L^2 \times B^0_{2,1}$. Later, Danchin and Paicu investigated in [25] the global well-posedness for any initial data (v^0, ρ^0) in $L^2 \times L^2$. The opposite case $\mu = 0$ and $\kappa > 0$ is also well-explored. Actually, Chae proved in [15] the global well-posedness for $(v^0, \rho^0) \in H^s \times H^s$ for $s > 2$ which was later improved by Hmidi and Keraani in [45] for critical Besov spaces, that is, $(v^0, \rho^0) \in B^{\frac{2}{p}+1}_{p,1} \times B^{-1+\frac{2}{p}}_{p,1} \cap L^r$, $r > 2$. The global existence in the framework of Yudovich solutions was accomplished in [26] by Danchin and Paicu for $(v^0, \rho^0) \in L^2 \times L^2 \cap B^{-1}_{\infty,1}$ and $\omega^0 \in L^r \cap L^\infty$ with $r \geq 2$. For other connected topics we refer the reader to [45, 49, 48, 51, 58, 64, 72].

The main focus of the current chapter is twofold. In the first part, we study the persistence regularity of the vortex patches for ($\mathrm{B}_{\mu,\kappa}$) for $\kappa = 1$, denoted simply by (B_μ). In the second part we shall deal with the strong convergence towards the limit system when the viscosity μ goes to zero. The limit system is nothing but the stratified Euler equations,

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \rho \vec{e}_2 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \rho + v \cdot \nabla \rho - \Delta \rho = 0 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \mathrm{div} v = 0, \\ (v, \rho)_{|t=0} = (v^0, \rho^0). \end{cases} \tag{B$_0$}$$

Before giving more details about our main contribution we shall review some aspects of the vortex patch problem for the viscous/inviscid incompressible fluid. Recall first the classical Navier-Stokes equations,

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = 0 \quad \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \text{div} v = 0, \\ v_{|t=0} = v^0. \end{cases} \tag{NS$_\mu$}$$

Notice that the incompressible Euler system (E), denoted sometimes by (NS_0), is given by

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0 \quad \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \text{div} v = 0, \\ v_{|t=0} = v^0. \end{cases} \tag{E}$$

We point out that the global existence of classical solutions for Euler system is based on the structure of the vorticity which is transported by the flow, that is,

$$\partial_t\omega + v\cdot\nabla\omega = 0.$$

This provides an infinite family of conservation laws and in particular we get for all $p\in[1,\infty]$

$$\|\omega(t)\|_{L^p} = \|\omega^0\|_{L^p}. \tag{3.1}$$

We mention that the conservation laws (3.1) served as a suitable framework for Yudovich [75] to relax the classical hyperbolic theory and show that (NS$_\mu$) and (E) are globally well-posed whenever $\omega^0\in L^1\cap L^\infty$. In this pattern, the velocity is no longer in the Lipschitz class but belongs to the log$-$Lipschitz space, denoted by LL[1]. It is known that with this regularity the associated flow Ψ is continuous with respect to $(t,x)-$variables and the vorticity can be recovered from its initial value according to the formula,

$$\omega(t,\Psi(t,x)) = \omega^0(x). \tag{3.2}$$

In particular, when the initial vorticity $\omega^0 = \mathbf{1}_{\Omega_0}$ is a vortex patch with Ω_0 being a regular bounded domain, then the advected vorticity remains a vortex patch relative to a domain $\Omega_t \triangleq \Psi(t,\Omega_0)$ which is homeomorphic to Ω_0. It is important to emphasize that the regularity persistence of the boundary does not follow from the general theory of Yudovich because the flow is not in general better than $C^{e^{-\alpha t}}$ where α depends on ω^0. This problem was solved by Chemin who proved in [19] that when the initial boundary is $C^{1+\varepsilon}$ then the boundary of the patch keeps this

[1]The space LL is the set of bounded functions u such that

$$\|u\|_{LL} \triangleq \sup_{0<|x-y|<1} \frac{|u(x)-u(y)|}{|x-y|\log\frac{e}{|x-y|}}.$$

regularity through the time. Broadly speaking, Chemin's strategy is entirely based on the control of Lipschitz norm of the velocity by means of logarithmic estimate of $\|\omega\|_{C^\varepsilon(X)}$ with $C^\varepsilon(X)$ is an anisotropic Hölder space associated to an adequate family of vector fields that capture the conormal regularity of the velocity (see section 4.2.5).

The study for the viscous case was initiated by Danchin in [22] who proved that if $\omega^0 = \mathbf{1}_{\Omega_0}$, such that the domain Ω_0 is $C^{1+\varepsilon}$ then the velocity v_μ is Lipschitz uniformly with respect to the viscosity μ. He also showed that the transported vorticity by the viscous flow Ψ_μ remains in the class $C^{1+\varepsilon'}, \forall \varepsilon' < \varepsilon$. Note that contrary to the Hölderian regularity, there is no loss of regularity in the Besov spaces $B^\varepsilon_{p,\infty}$, $\forall\, p < \infty$. For the borderline case $p = \infty$ Hmidi showed in [41] that this loss of regularity is artificial and his proof is mainly related to some smoothing effects for the transport-diffusion equation using Lagrangian coordinates. There is a large literature dealing with this subject and some connected topics and for more details we refer the reader to the papers [10, 29, 32, 34, 41] and the references therein.

It could be interesting to extend some of the foregoing results to the stratified Navier-Stokes system (B_μ). The investigation of this system with initial vorticity of patch type has been started recently in [52] for $\mu = 0$. It was proved that if the boundary of the initial patch is smooth enough then the velocity is Lipschitz for any positive time and the transported domain Ω_t preserves its initial regularity. In addition, the vorticity can be decomposed into a singular part which is a vortex patch term and a regular part, which is deeply related to the smoothing effect for density, i.e. $\omega(t) = \mathbf{1}_{\Omega_t} + \widetilde{\rho}(t)$. Later, Zerguine studied in [78] the same system but the usual dissipation operator $-\Delta$ is replaced by the critical fractional Laplacian $(-\Delta)^{\frac{1}{2}}$. He obtained sharper results compared to the incompressible Euler equations [19, 52] and describes the asymptotic behavior of the solutions for large time.

We are now ready to state the first main result, dealing with the global well-posedness for the system (B_μ) under a vortex patch initial data. More precisely, we have:

Theorem 3.1.1. *Let Ω_0 be a simply connected bounded domain such that its boundary $\partial\Omega_0$ is $C^{1+\varepsilon}$ with $0 < \varepsilon < 1$. Let $\omega^0_\mu = \mathbf{1}_{\Omega_0}$ and $\rho^0_\mu \in L^1 \cap L^\infty$ then the following assertions hold.*

(i) *The system (B_μ) admits a unique global solution (v_μ, ρ_μ) such that*

$$(v_\mu, \rho_\mu) \in L^\infty_{loc}(\mathbb{R}_+; \mathrm{Lip}) \times L^\infty_{loc}(\mathbb{R}_+; L^1 \cap L^\infty).$$

More precisely, there exists $C_0 \triangleq C(\varepsilon, \Omega_0) > 0$ such that, for all $\mu \in]0, 1[$ and for all $t \in \mathbb{R}_+$ we have

$$\|\nabla v_\mu(t)\|_{L^\infty} \leq C_0 e^{C_0 t \log^2(1+t)}. \tag{3.3}$$

(ii) *The boundary of the transported domain $\Omega_\mu(t) \triangleq \Psi_\mu(t, \Omega_0)$ is $C^{1+\varepsilon}$ for every $t \geq 0$ uniformly on μ, where Ψ_μ denotes the viscous flow associated to v_μ.*

Let us give a bunch of comments about Theorem 3.1.1 in the following few remarks.

Remark 3.1.2. *Compared to the incompressible Navier-Stokes system, we see that a Lipschitz norm of the velocity has a logarithmic growth for large time. This is due to the logarithmic factor in the growth of the vorticity, namely we have:*

$$\|\omega_\mu(t)\|_{L^\infty} \leq C_0 \log^2(1+t).$$

Remark 3.1.3. *When the viscosity μ is identically zero, we obtain the same result as in [52] for the stratified Euler system* (B_0)*, that is to say:*

$$\|\nabla v(t)\|_{L^\infty} \leq C_0 e^{C_0 t \log^2(1+t)}. \tag{3.4}$$

Now we shall briefly outline the ideas of the proof which is done in the spirit of the pioneering work of Chemin [19]. In order to get a bound for the quantity $\|\nabla v_\mu(t)\|_{L^\infty}$ we first show that the co-normal regularity of the vorticity $\partial_X \omega_\mu$ is controlled in $C^{\varepsilon-1}$, with $0 < \varepsilon < 1$. We then take advantage of the logarithmic estimate to derive the Lipschitz norm of the velocity, with X is a family of selected vector fields which satisfies the transport equation,

$$\partial_t X + v_\mu \cdot \nabla X = X \cdot \nabla v_\mu.$$

As it was pointed in [22, 41] the situation in the viscous case is more delicate than the inviscid one due to the Laplacian operator which does not commute with the family X. Actually, the evolution of the directional derivative $\partial_X \omega_\mu$ is governed by an inhomogeneous transport-diffusion equation,

$$(\partial_t + v_\mu \cdot \nabla - \mu\Delta)\partial_X \omega_\mu = -\mu[\Delta, X]\omega_\mu + \partial_X \partial_1 \rho_\mu, \tag{3.5}$$

where $[\Delta, X]$ denotes the commutator between Δ and X. Thus the difficulties reduce to understanding the terms $[\Delta, X]\omega_\mu$ and $\partial_X \partial_1 \rho_\mu$ which apparently need more regularity to be well-defined than what is initially prescribed. To circumvent the problem for the first term we shall use the formalism developed in [22, 41] for $2d-$incompressible Navier-Stokes system. However, to deal with the second term we find more convenient to diagonalize the system written in the vorticity-density formulation and introduce the coupled function $\Gamma_\mu \triangleq (1-\mu)\omega_\mu - \partial_1\Delta^{-1}\rho_\mu$ in the spirit of [50]. This function satisfies the following transport-diffusion equation,

$$\partial_t\Gamma_\mu + v_\mu\cdot\nabla\Gamma_\mu - \mu\Delta\Gamma_\mu = [\partial_1\Delta^{-1}, v_\mu\cdot\nabla]\rho_\mu \triangleq H_\mu.$$

By applying the directional derivative ∂_X to the last equation we find

$$(\partial_t + v_\mu\cdot\nabla - \mu\Delta)\partial_X\Gamma_\mu = -\mu[\Delta, X]\Gamma_\mu + \partial_X H_\mu.$$

At a formal level, and this will be justified rigorously as we shall see in the proofs, we see that H_μ is of order zero with respect to ρ_μ according to the smoothing effect of the singular operator $\partial_1\Delta^{-1}$. Thus instead of manipulating $\partial_X\partial_1\rho_\mu$ in the equation (4.3) which consumes two derivatives we need just to understand $\partial_X H_\mu$ which exhibits a good behavior on ρ_μ as it was revealed in [52].

The second part of this chapter is devoted to the inviscid limit problem which is in fact well-explored for the classical Navier-Stokes system (NS_μ). We mention that for smooth initial data the convergence towards Euler equations holds true and the rate of convergence in the energy space L^2 is bounded by μt, see [7] for initial data $v_0\in H^s$ with $s>4$. In [18], Chemin proved a strong convergence in L^2 for Yudovich's initial data and obtained that the rate is controlled by $(\mu t)^{\frac{1}{2}e^{-Ct}}$, which degenerating in time. To obtain a better result, Constantin and Wu [21] had to work under vortex patch structure and they obtained $(\mu t)^{\frac{1}{2}}$. Afterwards, Abidi and Danchin [1] improved this result and showed that the rate of convergence is exactly $(\mu t)^{\frac{3}{4}}$ which is proved to be optimal for the Rankine vortex.

Our second main result reads as follows.

Theorem 3.1.4. *Let (v_μ,ρ_μ), (v,ρ), (ω_μ,ρ_μ) and (ω,ρ) be the solutions of* (B_μ), (B_0), (VD_μ) *and* (VD_0) *respectively with the same initial data such that $\omega^0_\mu=\omega^0=\mathbf{1}_{\Omega_0}$, where Ω_0 is a $C^{1+\varepsilon}$ simply connected bounded domain. Then for all $t\geq 0, \mu\in]0,1[$ and $p\in[2,+\infty[$ we have:*

(i) $\|v_\mu(t)-v(t)\|_{L^p} + \|\rho_\mu(t)-\rho(t)\|_{L^p} \leq C_0 e^{e^{C_0 t\log^2(2+t)}}(\mu t)^{\frac{1}{2}+\frac{1}{2p}}.$

(ii) $\|\omega_\mu(t)-\omega(t)\|_{L^p} \leq C_0 e^{e^{C_0 t\log^2(1+t)}}(\mu t)^{\frac{1}{2p}}.$

Remark 3.1.5. *When ρ_μ^0 and ρ^0 are constants and $p = 2$ we get the result of Abidi and Danchin [1].*

The proof of Theorem 3.1.4 will be done using the approach of [1] by combining some classical ingredients like L^p-estimates, real interpolation results and some smoothing effects for the density and the vorticity.

The last result is dedicated to prove that $(\mu t)^{\frac{1}{2p}}$ is optimal for vortices in the case of Rankine initial data.

Theorem 3.1.6. *We assume that ρ_μ^0 and ρ^0 being constants and $\omega_\mu^0 = \omega^0 = \mathbf{1}_{\mathbb{D}}$ with $\mathbb{D}$ the unit disc. Then there exist two positive constants C_1 and C_2 independent on μ and t, such that for $\mu t \leq 1$, and $p \in [2, +\infty[$ we have:*

$$C_1(\mu t)^{\frac{1}{2p}} \leq \|\omega_\mu(t) - \omega(t)\|_{L^p} \leq C_2(\mu t)^{\frac{1}{2p}},$$

with C_1 and C_2 depending on p.

Note that the approach that we shall propose here is different from [1] which is specific for $p = 2$. The proof of Abidi and Danchin uses the explicit form of Fourier transform of the Rankine vortex given through Bessel function combined with its asymptotic behavior. Nevertheless, these tools are useless for $p \neq 2$ and the alternative is to make the computations in the physical variable using the explicit structure of the heat kernel.

For the reader's convenience, we provide a brief outline of this chapter. Section 2, starts with few important results about the Littlewood-Paley decomposition, para-differential calculus and some functional spaces. Moreover, we state some useful technical lemmas, in particular two smoothing effects estimates for transport-diffusion equations governing respectively the density and the vorticity evolution. Section 3, mainly treats the general version of Theorem 3.1.1. Section 4 is divided into two parts. The first one is dedicated to the upper bound rate of convergence. The second part deals with the optimality of the rate of convergence between the vortices. We end with an appendix where we give the proof of some technical propositions.

3.2 Tools

Before proceeding, we specify some of the notations we will constantly use during this work. We denote by C a positive constant which may be different in each occurrence but it does not depend on the initial data. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of type $X \leq CY$ with C independent of X and Y. The notation C_0 means a constant depend on the involved norms of the initial data.

3.2.1 Littlewood-Paley theory

Our results mostly rely on Fourier analysis methods based on a nonhomogeneous dyadic partition of unity with respect to the Fourier variable. The so-called Littlewood-Paley decomposition enjoying particularly "nice" properties. These properties are the basis for introducing the important scales of Besov and Hölder spaces and for their study.

Let $\chi \in \mathscr{D}(\mathbb{R}^2)$ be a reference cut-off function, monotonically decaying along rays and so that

$$\begin{cases} \chi \equiv 1 & \text{if } \|\xi\| \leq \frac{1}{2} \\ 0 \leq \chi \leq 1 & \text{if } \frac{1}{2} \leq \|\xi\| \leq 1 \\ \chi \equiv 0 & \text{if } \|\xi\| \geq 1. \end{cases}$$

Define $\varphi(\xi) \triangleq \chi(\frac{\xi}{2}) - \chi(\xi)$. We obviously check that $\varphi \geq 0$ and

$$\operatorname{supp} \varphi \subset \mathcal{C} \triangleq \{\xi \in \mathbb{R}^2 : \frac{1}{2} \leq \|\xi\| \leq 1\}.$$

Then we have the following elementary properties, see for example [?, ?].

Proposition 3.2.1. *Let χ and φ be as above. Then the following assertions are hold.*

(1) Decompositon of the unity:

$$\forall \xi \in \mathbb{R}^2, \quad \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1.$$

(2) Almost orthogonality in the sense of ℓ^2*:*

$$\forall \xi \in \mathbb{R}^2, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{q\geq 0} \varphi^2(2^{-q}\xi) \leq 1.$$

The Littlewood-Paley or cut-off operators are defined as follows.

Definition 3.2.2. *For every* $u \in \mathcal{S}'(\mathbb{R}^2)$*, setting*

$$\Delta_{-1}u \triangleq \chi(\mathrm{D})u, \quad \Delta_q u \triangleq \varphi(2^{-q}\mathrm{D})u \quad \text{if } q \in \mathbb{N}, \quad S_q u \triangleq \sum_{j\leq q-1} \Delta_j u \quad \text{for } q \geq 0.$$

Some properties of Δ_q and S_q are listed in the following proposition.

Proposition 3.2.3. *Let* $u, v \in \mathcal{S}'(\mathbb{R}^2)$ *we have*

(i) $|p-q| \geq 2 \Longrightarrow \Delta_p \Delta_q u \equiv 0,$

(ii) $|p-q| \geq 4 \Longrightarrow \Delta_q(S_{p-1}u\Delta_p v) \equiv 0,$

(iii) $\Delta_q, S_q : L^p \to L^p$ *uniformly with respect to* q *and* p*.*

(iv)
$$u = \sum_{q\geq -1} \Delta_q u.$$

Likewise the homogeneous operators $\dot{\Delta}_q$ and $\dot{S}_q$ are defined by

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q = \varphi(2^q D)u, \quad \dot{S}_q = \sum_{j\leq q-1} \dot{\Delta}_j v. \tag{3.6}$$

Now, we will recall the definition of the Besov spaces.

Definition 3.2.4. *For* $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$*. The inhomogeneous Besov space* $B^s_{p,r}$ *(resp. the homogeneous Besov space* $\dot{B}^s_{p,r}$*) is the set of all tempered distributions* $u \in \mathcal{S}'$ *(resp.* $u \in \mathcal{S}'_{|\mathbf{P}}$*) such that*

$$\|u\|_{B^s_{p,r}} \triangleq \Big(2^{qs}\|\Delta_q u\|_{L^p}\Big)_{\ell^r} < \infty.$$
$$\Big(\text{resp. } \|u\|_{\dot{B}^s_{p,r}} \triangleq \Big(2^{qs}\|\dot{\Delta}_q u\|_{L^p}\Big)_{\ell^r(\mathbb{Z})} < \infty\Big).$$

We have denoted by $\mathbf{P}$ *the set of polynomials.*

Remark 3.2.5. *We notice that:*

(1) If $s \in \mathbb{R}_+ \backslash \mathbb{N}$, the Hölder space noted by C^s coincides with $B^s_{\infty,\infty}$.

(2) $\big(C^s, \|\cdot\|_{C^s}\big)$ is a Banach space coincides with the usual Hölder space C^s with equivalent norms,

$$\|u\|_{C^s} \lesssim \|u\|_{L^\infty} + \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^s} \lesssim \|u\|_{C^s}. \tag{3.7}$$

(3) If $s \in \mathbb{N}$, the obtained space is so-called Hölder-Zygmund space and still denoted by $B^s_{\infty,\infty}$.

3.2.2 Paradifferential calculus

The well-known *Bony's* decomposition [12] enables us to split formally the product of two tempered distributions u and v into three pieces. In what follows, we shall adopt the following definition for paraproduct and remainder:

Definition 3.2.6. *For a given $u, v \in \mathcal{S}'$ we have*

$$uv = T_u v + T_v u + \mathscr{R}(u,v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad \mathscr{R}(u,v) = \sum_q \Delta_q u \widetilde{\Delta}_q v \quad \textit{and} \quad \widetilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

The mixed space-time spaces are stated as follows.

Definition 3.2.7. *Let $T > 0$ and $(s, \beta, p, r) \in \mathbb{R} \times [1,\infty]^3$. We define the spaces $L^\beta_T B^s_{p,r}$ and $\widetilde{L}^\beta_T B^s_{p,r}$ respectively by:*

$$L^\beta_T B^s_{p,r} \triangleq \Big\{u : [0,T] \to \mathcal{S}'; \|u\|_{L^\beta_T B^s_{p,r}} = \big\|\big(2^{qs}\|\Delta_q u\|_{L^p}\big)_{\ell^r}\big\|_{L^\beta_T} < \infty\Big\},$$

$$\widetilde{L}^\beta_T B^s_{p,r} \triangleq \Big\{u : [0,T] \to \mathcal{S}'; \|u\|_{\widetilde{L}^\beta_T B^s_{p,r}} = \big(2^{qs}\|\Delta_q u\|_{L^\beta_T L^p}\big)_{\ell^r} < \infty\Big\}.$$

The relationship between these spaces is given by the following embeddings. Let

$\varepsilon > 0$, *then*

$$\begin{cases} L^\beta_T B^s_{p,r} \hookrightarrow \widetilde{L}^\beta_T B^s_{p,r} \hookrightarrow L^\beta_T B^{s-\varepsilon}_{p,r} & \text{if } r \geq \beta, \\ L^\beta_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde{L}^\beta_T B^s_{p,r} \hookrightarrow L^\beta_T B^s_{p,r} & \text{if } \beta \geq r. \end{cases} \tag{3.8}$$

Accordingly, we have the following interpolation result.

Corollary 3.2.8. *Let* $T > 0$, $s_1 < s < s_2$ *and* $\zeta \in (0,1)$ *such that* $s = \zeta s_1 + (1-\zeta)s_2$. *Then we have*

$$\|u\|_{\widetilde{L}^a_T B^s_{p,r}} \leq C \|u\|^{\zeta}_{\widetilde{L}^a_T B^{s_1}_{p,\infty}} \|u\|^{1-\zeta}_{\widetilde{L}^a_T B^{s_2}_{p,\infty}}. \tag{3.9}$$

The following *Bernstein* inequalities describe a bound on the derivatives of a function in the L^b-norm in terms of the value of the function in the L^a-norm, under the assumption that the Fourier transform of the function is compactly supported. For more details we refer [5, 19].

Lemma 3.2.9. *There exists a constant* $C > 0$ *such that for* $1 \leq a \leq b \leq \infty$, *for every function* u *and every* $q \in \mathbb{N} \cup \{-1\}$, *we have*

(i)

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|S_q u\|_{L^a},$$

(ii)

$$C^{-k} 2^{qk} \|\Delta_q u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \leq C^k 2^{qk} \|\Delta_q u\|_{L^a}.$$

A noteworthy consequence of Bernstein inequality (i) is the following embedding:

$$B^s_{p,r} \hookrightarrow B^{\widetilde{s}}_{\widetilde{p},\widetilde{r}} \quad \text{whenever } \widetilde{p} \geq p,$$

with

$$\widetilde{s} < s - 2\Big(\frac{1}{p} - \frac{1}{\widetilde{p}}\Big) \quad \text{or} \quad \widetilde{s} = s - 2\Big(\frac{1}{p} - \frac{1}{\widetilde{p}}\Big) \quad \text{and} \quad \widetilde{r} \leq r.$$

3.2.3 Useful results

This paragraph is reserved to some useful properties freely used throughout this article. The most results concerning the system (VD_μ) rely strongly on a priori estimates in Besov spaces for the transport-diffusion equation:

$$\begin{cases} \partial_t a + v \cdot \nabla a - \mu \Delta a = f \\ a_{|t=0} = a^0. \end{cases} \tag{TD$_\mu$}$$

We start by the persistence of Besov regularity for (TD$_\mu$), whose proof may be found for example in [5].

Proposition 3.2.10. *Let $(s,r,p) \in]-1,1[\times[1,\infty]^2$ and v be a smooth divergence free vector-field. We assume that $a^0 \in B^s_{p,r}$ and $f \in L^1_{loc}(\mathbb{R}_+; B^s_{p,r})$. Then for every smooth solution a of (TD$_\mu$) and $t \geq 0$ we have*

$$\|a(t)\|_{B^s_{p,r}} \leq Ce^{CV(t)}\Big(\|a^0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^s_{p,r}}d\tau\Big),$$

with

$$V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$$

and C a constant which depends only on s and not on the viscosity. For the limit case

$$s = -1, r = \infty \text{ and } p \in [1,\infty] \quad \text{or} \quad s = 1, r = 1 \text{ and } p \in [1,\infty]$$

the above estimate remains true despite we change $V(t)$ by $Z(t) \stackrel{def}{=} \|v\|_{L^1_t B^1_{\infty,1}}$. In addition if $a = \mathrm{curl}\, v$, then the above estimate holds true for all $s \in [1,+\infty[$.

Next, we state the maximal smoothing effect result for (TD$_\mu$) in mixed time-space spaces, whose proof was developped in [44].

Proposition 3.2.11. *Let $s \in]-1,1[$, $(p_1,p_2,r) \in [1,+\infty]^3$ and v be a divergence free vector field belonging to $L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathrm{Lip})$. Then for every smooth solution a of (TD$_\mu$) we have*

$$\mu^{\frac{1}{r}}\|a\|_{\widetilde{L}^r_t B^{s+\frac{2}{r}}_{p_1,p_2}} \leq Ce^{CV(t)}(1+\mu t)^{\frac{1}{r}}\Big(\|a^0\|_{B^s_{p_1,p_2}} + \|f\|_{L^1_t B^s_{p_1,p_2}}\Big), \quad \forall t \in \mathbb{R}_+. \tag{3.10}$$

The asymptotic behavior in L^p-norm with $p \in [2,\infty]$ of every (ω_μ,ρ_μ) solution of (VD$_\mu$) is given by the following proposition. To be precise we have:

Proposition 3.2.12. *Let (ω_μ,ρ_μ) be a smooth solution of (VD$_\mu$) such that $\rho_0 \in L^1 \cap L^p$ and $\omega_0 \in L^2 \cap L^p$ with $p \in [2,\infty]$. Then for $t \geq 0$,*

$$\|\omega_\mu(t)\|_{L^p} + \|\nabla\rho_\mu\|_{L^1_t L^p} \leq C_0 \log^{2-\frac{2}{p}}(1+t).$$

Remark 3.2.13. *This property has been recently accomplished by [52] for Stratified Euler equations* (B_μ)*, with* $\mu = 0$*. We point out that the proof of such estimate remains available in our case with minor modifications due to the laplacien term, which has a good sign.*

We end this paragraph by the Calderón-Zygmund estimate which constitutes a deep statement of harmonic analysis.

Proposition 3.2.14. *Let* $p \in]1, \infty[$ *and* v *be a divergence-free vector field which its vorticity* $\omega \in L^p$*. Then* $\nabla v \in L^p$ *and*

$$\|\nabla v\|_{L^p} \leq c\frac{p^2}{p-1}\|\omega\|_{L^p}, \tag{3.11}$$

with c *being a universal constant.*

3.3 Smooth vortex patch problem

In this section we will give a detailed proof for the first main result stated in Theorem 3.1.1. We will inspire the general ideas from Chemin's result, we then follow the argument performed more recently by [52, 78] for Stratified Euler system. For this aim, we will state the general framework study of the vortex patch problem.

3.3.1 Vortex patch tool box

Before entering into details of the proof of the Theorem 3.1.1, we will state few important ingredients concerning the study of vortex patch problem. We will start with the concept of an admissible family of vector fields and some related properties, from which we will derive the notion of anisotropic Hölder space. At the end, we state the so-called *stationnary logarithmic estimate* which is the key step to prove that the velocity is a Lipschitz function.

Definition 3.3.1. *Let* $\varepsilon \in]0, 1[$*. A family of vector fields* $X = (X_\lambda)_{\lambda\in\Lambda}$ *is said to be admissible if and only if the following assertions hold.*

- *Regularity:*

$$\forall\lambda \in \Lambda \quad X_\lambda, \mathrm{div}X_\lambda \in C^\epsilon.$$

- *Non-degeneray:*

$$I(X) \triangleq \inf_{x\in\mathbb{R}^d} \sup_{\lambda\in\Lambda} \big|X_\lambda(x)\big| > 0. \tag{3.12}$$

Setting

$$\widetilde{\|}X_\lambda\|_{C^\varepsilon} \triangleq \|X_\lambda\|_{C^\varepsilon} + \|\mathrm{div} X_\lambda\|_{C^\varepsilon}. \tag{3.13}$$

Definition 3.3.2. *Let $X = (X_\lambda)_{\lambda\in\Lambda}$ be an admissible family. The action of each factor X_λ on $u \in L^\infty$ is defined as the directional derivative of u along X_λ by the formula,*

$$\partial_{X_\lambda} u = \mathrm{div}(uX_\lambda) - u\mathrm{div} X_\lambda.$$

The anisotropic Hölder spaces, denoted by $C^\varepsilon(X)$ are defined below.

Definition 3.3.3. *Let $\varepsilon \in]0,1[$ and X be an admissible family of vector fields. We say that $u \in C^\varepsilon(X)$ if and only if:*

- *$u \in L^\infty$ and satisfies*

$$\forall\, \lambda \in \Lambda,\ \partial_{X_\lambda} u \in C^{\varepsilon-1}, \quad \sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}} < +\infty.$$

- *$C^\varepsilon(X)$ is a normed space with*

$$\|u\|_{C^\varepsilon(X)} \triangleq \frac{1}{I(X)}\Big(\|u\|_{L^\infty} \sup_{\lambda\in\Lambda} \widetilde{\|}X_\lambda\|_{C^\varepsilon} + \sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}}\Big).$$

Now, let us take an initial family of vector-field $X_0 = (X_{0,\lambda})_{\lambda\in\Lambda}$ and define its time evolution $X_t = \big(X_{t,\lambda}\big)_{\lambda\in\Lambda}$ by

$$X_{t,\lambda}(x) \triangleq (X_{0,\lambda}\Psi)(t, \Psi^{-1}(t,x)), \tag{3.14}$$

that is X_t is the vector-field X_0 transported by the flow Ψ associated to v. From this definition the evolution family X_t satisfies the following transport equation.

Proposition 3.3.4. *Let v be a Lipschitzian vector-field, Ψ its flow and $X_t = (X_{t,\lambda})_{\lambda\in\Lambda}$ is the family defined by (4.12). Then the following equation holds true.*

$$\begin{cases} (\partial_t + v\cdot\nabla)X_{t,\lambda} = \partial_{X_{t,\lambda}} v & \textit{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ X_{t,\lambda|t=0} = X_{0,\lambda}. \end{cases} \tag{3.15}$$

To prove the Theorem 3.1.1, we state the following stationnary logarithmic estimate initially introduced by Chemin [19]. More precisely,

Theorem 3.3.5. *Let $\varepsilon \in]0,1[$ and $X = (X_\lambda)_{\lambda\in\Lambda}$ be a family of vector fields as in Definition 4.2.11. Let v be a divergence-free vector field such that its vorticity ω belongs to $L^2 \cap C^\varepsilon(X)$. Then there exists a constant C depending only on ε, such that*

$$\|\nabla v\|_{L^\infty} \leq C\bigg(\|\omega\|_{L^2} + \|\omega\|_{L^\infty}\log\Big(e + \frac{\|\omega\|_{C^\varepsilon(X)}}{\|\omega\|_{L^\infty}}\Big)\bigg). \tag{3.16}$$

We shall now make precise to the boundary regularity and the tangent space used in the proof of Theorem 3.1.1.

Definition 3.3.6. *Let $\varepsilon > 0$.*

1. *A closed curve Σ is said to be $C^{1+\varepsilon}-$regular if there exists $f \in C^{1+\varepsilon}(\mathbb{R}^2)$ such that Σ is locally a zero set of f, i.e., there exists a neighborhood V of Σ such that*
$$\Sigma = f^{-1}\{0\} \cap V, \quad \nabla f(x) \neq 0 \quad \forall x \in V. \tag{3.17}$$

2. *A vector field X with $C^\varepsilon-$regularity is said to be tangent to Σ if $X \cdot \nabla f_{|\Sigma} = 0$. The set of such vector fields will be denoted by $\mathcal{T}^\varepsilon_\Sigma$.*

Given a compact curve Σ of the class $C^{1+\varepsilon}$, $0 < \varepsilon < 1$. The co-normal space C^ε_Σ associated to Σ is defined by

$$C^\varepsilon_\Sigma \triangleq \big\{u \in L^\infty(\mathbb{R}^2); \forall X \in \mathcal{T}^\varepsilon_\Sigma,\ (\mathrm{div}X = 0) \Rightarrow \mathrm{div}(Xu) \in C^{\varepsilon-1}\big\}.$$

The following Danchin's result stated in [24], showing that C^ε_Σ contains the characteristic function of a bounded open domain surrounded by the curve Σ. More generally we have:

Proposition 3.3.7. *Let Ω_0 be a $C^{1+\varepsilon}-$bounded domain, with $0 < \varepsilon < 1$ and $f \in C^\varepsilon(\mathbb{R}^2)$, then we have:*

$$f\mathbf{1}_{\Omega_0} \in C^\varepsilon_\Sigma.$$

According to the previous proposition, we strive to give a general version of the Theorem 3.1.1 which allows to deal with more general structures than the vortex patches. Thus we have:

Theorem 3.3.8. *Let $0<\varepsilon<1, X_0$ be a family of admissible vector fields and v_μ^0 be a free-divergence vector field such that $\omega_\mu^0 \in L^2\cap C^\varepsilon(X_0)$. Let $\rho_\mu^0\in L^1\cap L^\infty$, then for $\mu\in]0,1[$ the system (B_μ) admits a unique global solution $(v_\mu,\rho_\mu)\in L^\infty_{loc}(\mathbb{R}_+;Lip)\times L^\infty(\mathbb{R}_+;L^1\cap L^\infty)$. More precisely:*

$$\|\nabla v_\mu(t)\|_{L^\infty}\le C_0e^{C_0t\log^2(1+t)}. \tag{3.18}$$

Furthermore, we have:

$$\|\omega_\mu(t)\|_{C^\varepsilon(X_t)}+\|\partial_{X_t}\psi_\mu(t)\|_{C^\varepsilon}\le C_0e^{\exp\{C_0t\log^2(2+t)\}}.$$

Proof. The most difficult point in the proof is to estimate suitably the quantity ω_μ in $C^\varepsilon(X_t)$ norm. For this aim, we shall use the following coupled function Γ_μ defined by $\Gamma_\mu=(1-\mu)\omega_\mu-\mathcal{L}\rho_\mu$, with $\mathcal{L}=\partial_1\Delta^{-1}$. After few computations, we obtain that Γ_μ evolves the following inhomogenous transport-diffusion equation:

$$(\partial_t+v_\mu\cdot\nabla-\mu\Delta)\Gamma_\mu=[\mathcal{L},v_\mu\cdot\nabla]\rho_\mu. \tag{3.19}$$

To simplify the presentation in what follows, we temporarily drop the viscosity parameter μ.
By virtue of (3.15) of the Proposition 3.3.4, one can check that the quantity $\partial_{X_{t,\lambda}}\Gamma$ satisfies the equation,

$$(\partial_t+v\cdot\nabla-\mu\Delta)\partial_{X_{t,\lambda}}\Gamma=X_{t,\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}-\mu[\Delta,X_{t,\lambda}]\Gamma. \tag{3.20}$$

According to [22, 41], the commutator $[\Delta,X_{t,\lambda}]$ can be decomposed as the sum of two terms in the following way:

$$\mu[\Delta,X_{t,\lambda}]\Gamma=F+\mu G,$$

with

$$F\triangleq 2\mu T_{\nabla X^i_{t,\lambda}}\partial_i\nabla\Gamma+2\mu T_{\partial_i\nabla\Gamma}\nabla X^i_{t,\lambda}+\mu T_{\Delta X^i_{t,\lambda}}\partial_i\Gamma+\mu T_{\partial_i\Gamma}\Delta X^i_{t,\lambda}.$$

and

$$G\triangleq 2\mathscr{R}(\nabla X^i_{t,\lambda},\partial_i\nabla\Gamma)+\mathscr{R}(\Delta X^i_{t,\lambda},\partial_i\Gamma).$$

Here, we have used Enstein's convention for the summation over the repeated indices. Thus the equation (3.20) takes the following form,

$$(\partial_t+v\cdot\nabla-\mu\Delta)\partial_{X_{t,\lambda}}\Gamma=X_{t,\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}-(F+\mu G),$$

Applying Theorem 3.38 page 162 in [5], one gets

$$
\begin{aligned}
\big\|\partial_{X_\lambda}\Gamma\big\|_{L^\infty_t C^{\varepsilon-1}} \quad &\leq \quad Ce^{CV(t)}\Big(\|\partial_{X_{0,\lambda}}\Gamma^0\|_{C^{\varepsilon-1}}+\|\partial_{X_\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}\|_{L^1_tC^{\varepsilon-1}}\\
&\quad +(1+\mu t)\|F\|_{L^\infty_t C^{\varepsilon-3}}+\mu\|G\|_{\widetilde{L}^1_tC^{\varepsilon-1}}\Big).
\end{aligned}
\tag{3.21}
$$

Recall from [5, 41] the following two inequalities

$$\|F\|_{L^\infty_t C^{\varepsilon-3}}\leq C\|\Gamma\|_{L^\infty_t L^\infty}\|X_\lambda\|_{L^\infty_t C^\varepsilon}.$$

and

$$\|G\|_{\widetilde{L}^1_tC^{\varepsilon-1}}\leq C\|\Gamma\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}\|X_\lambda\|_{L^\infty_t C^\varepsilon}.$$

Combining with (3.21), one finds

$$
\begin{aligned}
\big\|\partial_{X_\lambda}\Gamma\big\|_{L^\infty_t C^{\varepsilon-1}} \quad &\leq \quad Ce^{CV(t)}\Big(\|\partial_{X_{0,\lambda}}\Gamma^0\|_{C^{\varepsilon-1}}+\|\partial_{X_\lambda}\{[\mathcal{L},v\cdot\nabla]\rho\}\|_{L^1_tC^{\varepsilon-1}}\\
&\quad +(1+\mu t)\|\Gamma\|_{L^\infty_t L^\infty}\|X_\lambda\|_{L^\infty_t C^\varepsilon}+\mu\|\Gamma\|_{\widetilde{L}^1_tB^2_{\infty,\infty}}\|X_\lambda\|_{L^\infty_t C^\varepsilon}\Big).
\end{aligned}
\tag{3.22}
$$

• **Estimate of** $\big\|\partial_{X_{0,\lambda}}\Gamma^0\big\|_{C^{\varepsilon-1}}$. From the definition of the function Γ we have:

$$\big\|\partial_{X_{0,\lambda}}\Gamma^0\big\|_{C^{\varepsilon-1}}\leq\big\|\partial_{X_{0,\lambda}}\omega^0\big\|_{C^{\varepsilon-1}}+\big\|\partial_{X_{0,\lambda}}\mathcal{L}\rho^0\big\|_{C^{\varepsilon-1}}. \tag{3.23}$$

On the one hand, from Definition 4.2.13 we write

$$\big\|\partial_{X_{0,\lambda}}\omega^0\big\|_{C^{\varepsilon-1}}\lesssim\|\omega^0\|_{C^\varepsilon(X_0)}. \tag{3.24}$$

On the other hand, employing the fact C^ε is an algebra, then we obtain the general result

$$
\begin{aligned}
\|\partial_{X_\lambda}u\|_{C^{\varepsilon-1}} \quad &\leq \quad \|\mathrm{div}(uX_\lambda)\|_{C^{\varepsilon-1}}+\|u\ \mathrm{div}X_\lambda\|_{C^{\varepsilon-1}}\\
&\lesssim \quad \|uX_\lambda\|_{C^\varepsilon}+\|u\ \mathrm{div}X_\lambda\|_{L^\infty}\\
&\lesssim \quad \|u\|_{C^\varepsilon}\widetilde{\|}X_\lambda\|_{C^\varepsilon}.
\end{aligned}
\tag{3.25}
$$

Consequently

$$\big\|\partial_{X_{0,\lambda}}\mathcal{L}\rho^0\big\|_{C^{\varepsilon-1}}\lesssim\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon}\|\mathcal{L}\rho^0\|_{C^\varepsilon}.$$

Concerning $\|\mathcal{L}\rho^0\|_{C^\varepsilon}$, using the fact that $\mathcal{L}$ is of order -1. Then Bernstein's inequality yields for $p \geq \frac{2}{1-\varepsilon}$,

$$\begin{aligned} \|\mathcal{L}\rho^0\|_{C^\varepsilon} &\leq \|\mathcal{L}\rho^0\|_{L^\infty} + \sup_{q\in\mathbb{N}} 2^{q\varepsilon}\|\Delta_q\mathcal{L}\rho^0\|_{L^\infty} \\ &\lesssim \|\mathcal{L}\rho^0\|_{L^\infty} + \sup_{q\in\mathbb{N}} 2^{q(\varepsilon-1+2/p)}\|\Delta_q\rho^0\|_{L^p}. \end{aligned} \tag{3.26}$$

Furtheremore, $\mathcal{L}$ have a non local structure, i.e.,

$$\mathcal{L}\rho(t,x) \triangleq \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x_1-y_1)}{|x-y|^2}\rho(t,y)dy,$$

and so

$$|\mathcal{L}\rho(t,x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{|\rho(t,y)|}{|x-y|}dy = \Big(\frac{1}{2\pi|\cdot|}\star|\rho(t,\cdot)|\Big)(x).$$

Applying the convolution product properties and $\|\rho(t)\|_{L^1\cap L^\infty} \leq \|\rho^0\|_{L^1\cap L^\infty}$, we obtain

$$\begin{aligned} \|\mathcal{L}\rho(t)\|_{L^\infty} &\lesssim \|\rho(t)\|_{L^1\cap L^\infty} \\ &\lesssim \|\rho^0\|_{L^1\cap L^\infty}. \end{aligned} \tag{3.27}$$

Putting together (3.26) and (3.27). Then in view of $\Delta_q : L^p \to L^p$ is continuous and $L^p = [L^1, L^\infty]_{\frac{1}{p}}$, we deduce

$$\|\mathcal{L}\rho^0\|_{C^\varepsilon} \leq \|\rho^0\|_{L^1\cap L^\infty}.$$

Therefore

$$\big\|\partial_{X_{0,\lambda}}\mathcal{L}\rho^0\big\|_{C^{\varepsilon-1}} \leq C_0\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon} \tag{3.28}$$

More generally for $t>0$

$$\big\|\partial_{X_{t,\lambda}}\mathcal{L}\rho(t)\big\|_{C^{\varepsilon-1}} \leq C_0\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}, \tag{3.29}$$

Inserting (3.24) and (3.28) in (3.23) to get

$$\big\|\partial_{X_{0,\lambda}}\Gamma^0\big\|_{C^{\varepsilon-1}} \leq C_0\big(1+\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon}\big). \tag{3.30}$$

• **Estimate of** $\|\partial_{X_\lambda}\{[\mathcal{L}, v\cdot\nabla]\rho\}\|_{L^1_tC^{\varepsilon-1}}$. To estimate this term we write again in view of (3.25),

$$\|\partial_{X_\lambda}\{[\mathcal{L}, v\cdot\nabla]\rho\}\|_{L^1_tC^{\varepsilon-1}} \lesssim C\widetilde{\|}X_{t,\lambda}\|_{L^\infty_tC^\varepsilon}\Big\|[\mathcal{L}, v\cdot\nabla]\rho\Big\|_{L^1_tC^\varepsilon}.$$

Then in accordance with the Proposition 3.5.1 stated in appendix, the last estimate becomes

$$\|\partial_{X_{t,\lambda}}\{[\mathcal{L}, v\cdot\nabla]\rho\}\|_{L^1_t C^{\varepsilon-1}} \leq C_0 \widetilde{\|} X_{t,\lambda}\|_{L^\infty_t C^\varepsilon} t. \tag{3.31}$$

• **Estimate of** $\|\Gamma\|_{L^\infty_t L^\infty}$**.** By definition we have for $\mu \in]0,1[$,

$$\|\Gamma\|_{L^\infty_t L^\infty} \leq \|\omega\|_{L^\infty_t L^\infty} + \|\mathcal{L}\rho\|_{L^\infty_t L^\infty}. \tag{3.32}$$

Thanks to the Proposition 3.2.12 we have for $t>0$,

$$\|\omega(t)\|_{L^\infty} \leq C_0 \log^2(2+t).$$

Note that the term $\|\mathcal{L}\rho\|_{L^\infty_t L^\infty}$ will be done exactly as in (3.27). Then in view of the last estimate, (3.32) takes the form

$$\|\Gamma\|_{L^\infty_t L^\infty} \leq C_0 \log^2(2+t). \tag{3.33}$$

• **Estimate of** $\|\Gamma\|_{\widetilde{L}^1_t B^2_{\infty,\infty}}$. Applying the maximal smoothing effect (4.8) to the equation (3.19), it happens

$$\mu\|\Gamma\|_{\widetilde{L}^1_t B^2_{\infty,\infty}} \leq Ce^{CV(t)}(1+\mu t)\Big(\|\Gamma^0\|_{B^0_{\infty,\infty}} + \big\|[\mathcal{L}, v\cdot\nabla]\rho\big\|_{L^1_t B^0_{\infty,\infty}}\Big).$$

Using the fact $L^\infty \hookrightarrow B^0_{\infty,\infty}$ and $C^\varepsilon \hookrightarrow B^0_{\infty,\infty}$ for $\varepsilon>0$, it follows

$$\mu\|\Gamma\|_{\widetilde{L}^1_t B^2_{\infty,\infty}} \leq Ce^{CV(t)}(1+\mu t)\Big(\|\Gamma^0\|_{L^\infty} + \big\|[\mathcal{L}, v\cdot\nabla]\rho\big\|_{L^1_t C^\varepsilon}\Big),$$

and, in turn, using once more the Proposition 3.5.1, we get

$$\mu\|\Gamma\|_{\widetilde{L}^1_t B^2_{\infty,\infty}} \leq Ce^{CV(t)}(1+\mu t)\big(\|\Gamma^0\|_{L^\infty} + C_0 t\big), \tag{3.34}$$

For $\|\Gamma^0\|_{L^\infty}$, applying the same argument as in (3.27), we deduce

$$\|\Gamma^0\|_{L^\infty} \leq \|\omega^0\|_{L^\infty} + \|\rho^0\|_{L^1\cap L^\infty}$$

together with (3.34), it holds that for $\mu \in]0,1[$

$$\mu\|\Gamma\|_{\widetilde{L}^1_t B^2_{\infty,\infty}} \leq C_0 e^{CV(t)}(1+t)^2. \tag{3.35}$$

Plugging (3.30), (3.31), (3.33), (3.35) in (3.22), then after few computations we obtain for $\mu \in]0,1[$

$$\big\|\partial_{X_\lambda}\Gamma\big\|_{L^\infty_t C^{\varepsilon-1}} \;\leq\; C_0 e^{CV(t)}(1+t^2)\log^2(2+t)\big(1+\widetilde{\|} X_\lambda\|_{L^\infty_t C^\varepsilon}\big). \tag{3.36}$$

But,

$$\big\|\partial_{X_{t,\lambda}}\omega(t)\big\|_{C^{\varepsilon-1}} \leq \big\|\partial_{X_{t,\lambda}}\mathcal{L}\rho(t)\big\|_{C^{\varepsilon-1}} + \big\|\partial_{X_{t,\lambda}}\Gamma(t)\big\|_{C^{\varepsilon-1}}$$

combined with (3.27), (3.29) and (3.36) we get

$$\begin{aligned}\big\|\partial_{X_{t,\lambda}}\omega(t)\big\|_{C^{\varepsilon-1}} &\leq C_0 e^{CV(t)}(1+t^2)\log^2(2+t)\big(1+\widetilde{\|}X_\lambda\|_{L^\infty_t C^\varepsilon}\big) + C_0\widetilde{\|}X_\lambda\|_{L^\infty_t C^\varepsilon}\\ &\leq C_0 e^{CV(t)}(1+t^2)\log^2(2+t)\big(1+\widetilde{\|}X_\lambda\|_{L^\infty_t C^\varepsilon}\big).\end{aligned} \tag{3.37}$$

The term $\widetilde{\|}X_\lambda\|_{L^\infty_t C^\varepsilon}$ may be bounded by taking advantage of (3.15) and the Proposition 3.2.10, we thus have

$$\|X_{t,\lambda}\|_{C^\varepsilon} \leq Ce^{CV(t)}\bigg(\|X_{0,\lambda}\|_{C^\varepsilon} + \int_0^t e^{-CV(\tau)}\|X_{\tau,\lambda}v(\tau)\|_{C^\varepsilon}d\tau\bigg). \tag{3.38}$$

According to [5, 19], the quantity $\|\partial_{X_{t,\lambda}}v\|_{C^\varepsilon}$ satisfies,

$$\|\partial_{X_{t,\lambda}}v\|_{C^\varepsilon} \leq C\big(\|\partial_{X_{t,\lambda}}\omega\|_{C^{\varepsilon-1}} + \|\mathrm{div}X_{t,\lambda}\|_{C^\varepsilon}\|\omega(t)\|_{L^\infty} + \|X_{t,\lambda}\|_{C^\varepsilon}\|\nabla v(t)\|_{L^\infty}\big).$$

Plug the last estimate in (3.38) to obtain

$$\begin{aligned}\|X_{t,\lambda}\|_{C^\varepsilon} &\leq Ce^{CV(t)}\bigg(\|X_{0,\lambda}\|_{C^\varepsilon} + C\int_0^t e^{-CV(\tau)}\Big(\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}}\\ &\quad+\|\mathrm{div}X_{\tau,\lambda}\|_{C^\varepsilon}\|\omega(\tau)\|_{L^\infty} + \|X_{\tau,\lambda}\|_{C^\varepsilon}\|\nabla v(\tau)\|_{L^\infty}\Big)d\tau.\end{aligned} \tag{3.39}$$

To conclude, it is enough to treat the term $\mathrm{div}X_{t,\lambda}$. To do this, we apply "div" to (3.15) and using the fact $\mathrm{div}v = 0$, we deduce that $\mathrm{div}X_{t,\lambda}$ evolves the equation

$$(\partial_t + v\cdot\nabla)\mathrm{div}X_{t,\lambda} = 0.$$

Again the Proposition 3.2.10 gives

$$\|\mathrm{div}X_{t,\lambda}\|_{C^\varepsilon} \leq Ce^{CV(t)}\|\mathrm{div}X_{0,\lambda}\|_{C^\varepsilon}. \tag{3.40}$$

Combining (3.39) and (3.40), then (4.10) allows us to write

$$\begin{aligned}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} &\leq Ce^{CV(t)}\bigg(\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon}\big(1+\|\omega\|_{L^1_tL^\infty}\big)\\ &\quad+C\int_0^t e^{-CV(\tau)}\big(\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}} + \|X_{\tau,\lambda}\|_{C^\varepsilon}\|\nabla v(\tau)\|_{L^\infty}\big)d\tau\bigg).\end{aligned}$$

Then, the Proposition 3.2.12 implies

$$\begin{aligned}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} &\leq C_0 e^{CV(t)}\bigg(\log^2(2+t)+C\int_0^t e^{-CV(\tau)}\big(\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}} \\ &\qquad +\|X_{\tau,\lambda}\|_{C^\varepsilon}\|\nabla v(\tau)\|_{L^\infty}\big)d\tau\bigg),\end{aligned}$$

Gronwall's inequality asserts that

$$\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \leq C_0 e^{CV(t)}\bigg(\log^2(2+t)+C\int_0^t e^{-CV(\tau)}\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}}d\tau\bigg),$$

combined with (3.37), it holds for $t>0$

$$\begin{aligned}e^{-CV(t)}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} &\leq C_0(1+t)(1+t^2)\log^2(2+t) \\ &\qquad +C_0C\int_0^t(1+\tau^2)\log^2(2+\tau)e^{-CV(\tau)}\widetilde{\|}X_{\tau,\lambda}\|_{L^\infty_\tau C^\varepsilon}d\tau.\end{aligned}$$

Setting $\phi_1(t)=C_0(1+t)(1+t^2)\log^2(2+t)$ and $\phi_2(t)=C_0C(1+t^2)\log^2(2+t)$, then for $t>0$ the last estimate becomes

$$e^{-CV(t)}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}\leq\phi_1(t)+\int_0^t\phi_2(\tau)e^{-CV(\tau)}\widetilde{\|}X_{\tau,\lambda}\|_{L^\infty_\tau C^\varepsilon}d\tau.$$

Again Gronwall's inequality gives

$$e^{-CV(t)}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}\leq\phi_1(t)+\int_0^t\phi_1(\tau)\phi_2(\tau)e^{\int_\tau^t\phi_2(\tau')d\tau'}d\tau.$$

After a few computations we shall have for $t>0$

$$\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}\leq C_0e^{C_0t^3\log^2(2+t)}e^{CV(t)},$$

accordingly (3.37) becomes

$$\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}\leq C_0e^{C_0t^3\log^2(2+t)}e^{CV(t)}.$$

Putting together the last two estimates, we end up with

$$\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}+\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}\leq C_0e^{C_0t^3\log^2(2+t)}e^{CV(t)},\quad\forall\lambda\in\Lambda. \tag{3.41}$$

On the other hand, according to the Definition 4.2.13, we recall that:

$$\|\omega(t)\|_{C^\varepsilon(X_t)}=\frac{1}{I(X_t)}\Big(\|\omega\|_{L^\infty}\sup_{\lambda\in\Lambda}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}+\sup_{\lambda\in\Lambda}\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}\Big). \tag{3.42}$$

The required estimate for ω in $C^\varepsilon(X_t)$ norm follows by showing that X_t defined in (4.12) is a non degenerate family, that is to say, $I(X_t) > 0$. For that purpose, we derive $X_{t,\lambda} \circ \Psi(t,x) \triangleq \partial_{X_{0,\lambda}} \Psi(t,x)$ with respect to time and using the fact

$$\begin{cases} \frac{\partial}{\partial t}\Psi(t,x) = v(t,\Psi(t,x)) \\ \Psi(0,x) = x, \end{cases}$$

it follows

$$\begin{cases} \partial_t \partial_{X_{0,\lambda}} \Psi(t,x) = \nabla v(t,\Psi(t,x)) \partial_{X_{0,\lambda}} \Psi(t,x) \\ \partial_{X_{0,\lambda}} \Psi(0,x) = X_{0,\lambda}. \end{cases}$$

One deduce that this equation is a time reversible. Thus Gronwall's inequality asserts that

$$|X_{0,\lambda}(x)|^{\pm 1} \leq |\partial_{X_{0,\lambda}} \Psi(t,x)| e^{V(t)}.$$

In accordance with (3.12) and (4.12), one has

$$I(X_t) \geq I(X_0) e^{-V(t)} > 0. \tag{3.43}$$

Consequently, (3.41), (3.42) and (3.43) leading to

$$\|\omega(t)\|_{C^\varepsilon(X_t)} \leq C_0 e^{C_0 t^3 \log^2(2+t)} e^{CV(t)}. \tag{3.44}$$

Now, we are in position to apply the logarithmic estimate (4.14). By virtue of (3.44), the Proposition (3.2.12), the increasing of the function $(0,\infty) \ni \zeta \mapsto \zeta \log(e + a/\zeta)$ and the decreasing of $(0,\infty) \ni \zeta \mapsto \log(e + a/\zeta)$, it holds

$$\begin{aligned} \|\nabla v(t)\|_{L^\infty} &\leq C_0 \Bigg(\log(2+t) + \log^2(2+t) \log\left(e + \frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{\log^2(2+t)} \right) \Bigg) \\ &\leq C_0 \Big(\log(2+t) + t^3 \log^4(2+t) + \log^2(2+t) \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau \Big). \end{aligned}$$

Hence, the growth of the exponential function and Gronwall's inequality yield

$$\|\nabla v(t)\|_{L^\infty} \leq C_0 e^{C_0 t \log^2(2+t)}, \tag{3.45}$$

combining this estimate with (3.44), we get

$$\|\omega(t)\|_{C^\varepsilon(X_t)} \leq C_0 e^{\exp C_0 t \log^2(2+t)}.$$

To finalize, let us estimate $\partial_{X_{t,\lambda}} \Psi(t)$. First, we employ that $\partial_{X_{0,\lambda}} \Psi(t) = X_{t,\lambda} \circ \Psi(t)$

for every $\lambda \in \Lambda$, then by virtue of (4.5) we thus have

$$\begin{aligned}\|X_{t,\lambda} \circ \Psi(t)\|_{C^\varepsilon} &\leq \|X_{t,\lambda}\|_{C^\varepsilon} \|\nabla \Psi(t)\|^{\varepsilon}_{L^\infty} \\ &\leq \|X_{t,\lambda}\|_{C^\varepsilon} e^{CV(t)} \quad \forall \lambda \in \Lambda.\end{aligned}$$

Here we have used the classical estimate $e^{-CV(t)} \leq \|\nabla \Psi^{\pm 1}(t)\|_{L^\infty} \leq e^{CV(t)}$. Hence, (3.45) ensures that

$$\|X_{t,\lambda} \circ \Psi(t)\|_{C^\varepsilon} \leq C_0 e^{\exp C_0 t \log^2(2+t)}, \tag{3.46}$$

this concludes the proof. □

3.3.2 Proof of Theorem 3.1.1.

The proof of the Theorem 3.1.1 requires two principal steps:

(1) The velocity vector fields is a Lipschitz function globally in time, which immediately follows from Theorem 4.3.1.

(2) The persistence of Hölderian regularity in time of the transported patch, i.e., $\partial\Omega_t$ is a simple curve with $C^{1+\varepsilon}-$regularity given by the following scheme:

(2.i) Fabricate an initial admissible family $X_0 = (X_{0,\lambda})_{\lambda \in \{0,1\}}$, which enables us to show that $\mathbf{1}_{\Omega_0} \in C^\varepsilon(X_0)$ and parametrize its boundary $\partial\Omega_0$ by a simple curve.

(2.ii) The regularity of evolution family $X_t = (X_{t,\lambda})_{\lambda \in \{0,1\}}$ and the boundary $\partial\Omega_t$, with $\Omega_t = \Psi(t, \Omega_0)$.

(2.i) Since $\partial\Omega_0$ is a curve of the class $C^{1+\varepsilon}$. Consequently, (1) of the Definition 3.3.6 ensures the existence of a local chart (f_0, V_0), with V_0 is a neighborhood of $\partial\Omega_0$ such that

$$\begin{cases} f_0 \in C^{1+\varepsilon}(\mathbb{R}^2), \quad \nabla f_0(x) \neq 0 \quad \text{on } V_0 \\ \partial\Omega_0 = f_0^{-1}(\{0\}) \cap V_0, \end{cases}$$

On the other hand, let $\chi \in \mathscr{D}(\mathbb{R}^2), 0 \leq \chi \leq 1$ and

$$\text{supp } \chi \subset V_0, \quad \chi(x) = 1 \quad \forall x \in W_0,$$

where W_0 is a small neighborhood of $\partial\Omega_0$ such that $W_0 \Subset V_0$. Next, define for every $x \in \mathbb{R}^2$ the family $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ by:

$$X_{0,0}(x) = \nabla^\perp f_0(x) \quad \text{and} \quad X_{0,1}(x) = (1-\chi(x))\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.47}$$

It is worthwhile to examine the admissibility of the family $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$. First, we obviously check that $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ is non-degenerate, and that each component $X_{0,\lambda}$ and its divergence are in $C^\varepsilon(\mathbb{R}^2)$, then according to Definition 4.2.11, we conclude that $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ is an admissible family.
Second, $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ is a tangential family (see, (2)-Definition 3.3.6) with respect to $\Sigma = \partial\Omega_0$, i.e.,

$$X_{0,\lambda} \in \mathcal{T}^\varepsilon_{|\Sigma}, \quad \forall \lambda \in \{0,1\}.$$

Indeed, for the component $X_{0,0}$, clearly we have:

$$X_{0,0}(x)\cdot\nabla f_0(x) = \nabla^\perp f_0(x)\cdot\nabla f_0(x) = 0, \quad \forall x \in \partial\Omega_0,$$

while for the component $X_{0,1}$, using the fact $\chi \equiv 1$ on W_0, we immediately obtain

$$\begin{aligned} X_{0,1}(x)\cdot\nabla f_0(x) &= (1-\chi(x))\partial_1 f_0(x) \\ &= 0. \end{aligned}$$

(2.ii) For every $\lambda \in \{0,1\}$ and $x \in \mathbb{R}^2$, we set $X_{t,\lambda}(x) = \big(\partial_{X_{0,\lambda}}\Psi\big)\big(t,\Psi^{-1}(t,x)\big)$. Using the same argument as in (3.40), (3.41) and (3.43), we infer that (X_t) still remains non-degenerate for every $t \geq 0$, and that each $X_{t,\lambda}$ still has components and divergence in C^ε. This means that $X_t = (X_{t,\lambda})_{\lambda\in\{0,1\}}$ is an admissible family for all $t \geq 0$.
Now, we will parametrize the boundary $\partial\Omega_0$. To do this, let $x_0 \in \partial\Omega_0$ and define the curve γ^0 by the following ordinary differential equation

$$\begin{cases} \partial_\sigma\gamma^0(\sigma) = X_{0,0}(\gamma^0(\sigma)) \\ \gamma^0(0) = x_0. \end{cases}$$

By classical arguments we can see that γ^0 belongs to $C^{1+\varepsilon}(\mathbb{R},\mathbb{R}^2)$. A natural way to define the evolution parametrization of $\partial\Omega_t$ is to set for every $t \geq 0$,

$$\gamma(t,\sigma) \triangleq \Psi(t,\gamma^0(\sigma)).$$

Clearly that $\gamma(t,\cdot)$ is the transported of γ^0 by the flow Ψ. By applying the criterion differentiation with respect to σ, we readily get

$$\partial_\sigma \gamma(t,\sigma) = \big(\partial_{X_{0,0}}\Psi\big)(t,\, \gamma^0(\sigma)).$$

On the other hand, $\partial_{X_{0,0}}\Psi \equiv X_{0,0}\circ\Psi$, thus we have from estimate 3.46 of the Theorem 4.3.1 that $\partial_{X_{0,0}}\psi \in L^\infty_{loc}(\mathbb{R}_+; C^\varepsilon)$, accordingly $\gamma(t)$ belongs to $L^\infty_{loc}(\mathbb{R}_+; C^{1+\varepsilon})$. This tells us the regularity persistence of the boundary $\partial\Omega_t$ and so the proof of the Theorem 3.1.1 is accomplished.

3.4 The rate convergence

3.4.1 General statement

In this paragraph we are interested in the rate convergence between (v_μ, ρ_μ) and (v,ρ), the solutions of (B_μ) and (B_0). To be precise, we will provide a more general version of the Theorem 3.1.4. For this purpose, we state the following auxiliary result which shows that any vortex patch with smooth bounded domain belongs to $\dot{B}^{\frac{1}{p}}_{p,\infty}$.

Proposition 3.4.1. *Let Ω_0 be a $C^{1+\varepsilon}-$bounded domain, with $0<\varepsilon<1$, then the function $\mathbf{1}_{\Omega_0}$ belongs to $\dot{B}^{\frac{1}{p}}_{p,\infty}$, with $p\in[2,\infty[$.*

Proof. We follow the formalism performed in [61] with more details. Since Ω_0 is $C^{1+\varepsilon}-$bounded domain, then in view of $C^{1+\varepsilon}\hookrightarrow Lip$ we deduce that $\mathbf{1}_{\Omega_0}\in L^\infty\cap BV$, with BV[2] is the Banach space of functions with bounded variation. By means of the Proposition 3.5.2 stated in the appendix, we have

$$BV \hookrightarrow \dot{B}^1_{1,\infty}$$

[2] BV is the space of functions of bounded variations defined by

$$BV(\mathbb{R}^2) \triangleq \Big\{u\in L^1(\mathbb{R}^2) : \forall i=1,\ldots,2,\ \exists \lambda_i \in \mathscr{M}_b(\mathbb{R}^2,\mathbb{R}^2);\int_{\mathbb{R}^2} u\frac{\partial\varphi}{\partial x_i}dx = -\int_{\mathbb{R}^2}\varphi d\lambda_i \quad \forall \varphi\in\mathscr{D}(\mathbb{R}^2)\Big\}$$

equipped with the norm

$$\|u\|_{BV} \triangleq \|u\|_{L^1} + |Du|(\mathbb{R}^2),$$

where $|Du|(\mathbb{R}^2)$ is the total variation of measure Du.

In particular for $q \in \mathbb{Z}$
$$\|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^1} \lesssim 2^{-q}\|\mathbf{1}_{\Omega_0}\|_{BV},$$
combined with $L^p = [L^1, L^\infty]_{\frac{1}{p}}$, we deduce

$$\begin{aligned}
\|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^p} &\lesssim \|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^1}^{\frac{1}{p}}\|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^\infty}^{1-\frac{1}{p}} \\
&\lesssim 2^{-\frac{q}{p}}\|\mathbf{1}_{\Omega_0}\|_{BV}\|\mathbf{1}_{\Omega_0}\|_{L^\infty} \\
&\lesssim 2^{-\frac{q}{p}}\|\mathbf{1}_{\Omega_0}\|_{L^\infty\cap BV}.
\end{aligned}$$

Here we have used the fact that $\dot{\Delta}_q$ maps continuously L^∞ into it self. Thus we obtain for $q \in \mathbb{Z}$
$$2^{-\frac{q}{p}}\|\dot{\Delta}_q \mathbf{1}_{\Omega_0}\|_{L^p} \leq C\|\mathbf{1}_{\Omega_0}\|_{L^\infty\cap BV}.$$
Taking the supremum over $q \in \mathbb{Z}$, we finally obtain that $\mathbf{1}_{\Omega_0} \in \dot{B}^{\frac{1}{p}}_{p,\infty}$ □

Now, we state the general version of the Theorem 3.1.4. Roughly speaking we have:

Theorem 3.4.2. *Let (v_μ, ρ_μ) and (v, ρ) be the solutions of (B_μ) and (B_0) respectively with (v^0_μ, ρ^0_μ) and (v^0, ρ^0) their initial data. Let ω^0_μ, ω^0 be their vorticties with $\omega^0_\mu \in L^\infty \cap B^{\frac{1}{p}}_{p,\infty}$, $\omega^0 \in L^1 \cap L^\infty$ and $\rho^0, \rho^0_\mu \in L^1 \cap^p$. Setting $\Pi(t) = \|v_\mu - v\|_{L^p} + \|\rho_\mu - \rho\|_{L^p}$, then we have the following rate of convergence.*

$$\Pi(t) \leq Ce^{C(t+V_\mu(t)+V(t))}\Big(\Pi(0) + (\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)\big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\rho^0_\mu\|_{L^p}\big)\Big) \quad p \in [2, \infty[,$$

where
$$V_\mu(t) = \int_0^t \|\nabla v_\mu(\tau)\|_{L^\infty} d\tau, \quad V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau.$$

The proof of the previous Theorem requires the following interpolation result.

Proposition 3.4.3. *Let $(p, r, \eta) \in [1, \infty]^2 \times]-1, 1[$ and v_μ be a free divergence vector field depicted by the Biot-Savart law $v_\mu = \Delta^{-1}\nabla^\perp \omega_\mu$, i.e.,*

$$v_\mu(t, x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\omega_\mu(t, y)dy.$$

Then the following estimate holds true.

$$\|\Delta v_\mu\|_{L^r_t L^p} \leq C\|\omega_\mu\|^{\frac{1+\eta}{2}}_{\widetilde{L}^r_t B^\eta_{p,\infty}}\|\omega_\mu\|^{\frac{1-\eta}{2}}_{\widetilde{L}^r_t B^{2+\eta}_{p,\infty}}.$$

Proof. To prove this estimate, let $N \in \mathbb{N}$ be a parameter that will be chosen judiciously later. The fact $\Delta v_\mu = \nabla^\perp \omega_\mu$, interpolation in frequency and Bernstein inequality enable us to write

$$
\begin{aligned}
\|\Delta v_\mu\|_{L^r_t L^p} &\leq \sum_{q\leq N} \|\Delta_q \nabla^\perp \omega_\mu\|_{L^r_t L^p} + \sum_{q>N} \|\Delta_q \nabla^\perp \omega_\mu\|_{L^r_t L^p} \\
&\leq \sum_{q\leq N} 2^{q(1-\eta)} 2^{q\eta} \|\Delta_q \omega_\mu\|_{L^r_t L^p} + \sum_{q>N} 2^{q(-1-\eta)} 2^{q(2+\eta)} \|\Delta_q \omega_\mu\|_{L^r_t L^p} \\
&\leq 2^{N(1-\eta)} \|\omega_\mu\|_{\widetilde{L}^r_t B^\eta_{p,\infty}} + 2^{-N(1+\eta)} \|\omega_\mu\|_{\widetilde{L}^r_t B^{2+\eta}_{p,\infty}}.
\end{aligned}
\tag{3.48}
$$

Now, we choose N such that

$$
2^{N(1-\eta)} \|\omega_\mu\|_{\widetilde{L}^r_t B^\eta_{p,\infty}} \approx 2^{-N(1+\eta)} \|\omega_\mu\|_{\widetilde{L}^r_t B^{2+\eta}_{p,\infty}},
$$

whence

$$
2^{2N} \approx \frac{\|\omega_\mu\|_{\widetilde{L}^r_t B^{2+\eta}_{p,\infty}}}{\|\omega_\mu\|_{\widetilde{L}^r_t B^\eta_{p,\infty}}}. \tag{3.49}
$$

Inserting (3.49) in (3.48), we obtain the desired estimate and so the proof is completed. $\square$

Proof of the Theorem 3.4.2. We set $U = v_\mu - v$, $\Theta = \rho_\mu - \rho$ and $P = p_\mu - p$. We intend to estimate the quantity $\|U\|_{L^p} + \|\Theta\|_{L^p}$. To do this, making few computations we discover that U and Θ evolve the nonlinear equations,

$$
\begin{cases}
\partial_t U + (v_\mu \cdot \nabla) U - \mu \Delta v_\mu + \nabla P = \Theta e_2 - (U \cdot \nabla) v & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
\partial_t \Theta + (v_\mu \cdot \nabla)\Theta - \Delta \Theta = -U \cdot \nabla \rho & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
\mathrm{div} U = 0, \\
U_{|t=0} = U_0, \quad \Theta_{|t=0} = \Theta_0.
\end{cases}
\tag{$\widetilde{B}_\mu$}
$$

• **Estimate of $\|U(t)\|_{L^p}$.** Multiply the first equation in $(\widetilde{B}_\mu)$ by $U|U|^{p-2}$, and integrating by parts over the space variables $\mathbb{R}^2$, then in view of $\mathrm{div} v_\mu = \mathrm{div} v = 0$, it follows

$$
\begin{aligned}
\frac{1}{p}\frac{d}{dt}\|U(t)\|^p_{L^p} \leq\ & \int_{\mathbb{R}^2} \big|\nabla P \cdot U|U|^{p-2}\big| dx + \mu \int_{\mathbb{R}^2} \big|\Delta v_\mu \cdot U|U|^{p-2}\big| dx \\
&+ \int_{\mathbb{R}^2} \big|(U \cdot \nabla) v \cdot U|U|^{p-2}\big| dx + \int_{\mathbb{R}^2} \big|\Theta e_2 \cdot U|U|^{p-2}\big| dx.
\end{aligned}
$$

Hölder's inequality yields

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|U(t)\|_{L^p}^p &\leq \|\nabla P(t)\|_{L^p}\|U(t)\|_{L^p}^{p-1}+\mu\|\Delta v_\mu(t)\|_{L^p}\|U(t)\|_{L^p}^{p-1}\\ &+\|\nabla v\|_{L^\infty}\|U(t)\|_{L^p}^p+\|\Theta(t)\|_{L^p}\|U(t)\|_{L^p}^{p-1},\end{aligned}$$

so, integrating in time over $[0,t]$, we obtain

$$\begin{aligned}\|U(t)\|_{L^p} &\leq \|U_0\|_{L^p}+\int_0^t\|\nabla P(\tau)\|_{L^p}d\tau+\mu\int_0^t\|\Delta v_\mu(\tau)\|_{L^p}d\tau \\ &+\int_0^t\|\nabla v(\tau)\|_{L^\infty}\|U(\tau)\|_{L^p}d\tau+\int_0^t\|\Theta(\tau)\|_{L^p}d\tau.\end{aligned}\tag{3.50}$$

Concerning the term $\|\nabla P\|_{L^p}$, applying the "div" operator to the first equation of $(\widetilde{B}_\mu)$, one finds after easy algebraic computations

$$-\Delta P=\mathrm{div}\big(U\cdot\nabla(v_\mu+v)\big)+\partial_2\Theta,$$

then we have

$$-\nabla P=\nabla\Delta^{-1}\mathrm{div}\big(U\cdot\nabla(v_\mu+v)\big)+\nabla\Delta^{-1}\partial_2\Theta.$$

The boundedness of Riesz transform on L^p, $p\in]1,\infty[$ into it self leading to

$$\|\nabla P\|_{L^p}\lesssim\|U\|_{L^p}\big(\|\nabla v_\mu\|_{L^\infty}+\|\nabla v\|_{L^\infty}\big)+\|\Theta\|_{L^p}.$$

Inserting the above estimate into (3.50), we deduce that

$$\begin{aligned}\|U(t)\|_{L^p} &\lesssim \|U_0\|_{L^p}+\int_0^t\|U(\tau)\|_{L^p}\big(\|\nabla v_\mu(\tau)\|_{L^\infty}+\|\nabla v(\tau)\|_{L^\infty}\big)d\tau \\ &+\mu\int_0^t\|\Delta v_\mu(\tau)\|_{L^p}d\tau+\int_0^t\|\Theta(\tau)\|_{L^p}d\tau.\end{aligned}\tag{3.51}$$

• **Estimate of** $\|\Theta\|_{L^1_tL^p}$. Multiply the second equation in $(\widetilde{B}_\mu)$ by $\Theta|\Theta|^{p-2}$, and integrating by parts over $\mathbb{R}^2$. Then by virtue of $\mathrm{div}v_\mu=\mathrm{div}v=0$, it happens

$$\frac{1}{p}\frac{d}{dt}\|\Theta(t)\|_{L^p}^p+(p-1)\int_{\mathbb{R}^2}|\nabla\Theta(t)|^2|\Theta(t)|^{p-2}dx\leq\int_{\mathbb{R}^2}\big|U(t)\cdot\nabla\rho(t)\big||\Theta(t)|^{p-1}dx,$$

owing to Hölder's inequality, we shall have

$$\frac{1}{p}\frac{d}{dt}\|\Theta(t)\|_{L^p}^p+(p-1)\int_{\mathbb{R}^2}|\nabla\Theta(t)|^2|\Theta(t)|^{p-2}dx\leq\|U(t)\|_{L^p}\|\nabla\rho(t)\|_{L^\infty}\|\Theta(t)\|_{L^p}^{p-1}.$$

Since the second term of the left-hand side has a non-negative sign, one obtains

$$\frac{d}{dt}\|\Theta(t)\|_{L^p} \lesssim \|U(t)\|_{L^p}\|\nabla\rho(t)\|_{L^\infty}.$$

Integrating in time over $[0,t]$, we get

$$\|\Theta(t)\|_{L^p} \lesssim \|\Theta_0\|_{L^p} + \int_0^t \|U(\tau)\|_{L^p}\|\nabla\rho(\tau)\|_{L^\infty}d\tau. \tag{3.52}$$

Putting together (3.51) and (3.52), we readily get

$$\begin{aligned}\|U(t)\|_{L^p} + \|\Theta(t)\|_{L^p} \quad \lesssim \quad & \|U_0\|_{L^p} + \|\Theta_0\|_{L^p} + \int_0^t \|U(\tau)\|_{L^p}\big(\|\nabla v_\mu(\tau)\|_{L^\infty} \\ & +\|\nabla v(\tau)\|_{L^\infty}\big)d\tau + \mu\int_0^t \|\Delta v_\mu(\tau)\|_{L^p}d\tau \\ & + \int_0^t \|\Theta(\tau)\|_{L^p}d\tau + \int_0^t \|U(\tau)\|_{L^p}\|\nabla\rho(\tau)\|_{L^\infty}d\tau.\end{aligned}$$

Since $\Pi(t) \triangleq \|U(t)\|_{L^p} + \|\Theta(t)\|_{L^p}$, then after few caculations we find that

$$\begin{aligned}\Pi(t) \quad \lesssim \quad & \Pi(0) + \int_0^t \big(1 + \|\nabla v_\mu(\tau)\|_{L^\infty} + \|\nabla v(\tau)\|_{L^\infty} + \|\nabla\rho(\tau)\|_{L^\infty}\big)\Pi(\tau)d\tau \\ & +\mu\int_0^t \|\Delta v_\mu(\tau)\|_{L^p}d\tau.\end{aligned}$$

Using Gronwall's inequality, we can write

$$\Pi(t) \quad \lesssim \quad e^{Ct}e^{C(V_\mu(t)+V(t)+\|\nabla\rho\|_{L^1_tL^\infty})}\big(\Pi(0) + \mu\|\Delta v_\mu\|_{L^1_tL^p}\big). \tag{3.53}$$

We now turn to the estimate of the principal term $\mu\|\Delta v_\mu\|_{L^1_tL^p}$ which provides the desired rate of convergence. Take in Proposition 3.4.3 $\eta = \frac{1}{p}$ and $r = 1$ to obtain

$$\mu\|\Delta v_\mu\|_{L^1_tL^p} \leq \underbrace{\mu\|\omega_\mu\|^{\frac12+\frac{1}{2p}}_{\widetilde{L}^1_tB^{\frac1p}_{p,\infty}}}_{\mathrm{I}}\underbrace{\|\omega_\mu\|^{\frac12-\frac{1}{2p}}_{\widetilde{L}^1_tB^{2+\frac1p}_{p,\infty}}}_{\mathrm{II}}. \tag{3.54}$$

For the term I, applying the Hölder inequality in the time variable, we deduce that

$$\mathrm{I} \leq \mu t^{\frac12+\frac{1}{2p}}\|\omega_\mu\|^{\frac12+\frac{1}{2p}}_{\widetilde{L}^\infty_tB^{\frac1p}_{p,\infty}}.$$

Put $r=\infty, s=\frac{1}{p}, p_1=p, p_2=\infty$, Proposition 4.2.9 tell us

$$\|\omega_\mu\|_{\widetilde{L}^\infty_t B^{\frac{1}{p}}_{p,\infty}} \leq Ce^{CV_\mu(t)}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big).$$

Hence

$$\mathrm{I} \leq Ce^{CV_\mu(t)}\mu t^{\frac{1}{2}+\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big)^{\frac{1}{2}+\frac{1}{2p}}.$$

Concerning II, a new use of the Proposition 4.2.9 gives for $r=1, s=\frac{1}{p}, p_1=p, p_2=\infty$ the following

$$\|\omega_\mu\|_{\widetilde{L}^1_t B^{2+\frac{1}{p}}_{p,\infty}} \leq Ce^{CV_\mu(t)}\mu^{-1}(1+\mu t)\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big).$$

Accordingly, we infer

$$\mathrm{II} \leq Ce^{CV_\mu(t)}\mu^{\frac{1}{2p}-\frac{1}{2}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big)^{\frac{1}{2}-\frac{1}{2p}}.$$

Combining I and II, (3.54) becomes

$$\mu\|\Delta v_\mu\|_{L^1_t L^p} \lesssim Ce^{CV_\mu(t)}(\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big).$$

By means of the embedding $\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1} = L^1_t B^{\frac{1}{p}}_{p,1} \hookrightarrow L^1_t B^{\frac{1}{p}}_{p,\infty}$, the last estimate takes the form

$$\mu\|\Delta v_\mu\|_{L^1_t L^p} \lesssim Ce^{CV_\mu(t)}(\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\nabla\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1}}\Big)$$

together with (3.53), one obtains

$$\begin{aligned}\Pi(t) \quad \lesssim \quad & Ce^{C(t+V_\mu(t)+V(t)+\|\nabla\rho\|_{L^1_t L^\infty})}\Big(\Pi(0) + (\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} \\ & + \|\nabla\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1}}\Big)\Big).\end{aligned}$$

For the term $\|\nabla\rho\|_{L^1_t L^\infty}$ in the exponential, applying the Propositon 3.2.12 the last estimate takes the form

$$\Pi(t) \lesssim Ce^{C(t+V_\mu(t)+V(t))}\Big(\Pi(0) + (\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\nabla\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1}}\Big)\Big). \tag{3.55}$$

To end the proof of our claim, let us estimate $\|\nabla\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}}_{p,1}}$. Note that ∇ maps continuously $B^{1+\frac{1}{p}}_{p,1}$ into $B^{\frac{1}{p}}_{p,1}$, then the Proposition 4.2.9 combined with $L^p \hookrightarrow B^{\frac{1}{p}-1}_{p,1}$

gives for $p > 1$

$$\begin{aligned} \|\rho_\mu\|_{\widetilde{L}^1_t B^{\frac{1}{p}+1}_{p,1}} &\le Ce^{CV_\mu(t)}(1+t)\|\rho^0_\mu\|_{B^{\frac{1}{p}-1}_{p,1}} \\ &\le Ce^{CV_\mu(t)}(1+t)\|\rho^0_\mu\|_{L^p}. \end{aligned} \tag{3.56}$$

Plugging (3.56) in (3.55), we find that

$$\Pi(t) \lesssim Ce^{C(t+V_\mu(t)+V(t))}\Big(\Pi(0)+(\mu t)^{\frac{1}{2}+\frac{1}{2p}}(1+\mu t)^{\frac{1}{2}-\frac{1}{2p}}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}}+\|\rho^0_\mu\|_{L^p}\Big)\Big). \tag{3.57}$$

Hence the proof of the Theorem 3.4.2 is accomplished. □

3.4.2 Proof of Theorem 3.1.4

(i) Substituting (3.3) and (3.4) into (3.57) and the fact $\mathbf{1}_{\Omega_0} \in B^{\frac{1}{p}}_{p,\infty}$, it happens for $\mu \in]0,1[$

$$\Pi(t) \lesssim C_0 e^{e^{C_0 t \log^2(1+t)}}(\mu t)^{\frac{1}{2}+\frac{1}{2p}}.$$

(ii) To estimate $\omega_\mu - \omega$ in L^p-norm, using the definition of ω_μ and ω we shall have

$$\|\omega_\mu(t)-\omega(t)\|_{L^p} \le \|\nabla(v_\mu(t)-v(t))\|_{L^p}$$

combined with $B^0_{p,1} \hookrightarrow L^p$ and Bernstein inequality leads to

$$\|\omega_\mu(t)-\omega(t)\|_{L^p} \lesssim \|v_\mu(t)-v(t)\|_{B^1_{p,1}}. \tag{3.58}$$

On the other hand, let N be a fixed number that will be chosen later. Again Bernstein's inequality leading to

$$\begin{aligned} \|v_\mu(t)-v(t)\|_{B^1_{p,1}} &\le \sum_{q\le N} 2^q\|\Delta_q(v_\mu(t)-v(t))\|_{L^p} + \sum_{q>N} 2^{-\frac{q}{p}}2^{\frac{q}{p}}\|\Delta_q\nabla(v_\mu(t)-v(t))\|_{L^p} \\ &\lesssim 2^N\|v_\mu(t)-v(t)\|_{L^p} + \sup_{q\ge -1} 2^{\frac{q}{p}}\|\omega_\mu(t)-\omega(t)\|_{L^p}\sum_{q>N}2^{-\frac{q}{p}} \\ &\lesssim 2^N\|v_\mu(t)-v(t)\|_{L^p} + 2^{-\frac{N}{p}}\|\omega_\mu(t)-\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}. \end{aligned} \tag{3.59}$$

In the second line we have used the fact

$$\|\Delta_q\nabla(v_\mu(t)-v(t))\|_{L^p} \approx \|\Delta_q(\omega_\mu(t)-\omega(t))\|_{L^p}, \quad \forall q \in \mathbb{N}.$$

Taking

$$2^{N(1+\frac{1}{p})} \approx \frac{\|\omega_\mu(t)-\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}}{\|v_\mu(t)-v(t)\|_{L^p}}.$$

Then (3.58) and (3.59) lead us

$$\|v_\mu(t)-v(t)\|_{B^1_{p,1}} \lesssim \|v_\mu(t)-v(t)\|_{L^p}^{\frac{1}{p+1}} \|\omega_\mu(t)-\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}^{\frac{p}{1+p}},$$

whence (3.58) yields

$$\|\omega_\mu(t)-\omega(t)\|_{L^p} \leq \|v_\mu(t)-v(t)\|_{L^p}^{\frac{1}{p+1}} \|\omega_\mu(t)-\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}^{\frac{p}{p+1}},$$

in accordance with the Theorem 3.1.4, it holds

$$\|\omega_\mu(t)-\omega(t)\|_{L^p} \leq C_0 e^{e^{C_0 t\log^2(2+t)}} (\mu t)^{\frac{1}{2p}}(1+\mu t)\|\omega_\mu(t)-\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}.$$

To finalize, let us estimate $\|\omega_\mu(t)-\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}}$. To do this, using the persistence of Besov spaces explicitly formulated in the Proposition 3.2.10, one gets

$$\begin{aligned}
\|\omega_\mu(t)-\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}} &\leq \|\omega_\mu(t)\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}} \\
&\leq Ce^{C(V_\mu(t)+V(t))}\Big(\|\omega^0_\mu\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\omega^0\|_{B^{\frac{1}{p}}_{p,\infty}} + \|\nabla\rho_\mu\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}} \\
&\qquad + \|\nabla\rho\|_{L^1_t B^{\frac{1}{p}}_{p,\infty}}\Big).
\end{aligned}$$

The last two terms of the right-hand side stem from (3.56). Then thanks to (3.3) and (3.4), we end up with

$$\|\omega_\mu(t)-\omega(t)\|_{B^{\frac{1}{p}}_{p,\infty}} \leq C_0 e^{e^{C_0 t\log^2(2+t)}} (\mu t)^{\frac{1}{2p}}(1+\mu t).$$

This achieves the proof of the aimed estimate.

3.4.3 Optimality of the rate of convergence

In this paragraph we shall give the proof of Theorem 3.1.6 by showing that $(\mu t)^{\frac{1}{2p}}$ is optimal in L^p norm in the case of a circular vortex patch and ρ^0_μ and ρ^0 are constant densities.

Proof of Theorem 3.1.6. Since the initial data $\omega_\mu^0 = \omega^0 = \mathbf{1}_{\mathbb{D}}$ are radial then this structure is preserved in the evolution and thus

$$v_\mu \cdot \nabla \omega_\mu, \quad v \cdot \nabla \omega = 0.$$

Therefore the equation of ω_μ (resp. ω) takes the following form

$$\partial_t \omega_\mu - \mu \Delta \omega_\mu = 0, \quad \partial_t \omega = 0.$$

Recall that the solutions of the above equations are given by

$$\omega_\mu(t,x) = K_{\mu t} \star \omega_\mu^0(x), \quad \omega(t,x) = \omega^0(x), \tag{3.60}$$

where $K_{\mu t}$ is the heat kernel defined by

$$K_{\mu t}(x) \triangleq \frac{1}{4\pi\mu t} e^{-\frac{|x|^2}{4t}}$$

and satisfies

$$\int_{\mathbb{R}^2} K_{\mu t}(x) dx = 1.$$

On the other hand, setting $W(t,x) = \omega_\mu(t,x) - \omega(t,x)$. Then in view of (3.60), we have

$$W(t,x) = \int_{\mathbb{R}^2} K_{\mu t}(x-y)[\mathbf{1}_{\mathbb{D}}(y) - \mathbf{1}_{\mathbb{D}}(x)] dy.$$

For $|x| < 1$ we have

$$\begin{aligned} W(t,x) &= \int_{\{|y|\geq 1\}} K_{\mu t}(x-y) dy \\ &= \frac{1}{4\pi\mu t} \int_{\{|y|\geq 1\}} e^{-\frac{|x-y|^2}{4\mu t}} dy. \end{aligned}$$

Introduce $Z(t,x) = W(t, \sqrt{\mu t} x)$ and make the change of variables $y = \sqrt{\mu t} z$, one gets

$$Z(t,x) = \frac{1}{4\pi} \int_{\{|z|\geq \frac{1}{\sqrt{\mu t}}\}} e^{-\frac{|x-z|^2}{4}} dz, \quad |x| \leq \frac{1}{\sqrt{\mu t}}. \tag{3.61}$$

Let $\mu t \leq 1$, then

$$\begin{aligned} \|W(t)\|_{L^p(\mathbb{R}^2)} &\geq \|W(t)\|_{L^p(1-\sqrt{\mu t}\leq |x|\leq 1)} \\ &\geq (\mu t)^{\frac{1}{p}} \|Z(t)\|_{L^p(\frac{1}{\sqrt{\mu t}}-1\leq |x| \leq \frac{1}{\sqrt{\mu t}})}. \end{aligned} \tag{3.62}$$

Now, our task is to prove the following requirement,

$$\|Z(t)\|_{L^p(\frac{1}{\sqrt{\mu t}}-1\leq|x|\leq\frac{1}{\sqrt{\mu t}})} \geq C_2(\mu t)^{-\frac{1}{2p}}. \tag{3.63}$$

For this purpose, we plug the identity $|x-z|^2 \triangleq |x|^2+|z|^2-2\langle x,z\rangle$ into (3.61),

$$Z(t,x) = \frac{1}{4\pi}e^{-\frac{|x|^2}{4}}\int_{\{|z|\geq\frac{1}{\sqrt{\mu t}}\}} e^{-\frac{|z|^2}{4}+\frac{1}{2}\langle x,z\rangle}dz.$$

By rotation invariance, the above equation becomes

$$\begin{aligned} Z(t,x) &= \frac{1}{4\pi}e^{-\frac{|x|^2}{4}}\int_0^{2\pi}\int_{\frac{1}{\sqrt{\mu t}}}^{+\infty} e^{-\frac{r^2}{4}+\frac{1}{2}r|x|\cos\theta}rdrd\theta \\ &\geq \frac{1}{4\pi}e^{-\frac{|x|^2}{4}}\int_0^{\frac{\pi}{2}}\int_{\frac{1}{\sqrt{\mu t}}}^{+\infty} e^{-\frac{r^2}{4}+\frac{1}{2}r|x|\cos\theta}rdrd\theta. \end{aligned}$$

Since $\cos\theta \geq 1-\frac{\theta^2}{2}$ for $\theta\geq 0$, then we find

$$\begin{aligned} |Z(t,x)| &\geq \frac{1}{4\pi}\int_{\frac{1}{\sqrt{\mu t}}}^{+\infty} e^{-\frac{|x|^2}{4}-\frac{r^2}{4}+\frac{r|x|}{2}}\left(\int_0^{\frac{\pi}{2}} e^{-\frac{1}{4}r|x|\theta^2}d\theta\right)rdr \\ &= \frac{1}{4\pi}\int_{\frac{1}{\sqrt{\mu t}}}^{+\infty} e^{-\frac{1}{4}(|x|-r)^2}\left(\int_0^{\frac{\pi}{2}} e^{-\frac{1}{4}r|x|\theta^2}d\theta\right)rdr. \end{aligned}$$

Here, we have used Fubini's theorem. For the second integral of the right-hand side, using the change of variables $\alpha=\frac{1}{2}\sqrt{r|x|}\theta$, we get

$$|Z(t,x)| \geq \frac{1}{2\pi}\int_{\frac{1}{\sqrt{\mu t}}}^{\frac{2}{\sqrt{\mu t}}} e^{-\frac{1}{4}(|x|-r)^2}\left(\int_0^{\sqrt{r|x|}\frac{\pi}{4}} e^{-\alpha^2}\frac{d\alpha}{\sqrt{r|x|}}\right)rdr. \tag{3.64}$$

Since $r|x| \geq \frac{1}{\sqrt{\mu t}}\left(\frac{1}{\sqrt{\mu t}}-1\right) \approx \frac{1}{\mu t} \geq 1$, then we obtain that

$$\int_0^{\sqrt{r|x|}\frac{\pi}{4}} e^{-\alpha^2}\frac{d\alpha}{\sqrt{r|x|}} \geq \frac{1}{\sqrt{r|x|}}\int_0^{\frac{\pi}{4}} e^{-\alpha^2}d\alpha = \frac{c}{\sqrt{r|x|}}.$$

Consequently for $\frac{1}{\sqrt{\mu t}}-1\leq|x|\leq\frac{1}{\sqrt{\mu t}}$, the formula (3.64) takes the following form

$$|Z(t,x)| \geq C\int_{\frac{1}{\sqrt{\mu t}}}^{\frac{2}{\sqrt{\mu t}}} e^{-\frac{1}{4}(|x|-r)^2}\sqrt{\frac{r}{|x|}}dr.$$

But, $\frac{r}{|x|} \geq \frac{1}{\sqrt{\mu t}}\sqrt{\mu t} = 1$ and hence

$$|Z(t,x)| \geq C \int_{\frac{1}{\sqrt{\mu t}}}^{\frac{2}{\sqrt{\mu t}}} e^{-\frac{1}{4}(|x|-r)^2} dr.$$

Making the change of variables $k = r - |x|$, we readily get

$$|Z(t,x)| \geq C \int_{\frac{1}{\sqrt{\mu t}}-|x|}^{\frac{2}{\sqrt{\mu t}}-|x|} e^{-\frac{1}{4}k^2} dk.$$

However, $\frac{1}{\sqrt{\mu t}} - |x| \leq 1$ and $\frac{2}{\sqrt{\mu t}} - |x| \geq \frac{1}{\sqrt{\mu t}}$. This leads to

$$|Z(t,x)| \geq C \int_{1}^{\frac{1}{\sqrt{\mu t}}} e^{-\frac{1}{4}k^2} dk \geq C > 0.$$

Therefore, for $\frac{1}{\sqrt{\mu t}} - 1 \leq |x| \leq \frac{1}{\sqrt{\mu t}}$, it follows

$$|Z(t,x)| \geq C. \tag{3.65}$$

Taking the L^p-norm for (3.65) over the annulus $\frac{1}{\sqrt{\mu t}} - 1 \leq |x| \leq \frac{1}{\sqrt{\mu t}}$, it holds

$$\begin{aligned}
\|Z(t)\|_{L^p(\frac{1}{\sqrt{\mu t}}-1\leq|x|\leq\frac{1}{\sqrt{\mu t}})} &\geq C\left[\mathscr{L}\left(\frac{1}{\sqrt{\mu t}} - 1 \leq |x| \leq \frac{1}{\sqrt{\mu t}}\right)\right]^{\frac{1}{p}} \\
&\geq C\left[\pi\left(\frac{2}{\sqrt{\mu t}} - 1\right)\right]^{\frac{1}{p}} \\
&\geq \widetilde{C}(\mu t)^{-\frac{1}{2p}},
\end{aligned}$$

where $\mathscr{L}$ is the Lebesgue measure over $\mathbb{R}^2$. Hence,

$$\|Z(t)\|_{L^p(\frac{1}{\sqrt{\mu t}}-1\leq|x|\leq\frac{1}{\sqrt{\mu t}})} \geq C_1(\mu t)^{-\frac{1}{2p}}.$$

This leads to the desired estimate stated in (3.63). Combining the last estimate with (3.62), we end up with

$$\|W(t)\|_{L^p(\mathbb{R}^2)} \geq C_1(\mu t)^{\frac{1}{2p}}.$$

Now, the proof is completed. $\square$

3.5 Appendix

This section teats the detailed proof of two Propositions 3.5.1, 3.5.2 which are used respectively during the proof of Theorem 4.3.1 and Proposition 3.4.1.

Proposition 3.5.1. *Let $\varepsilon\ \in]0,1[, \rho$ be a smooth function and v be a smooth divergence-free vector field on $\mathbb{R}^2$ with vorticity ω. Assume that $v \in L^2, \omega \in L^2 \cap L^\infty$ and $\rho \in L^2 \cap L^p$, with $p > \frac{2}{1-\varepsilon}$. Then the following statement holds true,*

$$\big\| [\mathcal{L}, v \cdot \nabla] \rho \big\|_{C^\varepsilon} \leq C_0.$$

Proof. Recall from [52] the following commutator estimate,

$$\big\| [\mathcal{L}, v \cdot \nabla] \rho \big\|_{C^\varepsilon} \lesssim \|v\|_{L^2} \|\rho\|_{L^2} + \|\omega\|_{L^2 \cap L^\infty} \|\rho\|_{L^p}, \quad p > \frac{2}{1-\varepsilon}. \tag{3.66}$$

Let us estimate the first term of the right-hand side of (3.66). To do this, we apply the energy estimate for the velocity equation, we shall have

$$\|v(t)\|_{L^2} \leq \|v_0\|_{L^2} + \int_0^t \|\rho(\tau)\|_{L^2} d\tau.$$

A new use of [52] gives

$$(1+t)^{\frac{1}{2}} \|\rho(t)\|_{L^2} \lesssim \|\rho_0\|_{L^1 \cap L^2}$$

thus we obtain

$$\|v(t)\|_{L^2} \leq C_0 (1+t)^{\frac{1}{2}}.$$

Combining the last two estimates, we readily get

$$\|v(t)\|_{L^2} \|\rho(t)\|_{L^2} \leq C_0. \tag{3.67}$$

An usual interpolation inequality between the Lebesgue spaces yields for $p \in [2, +\infty[$

$$\begin{aligned} \|\rho(t)\|_{L^p} &\leq \|\rho(t)\|_{L^2}^{\frac{2}{p}} \|\rho_0\|_{L^\infty}^{1-\frac{2}{p}} \\ &\leq C_0 (1+t)^{-\frac{1}{p}}. \end{aligned} \tag{3.68}$$

Here we have used the maximum principle for the density equation. Putting together

(3.67), (3.68) and Proposition 3.2.12, we finally get

$$\begin{aligned}\big\|\big[\mathcal{L}, v\cdot\nabla\big]\rho\big\|_{C^\varepsilon} &\leq C_0 + C_0(1+t)^{-\frac{1}{p}}\log^2(2+t)\\ &\leq C_0.\end{aligned}$$

This completes the proof. □

For the reader's convenience we state the following classical result.

Proposition 3.5.2. *The following Sobolev embedding is hold.*

$$BV \hookrightarrow \dot{B}^1_{1,\infty}.$$

Proof. According to [59, 68] the equivalent norm to $\dot{B}^s_{p,r}$ is defined for $\ell \in \mathbb{N}^*$, $0 < s < \ell$ and $(p,r) \in [1,\infty]^2$ by

$$|||u|||_{\dot{B}^s_{p,r}} \triangleq \left(\int_{\mathbb{R}^N} |h|^{-sr}\|\boldsymbol{\Delta}^\ell_h f(x)\|^r_{L^p}\frac{dh}{|h|^N}\right)^{\frac{1}{r}}.$$

Here the difference operators $\boldsymbol{\Delta}^\ell_h$ are given by

$$\boldsymbol{\Delta}^1_h = \boldsymbol{\Delta}_h, \quad \boldsymbol{\Delta}^{\ell+1}_h = \boldsymbol{\Delta}_h\circ\boldsymbol{\Delta}^\ell_h \quad \forall \ell\in\mathbb{N}^*,$$

where $\boldsymbol{\Delta}_h$ is defined for every $u \in \mathcal{S}'(\mathbb{R}^N)$ and $h \in \mathbb{R}^N$ by

$$\boldsymbol{\Delta}_h u(x) \triangleq u(x+h) - u(x).$$

From (3.6), we have for $q\in\mathbb{Z}$ and $x\in\mathbb{R}^2$

$$\dot{\Delta}_q u(x) = 2^{2q}\int_{\mathbb{R}^2}\mathscr{F}^{-1}\varphi(2^q(x-y))u(y)dy,$$

with $\mathscr{F}^{-1}\varphi$ denotes the inverse Fourier of φ. As $\varphi(0) = 0$ then

$$\dot{\Delta}_q u(x) = 2^{2q}\int_{\mathbb{R}^2}\mathscr{F}^{-1}\varphi(2^q(x-y))\big(u(y)-u(x)\big)dy.$$

So, by making a change of variable $z = 2^d(x-y)$, we obtain

$$\begin{aligned}
\dot{\Delta}_q u(x) &= 2^{2q}\int_{\mathbb{R}^2} \mathscr{F}^{-1}\varphi(2^q(x-y))(u(y)-u(x))dy\\
&= \int_{\mathbb{R}^2} \mathscr{F}^{-1}\varphi(z)(u(x-2^{-q}z)-u(x))dz\\
&= \int_{\mathbb{R}^2} \mathscr{F}^{-1}\varphi(z)\boldsymbol{\Delta}_h u(x)dz, \quad h=-2^{-q}z.
\end{aligned}$$

Fubini's theorem implies

$$\|\dot{\Delta}_q u\|_{L^1} \le \int_{\mathbb{R}^2} \mathscr{F}^{-1}\varphi(z)\|\boldsymbol{\Delta}_h u\|_{L^1}dz. \tag{3.69}$$

We recall from Theorem 13.48 page 415 in [59] the following result

$$\begin{aligned}
\|\boldsymbol{\Delta}_h u\|_{L^1} &\le |h||Du|(\mathbb{R}^2)\\
&= 2^{-q}|z||Du|(\mathbb{R}^2).
\end{aligned}$$

Consequently

$$\|\boldsymbol{\Delta}_h u\|_{L^1} \le 2^{-q}|z|\|u\|_{BV}.$$

Inserting the last estimate in (3.69), we get for $q\in\mathbb{Z}$

$$\|\dot{\Delta}_q u(x)\|_{L^1} \le 2^{-q}\|u\|_{BV}\int_{\mathbb{R}^2} \mathscr{F}^{-1}\varphi(z)|z|dz.$$

By taking the supremum over $q\in\mathbb{Z}$, we obtain the aimed estimate. □

4 Local persistence of geometric structures for Boussinesq system with zero viscosity

This Chapter is the subject of the following publication:

H. Meddour: *Local persistence of geometric structure for Boussinesq system with zero viscosity*. Mathematicki Vesnik, Vol. 71, no. 4, (2019), 285–303.

4.1 Introduction

We are mainly concerned with studying the local well-posedness theory for the partial viscous Boussinesq system given by the coupled equations,

$$\begin{cases} \partial_t v_\kappa + v_\kappa \cdot \nabla v_\kappa + \nabla p_\kappa = \rho_\kappa \vec{e}_2, \quad t \geq 0,\ x \in \mathbb{R}^2 \\ \partial_t \rho_\kappa + v_\kappa \cdot \nabla \rho_\kappa - \kappa \Delta \rho_\kappa = 0, \\ \operatorname{div} v_\kappa = 0. \end{cases} \tag{B_κ}$$

It describes the evolution of stratified incompressible fluids in $\mathbb{R}^2$ under the influence of the gravity force which is proportional to ρ_κ in the direction $\vec{e}_2 = (0, 1)$; for the derivation of this model, see for instance [65]. Above, the velocity vector field $v_\kappa \in \mathbb{R}^2$ is solenoidal, $\pi_\kappa \in \mathbb{R}$ is the pressure and $\rho_\kappa \in \mathbb{R}_+$ is the density. The parameter $\kappa \geq 0$ denotes the molecular diffusivity of the fluid. We will consider the Cauchy problem to the Boussinesq system by prescribing the initial data

$$v_{\kappa|t=0} = v_\kappa^0, \quad \rho_{\kappa|t=0} = \rho_\kappa^0.$$

Note that if the initial density vanishes $\rho_\kappa^0 \equiv 0$ and therefore the system (B_κ) reduces to the classical Euler equations given by

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v^0. \end{cases} \tag{E}$$

One of the basic feature in the dynamics of Euler equations is related to the vorticity $\omega \triangleq \partial_1 v^2 - \partial_2 v^1$ which is transported by the flow,

$$\begin{cases} \partial_t \omega + v\cdot\nabla \omega = 0, \\ \omega_{|t=0} = \omega^0, \\ v = \mathscr{N}_2 \star \omega, \end{cases} \tag{4.1}$$

where, $\mathscr{N}_2$ is the Biot-Savart kernel defined by

$$\mathscr{N}_2(x) = \nabla^\perp E_2(x), \quad E_2(x) = \frac{1}{2\pi}\log\|x\|, \quad \nabla^\perp \triangleq (-\partial_2, \partial_1).$$

Let us denote by $\Psi = (\Psi_t)$ the flow (*particle-trajectory mapping*) associated to the time-dependent velocity vector field v, so that

$$\partial_t \Psi_t(x) = v(t, \Psi_t(x)), \quad \Psi_0(x) = x.$$

Thus the solution of (4.1) is determined explicitely by $\omega(t, \Psi_t(x)) = \omega^0(x)$ and admits in turn an infinite conservation laws. For example, all the L^p norms are time invariant, that is, $\|\omega(t)\|_{L^p} = \|\omega^0\|_{L^p}$ for $p \in [1,\infty]$. Under this pattern, Yudovich succeed in [75] to obtain global unique weak solutions for the system (E) whenever $\omega^0 \in L^1 \cap L^\infty$. Furthermore, the velocity vector field, which is not necessary Lipschitz, belongs to the class of $\log -$Lipschitz functions and the corresponding flow map Ψ is a planar homeomorphism. Notice that Yudovich's class encompasses vortex patches, that is, ω^0 is represented by the characteristic function of a bounded domain $\Omega_0 \subset \mathbb{R}^2$. This structure is preserved during the time, meaning that $\omega(t) = \mathbf{1}_{\Omega_t}$, with $\Omega_t = \Psi_t(\Omega_0)$ is the patch that moves with the flow.

In-depth study of vortex patches, whose dynamics is governed by the motion of closed curves in the complex plane, has led to several questions especially about the boundary regularity. A remarkable result in this way is due to Chemin [19] (see also P. Serfati [67]), ensures that when the boundary $\partial\Omega_0$ belongs to the Hölderian class $C^{1+\varepsilon}$, with $0 < \varepsilon < 1$, then the regularity of $\partial\Omega_t$ is shown to be retained over the time. Actually, Chemin's strategy requires essentially the control of the

Liscphitz norm of the velocity with respect to the co-normal regularity $\partial_{X_t}\omega$ of the vorticity in Hölder spaces $C^{\varepsilon-1}$ by means of logarithmic estimate. The choice of the family $X_t = (X_{t,\lambda})_{\lambda\in\Lambda}$ can be done in such way that it is non-degenerate and being tangential to $\partial\Omega_t$. The vector field $X_{t,\lambda}$ is the push-forward of $X_{0,\lambda}$ by the flow $\Psi(t)$,

$$(\partial_t + v\cdot\nabla)X_{t,\lambda} = \partial_{X_{t,\lambda}} v. \tag{4.2}$$

Those vector fields commute with the transport operator $\partial_t + v\cdot\nabla$, and consequently

$$(\partial_t + v\cdot\nabla)\partial_{X_{t,\lambda}}\omega = 0. \tag{4.3}$$

This allows to follow the tangential regularity of the vorticity which is a central step in the study of the vortex patch issue.

As the Boussinesq system (B_κ) is in some sense a perturbation of (E), it will be of interest to ask whether the known results for Euler equations can be extended to the Boussinesq system as well. The topic of local/global posedness for (B_κ) for $\kappa > 0$ has drawn great attention and widely studied during the last years. Particularly, worth mentioning that Chae showed in [15] that (B_κ) is globally well-posed whenever $(v^0,\rho^0)\in H^s\times H^s$, with $s>2$. This result was improved later by Hmidi and Keraani in [45], where they imposed that $(v^0,\rho^0)\in B_{p,1}^{1+\frac{2}{p}}\times B_{p,1}^{-1+\frac{2}{p}}\cap L^r$, with $r>2$. In the same fashion, Hmidi and Zerguine [51] established similar result in the setting of fractional laplacian $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha\in]1,2]$. In [26], Danchin and Paicu extended weak solutions of Yudovich's type to the system (B_κ). For further discussions about this subject, we refer to [2, 3, 9, 16, 17, 25, 33] and the references therein.

In this chapter we intend to conduct a detailed study of the vortex patch problem for the system (B_κ) and to investigate the convergence towards the inviscid system when the parameter κ goes to zero. Note that the limit system is simply obtained by taking $\kappa = 0$, that is

$$\begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = \rho\vec{e}_2, & t\geq 0, x\in\mathbb{R}^2\\ \partial_t\rho + v\cdot\nabla\rho = 0, \\ \operatorname{div} v = 0. \end{cases} \tag{B_0}$$

We point out that for the latter system local well-posedness can be implemented in various function spaces similarly to Euler equations. For instance, Chae and Nam showed in [16] that (B_0) is locally well-posed in Sobolev spaces H^s with $s>2$. This result was extended to critical Besov spaces $B_{p,1}^{1+\frac{2}{p}}$, $p\in]1,\infty[$ by Liu, Wang and

Zhang in [54]. The global existence of classical solutions is an outstanding open problem.

The study of the vortex patch problem for the system (B_κ) was done in [52] with $\kappa = 1$. It is shown that if the boundary of the initial vortex patch belongs to $C^{1+\varepsilon}$ for $0 < \varepsilon < 1$ then the velocity is a Lipschitz function globally in time and the transported patch, that is, Ω_t keeps its initial regularity. Furthermore, the vorticity is given by the decomposition $\omega(t) = \mathbf{1}_{\Omega_t} + \widetilde{\rho}(t)$, with $\widetilde{\rho}$ a smooth function. A similar result has been done recently in [78] for the system (B_κ) with critical fractional dissipation which has obtained a sharper result compared to the incompressible Euler equations [19]. In the same spirit, Hassainia and Hmidi [39] showed that the system (B_0) is locally well-posed whenever the initial patch has a regular/singular structure. The related subject about the aforementionned topics are selected in [27, 29, 32, 34, 38, 41, 45, 63] and the references therein.

At this stage, the first main result of this chapter is summarized in the following Theorem where we deal with local theory for the vortex patch problem uniformly with respect to the parameter κ. More accurately we have:

Theorem 4.1.1. *Let $\kappa \in [0, 1]$ and consider a bounded domain Ω_0 in $\mathbb{R}^2$ whose boundary $\partial\Omega_0$ is a Jordan curve of $C^{1+\varepsilon}-$regularity, with $0 < \varepsilon < 1$. Let v_κ^0 be a divergence-free vector field such that its vorticity $\omega_\kappa^0 = \mathbf{1}_{\Omega_0}$ and the initial density $\rho_\kappa^0 \in L^2 \cap C^{1+\varepsilon}$ with $\nabla\rho_\kappa^0 \in L^2$. Then there exists $T > 0$ independent of κ such that the system (B_κ) admits a unique local solution $(v_\kappa, \rho_\kappa) \in \Big(L^\infty\big([0, T]; \mathrm{Lip}(\mathbb{R}^2)\big)\Big)^2$. Furthermore, for all $t \in [0, T]$ the boundary $\partial\Omega_t$ is a Jordan curve of class $C^{1+\varepsilon}$, with $\Omega_t = \Psi_t(\Omega_0)$.*

Remark 4.1.2. *We note that the initial condition $\rho_\kappa^0 \in C^{1+\varepsilon}$ doesn't persist in time, that is $\rho_\kappa(t) \in C^{1+\varepsilon}$ being false in general for any positive time. Because the velocity field requires more regularity than the Lipschitz one.*

The main step in the proof of Theorem 4.1.1 is to get an estimate for the Lipschitz norm of the velocity locally in time uniformly on $\kappa \in [0, 1]$. For this purpose, we will employ the original Chemin's approach [19]. Thus we shall control $\|\nabla v_\kappa(t)\|_{L^\infty}$ with respect to the co-normal regularity of the vorticity $\partial_{X_t}\omega_\kappa$ in $C^{\varepsilon-1}$, with $0 < \varepsilon < 1$ by means of logarithmic estimate. The family of vector fileds $X_t = (X_{t,\lambda})_{\lambda\in\Lambda}$ obeys to the equation (4.2). The tangential derivative of the vorticity $\partial_{X_t}\omega_\kappa$ satisfies similarly to (4.3)

$$(\partial_t + v_\kappa \cdot \nabla)\partial_{X_{t,\lambda}}\omega_\kappa = \partial_{X_{t,\lambda}}\partial_1\rho_\kappa$$

This follows from the fact that the vorticity-density formulation of (B_κ) is given by

$$\begin{cases} \partial_t\omega_\kappa + v_\kappa\cdot\nabla\omega_\kappa = \partial_1\rho_\kappa, \quad t\geq 0,\ x\in\mathbb{R}^2\\ \partial_t\rho_\kappa + v_\kappa\cdot\nabla\rho_\kappa - \kappa\Delta\rho_\kappa = 0,\\ \operatorname{div} v_\kappa = 0, \end{cases} \tag{VD$_\kappa$}$$

Writing $\partial_{X_t}\partial_1\rho_\kappa = \partial_1(\partial_{X_t}\rho_\kappa) + [\partial_{X_t},\partial_1]\rho_\kappa$ and keeping in mind that the commutator behaves well, then the problem reduces to follow the regularity of $\partial_{X_t}\rho_\kappa$ in C^ε. It is straightforward that the quantity $\partial_{X_t}\rho_\kappa$ satisfies the following evolution equation

$$(\partial_t + v_\kappa\cdot\nabla - \kappa\Delta)X_{t,\lambda}\rho_\kappa = -\kappa[\Delta, X_{t,\lambda}]\rho_\kappa. \tag{4.4}$$

Observe that for the inviscid case, we check easily that the co-normal derivative of the density is transported by the flow which simplifies a lot the analysis see [39]. In our context the commutator term contributes with additional drawbacks. The remedy is to treat carefully the commutator using the maximal smoothing effect of the transport diffusion equation in the spirit of the approach developed in [22, 41].

Our second main result deals with the inviscid limit problem. To be precise we have:

Theorem 4.1.3. *Let (v_κ,ρ_κ) and (v,ρ) be the solutions of (B_κ) and (B_0) respectively with the same initial data given by Theorem 4.1.1. Then the following assertions hold true.*

(i) *For every $p\in[2,\infty]$*

$$\sup_{t\in[0,T]}\big(\|v_\kappa(t)-v(t)\|_{L^p} + \|\rho_\kappa(t)-\rho(t)\|_{L^p}\big) \leq C_0\kappa^{1/4+1/2p}.$$

(ii) *If Ψ_κ and Ψ denote the flow associated to v_κ and v respectively. Then we have*

$$\sup_{t\in[0,T]}\|\Psi_\kappa(t)-\Psi(t)\|_{L^\infty} \leq C_0\kappa^{1/4},$$

where $C_0 = C(\|\nabla\rho^0\|_{L^2\cap L^\infty}, T)$.

The proof of the above theorem relies on some classical L^p-estimates, the classical complex interpolation between Lebesgue spaces and the so-called Gagliardo Nerenberg inequality.

Outline of the chapter

Next section starts with a brief overview about the Littlewood-Paley theory, particularly the cut-off operators, paradifferential calculus. Thereafter, we undertake the concept of Besov, Hölder spaces and their connections with worthwhile lemmas concerning the persistence of Besov spaces and maximal regularity for a transport-diffusion equation. In Section 3, we state the general version of the Thoerem 4.1.1. For the sake of clarity, we divide its proof in several steps. Section 4 encloses the proof of Theorem 4.1.3.

4.2 Basic tools

This preparatory section comprises some basic tools that we shall freely used during this work. It starts with a short introduction to the Littlewood-Paley theory through the dyadic decomposition of unity, cut-off operators and Besov spaces. Afterwards, we state Bernstein's inequalities and Bony's decomposition which are required in particular, when it comes to the analysis of the commutator estimates. At the end, we state some technical lemmas freely used trough this work.

4.2.1 Notations

Throughout this chapter, we will adopt the following notations.
• We denote by C a positive constant which may be different in each occurrence but it does not depend on the initial data. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of the type $X \leq CY$ with C is independent of X and Y. The notation C_0 means a constant depending on the involved norms of the initial data.
• The space $\mathscr{S}'(\mathbb{R}^2)$ stands for the set of tempered distributions defined on $\mathbb{R}^2$.
• For any $u \in \mathscr{S}'(\mathbb{R}^2)$ both $\widehat{u}$ and $\mathscr{F}u$ (resp. $\mathscr{F}^{-1}u$) denote the Fourier transform (resp. Inverse Fourier transform) of u.
• For every $p \in [1, \infty]$, $\|\cdot\|_{L^p}$ denotes the norm in the Lebesgue space L^p.
• The norm in the mixed space time Lebesgue space $L^p([0,T], L^r(\mathbb{R}^2))$ is denoted by $\|\cdot\|_{L^p_T L^r}$.
• For any pair of operators P and Q, the commutator $[P, Q]$ is defined by $PQ - QP$.

- $B(0,R)$ denotes the ball centered at origin with radius $R>0$.
- $\mathscr{A}(0,R_1,R_2)$ represents the annulus centered at origin with radii $R_1<R_2$.

4.2.2 Brief review on the Littlewood-Paley theory

We start by the so-called Littlewood-Paley decomposition, based on a nonhomogeneous dyadic partition of unity with respect to the Fourier variables. For this purpose, let $\chi\in\mathscr{D}(\mathbb{R}^2)$ be a radial cut-off function, monotonically decaying along rays and so that

$$\chi(\xi)=\begin{cases}1 & \text{if } \|\xi\|\le\frac{1}{2}\\ 0 & \text{if } \|\xi\|\ge 1,\end{cases}$$

with $\operatorname{supp}\chi\subset B(0,1)$. Furthermore, define $\varphi(\xi)\triangleq\chi(\frac{\xi}{2})-\chi(\xi)$, $\varphi\ge 0$; thus we have $\operatorname{supp}\varphi\subset\mathscr{A}(0,R_1,R_2)$ and,

$$\forall\xi\in\mathbb{R}^2,\quad \chi(\xi)+\sum_{q\ge 0}\varphi(2^{-q}\xi)=1.$$

The Littlewood-Paley or frequency cut-off operators $(\Delta_q)_{q\ge -1}$ are defined for $u\in\mathcal{S}'(\mathbb{R}^2)$ by

$$\Delta_q u=\begin{cases}\chi(\mathrm{D})u & \text{if } q=-1\\ \varphi(2^{-q}\mathrm{D})u & \text{if } q\ge 0,\end{cases}$$

where in general case $f(\mathrm{D})$ stands the pseudo-differential operator $u\mapsto\mathscr{F}^{-1}(f\mathscr{F}u)$ with constant symbol. Also, the sequence $(S_q)_{q\ge 0}$ of lower frequencies is defined for $q\ge 0$ as follows

$$S_q u\triangleq\sum_{j\le q-1}\Delta_j u.$$

Few basic properties of the cut-off operators $(\Delta_q)_{q\ge -1}$ and $(S_q)_{q\ge 0}$ are listed in the following proposition.

Proposition 4.2.1. *Let $u,v\in\mathscr{S}'(\mathbb{R}^2)$. Then we have*

(i) $|p-q|\ge 2\Longrightarrow\Delta_p\Delta_q u\equiv 0$,

(ii) $|p-q|\ge 4\Longrightarrow\Delta_q(S_{p-1}u\Delta_p v)\equiv 0$,

(iii) $\Delta_q, S_q: L^p\to L^p$ *uniformly with respect to q and p.*

(iv)
$$u = \sum_{q \geq -1} \Delta_q u.$$

With these notations at our disposal, we now provide the definition of the inhomogeneous Besov space.

Definition 4.2.2. *For $(p, r, s) \in [1, +\infty]^2 \times \mathbb{R}$, the inhomogeneous Besov space $B^s_{p,r}$ is defined by*
$$B^s_{p,r} = \{u \in \mathscr{S}'(\mathbb{R}^2) : \|u\|_{B^s_{p,r}} < +\infty\},$$
where
$$\|u\|_{B^s_{p,r}} \triangleq \begin{cases} \left(\sum_{q \geq -1} 2^{rqs} \|\Delta_q u\|^r_{L^p}\right)^{1/r} & \text{if } r \in [1, +\infty[, \\ \sup_{q \geq -1} 2^{qs} \|\Delta_q u\|_{L^p} & \text{if } r = +\infty. \end{cases}$$

Remark 4.2.3. *We notice that:*

(i) If $s \in \mathbb{R}_+ \backslash \mathbb{N}$, the Hölder space noted by C^s coincides with $B^s_{\infty,\infty}$.

(ii) $\big(C^s, \|\cdot\|_{C^s}\big)$ is a Banach space coincides with the usual Hölder space C^s with equivalent norms,
$$\|u\|_{C^s} \lesssim \|u\|_{L^\infty} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^s} \lesssim \|u\|_{C^s}. \tag{4.5}$$

(iii) If $s \in \mathbb{N}$, the obtained space is so-called Hölder-Zygmund space and still noted by $B^s_{\infty,\infty}$.

4.2.3 Paradifferential calculus

The well-known *Bony's* decomposition [12] enables us to split formally the product of two tempered distributions u and v into three pieces. Specifically:

Definition 4.2.4. *For a given $u, v \in \mathscr{S}'$ we have*
$$uv = T_u v + T_v u + \mathscr{R}(u, v),$$
where
$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad \mathscr{R}(u, v) = \sum_q \Delta_q u \widetilde{\Delta}_q v,$$

with the notation

$$\widetilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

$T_u v$ is called paraproduct of v by u and $\mathscr{R}(u,v)$ the remainder term.

The mixed space-time spaces are stated as follows.

Definition 4.2.5. *Let $T > 0$ and $(\beta, p, r, s) \in [1,\infty]^3 \times \mathbb{R}$. The spaces $L^\beta_T B^s_{p,r}$ and $\widetilde{L}^\beta_T B^s_{p,r}$ are defined respectively by:*

$$L^\beta_T B^s_{p,r} \triangleq \Big\{ u : [0,T] \to \mathscr{S}'; \|u\|_{L^\beta_T B^s_{p,r}} = \big\| \big(2^{qs}\|\Delta_q u\|_{L^p}\big)_{\ell^r} \big\|_{L^\beta_T} < \infty \Big\},$$

$$\widetilde{L}^\beta_T B^s_{p,r} \triangleq \Big\{ u : [0,T] \to \mathscr{S}'; \|u\|_{\widetilde{L}^\beta_T B^s_{p,r}} = \big(2^{qs}\|\Delta_q u\|_{L^\beta_T L^p}\big)_{\ell^r} < \infty \Big\}.$$

The relationship between these spaces is given by the following embeddings. Let $\varepsilon > 0$, then

$$\begin{cases} L^\beta_T B^s_{p,r} \hookrightarrow \widetilde{L}^\beta_T B^s_{p,r} \hookrightarrow L^\beta_T B^{s-\varepsilon}_{p,r} & \text{if } r \geq \beta, \\ L^\beta_T B^{s+\varepsilon}_{p,r} \hookrightarrow \widetilde{L}^\beta_T B^s_{p,r} \hookrightarrow L^\beta_T B^s_{p,r} & \text{if } \beta \geq r. \end{cases} \tag{4.6}$$

Accordingly, we have the following interpolation result.

Corollary 4.2.6. *Let $T > 0$, $s_1 < s < s_2$ and $\eta \in]0,1[$ such that $s = \eta s_1 + (1-\eta)s_2$. Then we have*

$$\|u\|_{\widetilde{L}^\beta_T B^s_{p,r}} \leq C \|u\|^\eta_{\widetilde{L}^\beta_T B^{s_1}_{p,\infty}} \|u\|^{1-\eta}_{\widetilde{L}^\beta_T B^{s_2}_{p,\infty}}. \tag{4.7}$$

Now we shall state Bernstein's inequalities, see for instance [5, 19].

Lemma 4.2.7. *There exists a constant $C > 0$ such that for $1 \leq a \leq b \leq \infty$, for every function u and every $q \in \mathbb{N} \cup \{-1\}$, we have*

(i)

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q\left(k+2\left(\frac{1}{a}-\frac{1}{b}\right)\right)} \|S_q u\|_{L^a},$$

(ii)

$$C^{-k} 2^{qk} \|\Delta_q u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \leq C^k 2^{qk} \|\Delta_q u\|_{L^a}.$$

A noteworthy consequence of Bernstein's inequality (i) is the following embedding:

$$B^s_{p,r} \hookrightarrow B^{\widetilde{s}}_{\widetilde{p},\widetilde{r}} \quad \text{whenever } \widetilde{p} \geq p,$$

with
$$\widetilde{s} < s - 2\Big(\frac{1}{p} - \frac{1}{\widetilde{p}}\Big) \quad \text{or} \quad \widetilde{s} = s - 2\Big(\frac{1}{p} - \frac{1}{\widetilde{p}}\Big) \quad \text{and} \quad \widetilde{r} \leq r.$$

4.2.4 Useful results

The most results concerning the system (VD_κ) rely strongly on a priori estimates in Besov spaces for the transport-diffusion equation:

$$\begin{cases} \partial_t a + v \cdot \nabla a - \kappa \Delta a = f \\ a_{|t=0} = a^0. \end{cases} \tag{TD_κ}$$

We start by the persistence of Besov regularity for (TD_κ), whose proof may be found for example in [5, 44].

Proposition 4.2.8. *Let $(s, r, p) \in]-1, 1[\times [1, \infty]^2$ and v be a smooth divergence-free vector field. We assume that $a^0 \in B^s_{p,r}$ and $f \in L^1_{loc}(\mathbb{R}_+; B^s_{p,r})$. Then for every smooth solution a of (TD_κ) and $t \geq 0$ we have*

$$\|a(t)\|_{B^s_{p,r}} \leq Ce^{CV(t)}\Big(\|a^0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^s_{p,r}} d\tau\Big),$$

with
$$V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau$$
and C being a constant which depends only on s and not on κ.

Next, we state the maximal smoothing effect result for (TD_κ) in mixed time-space spaces, whose proof can be found in [5, 44].

Proposition 4.2.9. *Let $(s, p_1, p_2, r) \in]-1, 1[\times [1, +\infty]^3$ and v be a divergence- free vector field belonging to $L^1_{\mathrm{loc}}(\mathbb{R}_+; \mathrm{Lip})$. Then for every smooth solution a of (TD_κ) we have*

$$\kappa^{\frac{1}{r}}\|a\|_{\widetilde{L}^r_t B^{s+\frac{2}{r}}_{p_1,p_2}} \leq Ce^{CV(t)}(1+\kappa t)^{\frac{1}{r}}\Big(\|a^0\|_{B^s_{p_1,p_2}} + \|f\|_{L^1_t B^s_{p_1,p_2}}\Big), \quad \forall t \in \mathbb{R}_+. \tag{4.8}$$

We end this paragraph by the Calderón-Zygmund estimate which is a deep result of harmonic analysis.

Proposition 4.2.10. *Let $p \in]1,\infty[$ and v be a divergence-free vector field whose vorticity $\omega \in L^p$. Then $\nabla v \in L^p$ and*

$$\|\nabla v\|_{L^p} \le C\frac{p^2}{p-1}\|\omega\|_{L^p}, \tag{4.9}$$

with C being a universal constant.

4.2.5 Vortex patch tool box

In this section we state some aspects and properties about admissible family of vector fields often used in the definition of anisotropic Hölder spaces.

Definition 4.2.11. *Let $\varepsilon \in]0,1[$. A family of vector fields $X = (X_\lambda)_{\lambda\in\Lambda}$ is said to be admissible if and only if the following assertions hold.*

(i) Regularity: $X_\lambda, \mathrm{div}X_\lambda \in C^\varepsilon \quad \forall\lambda \in \Lambda.$

(ii) Non-degeneracy: $I(X) \triangleq \inf_{x\in\mathbb{R}^2}\sup_{\lambda\in\Lambda}\big|X_\lambda(x)\big| > 0.$

The class X is equipped with the norm

$$\widetilde{\|}X_\lambda\|_{C^\varepsilon} \triangleq \|X_\lambda\|_{C^\varepsilon} + \|\mathrm{div}X_\lambda\|_{C^\varepsilon}. \tag{4.10}$$

Definition 4.2.12. *Let $X = (X_\lambda)_{\lambda\in\Lambda}$ be an admissible family. The action of each member X_λ on $u \in L^\infty$ is defined as the directional derivative of u along X_λ by the formula*

$$\partial_{X_\lambda}u = \mathrm{div}(uX_\lambda) - u\,\mathrm{div}X_\lambda.$$

Now, we are in position to define the anisotropic Hölder spaces.

Definition 4.2.13. *Let $\varepsilon \in]0,1[$ and X be an admissible family of vector fields. We say that $u \in C^\varepsilon(X)$ if and only if $u \in L^\infty$ and satisfies for all $\lambda \in \Lambda$, $\partial_{X_\lambda}u \in C^{\varepsilon-1}$ and $\sup_{\lambda\in\Lambda}\|\partial_{X_\lambda}u\|_{C^{\varepsilon-1}} < +\infty$. The set $C^\varepsilon(X)$ is equipped with the canonical norm*

$$\|u\|_{C^\varepsilon(X)} \triangleq \frac{1}{I(X)}\Big(\|u\|_{L^\infty}\sup_{\lambda\in\Lambda}\widetilde{\|}X_\lambda\|_{C^\varepsilon} + \sup_{\lambda\in\Lambda}\|\partial_{X_\lambda}u\|_{C^{\varepsilon-1}}\Big). \tag{4.11}$$

Let v be a time-dependent Lipschitz vector field and $\Psi(t)$ its flow. The time evolution of a given initial family $X_0 = (X_{0,\lambda})_{\lambda\in\Lambda}$ is defined by,

$$X_{t,\lambda}(x) \triangleq X_{0,\lambda}\Psi(t, \Psi^{-1}(t, x)). \tag{4.12}$$

Notice that X_t is nothing but the push-forward of X_0 by the flow $\Psi(t)$, and from straightforward algebraic computations one finds that

$$\begin{cases} (\partial_t + v\cdot\nabla)X_{t,\lambda} = \partial_{X_{t,\lambda}} v & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2 \\ X_{t,\lambda|t=0} = X_{0,\lambda}. \end{cases} \tag{4.13}$$

One of the main feature of the family $(X_{t,\lambda})_{\lambda\in\Lambda}$ is its commutation with the transport operator $\partial_t + v\cdot\nabla$. This implies an important consequence about the dynamics of the tangential regularity of the vorticity subject to the system (VD_κ). Actually, one obtains easily the following result.

Proposition 4.2.14. *Let $(\omega_\kappa, \rho_\kappa)$ be a solution of the system (VD_κ) and $X_t \triangleq (X_{t,\lambda})_{\lambda\in\Lambda}$ be a family of vector fields satisfying the equations (4.13). Then we have*

$$\big(\partial_t + v_\kappa\cdot\nabla\big)\partial_{X_{t,\lambda}}\omega_\kappa = \partial_{X_{t,\lambda}}\partial_1\rho_\kappa.$$

The following result deals with a special logarithmic result involving striated regularity for the vorticity, see for instance [19].

Theorem 4.2.15. *Let $\varepsilon\in]0,1[$ and $X = (X_\lambda)_{\lambda\in\Lambda}$ be a family of vector fields as in the Definition 4.2.11. Let v be a divergence-free vector field such that its vorticity ω belongs to $L^2\cap C^\varepsilon(X)$. Then there exists a constant C depending only on ε, such that*

$$\|\nabla v(t)\|_{L^\infty} \le C\Bigg(\|\omega(t)\|_{L^2} + \|\omega(t)\|_{L^\infty}\log\bigg(e + \frac{\|\omega(t)\|_{C^\varepsilon(X)}}{\|\omega(t)\|_{L^\infty}}\bigg)\Bigg). \tag{4.14}$$

We end this section with the following geometric definition.

Definition 4.2.16. *Let $\varepsilon > 0$. A closed curve Σ is said to be of class $C^{1+\varepsilon}$, if there exists $f\in C^{1+\varepsilon}(\mathbb{R}^2)$ such that Σ is locally a zero set of f, i.e., there exists a neighborhood V of Σ such that*

$$\Sigma = f^{-1}(\{0\})\cap V, \quad \nabla f(x)\neq 0 \quad \forall x\in V. \tag{4.15}$$

4.3 Smooth vortex patch

This section cares with more general class of initial data than the vortex patches stated in Theorem 4.1.1. This theorem is a consequence of the following one.

Theorem 4.3.1. *Let $\kappa \in [0,1]$, $\varepsilon \in]0,1[$ and take an admissible family of vector fields $X_0 = (X_{0,\lambda})_{\lambda\in\Lambda}$ according to the Definition 4.2.11. Let v_κ^0 be the initial velocity with $\mathrm{div} v_\kappa^0 = 0$, and such that its vorticity $\omega_\kappa^0 \in L^2 \cap C^\varepsilon(X_0)$. Assume that the initial density $\rho_\kappa^0 \in L^2 \cap C^{1+\varepsilon}(X_0)$ with $\nabla\rho_\kappa^0 \in L^2$. Then there exist a time $T > 0$ independent of κ and a unique solution (v_κ, ρ_κ) for the system (B_κ), such that*

- $v_\kappa \in L^\infty\big([0,T]; \mathrm{Lip}(\mathbb{R}^2)\big)$ *and* $\omega_\kappa \in L^\infty\big([0,T]; L^2 \cap L^\infty\big)$;
- $\rho_\kappa \in L^\infty\big([0,T]; L^2 \cap \mathrm{Lip}(\mathbb{R}^2)\big)$.

Moreover, the family of vector fields transported by the flow defined in (4.12) still remains, at every time, admissible of the class C^ε and

$$\rho_\kappa(t) \in C^{1+\varepsilon}(X_t), \quad \omega_\kappa(t) \in C^\varepsilon(X_t). \tag{4.16}$$

We emphasize that the estimates of the solution in the above spaces are uniform with respect to $\kappa \in [0,1]$.

The proof of Theorem 4.3.1 follows several steps that will be detailed in the following subsections. To simplify the notations, we will omit the index κ.

4.3.1 A priori estimates for the vorticity and density

We intend to establish the following elementary persistence results on weak regularities.

Proposition 4.3.2. *Let (v,ρ) be a smooth solution of the system (B_κ) defined on $[0,T]$. Then, for every $p \in [1,\infty]$ and $t \in [0,T]$ the following assertions hold.*

(i) $\|\nabla\rho(t)\|_{L^p} \le \|\nabla\rho^0\|_{L^p} e^{CV(t)}$,

(ii) $\kappa\|\nabla\rho\|_{\widetilde{L}^1_t B^2_{\infty,\infty}} \le C(1+\kappa t)\|\nabla\rho^0\|_{L^\infty} e^{CV(t)}$,

(iii) $\|\omega(t)\|_{L^p} \leq \|\omega^0\|_{L^p} + \|\nabla\rho^0\|_{L^p} e^{CV(t)} t,$

with the notation

$$V(t) = \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau.$$

Proof. **(i)** Applying the partial derivative ∂_j to the density equation of (B_κ), one obtains

$$\partial_t\partial_j\rho + v\cdot\nabla(\partial_j\rho) - \kappa\Delta(\partial_j\rho) = -\partial_j v\cdot\nabla\rho \tag{4.17}$$

from which we infer the following classical estimate,

$$\|\nabla\rho(t)\|_{L^p} \leq \|\nabla\rho^0\|_{L^p} + \int_0^t \|\nabla\rho(\tau)\|_{L^p}\|\nabla v(\tau)\|_{L^\infty} d\tau.$$

Gronwall's inequality ensures that

$$\|\nabla\rho(t)\|_{L^p} \leq \|\nabla\rho^0\|_{L^p} e^{V(t)}.$$

(ii) Applying Proposition 4.2.9 to (4.17), one obtains

$$\kappa\|\nabla\rho(t)\|_{\widetilde{L}^1_t B^2_{\infty,\infty}} \leq Ce^{CV(t)}(1+\kappa t)\Big(\|\nabla\rho^0\|_{B^0_{\infty,\infty}} + \int_0^t \|\nabla v(\tau)\cdot\nabla\rho(\tau)\|_{B^0_{\infty,\infty}} d\tau\Big).$$

Then using the embedding $L^\infty \hookrightarrow B^0_{\infty,\infty}$, one gets

$$\begin{aligned}
\kappa\|\nabla\rho(t)\|_{\widetilde{L}^1_t B^2_{\infty,\infty}} &\leq Ce^{CV(t)}(1+\kappa t)\Big(\|\nabla\rho^0\|_{L^\infty} + \int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\nabla\rho(\tau)\|_{L^\infty} d\tau\Big)\\
&\leq Ce^{CV(t)}(1+\kappa t)\big(\|\nabla\rho^0\|_{L^\infty} + \|\nabla\rho\|_{L^\infty_t L^\infty}\big).
\end{aligned}$$

Inserting the estimate **(i)** for $p=\infty$ into the last quantity of the above inequality, we finally get

$$\kappa\|\nabla\rho(t)\|_{\widetilde{L}^1_t B^2_{\infty,\infty}} \leq C(1+\kappa t)\|\nabla\rho^0\|_{L^\infty} e^{CV(t)}. \tag{4.18}$$

(iii) The L^p-estimate for the vorticity can be derived without any difficulty from the first equation of (VD_κ),

$$\|\omega(t)\|_{L^p} \leq \|\omega^0\|_{L^p} + \int_0^t \|\nabla\rho(\tau)\|_{L^p} d\tau$$

that we combine with **(i)** in order to get the desired estimate. □

4.3.2 A priori estimates for the co-normal regularity of the density

The main result of this paragraph is to prove the persistence of the tangential regularity for the density. This latter unknown obeys to the following transport-diffusion equation

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho - \kappa \Delta \rho = 0 \\ \rho_{|t=0} = \rho^0. \end{cases} \tag{TD_κ}$$

Proposition 4.3.3. *Let v be a smooth free-divergence vector field and $X_t = (X_{t,\lambda})_{\lambda\in\Lambda}$ be the family defined in (4.12). Assume that ρ is a smooth solution of (TD_κ), then for every $t \geq 0$ we have*

$$\|\partial_{X_{t,\lambda}} \rho(t)\|_{C^\varepsilon} \lesssim e^{CV(t)}(1+\kappa t)\big(\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon} + \|\nabla\rho^0\|_{L^\infty}\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}\big). \tag{4.19}$$

Proof. Applying the directional derivative $\partial_{X_{t,\lambda}}$ to (TD_κ), one gets

$$\begin{cases} \partial_t \partial_{X_{t,\lambda}} \rho + v \cdot \nabla \partial_{X_{t,\lambda}} \rho - \kappa \Delta \partial_{X_{t,\lambda}} \rho = -\kappa[\Delta, X_{t,\lambda}]\rho \\ \partial_{X_{t,\lambda}} \rho_{|t=0} = \partial_{X_{0,\lambda}} \rho^0, \end{cases} \tag{4.20}$$

where $[\Delta, X_{t,\lambda}]$ stands for the commutator between Δ and $X_{t,\lambda}$. According to [22, 41], the commutator $\kappa[\Delta, X_{t,\lambda}]\rho$ can be decomposed as follows

$$\kappa[\Delta, X_{t,\lambda}]\rho = F + \kappa G,$$

where

$$\begin{aligned} F &= 2\kappa\mathscr{R}(\nabla X^i_{t,\lambda}, \partial_i \nabla\rho) + \kappa\mathscr{R}(\Delta X^i_{t,\lambda}, \partial_i\rho) \\ &:= \kappa F_1 + \kappa F_2 \end{aligned}$$

and

$$G = 2T_{\nabla X^i_{t,\lambda}} \partial_i \nabla \rho + 2T_{\partial_i \nabla\rho} \nabla X^i_{t,\lambda} + T_{\Delta X^i_{t,\lambda}} \partial_i \rho + T_{\partial_i\rho} \Delta X^i_{t,\lambda}.$$

To bound $\partial_{X_{t,\lambda}}\rho$ in C^ε we apply Theorem 2 page 1461 in [41] to (4.20) which implies that

$$\|\partial_{X_{t,\lambda}} \rho(t)\|_{C^\varepsilon} \leq Ce^{CV(t)}\Big(\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon} + \|F\|_{\widetilde{L}^1_t C^\varepsilon} + (1+\kappa t)\|G\|_{\widetilde{L}^\infty_t C^{\varepsilon-2}}\Big) \tag{4.21}$$

- **Estimate of $\|F\|_{\widetilde{L}^1_t C^\varepsilon}$.** Using Bernstein's inequality, one gets

$$\begin{aligned}\|\Delta_q F_1\|_{L^1_t L^\infty} &\leq C\sum_{j\geq q-4}\|\Delta_j \nabla X\|_{L^\infty_t L^\infty}\|\Delta_j\partial_i\nabla\rho\|_{L^1_t L^\infty} \\ &\leq C\sum_{j\geq q-4}\|\Delta_j X\|_{L^\infty_t L^\infty}2^{2j}\|\Delta_j\nabla\rho\|_{L^1_t L^\infty} \\ &\leq C\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}\|\nabla\rho\|_{\widetilde{L}^1_t B^2_{\infty,\infty}}2^{-q\varepsilon}.\end{aligned} \tag{4.22}$$

Multiplying both sides by $2^{q\varepsilon}$ and taking the supremum over q, it holds

$$\|F_1\|_{\widetilde{L}^1_t C^\varepsilon} \leq C\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}\|\nabla\rho\|_{\widetilde{L}^1_t B^2_{\infty,\infty}}. \tag{4.23}$$

The estimate of F_2 can be done in a similar way and one finds that

$$\|F_2\|_{\widetilde{L}^1_t C^\varepsilon} \leq C\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}\|\nabla\rho\|_{\widetilde{L}^1_t B^2_{\infty,\infty}}. \tag{4.24}$$

Finally, combining (4.23) and (4.24), we end up with

$$\|F\|_{\widetilde{L}^1_t C^\varepsilon} \leq C\kappa\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}\|\nabla\rho\|_{\widetilde{L}^1_t B^2_{\infty,\infty}}. \tag{4.25}$$

- **Estimate of $\|G\|_{\widetilde{L}^\infty_t C^{\varepsilon-2}}$.** From the definition we have the splitting

$$\begin{aligned}G &= 2T_{\nabla X^i_{t,\lambda}}\partial_i\nabla\rho + 2T_{\partial_i\nabla\rho}\nabla X^i_{t,\lambda} + T_{\Delta X^i_{t,\lambda}}\partial_i\rho + T_{\partial_i\rho}\Delta X^i_{t,\lambda} \\ &= 2G_1 + 2G_2 + G_3 + G_4.\end{aligned}$$

We start by estimating G_1 in $\widetilde{L}^\infty_t C^{\varepsilon-2}$. For every $q\geq -1$ we have from Bernstein's inequality

$$\begin{aligned}\|\Delta_q G_1\|_{L^\infty} &\leq \sum_{|j-q|\leq 4}\|\Delta_q\big(S_{j-1}\nabla X\Delta_j\partial_i\nabla\rho\big)\|_{L^\infty} \\ &\leq C\sum_{|j-q|\leq 4}2^j\|S_{j-1}\nabla X\|_{L^\infty}\|\Delta_j\nabla\rho\|_{L^\infty}.\end{aligned}$$

Multiplying both sides by $2^{q(\varepsilon-2)}$ and using once again Bernstein's inequality we deduce that

$$\begin{aligned}2^{q(\varepsilon-2)}\|\Delta_q G_1\|_{L^\infty} &\leq C\sum_{|j-q|\leq 4}2^j 2^{q(\varepsilon-2)}\sum_{l\leq j-2}\|\Delta_l\nabla X\|_{L^\infty}\|\Delta_j\nabla\rho\|_{L^\infty} \\ &\leq C\sum_{|j-q|\leq 4}2^{q(\varepsilon-2)}2^j\|\Delta_j\nabla\rho\|_{L^\infty}\sum_{l\leq j-2}2^{l(1-\varepsilon)}2^{l\varepsilon}\|\Delta_l X\|_{L^\infty} \\ &\leq C\|X\|_{C^\varepsilon}\sum_{|j-q|\leq 4}2^{(j-q)(2-\varepsilon)}\|\Delta_j\nabla\rho\|_{L^\infty}.\end{aligned}$$

Since $L^\infty \hookrightarrow B^0_{\infty,\infty}$, then

$$\|G_1\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon}\|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{4.26}$$

The estimate of G_2 is quite similar to G_1 and one obtains

$$\|G_2\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon}\|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{4.27}$$

As to G_3 we write for every $q \ge -1$

$$\begin{aligned}
\|\Delta_q G_3\|_{L^\infty} &\le C\sum_{|j-q|\le 4}\|\Delta_q\big(S_{j-1}\Delta X\Delta_j\partial_i\rho\big)\|_{L^\infty}\\
&\le C\sum_{|j-q|\le 4}\|S_{j-1}\Delta X\|_{L^\infty}\|\Delta_j\nabla\rho\|_{L^\infty}\\
&\le C\|\nabla\rho\|_{L^\infty}\sum_{|j-q|\le 4}\sum_{l\le j-2}2^{2l}\|\Delta_l X\|_{L^\infty}.
\end{aligned}$$

Multiplying both sides by $2^{q(\varepsilon-2)}$, we obtain

$$\begin{aligned}
2^{q(\varepsilon-2)}\|\Delta_q G_3\|_{L^\infty} &\le C\|X\|_{C^\varepsilon}\|\nabla\rho\|_{L^\infty}\sum_{\substack{|j-q|\le 4\\ l\le j-2}}2^{(q-l)(\varepsilon-2)}\\
&\le C\|X\|_{C^\varepsilon}\|\nabla\rho\|_{L^\infty}.
\end{aligned}$$

Consequently,

$$\|G_3\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon}\|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{4.28}$$

The estimate of G_4 is quite similar to the preceding ones

$$\|G_4\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon}\|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{4.29}$$

Putting together (4.26), (4.27), (4.28) and (4.29), we get

$$\|G\|_{\widetilde{L}^\infty_t B^{\varepsilon-2}_{\infty,\infty}} \le C\|X\|_{L^\infty_t C^\varepsilon}\|\nabla\rho\|_{L^\infty_t L^\infty}. \tag{4.30}$$

Now, substituting (4.25) and (4.30) in (4.21), we end up with

$$\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon} \le Ce^{CV(t)}\big(\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}+\kappa\|\nabla\rho\|_{\widetilde{L}^1_t B^2_{\infty,\infty}}\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}+(1+\kappa t)\|\nabla\rho\|_{L^\infty_t L^\infty}\|X\|_{L^\infty_t C^\varepsilon}\big). \tag{4.31}$$

By invoking **(i)**-**(ii)** of the Proposition 4.3.2 we obtain

$$\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon} \le Ce^{CV(t)}\big(\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon} + (1+\kappa t)\|\nabla\rho^0\|_{L^\infty}\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}\big).$$

Hence

$$\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon} \le C_0 e^{CV(t)}(1+\kappa t)\big(1+\|X\|_{\widetilde{L}^\infty_t C^\varepsilon}\big).$$

□

4.3.3 A priori estimates for the co-normal regularity $\partial_{X_{t,\lambda}}\omega$

In this paragraph we shall focus on the estimate of the conormal regularity $\partial_{X_{t,\lambda}}\omega$ in the Hölder space $C^{\varepsilon-1}$. For this aim we prove

Proposition 4.3.4. *Let (v,ρ) be any smooth solution of the system* (B_κ) *on $[0,T]$, and take any time dependent family of vector field $X_t=(X_{t,\lambda})_{\lambda\in\Lambda}$ transported by the flow of v. Then we have for all $t\in[0,T]$, $\lambda\in\Lambda$*

(i) $I(X_{t,\lambda}) \ge I(X_{0,\lambda})e^{-V(t)}$,

(ii) $\|\mathrm{div} X_{t,\lambda}\|_{C^\varepsilon} \le \|\mathrm{div} X_{0,\lambda}\|_{C^\varepsilon} e^{CV(t)}$ *for every* $\lambda\in\Lambda$,

(iii) $\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}} + \widetilde{\|} X_{t,\lambda}\|_{C^\varepsilon} \le C\Big(\|\partial_{X_{0,\lambda}}\omega^0\|_{C^{\varepsilon-1}} + \widetilde{\|} X_{0,\lambda}\|_{C^\varepsilon} + \|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}\Big)e^{C\Phi(t)}$,

with

$$\Phi(t) := (t+\kappa t^2)\|\nabla\rho^0\|_{L^\infty}e^{CV(t)} + V(t) + t$$

Proof. **(i)** Let us bound from below $I(X_{t,\lambda})$ by applying the time derivative to $\partial_{X_{0,\lambda}}\Psi(t,x)$ and invoking the fact

$$X_{t,\lambda}(\Psi(t,x)) = \partial_{X_{0,\lambda}}\Psi(t,x)$$

and $\partial_t\Psi(t,x) = v(t,\psi(t,x))$ with $\Psi(0,x)=x$. We deduce that

$$\partial_t\partial_{X_{0,\lambda}}\Psi(t,x) = \nabla v(t,\Psi(t,x))\cdot\partial_{X_{0,\lambda}}\Psi(t,x), \quad \partial_{X_{0,\lambda}}\Psi(0,x) = X_{0,\lambda}.$$

The time reversibility of this equation combined with Gronwall's inequality tell us

$$|X_{0,\lambda}(x)| \le e^{V(t)}\big|\partial_{X_{0,\lambda}}\Psi(t,x)\big|, \quad \forall(\lambda,x)\in\Lambda\times\mathbb{R}^2.$$

In view of (ii)-Definition 4.2.11 we get the desired estimate.
(ii) Applying "div" operator to (4.13), an easy computation combined with $\mathrm{div}v=0$

shows us that $\mathrm{div}X_{t,\lambda}$ satisfies

$$(\partial_t + v\cdot\nabla)\mathrm{div}X_{t,\lambda} = 0.$$

Using Proposition 4.2.8 yields

$$\|\mathrm{div}X_{t,\lambda}\|_{C^\varepsilon} \le Ce^{CV(t)}\|\mathrm{div}X_{0,\lambda}\|_{C^\varepsilon}. \tag{4.32}$$

(iii) To bound $\partial_{X_{t,\lambda}}\omega$ in $C^{\varepsilon-1}$, we first recall from Proposition 4.2.14 that

$$(\partial_t + v\cdot\nabla)\partial_{X_{t,\lambda}}\omega = \partial_{X_{t,\lambda}}\partial_1\rho.$$

In accordance with Proposition 4.2.8, we readily get

$$\big\|\partial_{X_{t,\lambda}}\omega(t)\big\|_{C^{\varepsilon-1}} \le Ce^{CV(t)}\Big(\big\|\partial_{X_{0,\lambda}}\omega^0\big\|_{C^{\varepsilon-1}} + \int_0^t e^{-CV(\tau)}\|\partial_{X_{\tau,\lambda}}\partial_1\rho(\tau)\|_{C^{\varepsilon-1}}d\tau\Big). \tag{4.33}$$

Let us estimate $\|\partial_{X_{\tau,\lambda}}\partial_1\rho(\tau)\|_{C^{\varepsilon-1}}$. To do this, starting by the identity

$$\partial_{X_{\tau,\lambda}}\partial_1\rho = \partial_1(\partial_{X_{\tau,\lambda}}\rho) - \partial_{\partial_1 X_{\tau,\lambda}}\rho$$

combined with the following estimate proved in Corollary 1 of [39]

$$\|\partial_j X\cdot\nabla f\|_{C^{\varepsilon-1}} \le C\|\nabla f\|_{L^\infty}\widetilde{\|}X\|_{C^\varepsilon}$$

one obtains in particular

$$\|\partial_{X_{t,\lambda}}\partial_1\rho(t)\|_{C^{\varepsilon-1}} \le \|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon} + \|\nabla\rho(t)\|_{L^\infty}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}.$$

Plugging the last estimate into (4.33), it follows

$$\begin{aligned}\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}} \;\le\; & Ce^{CV(t)}\Big(\|\partial_{X_{0,\lambda}}\omega^0\|_{C^{\varepsilon-1}} + \int_0^t e^{-CV(\tau)}\|\partial_{X_{\tau,\lambda}}\rho(\tau)\|_{C^\varepsilon}d\tau \\ & + \int_0^t e^{-CV(\tau)}\|\nabla\rho(\tau)\|_{L^\infty}\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon}d\tau\Big).\end{aligned} \tag{4.34}$$

For the term $\|\partial_{X_{t,\lambda}}\rho(t)\|_{C^\varepsilon}$, we may apply (4.31) and (4.18) and therefore (4.34) becomes

$$\begin{aligned}e^{-CV(t)}\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}} \;\lesssim\; & \|\partial_{X_{0,\lambda}}\omega^0\|_{C^{\varepsilon-1}} + \|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}t \\ & + \int_0^t e^{-CV(\tau)}(1+\kappa\tau)\big(\|\nabla\rho^0\|_{L^\infty} + \|\nabla\rho\|_{L^\infty_\tau L^\infty}\big)\widetilde{\|}X_\lambda\|_{L^\infty_\tau C^\varepsilon}d\tau.\end{aligned} \tag{4.35}$$

To estimate $\widetilde{\|}X_\lambda\|_{L^\infty_t C^\varepsilon}$, we apply again Proposition 4.2.8 to (4.13),

$$\|X_{t,\lambda}\|_{C^\varepsilon} \le Ce^{CV(t)}\Big(\|X_{0,\lambda}\|_{C^\varepsilon} + \int_0^t e^{-CV(\tau)}\|\partial_{X_{\tau,\lambda}} v(\tau)\|_{C^\varepsilon} d\tau\Big).$$

As to $\|\partial_{X_{t,\lambda}} v(t)\|_{C^\varepsilon}$ we use the following estimate proved in [5, 19],

$$\|\partial_{X_{t,\lambda}} v(t)\|_{C^\varepsilon} \lesssim \|\nabla v(t)\|_{L^\infty}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} + \|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}}.$$

That we get

$$\|X_{t,\lambda}\|_{C^\varepsilon} \le Ce^{CV(t)}\bigg(\|X_{0,\lambda}\|_{C^\varepsilon}+\int_0^t e^{-CV(\tau)}\big(\|\nabla v(\tau)\|_{L^\infty}\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon}+\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}}\big)d\tau\bigg).$$

Since $\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} = \|X_{t,\lambda}\|_{C^\varepsilon} + \|\mathrm{div} X_{t,\lambda}\|_{C^\varepsilon}$, then the last estimate combined with **(ii)** provides

$$e^{-CV(t)}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \lesssim \widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon}+\int_0^t e^{-CV(\tau)}\big(\|\nabla v(\tau)\|_{L^\infty}\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon}+\|\partial_{X_{\tau,\lambda}}\omega(\tau)\|_{C^{\varepsilon-1}}\big)d\tau. \tag{4.36}$$

Adding (4.35) and (4.36) and setting $\Pi(t) := e^{-CV(t)}\big(\|\partial_{X_{t,\lambda}}\omega(t)\|_{C^{\varepsilon-1}} + \widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}\big)$, we find

$$\Pi(t) \lesssim \Pi(0)+\|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}t+\int_0^t \big((1+\kappa\tau)(\|\nabla\rho^0\|_{L^\infty}+\|\nabla\rho\|_{L^\infty_\tau L^\infty})+\|\nabla v(\tau)\|_{L^\infty}+1\big)\Pi(\tau)d\tau.$$

Using Gronwall's inequality we obtain

$$\Pi(t) \lesssim \big(\Pi(0) + \|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}\big)e^{(1+\kappa t)t\|\nabla\rho^0\|_{L^\infty}+(1+\kappa t)t\|\nabla\rho\|_{L^\infty_t L^\infty}+CV(t)+Ct}.$$

Finally, from Proposition 4.3.2-(**i**) we deduce that

$$\Pi(t) \lesssim \big(\Pi(0) + \|\partial_{X_{0,\lambda}}\rho^0\|_{C^\varepsilon}\big)e^{(1+\kappa t)t\|\nabla\rho^0\|_{L^\infty}e^{CV(t)}+CV(t)+Ct}. \tag{4.37}$$

□

4.3.4 Regularity persistence

This part is concerned with the regularity persistence of the prescribed initial regularity. The basic ingredient is to get an estimate for the Lipschitz norm of the velocity for short time. The main result is the following.

Proposition 4.3.5. *Under the assumptions of the Theorem 4.3.1, the solution (v, ρ) of (B_κ) can be defined on an interval $[0, T]$ such that T is related to the size of the initial data with the persistence result:*

$$\forall t \in [0, T], \quad \|\omega(t)\|_{L^2 \cap L^\infty} + \|\omega(t)\|_{C^\varepsilon(X_t)} + \|\nabla v(t)\|_{L^\infty} + \widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \leq C_0, \tag{4.38}$$

with C_0 a constant depending on the initial data.

Proof. The basic ingredient of the proof is to get an a priori estimate for the Lipschitz norm of the velocity over a time interval $[0, T]$ that can be quantified with respect to the initial data. By virtue of Proposition 4.3.2-**(iii)** and Proposition 4.3.4-**(iii)** we deduce that

$$\|\partial_{X_{t,\lambda}} \omega(t)\|_{C^{\varepsilon-1}} + \|\omega(t)\|_{L^\infty} \widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \leq C_0 e^{C\Phi(t)},$$

with the estimate

$$0 \leq \Phi(t) \leq C_0(1 + t^2) e^{CV(t)}. \tag{4.39}$$

Therefore combining this estimate with the definition 4.2.13 and Proposition 4.3.4-**(i)** yields

$$\|\omega(t)\|_{C^\varepsilon(X_t)} \leq C_0 e^{C\Phi(t)}. \tag{4.40}$$

Thus plugging this estimate into the logarithmic estimate (4.14) and using Proposition 4.3.2 we find

$$\|\nabla v(t)\|_{L^\infty} \leq C\Big(\|\omega^0\|_{L^2} + t\|\nabla \rho^0\|_{L^2} e^{CV(t)}\Big) + C\|\omega(t)\|_{L^\infty} \log\left(e + \frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{\|\omega(t)\|_{L^\infty}}\right). \tag{4.41}$$

As the function $x \in]0, +\infty[\mapsto x \log(e + a/x)$ is strictly increasing and $x \in]0, +\infty[\mapsto \log(e + a/x)$ is strictly decreasing then we obtain

$$\|\omega(t)\|_{L^\infty} \log\left(e + \frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{\|\omega(t)\|_{L^\infty}}\right) \leq C_0(1 + t) e^{CV(t)} \log\left(e + \frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{C_0}\right)$$

Notice that we have used the following estimate which follows from Proposition 4.3.2-**(iii)**

$$\|\omega(t)\|_{L^\infty} \leq C_0(1 + t) e^{CV(t)}.$$

Consequently (4.41) becomes

$$\|\nabla v(t)\|_{L^\infty} \leq C_0(1 + t) e^{CV(t)} + C_0(1 + t) e^{CV(t)} \log\left(e + \frac{\|\omega(t)\|_{C^\varepsilon(X_t)}}{C_0}\right) \tag{4.42}$$

Applying (4.39) and (4.40) we get

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty} &\leq C_0(1+t)e^{CV(t)} + C_0(1+t)e^{CV(t)}\big(1+\Phi(t)\big)\\ &\leq C_0(1+t^2)e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau}. \end{aligned} \tag{4.43}$$

From this we deduce the existence of $T > 0$ depending on the initial data through C_0 such that

$$\forall t \in [0, T], \quad \|\nabla v(t)\|_{L^\infty} \leq 2C_0, \tag{4.44}$$

which implies in turn that all the involved norms are bounded over the time interval $[0, T]$. The proof of Theorem 4.3.1 follows easily from Proposition 4.3.5. Indeed, up to now we have established the suitable a priori estimates required for the regularity persistence which are enough to construct a unique solution for short time. This latter part concerning the construction of the solutions is classical and is well-detailed in various references such as [19, 39]. □

4.3.5 Proof of Theorem 4.1.1.

The proof of Theorem 4.1.1 follows from Theorem 4.3.1. To see this, it suffices to build an initial admissible family $X_0 = (X_{0,\lambda})_{0\leq\lambda\leq 1}$ such that $\mathbf{1}_{\Omega_0} \in C^\varepsilon(X_0)$ and to check the regularity persistence of the boundary. This is very classical and was done first in [19], and for the convenience of the reader we shall reproduce here the basic ingredients. Since the initial boundary $\partial\Omega_0$ is a Jordan curve in the class $C^{1+\varepsilon}$, then according to the definition 4.2.16, there exists a local chart (f_0, V_0), with V_0 is a neighborhood of $\partial\Omega_0$ such that

$$\begin{cases} f_0 \in C^{1+\varepsilon}(\mathbb{R}^2), \quad \nabla f_0(x) \neq 0 \quad \text{on } V_0 \\ \partial\Omega_0 = f_0^{-1}(\{0\}) \cap V_0. \end{cases}$$

On the other hand, take $\chi \in \mathscr{D}(\mathbb{R}^2)$, with $0 \leq \chi \leq 1$ and

$$\operatorname{supp} \chi \subset V_0, \quad \chi(x) = 1 \quad \forall x \in W_0,$$

where W_0 is a small neighborhood of $\partial\Omega_0$ strictly contained in V_0. Next, define the family $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ by:

$$X_{0,0}(x) = \nabla^\perp f_0(x) \quad \text{and} \quad X_{0,1}(x) = (1-\chi(x))\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.45}$$

We observe that $X_0 = (X_{0,\lambda})_{\lambda\in\{0,1\}}$ is non-degenerate, and each member $X_{0,\lambda}$ with its divergence belong to $C^\varepsilon(\mathbb{R}^2)$, then according to the Definition 4.2.11, we conclude that X_0 is an admissible family. Moreover, from the identity

$$\nabla\omega_0(x) = -\vec{n}(x)d\sigma_{\partial\Omega_0}$$

with $\vec{n}$ being the outward unit normal vector to the boundary and $d\sigma_{\partial\Omega_0}$ is the arc-length measure on $\partial\Omega_0$, we check easily that

$$\forall\lambda\in\{0,1\},\quad X_{0,\lambda}(x)\cdot\nabla\omega_0(x) = 0.$$

In addition, since $\rho^0\in C^{1+\varepsilon}(\mathbb{R}^2)$ then $\rho^0\in C^{1+\varepsilon}(X_0)$. Consequently, in view of the Theorem 4.3.1, the system (B_κ) is locally well-posed, with the persistence regularity detailed in Proposition 4.3.5. i.e., there exists a unique local solution $(v_\kappa,\rho_\kappa)\in \Big(L^\infty\big([0,T];\mathrm{Lip}(\mathbb{R}^2)\big)\Big)^2$ for (B_κ).
Now, it remains to check the regularity of the transported boundary $\partial\Omega_t$. We parametrize the boundary $\partial\Omega_0$ by defining the periodic curve $\gamma^0\in C^{1+\varepsilon}([0,2\pi];\mathbb{R}^2)$ as the solution of the following ordinary differential equation

$$\begin{cases}\partial_\sigma\gamma^0(\sigma) = X_{0,0}(\gamma^0(\sigma))\\ \gamma^0(0) = x_0,\quad x_0\in\partial\Omega_0.\end{cases}$$

To define the evolution parametrization of $\partial\Omega_t$, we simply set for $t\geq 0$,

$$\gamma(t,\sigma)\triangleq\Psi(t,\gamma_0(\sigma)).$$

Clearly, the curve $\gamma(t,\cdot)$ is the transport of γ^0 by the flow Ψ_t and by the criterion of differentiation with respect to σ, we readily get

$$\partial_\sigma\gamma(t,\sigma) = \big(\partial_{X_{0,0}}\Psi\big)(t,\,\gamma^0(\sigma)).$$

Since $\partial_{X_{0,0}}\Psi(t)\equiv X_{t,0}\circ\Psi(t)$, with $X_{t,0}$ is the push-forward of $X_{0,0}$ by the flow $\Psi(t)$, then in view of the classical estimate

$$\begin{aligned}\|X_{t,0}\circ\Psi(t)\|_{C^\varepsilon} &\leq \|X_{t,0}\|_{C^\varepsilon}\|\nabla\Psi(t)\|_{L^\infty}^{\varepsilon}\\ &\leq \|X_{t,0}\|_{C^\varepsilon}e^{CV(t)}\\ &\leq C_0,\end{aligned}$$

where we have used the fact $\|\nabla\Psi(t)\|_{L^\infty}\leq e^{CV(t)}$ and the estimates of Proposition 4.3.5. Therefore $\partial_{X_{0,0}}\Psi(t)\in L^\infty([0,T];\,C^\varepsilon)$ and $t\in[0,T]\mapsto\gamma_t$ belongs to

$L^\infty([0,T];C^{1+\varepsilon}([0,2\pi];\mathbb{R}^2))$. This concludes the regularity persistence of the boundary $\partial\Omega_t$ and so the proof of the Theorem 4.1.1 is finished.

4.4 Inviscid limit for velocities and densities

This section cares essentially with the proof of the Theorem 4.1.3.

4.4.1 Proof of Theorem 4.1.3

(i) Setting $U=v_\kappa-v$, $\Theta=\rho_\kappa-\rho$ and $P=\pi_\kappa-\pi$. Then, a straightforward computation provides that (U,Θ,P) satisfies,

$$\begin{cases} \partial_t U+v_\kappa\cdot\nabla U+\nabla P=\Theta e_2-U\cdot\nabla v & (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \partial_t\Theta+v_\kappa\cdot\nabla\Theta-\kappa\Delta\Theta=-U\cdot\nabla\rho+\kappa\Delta\rho & (t,x)\in\mathbb{R}_+\times\mathbb{R}^2,\\ \mathrm{div}\, U=0, & \\ U_{|t=0}=U_0,\quad \Theta_{|t=0}=\Theta_0. & \end{cases} \tag{4.46}$$

• *First case:* $p=2$. Dotting $U-$equation (resp. $\Theta-$equation) by U (resp. Θ) respectively. After some integration by parts and the convective terms integrate to zero, due to the fact $\mathrm{div}\, v_\kappa=\mathrm{div}\, v=0$. Thus we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 &\le \|\Theta(t)\|_{L^2}\|U(t)\|_{L^2}+\|U(t)\|_{L^2}^2\|\nabla v(t)\|_{L^\infty} \\ &\le \frac{1}{2}\big(\|\Theta(t)\|_{L^2}^2+\|U(t)\|_{L^2}^2\big)+\|U(t)\|_{L^2}^2\|\nabla v(t)\|_{L^\infty} \\ &\le C\big(\|\nabla v(t)\|_{L^\infty}+1\big)\big(\|\Theta(t)\|_{L^2}^2+\|U(t)\|_{L^2}^2\big) \end{aligned} \tag{4.47}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Theta(t)\|_{L^2}^2+\kappa\|\nabla\Theta(t)\|_{L^2}^2 &\le \kappa\|\nabla\rho(t)\|_{L^2}\|\nabla\Theta(t)\|_{L^2}+\|U(t)\|_{L^2}\|\nabla\rho(t)\|_{L^\infty}\|\Theta(t)\|_{L^2} \\ &\le \frac{\kappa}{2}\|\nabla\rho(t)\|_{L^2}^2+\frac{\kappa}{2}\|\nabla\Theta(t)\|_{L^2}^2+\frac{1}{2}\|\nabla\rho(t)\|_{L^\infty}\big(\|U(t)\|_{L^2}^2+\|\Theta(t)\|_{L^2}^2\big). \end{aligned}$$

Here, we have used two times Young's inequality. Carrying over the term $\frac{\kappa}{2}\|\nabla\Theta(t)\|_{L^2}^2$, to the left-hand side, thus we have

$$\frac{1}{2}\frac{d}{dt}\|\Theta(t)\|_{L^2}^2\le\frac{\kappa}{2}\|\nabla\rho(t)\|_{L^2}^2+\frac{1}{2}\|\nabla\rho(t)\|_{L^\infty}\big(\|U(t)\|_{L^2}^2+\|\Theta(t)\|_{L^2}^2\big). \tag{4.48}$$

Collecting (4.47) and (4.48), and define $\Sigma(t)=\|U(t)\|_{L^2}^2+\|\Theta(t)\|_{L^2}^2$, it happens

$$\frac{d}{dt}\Sigma(t)\le\kappa\|\nabla\rho(t)\|_{L^2}^2+\Sigma(t)\big(\|\nabla v(t)\|_{L^\infty}+\|\nabla\rho(t)\|_{L^\infty}+1\big).$$

Gronwall's inequality then implies

$$\Sigma(t)\le e^{V(t)+\|\nabla\rho\|_{L_t^1L^\infty}+t}\big(\Sigma(0)+\kappa\|\nabla\rho\|_{L_t^1L^2}^2\big). \tag{4.49}$$

For the two terms $\|\nabla\rho\|_{L_t^1L^\infty}$ and $\|\nabla\rho\|_{L_t^1L^2}^2$, we employ the proposition 4.3.2-**(i)** (which remains true for $\kappa=0$) for $p=\infty$ and $p=2$. Then, in view of $\Sigma(0)=0$ we have

$$\Sigma(t)\le\kappa e^{C(V(t)+\|\nabla\rho^0\|_{L^\infty}(e^{CV(t)}+1)t)}\|\nabla\rho^0\|_{L^2}^2t. \tag{4.50}$$

Even though, all the involved norms are bounded over $[0,T]$, we infer that

$$\sup_{t\in[0,T]}\big(\|v_\kappa(t)-v(t)\|_{L^2}+\|\rho_\kappa(t)-\rho(t)\|_{L^2}\big)\le C_0\kappa^{1/2}. \tag{4.51}$$

• *Second case:* $2<p\le\infty$. Using in general case the following classical complex interpolation

$$\|f\|_{L^p}\le C\|f\|_{L^2}^{2/p}\|f\|_{L^\infty}^{1-2/p}.$$

Then, in view of (4.51) we deduce that

$$\|v_\kappa(t)-v(t)\|_{L^p}+\|\rho_\kappa(t)-\rho(t)\|_{L^p}\le C_0\kappa^{\frac{1}{p}}\big(\|v_\kappa(t)-v(t)\|_{L^\infty}^{1-2/p}+\|\rho_\kappa(t)-\rho(t)\|_{L^\infty}^{1-2/p}\big) \tag{4.52}$$

To get bound the two last quantities, employing in general case the so-called Gagliardo-Nirenberg inequality

$$\|f\|_{L^\infty}\lesssim\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^\infty}^{1/2}. \tag{4.53}$$

Thus we get in view of (4.44), (4.51) and proposition4.3.2-**(i)** that

$$\begin{aligned}\|v_\kappa(t)-v(t)\|_{L^\infty}&\le\|v_\kappa(t)-v(t)\|_{L^2}^{1/2}\|\nabla v_\kappa(t)-\nabla v(t)\|_{L^\infty}^{1/2}\\&\le\|v_\kappa(t)-v(t)\|_{L^2}^{1/2}\big(\|\nabla v_\kappa(t)\|_{L^\infty}+\|\nabla v(t)\|_{L^\infty}\big)^{1/2}\\&\le C_0\kappa^{1/4}.\end{aligned}$$

Similarly

$$\begin{aligned}
\|\rho_\kappa(t)-\rho(t)\|_{L^\infty} &\leq \|\rho_\kappa(t)-\rho(t)\|_{L^2}^{1/2}\|\nabla\rho_\kappa(t)-\nabla\rho(t)\|_{L^\infty}^{1/2}\\
&\leq \|\rho_\kappa(t)-\rho(t)\|_{L^2}^{1/2}\big(\|\nabla\rho_\kappa(t)\|_{L^\infty}+\|\nabla\rho(t)\|_{L^\infty}\big)^{1/2}\\
&\leq C_0\kappa^{1/4}\|\nabla\rho^0\|_{L^\infty}^{1/2}e^{C(V_\kappa(t)+V(t))}\\
&\leq C_0\kappa^{1/4}.
\end{aligned}$$

Plugging the last two estimates in (4.52), then with the notation $C_0 = C(\|\nabla\rho^0\|_{L^2\cap L^\infty}, T)$, we could obtain for $p\in[2,\infty]$

$$\sup_{t\in[0,T]}\big(\|v_\kappa(t)-v(t)\|_{L^p}+\|\rho_\kappa(t)-\rho(t)\|_{L^p}\big)\leq C_0\kappa^{1/4+1/2p} \tag{4.54}$$

this finishes the proof of **(i)**.

(ii) Recall that

$$\Psi_\kappa(t,x)=x+\int_0^t v_\kappa(\tau,\Psi_\kappa(\tau,x))d\tau,\quad \Psi(t,x)=x+\int_0^t v(\tau,\Psi(\tau,x))d\tau$$

We intend to prove that $(\Psi_\kappa)_\kappa$ converges uniformly towards Ψ locally in time when κ goes to 0. To do this, we have for every $\kappa>0$

$$|\Psi_\kappa(t,x)-\Psi(t,x)|\leq \underbrace{\int_0^t \big|v_\kappa\big(\tau,\Psi_\kappa(\tau,x)\big)-v\big(\tau,\Psi_\kappa(\tau,x)\big)\big|d\tau}_{\text{(I)}}+\underbrace{\int_0^t \big|v\big(\tau,\Psi_\kappa(\tau,x)\big)-v\big(\tau,\Psi(\tau,x)\big)\big|d\tau}_{\text{(II)}} \tag{4.}$$

The term (I) comes immediately from (4.54), that is for $t\in[0,T]$

$$\text{(I)}\leq C_0\kappa^{1/4}.$$

Concerning (II), using the following general estimate

$$\begin{aligned}
\big|f\circ\Psi_\kappa-f\circ\Psi\big| &= \frac{\big|f\circ\Psi_\kappa-f\circ\Psi\big|}{|\Psi_\kappa-\Psi|}|\Psi_\kappa-\Psi|\\
&\leq \|\nabla f\|_{L^\infty}\|\Psi_\kappa-\Psi\|_{L^\infty}.
\end{aligned}$$

Thus we have

$$\text{(II)}\leq\int_0^t\|\nabla v(\tau)\|_{L^\infty}\|\Psi_\kappa(\tau)-\Psi(\tau)\|_{L^\infty}d\tau.$$

Adding (I) and (II) and insert them in (4.55), we shall get for $x \in \mathbb{R}^2$

$$|\Psi_\kappa(t,x) - \Psi(t,x)| \lesssim C_0 \kappa^{1/4} + \int_0^t \|\nabla v(\tau)\|_{L^\infty} \|\Psi_\kappa(\tau) - \Psi(\tau)\|_{L^\infty} d\tau. \tag{4.56}$$

Gronwall's inequality implies for every $t \in [0,T]$

$$\|\Psi_\kappa(t) - \Psi(t)\|_{L^\infty} \lesssim C_0 \kappa^{1/4},$$

which achieves the proof of **(ii)**, so the Theorem 4.1.3.

5 Appendix

5.1 Description of the models

5.1.1 Effects of rotation and stratification

The atmosphere and ocean are shallow layers of fluid on a sphere. Shallow or superficial because their thickness is much less than their horizontal extent. Their motion is strongly influenced by two effects: rotation and stratification.In this subsection, we consider how the equations of motion are affected by these effects. First, we consider some elementary effects of rotation on a fluid and derive the Coriolis and centrifugal forces. Then, we discuss the Boussinesq approximation to the equations of motion that are appropriate for large-scale flow in the ocean.

Example. In most geophysical systems, the fluid density varies, but not greatly, around a mean value. For example, the average temperature and salinity in the ocean are $T = 4°C$ and $S = 34.7$, to which corresponds a density $\rho = 1028kg/m^3$ at surface pressure. Variations in density within one ocean basin rarely exceed $3kg/m^3$. Even in estuaries where fresh river waters $S = 0$ ultimately turn into salty seawater's $S = 34.7$, the relative density difference is less than $3/100$.

By contrast, the air of the atmosphere becomes gradually more rarefied with altitude, and its density varies from a maximum at ground level to nearly zero at great heights, thus covering a $100/100$ range of variations. Most of the density changes, however, can be attributed to hydrostatic pressure effects, leaving only a moderate variability caused by other factors. Furthermore, weather patterns are confined to the lowest layer, the troposphere approximately $10km$ thick, within which the density variations responsible for the winds are usually no more than $5/100$.

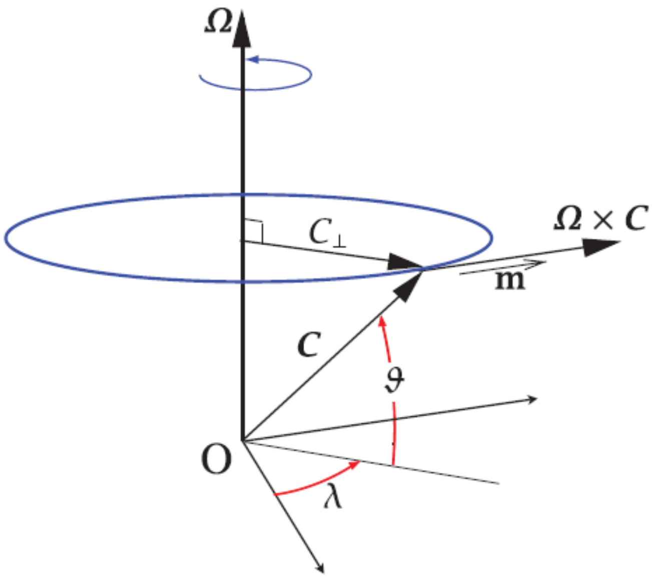
5.1.2 Equations of motion in rotating frame

Newton's second law of motion, that the acceleration of a body is proportional to the imposed force divided by the body's mass, applies in so-called inertial frames of reference, that is to say, frames that are stationary or moving only with a constant rectilinear velocity relative to the distant galaxies. Now, the earth spins round its own axis with a period of almost 24 hours and so the surface of the earth manifestly is not an inertial frame. Nevertheless, it is very convenient to describe the flow relative to earth's surface (which in fact is moving at speeds of up to a few hundreds of meters per second), rather than in some inertial frame. This necessitates recasting the equations into a form that is appropriate in a rotating frame of reference.

5.1.3 Rate of change of vector

Consider first a vector $\mathbf{C}$ of constant length rotating relative to an inertial frame at a constant angular velocity Ω. Then in rotating frame with that same angular velocity it appears stationary and constant. If in small interval of time δt the vector $\mathbf{C}$ rotates through a small angle $\delta\lambda$ then the change in $\mathbf{C}$ (see Figure 1.), is given by

$$\delta\mathbf{C} = |\mathbf{C}| \cos\vartheta \delta\lambda \mathbf{m}, \tag{5.1}$$



where $\mathbf{m}$ is the unit vector in the direction of change of $\mathbf{C}$, which is perpendicular to both $\mathbf{C}$ and Ω. But the rate of change of the angle λ is just, by definition, the angular velocity so that $\delta\lambda = |\Omega|\delta t$ and

$$\delta\mathbf{C} = |C||\Omega| \sin\hat{\vartheta}\mathbf{m}\delta t = \Omega \times \mathbf{C}\ \delta t, \tag{5.2}$$

using the definition of the vector cross product $\times$, where $\hat{\vartheta} = \frac{\pi}{2} - \vartheta$ is the angle between Ω and $\mathbf{C}$. Thus

$$\left(\frac{d\mathbf{C}}{dt}\right)_{\mathbf{I}} = \Omega \times \mathbf{C}, \tag{5.3}$$

where the left-hand side is the rate of change of $\mathbf{C}$.

Now consider a vector $\mathbf{B}$ that changes in the inertial frame. In a small time δt the change in $\mathbf{B}$ is related to the change seen in the inertial frame by

$$(\delta\mathbf{B})_I = (\delta\mathbf{B})_R + (\delta\mathbf{B})_{\mathrm{rot}} \tag{5.4}$$

where the terms are, respectively, the change seen in the inertial frame, the change due to the vector itself changing as measured in the rotating frame, and the change due to the rotation. Using (5.2), we can write $(\delta\mathbf{B})_{\mathrm{rot}} = \Omega \times \mathbf{B}\delta t$, and so the rates of change of the vector $\mathbf{B}$ in the rates of change of the vector $\mathbf{B}$ in the inertial and rotating frames are related by

$$\left(\frac{d\mathbf{B}}{dt}\right)_I = \left(\frac{d\mathbf{B}}{dt}\right)_R + \Omega \times \mathbf{B}. \tag{5.5}$$

This relation applies to a vector $\mathbf{B}$ that, as measured at any one time, is the same in both inertial and rotating frames.

5.1.3.1 Velocity and acceleration in rotating frame

The velocity of a body is not measured to be the same in the inertial and rotating frames, so care must be taken when applying (5.5) to velocity. First apply (5.5) to $\mathbf{r}$, the position of a particle. We get

$$\left(\frac{d\mathbf{r}}{dt}\right)_I = \left(\frac{d\mathbf{r}}{dt}\right)_R + \Omega \times \mathbf{r}, \tag{5.6}$$

or

$$\mathbf{v}_I = \mathbf{v}_R + \Omega \times \mathbf{r}. \tag{5.7}$$

Above, $\mathbf{v}_R$ and $\mathbf{v}_I$ refer to the relative and inertial velocity, respectively, and (5.7) connect the two later quantities. Using once again (5.5) for the time variation of $\mathbf{v}_R$, we must obtain

$$\left(\frac{d\mathbf{v}_R}{dt}\right)_I = \left(\frac{d\mathbf{v}_R}{dt}\right)_R + \Omega \times \mathbf{v}_R, \tag{5.8}$$

or, we employ (5.7), to get

$$\left(\frac{d}{dt}(\mathbf{v}_I - \boldsymbol{\Omega} \times \mathbf{r})\right)_I = \left(\frac{d\mathbf{v}_R}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{v}_R, \tag{5.9}$$

since

$$\left(\frac{d\mathbf{v}_I}{dt}\right)_I = \left(\frac{d\mathbf{v}_R}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{v}_R + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} + \boldsymbol{\Omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_I. \tag{5.10}$$

So, in view of

$$\left(\frac{d\mathbf{r}}{dt}\right)_I = \left(\frac{d\mathbf{r}}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{r} = \mathbf{v}_R + \boldsymbol{\Omega} \times \mathbf{r}, \tag{5.11}$$

if we suppose that the rate of rotation is constant, (5.10) becomes

$$\left(\frac{d\mathbf{v}_R}{dt}\right)_R = \left(\frac{d\mathbf{v}_I}{dt}\right)_I - 2\boldsymbol{\Omega} \times \mathbf{v}_R - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}). \tag{5.12}$$

This equation may be interpreted as follows. The term on the left-hand side is the rate of change of the relative velocity as measured in the rotating frame. The first term on the right-hand side is the rate of change of the inertial velocity as measured in the inertial frame. The second and third terms on the right-hand side are the *Coriolis force* and the *centrifugal force* per unit mass.

Centrifugal force

If $\mathbf{r}^\perp$ is the perpendicular distance from the axis of rotation (see Figure.1 and substitute $\mathbf{r}$ for $\mathbf{C}$), then, because Ω is perpendicular to $\mathbf{r}^\perp$, we get $\Omega \times \mathbf{r} = \Omega \times \mathbf{r}^\perp$. Then, the fact that $\Omega \times (\Omega \times \mathbf{r}^\perp) = (\Omega \cdot \mathbf{r}^\perp)\Omega - (\Omega \cdot \Omega)\mathbf{r}^\perp$ and noting that the first term vanishes, we deduce that the centrifugal force per unit mass is expressed as follows

$$F_{ce} = -\Omega \times (\Omega \times \mathbf{r}) = \Omega^2 \mathbf{r}^\perp. \tag{5.13}$$

Sometimes, the centrifugal force is written as the gradient of a scalar potential

$$F_{ce} = -\nabla \Phi_{ce}, \tag{5.14}$$

where $\Phi_{ce} = -\frac{\Omega^2(\mathbf{r}^\perp)^2}{2} = -\frac{(\Omega \times \mathbf{r}^\perp)^2}{2}$.

Coriolis force

The Coriolis force per unit mass is expressed by

$$F_{Co} = -2\Omega \times \mathbf{v}_R. \tag{5.15}$$

It plays a central role in much of geophysical fluid dynamics. Some basic properties of Coriolis force are cited bellow.

(i) There is no Coriolis force on bodies that are stationary in the rotating frame.

(ii) The Coriolis force acts to deflect moving bodies at right angles to their direction of travel.

(iii) The Coriolis force does no work on a body because it is perpendicular to the velocity, so

$$\mathbf{v}_R \cdot (\Omega \times \mathbf{v}_R) = 0.$$

Momentum equation in rotating frame

Since (5.11) simply relates the accelerations of a particle in the inertial and rotating frames, then in the rotating frame of reference, the momentum equation is reads as follows

$$\frac{D\mathbf{v}}{Dt} + 2\Omega \times \mathbf{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \tag{5.16}$$

where we have integrated the centrifugal term into the potential Φ and we have dropped the subscript R. Henceforth, unless we need to be explicit, all velocities without a subscript will be considered to be relative to the rotating frame.

Mass and tracer conservation in a rotating frame

Let ϕ be a scalar field which obeys in the inertial frame

$$\frac{D\phi}{Dt} + \phi\nabla \cdot \mathbf{v}_I = 0. \tag{5.17}$$

Now, observes in both rotating and inertial frame measure the same value of ϕ. Furthermore, $\frac{D\phi}{Dt}$ is the rate of change of ϕ associated with a material parcel, and hence is reference frame invariant. Thus, we can write

$$(\frac{D\phi}{Dt})_R = (\frac{D\phi}{Dt})_I, \tag{5.18}$$

where $(\frac{D\phi}{Dt})_R = (\frac{\partial\phi}{\partial t})_R + \mathbf{v}_R \cdot \nabla\phi$ and $(\frac{D\phi}{Dt})_I = (\frac{\partial\phi}{\partial t})_I + \mathbf{v}_I \cdot \nabla\phi$ and the local temporal derivatives $(\frac{\partial\phi}{\partial t})_R$ and $(\frac{\partial\phi}{\partial t})_I$ are evaluated at fixed locations in the rotating and inertial frames, respectively. Using (5.7), one gets

$$\nabla \cdot \mathbf{v}_I = \nabla \cdot (\mathbf{v}_R + \Omega \times \mathbf{r}) = \nabla \cdot \mathbf{v}_R, \tag{5.19}$$

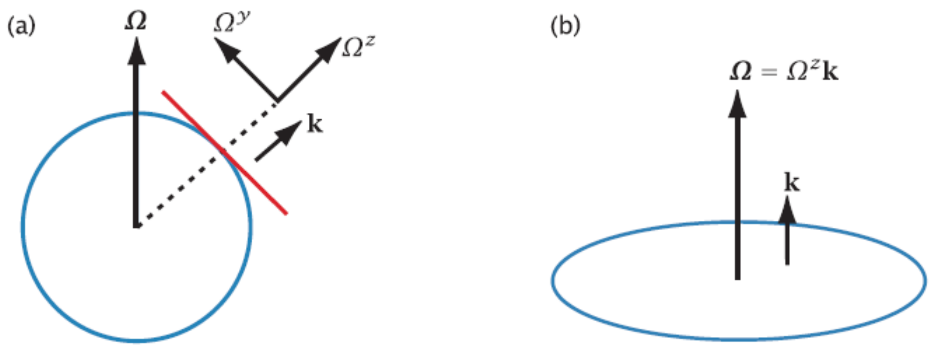
because $\nabla \cdot (\Omega \times \mathbf{r}) = 0$. Thus, collecting (5.18) and (5.19), (5.17) is equivalent to

$$\frac{D\phi}{Dt} + \phi\nabla \cdot \mathbf{v}_R = 0. \tag{5.20}$$

This means that, the equation for the evolution of a scalar field whose measured value is the same in rotating and inertial frames is unchanged by the presence of rotation. In particular, the mass conservation equation is unchanged by the presence of rotation.

Cartesian approximations: the tangent plane

The f-plane



Though the rotation of the earth is central for many dynamical phenomena, the sphericity of the earth is not always so, this is especially true for phenomena on a scale somewhat smaller than global where the use of spherical coordinates becomes awkward, and it is more convenient to use a locally Cartesian representation of the equations. Referring to the red line in **Figure 2.** we will define a plane tangent to the surface of the earth at a latitude ϑ_0, and then use a Cartesian coordinate system (x, y, z) to describe motion on that plane. For small excursions on the plane, $(x, y, z) = (a\lambda \cos\vartheta_0, a(\vartheta - \vartheta_0), z)$. Then, the velocity is $\mathbf{v} = (u, v, w)$, so that u, v and w are the components of the velocity in the tangent plane. Thus, the momentum equations for the flow in this plane are expressed by

$$\frac{\partial u}{\partial t} + (\mathbf{v} \cdot \nabla)u + 2(\Omega^y w - \Omega^z v) = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{5.21}$$

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)v + 2(\Omega^z u - \Omega^x w) = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{5.22}$$

$$\frac{\partial w}{\partial t} + (\mathbf{v} \cdot \nabla)w + 2(\Omega^x v - \Omega^y u) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{5.23}$$

where the rotation vector $\Omega = \Omega^x i + \Omega^y j + \Omega^z k$ and $\Omega^x = 0$, $\Omega^y = \Omega\cos\vartheta_0$ and $\Omega^z = \Omega\sin\vartheta_0$. If we make the traditional approximation, and so ignore the components of Ω not in the direction of the local vertical, then the above equations become

$$\frac{Du}{Dt} - f_0 v = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{5.24}$$

where $f_0 = 2\Omega^z = 2\Omega\sin\vartheta_0$. Defining the horizontal velocity vector $\mathbf{u} = (u, v, 0)$, the first two equations may be written as

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f}_0 \times \mathbf{u} = -\frac{1}{\rho}\nabla_z p, \tag{5.25}$$

where $\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{u}$ and $\mathbf{f_0} = 2\Omega\sin\vartheta_0 k = f_0 k$ and k is the direction perpendicular to the plane.

The β-plane approximation

The magnitude of the vertical component of rotation varies with latitude, and this has important dynamical consequences. We can approximate this effect by allowing the effective rotation vector to vary. Thus, for small variations in latitude, we note that

$$f = 2\Omega\sin\vartheta = 2\Omega\sin\vartheta_0 + 2\Omega(\vartheta - \vartheta_0)\cos\vartheta_0,$$

then on the tangent plane we may simulate this by allowing the Coriolis parameter to vary as

$$f = f_0 + \beta y,$$

where $f_0 = 2\Omega \sin \vartheta_0$ and $\beta = \frac{\partial f}{\partial y} = \frac{2\Omega \cos \vartheta_0}{a}$. This important approximation is known as the β-plane approximation which captures the most important dynamical effects of sphericity. The β-plane horizontal momentum equations are read as follows

$$\frac{D\mathbf{u}}{Dt} + \mathbf{f} \times \mathbf{u} = -\frac{1}{\rho} \nabla_z p, \tag{5.26}$$

where $\mathbf{f} = (f_0 + \beta y)k$. The mass conservation equations in the β-plane approximation are the same as the usual Cartesian $\mathbf{f}-$plane equations.

5.1.4 Equations for a stratified ocean: the Boussinesq approximation

The density variations in the ocean are quite small compared to the mean density, and we may exploit this to derive the motion equations. Let us first examine how much density does vary in the ocean.

Variation of density in the ocean

The variations of density in the ocean are due to three effects: the compression of water by pressure (still denoted by $\Delta_p \rho$), the thermal expansion of water if its temperature changes ($\Delta_T \rho$), and the saline contraction if its salinity changes($\Delta_S \rho$). How big are these? An appropriate equation of state to approximately evaluate these effects is the linear one

$$\rho = \rho_0 [1 - \beta_T (T - T_0) + \beta_S (S - S_0) + \beta_p p], \tag{5.27}$$

where $\beta_T \approx 2 \times 10^{-4} K^{-1}$, $\beta_S \approx 10^{-3} g/kg^{-1}$ and $\beta_p = 1/(\rho_0 c_s^2) \approx 4.4 \times 10^{-10} Pa^{-1}$, with $c_s \approx 1500 ms^{-1}$.

5.1.5 The Boussinesq equations

The Boussinesq equations are a set of equations that exploit the smallness of density variations in liquids. First, let us note

$$\rho = \rho_0 + \delta\rho(x, y, z, t) = \rho_0 + \hat{\rho}(z) + \rho'(x, y, z, t) = \tilde{\rho}(z) + \rho'(x, y, z, t), \tag{5.28}$$

where ρ_0 is a constant and we assume that

$$|\hat{\rho}|, |\rho'|, |\delta\rho << \rho_0. \tag{5.29}$$

We need to assume that $|\rho'| << |\hat{\rho}|$, but this is usually the case of ocean. The horizontal gradients (gradients at constant z, ∇_z) satisfy $\nabla_z p = \nabla_z p' = \nabla_z \delta p$. To obtain the Boussinesq equations we will just use the first expression of (5.28). Associated with the reference density is a reference pressure that is defined to be hydro-static balance with it. That is to say

$$p = p_0(z) + \delta p(x, y, z, t), \tag{5.30}$$

where $|\delta p| << p_0$ and

$$\frac{dp_0}{dz} = -g\rho_0. \tag{5.31}$$

5.1.5.1 Momentum equations

Let $\rho = \rho_0 + \delta\rho$, then the momentum equation reads as follows

$$(\rho_0 + \delta\rho)\big(\frac{D\mathbf{v}}{Dt} + 2\Omega \times \mathbf{v}\big) = -\nabla\delta p - \frac{\partial p_0}{\partial z}\mathbf{k} - g(\rho_0 + \delta\rho)\mathbf{k}. \tag{5.32}$$

If $\frac{\delta\rho}{\rho_0} << 1$, then we may neglect the term $\delta\rho$ on the left-hand side and the equation becomes

$$\frac{D\mathbf{v}}{Dt} + 2\Omega \times \mathbf{v} = -\nabla\Phi + b\mathbf{k}, \tag{5.33}$$

where $\Phi = -\frac{\delta p}{\rho_0}$ and $b = -\frac{g\delta\rho}{\rho_0}$ is the buoyancy. We should not neglect the term $g\delta\rho$, for there is no reason to believe it to be small: $\delta\rho$ may be small, but g is big. Equation (5.33) is the momentum equation in the Boussinesq approximation, and it is common to say that the Boussinesq approximation ignores all variations of density of fluid in the momentum equation, except when associated with the gravitational term. For most large-scale motions in the ocean the deviation pressure and density fields are also approximately in hydro-static balance. Hence, in this case the vertical component of (5.33) becomes

$$\frac{\partial \Phi}{\partial z} = b. \tag{5.34}$$

A condition for (5.34) to hold is that vertical accelerations are small compared to $\frac{g\delta\rho}{\rho_0}$, and not compared to the acceleration due to the gravity itself.

5.1.5.2 Mass continuity

The mass continuity equation is given by

$$\frac{D\rho}{Dt} + (\rho_0 + \delta\rho)\nabla \cdot \mathbf{v} = 0. \tag{5.35}$$

Provided that time scales advectivily (witch means that $\frac{D}{Dt}$ scales in the same way as $\mathbf{v} \cdot \nabla$), then we may approximate this equation by

$$\nabla \cdot \mathbf{v} = 0. \tag{5.36}$$

which is the same as that for a constant density fluid. This absolutely does not allow one to go back and use (5.35) to say that $\frac{D\delta\rho}{Dt} = 0$, the evolution of density is given by the thermodynamic equation in conjunction with an equation of state, and this should not be confused with the mass conservation equation.

5.1.5.3 Thermodynamic equation and equation of state

The Boussinesq equations are closed by the addition of an equation of state, a thermodynamic equation and a salinity equation. If we neglect the salinity, the thermodynamic equation is given by

$$\frac{D\rho}{Dt} - \frac{1}{c_s^2}\frac{Dp}{Dt} = -\dot{Q}(\frac{\rho_0\beta_T}{c_p}) \approx -c_p/(T\rho_0\beta_T), \tag{5.37}$$

where $\dot{Q}$ is the heating rate per unit mass, with the oceanic values $c_p \approx 4 \times 10^3 Jkg^{-1}K^{-1}$, $\beta_T \approx, 2 \times 10^{-4} K^{-1}$ and $c_s \approx 1500 ms^{-1}$. Using (5.28) and (5.30), (5.37) can be written as

$$\frac{D\delta\rho}{Dt} - \frac{1}{c_s^2}\frac{Dp_0}{Dt} = -\dot{Q}(\frac{\rho_0\beta_T}{c_p}), \tag{5.38}$$

or, using (5.31), it holds that

$$\frac{D}{Dt}(\delta\rho + \frac{\rho_0 g}{c_s^2}z) = -\dot{Q}(\frac{\rho_0\beta_T}{c_p}). \tag{5.39}$$

The term in brackets on left-hand side is the potential density. The severest approximation to this is to neglect the second term there, and noting that $b = -\frac{g\delta\rho}{\rho_0}$, one gets

$$\frac{Db}{Dt} = \dot{b}, \tag{5.40}$$

where $\dot{b} = g\beta_T\dot{Q}/c_p$.

Hence, the equations of momentum, mass continuity and thermodynamic form a closed set, called the simple *Boussinesq equations.*

Remark 5.1.1. **(i)** *In the ocean, the compressibility effect can be important and it is convenient to write the thermodynamic equation as*

$$\frac{Db_\sigma}{Dt} = \dot{b}_\sigma, \tag{5.41}$$

where b_σ is the potential buoyancy, with

$$b_\sigma = -g\frac{\delta\rho_\theta}{\rho_0} = -\frac{g}{\rho_0}(\delta\rho + \frac{\rho_0 g z}{c_s^2}) = b - g\frac{z}{H_\rho}, \tag{5.42}$$

with $H_\rho = c_s^2/g$. Buoyancy itself is obtained from $\underline{\sigma}$ by the equation of state $b = b_\sigma + g\frac{z}{H_\rho}$.

(ii) *In many applications we may need to use a still more accurate equation of state. In that case we replace(5.40) by the thermodynamic equations*

$$\frac{D\Theta}{Dt} = \dot{\Theta}, \quad \frac{DS}{Dt} = \dot{S}, \tag{5.43}$$

where Θ is an appropriate thermodynamic state variable, such as potential enthalpy or entropy, S is salinity, and an equation of state then gives the buoyancy. The equation of state has the general form $b = b(\Theta, S, p)$.

Mean stratification and the buoyancy frequency

The processes that cause density to vary in the vertical often differ from those that cause it to vary in the horizontal. For this reason it is sometimes useful to write

$\rho = \rho_0 + \hat{\rho}(z) + \rho'(x,y,z,t)$ and define $\tilde{b}(z) \equiv -g\frac{\hat{\rho}}{\rho_0}$ and $b' \equiv -g\frac{\rho'}{\rho_0}$. Using the hydrostatic equation to evaluate pressure, the thermodynamic equation (5.37) becomes,

$$\frac{Db'}{Dt} + N^2 w = 0, \tag{5.44}$$

where $\frac{D}{Dt}$ remains a three-dimensional operator and

$$N^2(z) = \left(\frac{d\tilde{b}}{dz} - \frac{g^2}{c_s^2}\right) = \frac{d\tilde{b_\sigma}}{dz}, \tag{5.45}$$

where $\tilde{b_\sigma} = \tilde{b} - g\frac{z}{H_\rho}$. The quantity N^2 is a measure of the mean stratification of the fluid, and is equal to the vertical gradient of the mean potential buoyancy. N is known as the buoyancy frequency. Equations (5.44) and (5.45) also hold in the simple Boussinesq equations, but with $c_s^2 = \infty$.

Summary of Boussinesq equations

The simple Boussinesq equations for an inviscid fluid are expressed by

- momentum equations $\frac{Dv}{Dt} + f \times v = -\nabla\phi + \Theta k$,
- mass conservation div$v = 0$,
- buoyancy equation $\frac{Db}{Dt} = \dot{b}$,

A more general form replaces the buoyancy equation by

- thermodynamic equation $\frac{D\Theta}{Dt} = \dot{\Theta}$,
- salinity equation $\frac{DS}{Dt} = \dot{S}$,
- equation of state $b = b(\Theta, S, z)$.

To better understand this topic, we refer to Beckers and Roisin [8], Geoffrey [35], Pedlosky [65], Turner [69] and the references therein.

5.2 A particular case: 2d-Boussinesq equations

After this narrative, detailed of 3d-Boussinesq equations, where the rotation and stratification are the pivotal hypothesis for the fluid motion, now, and in what follows, we restrict ourselves to the Boussinesq equations in space dimension two. In this case the parameter of Coriolis force $\mathbf{f} = 0$ and $v(t,x) = (v^1(t,x), v^2(t,x))$, is the distribution of the fluid velocity localized at $x \in \mathbb{R}^2$ at a time $t \in \mathbb{R}_+$. More precisely, the set of Boussinesq equations take the following expression:

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p + \rho \vec{e}_2 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \rho + v \cdot \nabla \rho = 0 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \mathrm{div} v = 0. \end{cases} \tag{5.46}$$

Mathematically, the 2d-Boussinesq equations serve as a lower-dimensional model of the 3d-hydrodynamics equations. In fact, the 2d-Boussinesq equations retain some key features of the 3d-Euler and Navier-Stokes equations such as the vortex stretching mechanism. The inviscid 2d-Boussinesq equations are formally identical to the Euler equations for the 3d-axisymmetric swirling flows. The fundamental issue of whether classical solutions to the 3d-Euler and Navier-Stokes equations can develop finite time singularities remains outstandingly open and the study of the 2d-Boussinesq equations may shed light on this problem.

Let us recall that the velocity vector field v is said to be axisymmetric if it is decomposed in cylindrical coordinates (e_r, e_θ, r_z) [1] as follows

$$v(t,r,\theta,z) = v^r(r,z,t)e_r + v^\theta(r,z,t)e_\theta + v^z(r,z,t)e_z.$$

The corresponding vorticity $\omega = curl v = \nabla \times v$ associated to such velocity takes the form

$$\omega = -\partial_z v^\theta e_r + \omega^\theta e_\theta + \frac{1}{r}\partial_r(r v^\theta) e_z, \tag{5.47}$$

where $\omega^\theta = \partial_z v^r - \partial_r v^z$. Observing that $\partial_\theta e_r = \partial_r e_\theta = 0$, we can find that the component v^θ of the velocity and the vorticity equations in this case are expressed by

$$\begin{cases} \frac{\widetilde{D}}{Dt} v^\theta & = -\frac{v^r}{r} v^\theta, \\ \frac{\widetilde{D}}{Dt} \omega^\theta & = \frac{1}{r}(\omega^\theta v^r - v^\theta \omega^r) + \omega^r \partial_r v^\theta + \omega^z \partial_z v^\theta, \end{cases} \tag{5.48}$$

[1] $x = (x_1, x_2, z) \in \mathbb{R}^3, e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0)^t, e_\theta = (\frac{x_2}{r}, -\frac{x_1}{r}, 0)^t, e_z = (0,0,1)^t$ and $r = \sqrt{x_1^2 + x_2^2}$

where $\frac{\widetilde{D}}{Dt} = \partial_t + v^r\partial_r + v^z\partial_z$.
Thus, one can deduce the following equations

$$\begin{cases} \frac{\widetilde{D}}{Dt}(rv^\theta) & = 0 \\ \frac{\widetilde{D}}{Dt}(\frac{\omega^\theta}{r}) & = -\frac{1}{r^2}\omega^r v^\theta + \frac{1}{r}\omega^r\partial_r v^\theta + \frac{1}{r}\omega^z\partial_z v^\theta \\ & = -\frac{1}{r^4}\partial_z\big((rv^\theta)^2\big). \end{cases} \tag{5.49}$$

Hence, the axial vorticity ω^θ intensifies through gradients of the quantity $(rv^\theta)^2$ which moves with the flow, that is to say

$$\frac{\widetilde{D}}{Dt}(rv^\theta)^2 = 0 \tag{5.50}$$

Let us return to the 2d-Boussinesq equations and show how it models the 3d-Euler equations axisymmetric with swirl mentioned above. Applying the *curl* operator to the first equation of (5.46) and taking into account that $\mathrm{div} v = 0$, the vorticity-density formulation associated to (5.46) is expressed as follows

$$\begin{cases} \partial_t\omega + v\cdot\nabla\omega = \partial_1\rho & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \partial_t\rho + v\cdot\nabla\rho = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ v = \nabla^\perp\Delta^{-1}\omega. \end{cases} \tag{5.51}$$

or

$$\begin{cases} \frac{D}{Dt}\omega = \partial_1\rho & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ \frac{D}{Dt}\rho = 0 & \text{if } (t,x)\in\mathbb{R}_+\times\mathbb{R}^2, \\ v = \nabla^\perp\Delta^{-1}\omega. \end{cases} \tag{5.52}$$

A direct observation leads to show that there is analogy between (5.49), (5.50) and (5.52). However, we have correspondence between the previous sets of equations as follows

$$r \longleftrightarrow x_2, \quad x^z \longleftrightarrow x_1,$$

$$v^r \longleftrightarrow v^2, \quad v^z \longleftrightarrow v^1,$$

$$\omega^\theta \longleftrightarrow \omega, \quad (rv^\theta)^2 \longleftrightarrow \rho.$$

With this correspondence, we observe that the 2d-Boussinesq equations are formally identical to the 3d-Euler equations axisymmetric with swirl. This provides us to evaluate all external variable coefficients in (5.49) and (5.50) at $r = 1$. Consequently, away from the axis of singularities $r = 0$ for swirling flows, one can expect the qualitative behavior of solutions for the two system of equations to be identical. We recall, to name but few, that E and Shou [31] exploited this analogy to study the nonlinear development of potential singularities in the 2d-Boussinesq equations with initial data analogous to those imposed by Grauer and Sideris [36] or Pumir and Siggia [66] for 3d-axisymmetric Euler equations. For a more detailed explanation, we can see Bertozzi and Majda [11], Wu [74] and other references related to this subject.

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ملخص:
تتعلق الأطروحة الحالية بدراسة الاستمرار الشامل/المحلي لصنفين من معادلات بوسينسك ثنائية الأبعاد. من ناحية معادلات بوسينسك ذات لزوجة كاملة، نبرهن أنه من أجل دوران إبتدائي عبارة عن رقعة دوامة ملساء فإن السرعة المرفقة بها هي دالة ليبشتز شاملة بالنسبة للزمن، كما أن الرقعة التي تتحرك وفق الزمن عبر التدفق المرفق للسرعة، تحافظ على انتظامها الأولي. بالإضافة إلى ذلك، ننشئ النهاية غير اللزجة لما معامل اللزوجة يؤول إلى الصفر ونبرهن أن معدل التقارب الذي تم الحصول عليه هو الأمثل في حالة دوامة رانكين. فيما يخص معادلات بوسينسك ذات لزوجة معدومة، نتعامل مع نفس المسألة، في هذه الحالة نثبت أن السرعة هي دالة ليبشتز محلية بالنسبة للزمن. ندرس أيضا النهاية غير اللزجة لما معامل الانبعاث يؤول إلى الصفر، وهذا يمكننا من قياس الفرق بين السرعات، الكثافات والتدفقات المرفقة.

الكلمات المفتاحية: معادلات بوسينسك، رقعة دوامة، معدل التقارب، الاستمرار الشامل/المحلي، النهاية غير اللزجة.

Abstract:
The current dissertation presents the investigation of the global/local persistence of geometric structures for two kinds of bi-dimensional Boussinesq system. In broad terms for the full viscous Boussinesq equations, we exhibit as soon as the initial vorticity is a smooth vortex patch, then the related velocity is a Lipschitzian function globally in time and the patch that moves through the time keeps its initial regularity. We also establish the inviscid limit when the viscosity vanishes and we clear that the obtained rate of convergence is optimal in the case of the Rankine vortex. Regarding the Boussinesq system with zero viscosity, we treat the same as before, yet the velocity is only a Lipschitzian function locally in time. We also provide the inviscid limit whenever the diffusivity vanishes. This enables us to measure the difference between velocities, densities and the corresponding flows.

Key words: Boussinesq system, vortex patches, rate of convergence, global/local well-posedness, inviscid limit.

Résumé :
La thèse actuelle porte sur la persistance globale/locale des structures géométriques pour deux types d'équations de Boussinesq bidimensionnel. En premier lieu, nous démontrons pour le système de Boussinesq complétement visqueux que si la vorticité initiale a une structure de poche régulière, alors la vitesse associée est une fonction Lipschitzienne globalement en temps et et le transporté de la poche initiale par le flot associé à la vitesse, préserve sa régularité initiale au cours du temps. Nous établissons également la limite non visqueuse lorsque la viscosité tend vers zéro et nous précisons que le taux de convergence obtenu est optimal dans le cas des poches de type Rankine. En ce qui concerne le système de Boussinesq partiellement visqueux, nous prouvons la persistance locale en temps des structures géométriques des solutions pour des donnés initiales de même types. Nous fournissons également la limite non visqueuse quand la diffusivité tend vers zéro. Cela nous permet de mesurer la différence entre les vitesses, les densités et les flots associés.

Mots clés : Système de Boussinesq, poches de tourbillons, taux de convergence, régularité globale/locale, limite non visqueuse.