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Nabila Abdessemed

Entitled

Resolution of semidefinite linear complementarity problem (SDLCP) using new interior point methods based on kernel fonctions .

Before a jury composed of

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I dedicate this thesis especially

To

The first to encourage and supported me to go so far in my studies

My parents,

My husband, my daughters,

And

My sisters, my brothers

To

Who helped me

Faiza and Abdelmalek

Abdessemed Nabila

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Notation

$\mathbb{R}^n$: Space of n-dimentional real vectors,

$\mathbb{R}^n_+$: Positive orthant of $\mathbb{R}^n$,

$\mathbb{R}^{m\times n}$: Vector space of real matrices of size $(m \times n)$,

$0_{m\times n}$: Zero matrix $(m \times n)$,

I : Identity matrix of order n,

A^T : Transpose of matrix $A \in \mathbb{R}^{m\times n}$,

A^{-1} : Inverse of a regular matrix A,

a_{ij} : Matrix element of $A \in \mathbb{R}^{m\times n}$,

$A^{\frac{1}{2}}$: Square root of the matrix $A \succ 0$,

$A \succeq 0 (A \succ 0)$: A is a positive semidefinite (positive definite) matrix,

$A \preceq 0 (A \prec 0)$: A is a negative semidefinite (negative definite) matrix,

$\lambda_i(A)$: The ith eigenvalue of $A \in \mathbb{R}^{n\times n}$,

$Sp(A)$: The spectral of the matrix A,

$diag(x)$: The diagonal matrix X with $X_{ii} = x_{ii}$,

$Tr(A)$: Trace of matrix A $\in \mathbb{R}^{n\times n}$($Tr(A) = \sum a_{ii} = \sum \lambda_i(A)$),

$\rho(A)$: The spectral ray of A($\rho(A) = max_i \mid \lambda_i(A) \mid$),

$det(A)$: The determining of $A \in \mathbb{R}^{n\times n}$($det(A) = \Pi_i \lambda_i(A)$),

$S^n = \left\{ X : X \in \mathbb{R}^{n\times n},\ X = X^T \right\}$,

$S^n_+ = \{ X : X \in \mathbb{S}^n,\ X \succeq 0 \}$,

$S^n_{++} = \{ X : X \in \mathbb{S}^n,\ X \succ 0 \}$,

$A \bullet B = Tr(A^T B)\ \forall A,\ B \in \mathbb{R}^{n\times n}$,

$\| A \|_F^2 = Tr(AA^T) = \sum_i \sum_j a_{ij}^2$,

Terminology

LCP : Linear complementarity problem,

SDP : Semidefinite programming,

N T : Nesterov-Todd,

T C : Trajectory center,

inner : Inner iteration,

outer : Outer iteration,

SVD : Singular value decomposition,

K.K.T : Karush-Kuhn-Tucker,

IPMS : Interior point methods,

SDLCP : Semidefinite linear complementarity problem,

SDLS : Semidefinite least square,

NS-SDLS : Non symmetric semidefinite least square,

LMI-LS : Linear matrix inequalities least squares problem

GUS-property : globally uniquely solvable property.

Contents

Introduction

The (SDLCP) is a generalization of the (LCP) and the (SDP), in the sense that the variables are positive semi definite symmetric matrices instead of being positive vectors. It was introduced by Kojima et al [19] for the first time,

So in recent years, this problem gained the attention of researchers from the theoretical and practical applications, especially to control theory, quantum chemistry, combinatorial optimization, architecture, and electrical engineering.

IPMS are considered as the best instrument to resolve a (LO) problems and can be extended to (CP), (SDO), and (SDLCP).

The methods fall into four categories:

(1)- Affine.

(2)-Projective.

(3)-potential reduction.

(4)-Center path.

Currently, the primal-dual center path stands out as one of most appealing approaches to resolve a wide rang of optimization problems, because these algorithms admit a good theoretical behavior such as polynomial complexity.

Also, Bai et al [6] are introduce these methods for (LO) and Elghami [11] for (SDO), then it's extended by many authors for different problems in mathematical programming ([35], [42], [44]).

The majority of primal-dual (IPMS) that have polynomial complexity for (LP) apply the classical logarithmic function as a barrier function. But currently there is a gap between the

practical behavior of these algorithms and the theoretical results of performance, especially for the large step algorithms for which the theoretical limit of iterations is $O(nlog\frac{n}{\epsilon})$. In practice, large step algorithms are more efficient than small step algorithms for which the theoretical iteration bound is only $O(\sqrt{n}log\frac{n}{\epsilon})$.
These large gap between theory and practice relates to the drawback of interior point methods.

We shown that the kernel function has an important role in generating a new design of primal-dual IP algorithm, in particular for determining new search directions and new proximity functions for analyzing the complexity of these algorithms.

The objective of this work is the theoretical and numerical study of a central trajectory method whose directions are not orthogonal and with the introduction of a new kernel function for the resolution of a (SDLCP).
This new kernel function is defined by :

$$g(x) = \frac{1}{2}(2x^2 + \frac{1}{x^2} - 5) + e^{\frac{1}{x}-1}.$$

Moreover, We get the best known iteration bound for the algorithm with large-update method, namely $O(\sqrt{n}(logn)^2log\frac{n}{\epsilon})$, under a special choice of parameters $(\alpha,\ \theta,\ \mu)$. The implementation of the algorithm showed a great improvement concerning the time and the number of iterations. [1]

This thesis is composed of five chapters:
The first chapter has an introductory character, it presents the fundamental notions of matrix calculus, convex analysis, the optimality conditions of a mathematical program and the results which will be useful in the following of the thesis.
The second chapter is devoted to an in-depth theoretical study of the problem semidefinite linear complementarity.
In the third chapter, we develop a large step primal-dual interior point algorithm based on the classic directions. In order to determine these directions, we use the (N-T) symmetrization

scheme.

The fourth most important chapter is devoted to the resolution of the (SDLCP) by the primal-dual interior points method based on a new kernel function, we established a detailed analysis complexity of the algorithm.
The purpose of the last chapter is to present numerical results from the application of primal-dual interior point algorithm based on a new kernel function on (SDLCP).
We end this thesis with a conclusion and perspectives.

Chapter 1

Matrix calculation and convex analysis

1.1 Matrix calculation

In this part, we present several known results on norms, symmetric matrices and semi definite matrices.

1.1.1 Scalar product and norms

These definitions are similar to the ones obtained in [15].

Definition 1.1. *The scalar product of Z and R of $\mathbb{R}^n$ is exposed by:*

$$< Z, R > = \sum_{i=1}^{n} z_i r_i = Z^T R.$$

Similarly, we define a scalar product on the set of real square matrices

Definition 1.2. *Let $M, N \in \mathbb{R}^{n\times n}$, the scalar product of M and N is:*

$$M \bullet N = trace(M^T N) = \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij} n_{ij} = N \bullet M.$$

We recall that $Trace(.)$ is the trace of $\mathbb{R}^{n\times n}$ matrix.

We note that the trace is a linear function, moreover it verifies these properties:

(1) $\forall M, N \in \mathbb{R}^{n\times n} : trace(MN) = trace(NM)$,

(2) $\forall M \in \mathbb{R}^{n\times n} : trace(M) = trace(M^T)$,

(3) $\forall M, N \in \mathbb{R}^{n\times n} : M \sim N \Rightarrow trace(M) = trace(N)$.

Let us now state the notion of vector norm [16].

Definition 1.3. *The vector norm is an application of* $\mathbb{R}^n$ *in* $\mathbb{R}_+$, *denoted by* $\| \,.\, \|$ *and satisfies these conditions:*

(1) $\forall Z \in \mathbb{R}^n : \| Z \| = 0 \Leftrightarrow Z = 0$,

(2) $\forall Z \in \mathbb{R}^n : \| \lambda Z \| = | \lambda | \| Z \|$,

(3) $\forall Z,\ R \in \mathbb{R}^n : \| Z + R \| \leq \| Z \| + \| R \|$.

Definition 1.4. *Let* $M, N \in \mathbb{R}^{n\times n}$ *, the application* $\| \,.\, \|: \mathbb{R}^{n\times n}$ *in* $\mathbb{R}_+$ *is a matrix norm if it verifies these conditions:*

(1) $\| M \| = 0 \Leftrightarrow M = 0,\ \forall M \in \mathbb{R}^{n\times n}$,

(2) $\| \alpha M \| = | \alpha | \| M \|,\ \forall \alpha \in \mathbb{R}$,

(3) $\| M + N \| \leq \| M \| + \| N \|,\ \forall M,\ N \in \mathbb{R}^{n\times n}$,

(4) $\| MN \| \leq \| M \| \| N \|,\ \forall M,\ N \in \mathbb{R}^{n\times n}$.

Note that the usual matrix norms for any $M \in \mathbb{R}^{n\times n}$ are :

$$\| M \|_1 = max_{k=1}^{n}(\sum_{l=1}^{n} | m_{lk} |),$$

$$\| M \|_\infty = max_{l=1}^{n}(\sum_{k=1}^{n} | m_{lk} |),$$

and

$$\| M \|_2 = \sqrt{\rho(MM^T)}.$$

The last norm is called the spectral norm. We will also use the norm of Frobenius:

$$\| M \|_F = \sqrt{M \bullet M} = \sqrt{\sum_{l=1}^{n}\sum_{k=1}^{n}(m_{lk})^2}.$$

If $M = M^T$, we have

$$\| M \|_F = \sqrt{trace(M^T M)} = \sqrt{trace((M)^2} = \sqrt{\sum_{l=1}^{n} (\gamma_l)^2},$$

and

$$\| M \|_2 = \sqrt{\rho(M^2)} = max_{i=1}^{n} \mid \gamma_i \mid, \quad \| M \|_2 \leq \| M \|_F \leq \sqrt{n} \| M \|_2 .$$

and for any matrix norm we have:

$$\rho(M) \leq \| M \| .$$

Now, we give some important properties :

(1) $\forall Z,\ R \in S^n$:

$$trace(ZR) = trace(RZ),$$

(2) If M is reversible matrix, then

$$trace(MZM^{-1}) = trace(Z).$$

Definition 1.5. *Let $Z \in S^n$*

(1) $Z \in S^n_+$ *or* $Z \succeq 0$ *if* $\forall u \in R^n,\ u^T Z u \geq 0,$

(2) $Z \in S^n_{++}$ *or* $Z \succ 0$ *if* $\forall u \in \mathbb{R}^n, u \neq 0, u^T Z u > 0.$

Theorem 1.1. *Let $M \in S^n$, these conditions are equivalents :*

(1) $M \in S^n_+$ (*resp* $M \in S^n_{++}$),

(2) $\gamma_{min}(M) \geq 0$ ($\gamma_{min}(M) > 0$),

(3) $\exists P \in \mathbb{R}^{n \times n} : M = PP^T$ (*resp* $\exists P \in \mathbb{R}^{n \times n} : rg(P) = n,\ M = PP^T$).

Theorem 1.2. *Let $M \in S^n_{++}$, then $\exists ! N \in S^n_{++}$ such that $M = N^2$, Moreover $N = M^{\frac{1}{2}}$ is the square root of M.*

Lemma 1.1. *[23]*

Let $M, N \in S^n_{++}$, the next assertions are equivalent :

(1) $M \bullet N = 0,$

(2) $MN = 0,$

(3) $\frac{1}{2}(MN + NM) = 0.$

Lemma 1.2. *[23]*

Let $M, N \in S^n_{++}$, *then*

$$\lambda_{min}(M)\gamma_{max}(N) \leq \gamma_{min}(M)Trace(N) \leq M \bullet N \leq \gamma_{max}(M)Trace(N) \leq n\gamma_{max}(M)\gamma_{max}(N).$$

Theorem 1.3. *[45]*

Let $M, N \in S^n$, *we have :*

(1) $trace(MN) \geq 0 \ \forall M, N \succeq 0.$

(2) $\forall M, N \succeq 0, trace(MN) = 0 \Longleftrightarrow MN = MN.$

(3) $\forall M, N \in S^n$ *and* $trace(MN) \geq 0, \forall N \succeq 0 \Longrightarrow M \succeq 0.$

Definition 1.6. *[45]*

(1) $M \in \mathbb{R}^{n\times n}$ *is a* ***normal*** *matrix if* $MM^T = M^TM$ *and moreover if* $MM^T = I$ *then* M *is said to be* ***orthogonal****.*

(2) A normal matrix M *is said to be* ***positively stable*** $\Longleftrightarrow$ M *is positive define .*

(3) Any matrix $M \in S^n$ *is diagonalizable, i.e,* $\exists U$ *an orthogonal matrix such that :* $M = UDU^T$ *with*

$$D = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \gamma_n \end{pmatrix}.$$

where γ_i *are the eigenvalues of* M*.*

(4) If $M \succeq 0 \Longrightarrow UMU^T \succeq 0$ *for any orthogonal matrix* U*.*

(5) $M,\ N \in S^n$. If $MN = NM$ (M and N are commutative matrices), then there is an orthogonal matrix U and two diagonal matrices E and D such that :

$$M = UDU^T \text{ and } N = UEU^T.$$

Definition 1.7. *[45]*
Let $M \in S^n$, if $M \succeq 0$ where $M = UDU^T$ the matrix $\sqrt{M}$ is defined by

$$\sqrt{(M)} = U \begin{pmatrix} \sqrt{\gamma_1} & 0 & \dots & 0 \\ 0 & \sqrt{\gamma_2} & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sqrt{\gamma_n} \end{pmatrix} U^T,$$

where γ_i are the eigenvalues of M (with U orthogonal matrix and $D = Diag((\gamma_1, \gamma_i, ..., \gamma_n)^T)$. The $\sqrt{(M)}$ matrix is called the square root of the M which is furthermore $\sqrt{(M)} \in S^n_+$.

1.1.2 Singular value decomposition

The SVD allows the generalization of the notion of eigenvalues to rectangular matrices $(m \times n)$ and the construction of generalized inverses [7], [40]

Lemma 1.3. *Let $M \in \mathbb{R}^{m\times n}$, then the matrix $MM^T \in S^n_+$.*

Definition 1.8. *Let $M \in \mathbb{R}^{m\times n}$. The singular values of a matrix M are the square roots of the eigenvalues of MM^T.*

Theorem 1.4. *Let $M \in R^{m\times n}$ with the eigenvalues of MM^T not all zero. There are three unitary matrices $U \in R^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ and $W \in \mathbb{R}^{m\times n}$ such that :*

(1) $W = U^T MV$.

(2) The matrix

$$W = \begin{pmatrix} D & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)(n-r)} \end{pmatrix},$$

where

$$D = \begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma_3 & 0 & 0 \\ 0 & 0 & 0 & \gamma_4 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \gamma_r \end{pmatrix}.$$

with $r = min(m, n)$ *and* $\sigma_1 > \sigma_2 > .... > \sigma_r > 0$ *are the nonzero singular values of* M.

1.2 Convex analysis

1.2.1 Convex sets and functions

Definition 1.9. *[18]*
Let $\Omega \subset \mathbb{R}^n$, *we said that* Ω *is affine if*

$$\forall Z, R \in \Omega, \quad \forall \gamma \in \mathbb{R} : (1-\gamma)Z + \gamma R \in \Omega.$$

Let $\hbar : \mathbb{R}^n \to \mathbb{R}^m$ *be a transformation.* $\hbar$ *is affine if*

$$\hbar((1-\gamma)Z + \gamma R) = (1-\gamma)\hbar(Z) + \gamma\hbar(R), \ \forall Z, R \in \mathbb{R}^n, \quad \forall \gamma \in \mathbb{R}.$$

Definition 1.10. *[18]*

(1) A subset $\Omega \in \mathbb{R}^n$. *We said that* Ω*is convex set if*

$$\forall Z, R \in \Omega, \ \forall \gamma \in [0,1] : (1-\gamma)x + \gamma y \in \Omega.$$

(2) We said that $\varphi : \Omega \longrightarrow \mathbb{R}$ *is a convex function if*

$$\varphi(\beta Z + (1-\beta)R) \leq \beta\varphi(Z) + (1-\beta)\varphi(R) \ \ \forall Z, R \in \Omega, \ \forall \beta \in [0,1].$$

(3) A function φ *is said to be concave if and only if* $-\varphi$ *is convex. And if the inequality is strict then* φ *is said to be strictly convex* $\forall Z \neq R$.

1.2.2 Convex and self adjoint cones

Definition 1.11. *[46]*
A subset $K \subset \mathbb{R}^n$, K is a cone if

$$\mathbb{R}_+^* K \subseteq K,$$

that's to say

$$\forall Z \in K, \forall \lambda > 0, \lambda Z \in K.$$

If $K \cap (-K) = 0$, K is said to be a pointed cone, with $-K = \{-z : z \in K\}$. If K is convex, it is called a convex cone .

Example 1.1. *S_+^n is a pointed convex cone of S^n.*

Theorem 1.5. *[27]*
Let $\Phi \in C^2(C)$, C is an open convex of $\mathbb{R}^n$ or $(C = \mathbb{R}^n)$. Then these assertions are equivalent:

(1) *f is convex on C,*

(2) $\Phi(Z) - \Phi(R) \ \geq < \bigtriangledown\Phi(Z), R - Z >, \ \forall Z, \ R \in C,$

(3) *The hessian matrix $H(Z) = \bigtriangledown^2\Phi(Z)$ is positive semi definite in C.*

Proposition 1.6. *[8]*
S_+^n is a self adjoint cone in S^n i.e $(S_+^n)^* = S_+^n$.

1.3 Mathematical programming

A mathematical program is an optimization problem which consists in finding a solution of the problem which maximizes or minimizes a given function under a set of constraints.
In general, a mathematical program is defined by :

$$(PM) \begin{cases} Min \Phi(Z) \\ s.t. \ Z \in C, \end{cases}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a continuous is called objective function, $\emptyset \neq C \subseteq \mathbb{R}^n$ is the set of constraints.

Often C is presented as follows :

$$C = \{Z \in \mathbb{R}^n : \ g_\ell(Z) \leq 0, \ \ell = 1, 2, ..., t, \ h_m(Z) = 0, \ m = 1, 2, ..., k\} \tag{1.1}$$

with g_ℓ, h_m are functions of $\mathbb{R}^n \to \mathbb{R}$.

If $C = \mathbb{R}^n$, this problem is called unconstrained optimization problem.

Saturated (or active)constraint

A constraint with inequality $g_\ell(z) \leq 0 \quad \forall \ell$, is said to be saturated (or active) in $z_0 \in C$ if

$$g_\ell(z_0) = 0 \ \ \forall \ell = 1, 2, ..., t.$$

A constraint of equality $h_m(z) = 0$ is by definition saturated at any point $z \in C$.

Feasible solution

We said that z_0 is a feasible solution of (PM) if he checked the constraints $z_0 \in C$.

Local optimal solution

Any point $z^* \in C$ satisfying

$$\Phi(z^*) \leq \Phi(z), \forall z \in C \cap \Gamma,$$

where Γ is a neighborhood of z^*, is called the **local** optimal solution.

The set of local solutions of (PM) is:

$$locmin_{z \in C} \Phi(z).$$

Global optimal solution

Any satisfying $z^* \in C$ point

$$\Phi(z^*) \leq \Phi(z), \quad for\ all z \in C,$$

is called the **global** optimal solution.

The set of global optimal solutions of (PM) is:

$$argmin_{z \in C} \Phi(z)$$

We always have :

$$argmin_{z\in C}\Phi(z) \subseteq locmin_{z\in C}\Phi(z).$$

Classifications

We classify the problem (PM) from two main properties namely the convexity and the differentiability of the objective function and the constraints.

(1)- (PM) is differentiable if the functions $\Phi,\ g_\ell,\ h_m$ are differentiable,

(2)- (PM) is convex if $\Phi,\ g_\ell$ are convex and h_m are affine.

Not that if (PM) is convex, then any local optimum is a global optimum.

Qualification of constraints

constraints qualification is satisfied for any $Z^* \in C$ in one of three cases :

(1)- The constraints are affine (C is a convex polyhedron)

(2)- Slater's condition: If C is convex (i.e g_ℓ are convex and h_m are affine).

(3)- The magazarian-Fromowitz condition: If the gradient of the saturated constraints in $Z^* \in C$ are linearly independent.

Principal results of existence and uniqueness

These results are analogous to the ones obtained in [27]

Theorem 1.7. *(Weirstrass) If Φ is a continuous function on $C \subseteq \mathbb{R}^n,\ C$ is a compact (closed and bounded), then (PM) admits at least one optimal solution $Z^* \in C$.*

Corollary 1.1. *If $C \subseteq \mathbb{R}^n$ is nonempty and closed and if Φ is a continuous and coercive on D ($\Phi(z) \to +\infty$ when $\|z\| \to +\infty$), then (PM) admits at least one optimal solution .*

Remark 1.1. *- If Φ is strictly convex and C is a convex, then if (PM) admits an optimal solution, the solution is unique.*

- Strict convexity ensures the uniqueness of solution and not the existence of the latter.

Optimality conditions

The following **Karush-Kuhn-Tucker-Lagrange** theorem gives a necessary condition for optimality (first -order condition).

Theorem 1.8. *Suppose that the function Φ, g_ℓ, h_m are C^1 in a neighborhood of $Z^* \in D$ and that the constraints satisfy one of the three qualification conditions above. If Φ has a local minimum in Z^* over D then there exist $\lambda_\ell \in \mathbb{R}^t_+$ and $\mu_m \in \mathbb{R}^k$ such that :*

$$\textbf{\textit{(K.K.T)}} \begin{cases} \nabla\Phi(Z^*) + \sum_{\ell=1}^{t} \lambda_\ell \nabla g_\ell(Z^*) + \sum_{m=1}^{k} \mu_m \nabla h_m(Z^*) = 0 \\ \lambda_\ell g_\ell(Z^*) = 0, \forall \ell = 1, ..., t \\ h_m(Z^*) = 0, \forall m = 1, ..., k \end{cases} \tag{1.2}$$

*The quantities λ_ℓ and μ_m, are called **Karush-Kuhn-Tucker-Lagrange** multipliers.*

Remark 1.2. *- If the constraints of (PM) are not qualified in Z^*, the conditions of **(K.K.T)** do not apply.*
*- If (PM) is convex, the **(K.K.T)** are both necessary and sufficient so that Z^*, is a global minimum.*

Newton method for nonlinear system

The principle of which is given a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ twice continuously differentiable and let $J(z)$ be the Jacobian matrix of Φ where :

$$J(z) = \begin{pmatrix} \frac{\partial \Phi_1}{\partial z_1} & . & . & \frac{\partial \Phi_1}{\partial z_n} \\ . & . & . & \\ . & . & . & \\ \frac{\partial \Phi_n}{\partial z_1} & . & . & \frac{\partial \Phi_n}{\partial z_n} \end{pmatrix},$$

The system $\Phi(z) = 0$, with z^0 is a given vector of $\mathbb{R}^n$, Newton's iteration formula at iteration (k) is given by :

$$z^{(\ell+1)} = z^{(\ell)} - (J(z^{(\ell)}))^{-1}\Phi(z^{(\ell)}), \ell = 0, 1, ...,$$

where $(J(z^{(\ell)}))^{-1}$ is the reverse matrix of $J(z^{(\ell)})$. So we get :

$$z^{(\ell+1)} = z^{(\ell)} + \triangle\, z^{(\ell)},$$

where $\triangle z^{(\ell)}$ is the solution (direction of Newton) of the linear system :

$$J(z^{(\ell)}) \triangle z^{(\ell)} = -\Phi(z^{(\ell)}),$$

If z^0 is sufficiently inside the set of solutions of Φ, then this sequence converges to a root of $\Phi(z) = 0$.

The choice of Newton's method is very important from the point of view of interior point methods because of its numerical efficiency.

1.4 Primal-dual IPMs of central trajectory

The primal dual are the most efficient among the IPMs. Within the framework of mathematical programming with constraints, we designate by method of interior points, any iterative procedure of resolution generating a series of points belonging to the relative interior of the realizable domain and converting towards an optimal solution. Similarly, we call barrier function any function f which verifies,

(1) Φ has finite values inside the relative realizable domain,

(2) $\Phi(z) \longrightarrow \infty$ when z approaches the boundary. These methods are renowned for their polynomial complexity, their speed and efficiency and have proven to be real competitors of classical methods (Simplex, Lemke pivoting, ..)

The literature on these methods has experienced a great expansion has been enriched with several classes and variants in order to reduce complexity and improve numerical efficiency. There are in practically four categories of methods of interior points: the projective methods, the affine methods, the potential reduction, and central trajectory.

In this thesis we present only the methods of central trajectory which will be subsequently used.

1.4.1 Central trajectory methods

The general idea of central trajectory methods consists in following particular path (known as centers or central path), taking Newton's direction of displacement. In other words, the

algorithm generates a strictly feasible sequence (Z^k, R^k) and a sequence μ^k which expresses the condition of complementarity, the latter is verified when μ^k tends to zero.
The principle of central path tracking methods is to define a certain neighborhood around the central path and to make the iterates evolve inside this neighborhood while progressing towards the solution.

1.4.2 Description of the method in case of (SDLCP)

We have seen in the following that the resolution of the (SDLCP) amounts to solving the following perturbed non linear subsystems

$$\begin{pmatrix} \hbar(Z)+Q-R \\ ZR \end{pmatrix} = \begin{pmatrix} 0 \\ \mu I \end{pmatrix} \tag{1.3}$$

The word " under system" is linked to the parameter μ because at each fixed μ we have a system to solve and the solution of (SDLCP) is obtained when $\mu \longrightarrow 0$.
The perturbed complementary equation is the 2^{nd} equation of the system. The previous system can be expressed as a nonlinear equation $F(Z, R) = 0$, where

$$F : S^n \times S^n \longrightarrow S^n \times \mathbb{R}^{n\times n}$$

such that:

$$(Z, R) \longmapsto (\ \hbar(Z) + Q - R, \quad ZR - \mu I\ ).$$

To apply Newton's method to solve the previous system, it is necessary to symmetrize the 2^{nd} equation of the system, because the $dim(S^n \times S^n)$ (the departure space) is different from the $dim(S^n \times \mathbb{R}^{n\times n})$ (the arrival space).
We define a symmetrization operator Ω_T, for any regular matrix T as follows :

$$\Omega_T : \mathbb{R}^{n\times n} \longrightarrow S^n$$

$$\Omega_T(S) = \tfrac{1}{2}(\ TST^{-1} + (TST^{-1})^T\ ),\ \forall S \in \mathbb{R}^{n\times n}.$$

Putting $M = ZR$, we obtain the following equivalent symmetric system :

$$\begin{pmatrix} \hbar(Z)+Q-R \\ \Omega_T(ZR)-\mu I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.4}$$

Hence, the following Newton's system :

$$\begin{cases} \hbar(\triangle\ Z)\ = \triangle\ R \\ \Omega_T(\triangle\ ZR + Z\ \triangle\ R)\ =\ \mu I, \end{cases} \tag{1.5}$$

which gives it a solution, that is $(\triangle\ Z,\ \triangle\ R)$ totally symmetric ($(Z, R) \in S^n \times S^n$).

Chapter 2

Existence and uniqueness of solution

We give definitions and some properties related to the linear transformation

2.1 Definition of the problem

The (SDLCP) is to find a pair $(Z, R) \in S^n \times S^n$ such that :

$$Z,\ R \in S^n_+,\ R \ = \ \hbar(Z) + Q,\ \ and\ \ Z \bullet R \ = \ Tr(ZR) \ = \ 0, \tag{2.1}$$

where $Z \bullet R$ denote the trace of matrix product ZR, $Q \in S^n$ and $\hbar : S^n \to S^n$ is a linear transformation.

2.2 Definitions and some properties

These definitions are analogous to the ones obtained in [52].

In this part, we assume that $\hbar : S^n \to S^n$ is a linear transformation,

Definition 2.1.

(1) $\hbar$ is a $P-$transformation if

$$Z \in S^n, \quad Z\hbar(Z) = \hbar(Z)Z, \quad and \quad Z\hbar(Z) \preceq 0 \Rightarrow Z = 0.$$

(2) $\hbar$ is a P_0-transformation if $(\hbar + \varepsilon I)$ is $P-$transformation for all $\varepsilon > 0$, I is identity transformation on S^n.

(3) $\hbar$ is a P_2-transformation if

$$Z \succeq 0, \; R \succeq 0, \; (Z - R)[\hbar(Z) - \hbar(R)](Z + R) \preceq 0 \Rightarrow Z = R.$$

(4) $\hbar$ is a monotone transformation (strictly monotone) if

$$< \hbar(Z), Z > \geq 0 \; (> 0), \; for \; all \; matrix \; Z \in S^n \; (Z \neq 0).$$

Definition 2.2.

(1) $\hbar$ is a strictly semi monotone transformation $(S.S.M)$ (or $\hbar$ is $E-$ transformation) if

$$Z \succeq 0, \quad Z\hbar(Z) = \hbar(Z)Z, \quad and \quad Z\hbar(Z) \preceq 0 \Rightarrow Z = 0.$$

(2) If $(\hbar + \varepsilon I)$ is strictly semi monotone transformation for all $\varepsilon > 0$, then $\hbar$ is semi monotone transformation $(S.M)$ (or$\hbar$ is E_0-transformation).

Definition 2.3.

(1) $\hbar$ has the R_0- property, if $(SDLCP)(\hbar, \; 0)$ has a unique solution is zero matrix .

(2) $\hbar$ has the $Q-$ property, if the problem $(SDLCP)(\hbar, \; Q)$ has a solution $\forall Q \in S^n$.

Definition 2.4. *We say that $\hbar$ has*

*(1) (**GUS**-property) if $(SDLCP)(\hbar,\ Q)$ has only one solution $\forall Q \in S^n$,.*

*(2) **Column suficiency**- property if $\forall\ Q \in S^n$, the problem $(SDLCP)(\hbar,\ Q)$ has a convex (possibly empty) solution set.*

*(3) **Cross commutative**- property if $\forall\ Q \in S^n$, solutions Z_1 and Z_2 of $(SDLCP)(\hbar,\ Q)$ the following holds*

$$Z_1R_2 = R_2Z_1 \quad and \quad Z_2R_1 = R_1Z_2,$$

where $R_i = \hbar(Z_i) + Q,\ i = 1, 2.$

Definition 2.5. *The transpose of $\hbar$ is $\hbar^T : S^n \to S^n$ defined by*

$$<\ \hbar(Z),\ R\ > = <\ Z,\ \hbar^T(R)\ > \ \text{for all}\ Z,\ R \in S^n,$$

$\hbar$ is self adjoint transformation on S^n if $\hbar = \hbar^T$ and normal if $\hbar\hbar^T = \hbar^T\hbar$.

Theorem 2.1.

(1) $\hbar$ is a monotone transformation ($<\ \hbar(Z),\ Z\ > \geq\ 0$ for all $Z \in S^n$),

(2) $\hbar$ is column sufficient,

(3) $\hbar$ has the cross commutative-property,

then

$$(1) \Rightarrow (2) \Rightarrow (3)$$

2.3 Existence and uniqueness of solution

The existence of the solution is given by the Theorem of Karamardian [17], [52] .

Theorem 2.2. *$(SDLCP)(\hbar,\ Q)$ has a solution $\forall Q \in S^n$, If $(SDLCP)(\hbar,\ 0)$ and $(SDLCP)(\hbar,\ I)$, have only one solution (namely zero) for $I \in S^n$.*

Theorem 2.3. *Suppose that* $\hbar$ *has the P-property then* $(SDLCP)(\hbar,\ Q)$ *has a solution* $\forall Q \in S^n$.

Remark 2.1. *The p−property of the transformation ensures the existence of the solution but not the uniqueness, contrary to (LCP) problem where the p−property ensures the existence and uniqueness of the solution.*

Theorem 2.4. *For a linear transformation* $\hbar : S^n \to S^n$*, the following assertions are equivalent:*

(1) $\forall\ Q \in S^n, (SDLCP)(\hbar,\ Q)$ *has only one solution,*

(2) $\hbar$ *has the P-property and cross commutative-property,*

(3) $\hbar$ *has GUS- property.*

Theorem 2.5. *If* $\hbar$ *is monotone, then the set of solutions of* $(SDLCP)(\hbar,\ Q)$ *is convex (possibly empty,)*$\forall Q \in S^n$*. the problem*

Corollary 2.1. *If* $\hbar$ *is linear monotone transformation, then*

$$\hbar \text{ has } GUS\text{-property} \Longleftrightarrow \hbar \text{ has } P\text{-property.}$$

Theorem 2.6. *If* $\hbar$ *is strictly monotone transformation, then* $\hbar$ *has GUS-property.*

Theorem 2.7. *For the linear transformation* $\hbar$*, we have the following assertions*

$$\text{strictly monotone property} \Rightarrow P_2 \text{ -property} \Rightarrow GUS\text{- property}$$

Theorem 2.8. *Let* $\hbar$*, be a transformation which has the P- property.*

If $\hbar$ *is self adjoint , then* $\hbar$ *is strictly monotone and the following implication are satisfied*

$$\text{strictly monotone property} \Longleftrightarrow P_2\text{-property} \Longleftrightarrow GUS\text{-property}$$

2.4 Study of some formes

2.4.1 Lyapunov transformation

The **Lyapunov** transformation associated with square matrix M is defined by

$$\hbar_M(Z) = MZ + ZM^T$$

Theorem 2.9. *[14] Consider the Lyapunov transformation $\hbar_M$, with a given square matrix M. Then these assertions are equivalent :*

(1) $\hbar_M$ has GUS- property,

(2) M is positive stable and positive semi definite.

Theorem 2.10. *$\hbar_M$ has P_2- property $\Longleftrightarrow$ M is positive definite matrix.*

Theorem 2.11. *We assume that M is normal matrix, then we have*

$$\hbar_M \text{ has strictly monotone property} \Leftrightarrow \hbar_M \text{ has GUS-property} \Leftrightarrow \hbar_M \text{ has P-property.}$$

2.4.2 Stein transformation

The transformation of **Stein** is defined by

$$S_M(Z) \;=\; MZ + ZM^T \quad \text{for all } M \in \mathbb{R}^{n\times n}$$

Theorem 2.12. *Let $M \in \mathbb{R}^{n\times n}$ is normal matrix, then we have*

$$\text{strictly monotone} \Leftrightarrow GUS \Leftrightarrow P.$$

2.4.3 Two-sided transformation

The transformation of **two-sided** is defined by

$$M_A(X) \;=\; AXA^T \quad \text{for all } A \in \mathbb{R}^{n\times n}$$

Theorem 2.13. *[14]*

(1) $M_A(X)$ has GUS- property,

(2) $A \succ 0$ or $A \prec 0$.

2.5 The equivalence between the (SDLCP) and some mathematical problems

In this section we give some examples for (SDLCP) problem

2.5.1 LCP is a SDLCP

The standard (LCP) is defined by :
Find Z and $R \in \mathbb{R}^n$:

$$R = NZ + q, \; Z, \; R \in \mathbb{R}^n_+ \; \; and \; \; Z^T R = 0. \tag{2.2}$$

Indeed, the (LCP) is a (SDLCP), if we give $N \in \mathbb{R}^n \times \mathbb{R}^n$ and $q \in \mathbb{R}^n$ and $\hbar$ is

$$\hbar_N(Z) = Diag(Ndiag(Z)),$$

where $diag(Z)$ is a vector whose components are the diagonal elements of matrix Z and $Diag(Z)$ is a diagonal matrix whose diagonal elements are the components of the vector Z, we obtain $(SDLCP)(\hbar_N, \; Diag(q))$:
Find $Z \in S^n$:

$$Z \in S^n_+, \; R = \hbar_N(Z) + Diag(q) \in S^n_+, \; \; and \; < Z, R > = \; 0, \tag{2.3}$$

and we have these results

Z is a solution of $(SDLCP)(\hbar_M, \; Diag(q))$, if $diag(Z)$ is a solution of $(LCP)(N, \; q)$. Conversely if Z is a solution of $(LCP)(N, \; q)$, then $Diag(Z)$ is a solution of $(SDLCP)(\hbar_N, \; Diag(q))$.

2.5.2 Geometric SDLCP

([8]) In (1997) kojima, Shindoh and Hara presented the Geometric (SDLCP)(F) in their work, under the name of geometric semi definite linear complemtarity problem defined as follows :

$$Find \; the \; pair \; \; (Z, R) \in F \cap (S^n_+ \times S^n_+) : \; < Z, R > = \; 0, \tag{2.4}$$

where F is any affine subset of $S^n_+ \times S^n_+$, of dimension $\frac{n(n+1)}{2}$. In this case the linear transformation $\hbar_M$ is not given explicitly. This version of (SDLCP) includes semi definite linear programming (SDP). The (SDP) is a recent and important problem because on the one hand it contains linear programming (LP), and on the other hand it has many applications such as control theory, quantum chemistry and combinatorial optimization.

The linear semi definite problem (LCP) in primal form is an optimization problem:

$$\begin{cases} min\ _X\ trace(CX) \\ s.t < A_i, X > = \ b_i, \quad i = 1, 2, ..., m, \quad X \in S^n_+, \end{cases} \tag{2.5}$$

where $b \in \mathbb{R}^m, \quad C,\ A_i \in S^n, \quad i = 1, 2, ..., m.$

and it's dual is

$$\begin{cases} max\ _{(y,Z)}\ b^T y \\ s.t \sum_i^m y_i A_i + Z \ = \ C, \quad Z \in S^n_+, \end{cases} \tag{2.6}$$

where $Z \in R^m$.

Let X and (Z, y) such that $X \succeq 0$ and $Z \succeq 0$ are solutions of (P) and (D) respectively, then (X, Z) is a solution of geometric $(SDLCP)$:

Find the pair $(X, Z) \in \mathcal{F} \cap (S^n_+ \times S^n_+) : < X, Z > = \ 0$, with the affine set F is given by :

$$\mathcal{F} \ = \ \{\ (X, Z) \in S^n \times S^n :< A_i, X > = \ b_i, \quad i = 1, 2, ..., m,\ \sum_i^m y_i A_i + Z \ = \ C,\ for\ y \in R^m\ \},$$

where $A_i,\ C \in S^n,\ b \in \mathbb{R}^m$. Hence the resolution of (P) and (D) are equivalent to the resolution of the previous geometric (SDLCP). It is clear that (SDLCP) is a geometric (SDLCP).

Now consider an affine subset $\mathcal{F}$ of $S^n \times S^n$ of dimension $\frac{n(n+1)}{2}$ and the geometric $(SDLCP)(\ \mathcal{F}\ )$ associated with $\mathcal{F}$. Assume without loss of generality that :

$$\mathcal{F} \ = \ \{\ (Z, R) \in S^n \times S^n, \quad \hbar_1(X) + \hbar_2(X) \ = \ B\ \},$$

where $\hbar_1,\ \hbar_2 : S^n \to S^n$ are two linear transformations and B is a matrix of S^n.

We define $\hbar : S^{3n} \to S^{3n},\ Q \in S^{3n}$ by

$$\hbar\left(\begin{bmatrix} X & * & * \\ * & Y & * \\ * & * & Z \end{bmatrix}\right) = \begin{bmatrix} Y & 0 & 0 \\ 0 & \hbar_1(Z)+\hbar_2(R) & 0 \\ 0 & 0 & -\hbar_1(Z)-\hbar_2(R) \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -B & 0 \\ 0 & 0 & B \end{bmatrix}$$

For

$$W = \begin{pmatrix} X & * & * \\ * & Y & * \\ * & * & Z \end{pmatrix}$$

solution of $(SDLCP)(\hbar,\ Q)$, then $(Z,\ R)$ solution of geometric $(SDLCP)(\mathcal{F})$.
If $(Z,\ R)$ solution of geometric $(SDLCP)(\mathcal{F})$ then

$$W = \begin{pmatrix} X & * & * \\ * & Y & * \\ * & * & Z \end{pmatrix}$$

is a solution of $(SDLCP)(\hbar,\ Q)$. Finaly we deduce that the resolution of $(SDLCP)(\mathcal{F})$ is equivalent to that of $(SDLCP)(\hbar,\ Q)$.

2.5.3 SDLCP Per block

Let $n_1, n_2, ..., n_k$ be integers fixed and S^n the set of real symmetric square matrices of order $n_i,\ i = 1, ..., k$.
Let S the set of all diagonal matrices per block $X_i \in S^{n_i},\ i = 1, ..., k$ defined by :

$$\mathcal{S} \ = \ \{ \ X : \ X_i \in S^{n_i}, \ i \ = \ 1, ..., \ k \ \}.$$

Where

$$X = \begin{pmatrix} X_1 & 0 & .... & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & X_k \end{pmatrix}$$

S is a Hilbert space endowed with the following inner product :

$$< Z, R > = \ Tr(ZR) \ = \ \sum_{i}^{k} Tr(Z_i R_i).$$

Let S_i the close convex cone of matrices positive semi definite in S and consider the following problem :

$$Find\ Z \in S \ \ such\ that \ \ Z \in S_+, \ R \ = \ \hbar(Z) + Q \in S_+ \ and \ \ < Z, R > = \ 0, \qquad (2.7)$$

where $\hbar : S^n \to S^n$ is a given linear transformation and $Q \in S$. This problem includes the (LCP) in the case where $n_i = 1$ for all i. Moreover the (SDLCP) is a special case of this problem .
reformulated the last system as an (SDLCP).
Indeed, let $m = n_1 + n_2 + ... + n_k$ and $\hbar : S^m \to S^m$ is a linear transformation such that :

$$\hbar \begin{pmatrix} Z_1 & * & ... & * \\ * & Z_2 & \ddots & \vdots \\ \vdots & * & \ddots & * \\ * & * & * & Z_k \end{pmatrix} = \begin{pmatrix} \hbar_1(Z_1) & 0 & \dots & 0 \\ 0 & \hbar_2(Z_2) & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \hbar_k(X_k) \end{pmatrix}.$$

where $\hbar_i(Z_i)$ is the i-th block matrix of the matrix $\hbar(Z),\ \hbar_i : S^{n_i} \to S^{n_i},\ i = 1, 2, ..., k$ for a Q matrix.

If Z is a solution of (2.7) problem, then Z is a solution of the following $(SDLCP)(\hbar, Q)$:

Find $Z \in S^m$ such that $Z \in S^m_+,\ R \ = \ \hbar(Z) + Q \in S^m_+$ and $< Z,\ R > = \ 0,$

Now, let Z a solution of $(SDLCP)(\hbar, Q)$, then matrix

$$Z = \begin{pmatrix} Z_1 & 0 & \dots & 0 \\ 0 & Z_2 & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & Z_k \end{pmatrix}.$$

is a solution of (2.7) where Z_i are sub matrices of Z whose row and column index are taken from the set :

$$\alpha = \left\{ \sum_{k=1}^{i} n_{k-1} + 1, \sum_{k=1}^{i} n_{k-1} + 2, ..., \sum_{k=1}^{i} n_{k-1} + n_i \right\},$$

with $n_0 = 0$.

The following three problems are analogous to the ones obtained in [23]

2.5.4 SDLS is a SDLCP

The semidefinite least squares problem (SDLS) is a convex optimization problem defined by :

$$Min_{X \in S^n_+} \frac{1}{2} \|MX - N\|^2_F,$$

where $M,\ N \in R^{n\times n}$ and $X \in S^n$.

It is demonstrated if the matrix M is full rank, the problem (SDLS) has a unique global minimizer given by the **(K. K. T)** equations

$$\begin{cases} M^T(MX - N) \ = \ Z, \\ \frac{1}{2}(ZZ^T) \ = \ Y, \\ < X, Y > = \ 0,\ X,\ Y \ \succeq \ 0. \end{cases} \tag{2.8}$$

Letting $Q \in S^n$ and $\hbar$ be defined as :

$$\hbar(X) \ = \ M^T MX + XM^T M \ \ and \ \ Q \ = \ -\frac{1}{2}(\ M^T N + N^T M\ ),$$

and letting

$$\mathcal{F} = \{ \ (Z, \ R) \in S^n \times S^n, \ R \ = \ \hbar(Z) + Q \ \} .$$

We note that (Z, R) satisfies the **K.K.T** equations $\Longleftrightarrow$ (Z, R) satisfies

$$(Z, R) \in \mathcal{F}, \ Z, \ R \ \succeq \ 0 \ \ and \ \ < Z, R > = \ 0. \tag{2.9}$$

The transformation $\hbar$ is linear on S^n, where indeed (SDLS) is a (SDLCP).

2.5.5 NS-SDLS is a an SDLCP

The non symmetric semi definite least squares problem (NS-SDLS) is also a convex optimization problem defined as follows :

$$(NS - SDLS) \ \ Min_{X \in D} \tfrac{1}{2} \|AX - B\|_F^2,$$

where

$$D \ = \ \left\{ X \in R^{\ n \times n} : \tfrac{1}{2}(\ X + X^T \ ) \ \succeq \ 0 \right\} .$$

With $\frac{1}{2}(\ X + X^T \ )$ is the symmetric part of the matrix X and $A, \ B \in R^{\ n \times n}$.
This problem admits a unique minimizer on the set D if the matrix A has full rank, the problem(NS-SDLS) is equivalent to finding the symmetric matrices $Y, \ Z$ that satisfy the following equations of **(K.K.T)**

$$\begin{cases} A^T(\ AX - B \ ) \ = \ Z, \\ \frac{1}{2}(\ X + X^T \ ) \ = \ Y, \\ < Z, \ Y > = \ 0, \ \ \ Z, \ Y \ \succeq \ 0. \end{cases} \tag{2.10}$$

Then (NS-SDLS) problem is equivalent to this (SDLCP) :

find the pair $(Z, \ Y)$ such that :

$$(Z, \ Y) \in \mathcal{F}, \ Z, \ Y \ \succeq \ 0 and < \ Z, \ Y > = \ 0.$$

with

$$\mathcal{F} \ = \ \{ \ (Z, \ Y) \in S^n \times S^n, \ Y \ = \ \hbar(Z) + Q \ \} .$$

where

$$\hbar(Z) \;=\; \frac{1}{2}(A^TA)^{-1}Z + Z(A^TA)^{-1} \;\; and \;\; Q \;=\; \frac{1}{2}((A^TA)^{-1}A^TB + B^TA(A^TA)^{-1}).$$

It is easy also to verify that L is a linear transformation on S^n in himself.

If $(\ Z,\ Y\ )$ is a solution of (SDLCP) then $X \;=\; (A^TA)^{-1}(Z + A^TB)$.

2.5.6 LMI-LS is a an SDLCP

The least squares problem of linear matrix inequalities (LMI-LS) is also a convex optimization problem defined by:

$$(LMI - LS)\;\; Min_{\kappa x \preceq C}\; \tfrac{1}{2}\|Ax - b\|\,_2^2,$$

where

$\kappa x \;=\; \sum_{i=1}^{k} x_i K_i,\; \kappa : \mathbb{R}^k \to S^n,\; K_i \in S^n(i = 1, k),\; C \in S^n,\; A \in \mathbb{R}^{m\times k},\; b \in \mathbb{R}^m \; and \; x \in \mathbb{R}^k.$

Under the following conditions:

(1) The matrix A has full rank,

(2) The set $\{\ x \in \mathbb{R}^n : \kappa x \prec C\ \}$ is not empty.

(LMI-LS) has a unique minimizer x characterized by the equations of **K.K.T**

$$\begin{cases} A^T(Ax - b) + \kappa^* Z \;=\; 0, \\ \kappa x + Y \;=\; C, \\ < Y, Z > \,=\; 0,\; Z,\; Y \;\succeq\; 0. \end{cases} \tag{2.11}$$

Where $\kappa^* : S^n \to \mathbb{R}^k$ is the adjoint transformation of κ defined by $<\ Y,\ \kappa x\ > \,=\, <\ \kappa^* Y,\ x\ >$.

Then the system defines a (SDLCP) as follows :

find the pair $(\ Z,\ Y\ )$ such that

$$(\ Z,\ Y\ ) \in \mathcal{F},\; Z,\; Y \;\succeq\; 0 \; and \;\; <\ Z,\ Y\ > \,=\; 0.$$

where

$$\mathcal{F} \;=\; \{\ (\ Z,\ Y\ ) \in S^n \times S^n,\; Y \;=\; \hbar(Z) + Q\ \}\,.$$

with

$$\hbar(Z) \ = \ \kappa(A^TA)^{-1}\kappa^*Z \ \ and \ \ Q \ = \ C - \kappa(A^TA)^{-1}(A^Tb).$$

For ($Z, \ Y$) a solution of (SDLCP), then $x \ = \ (A^TA)^{-1}(A^Tb - \kappa^*Z)$ is a solution of (LMI-LS).

The transformation L is linear on S^n.

Remark 2.2. *We show that the linear transformations associated with (NS-SDLS)and (LMI-LS) are the Lyapunov transformations $L_B(X) \ = \ BX + XB^T$ with $B \ = \ A^TA$ and $B \ = \ (A^TA)^{-1}$ respectively.*

2.6 Applications

2.6.1 Linear differential systems

Theorem 2.14. *(Lyapunov) [52] Let A be a real square matrix and M a stable positive matrix, then A is positive stable $\Longleftrightarrow$ M positive definite where :*

$$XA + A^TX \ = \ M$$

The differential system is stable if A is positive stable (the real part of the eigenvalues of A is positive).

This is verified by Gowda and Song [52] by proving that

$$\forall Q \in S^n, \ \exists X \in S^n$$

where $X \ \succeq \ 0$ such that :

$$AX + XA^T + Q \ \succeq \ 0 \ \ and \ \ X[AX + XA^T + Q] \ = \ 0,$$

which represents a semi definite linear complementarity problem associated with the Lyapunov transformation:

$$L_A(X) \ = \ XA + XA^T$$

2.6.2 Dynamic systems

Continuous dynamical systems

$$\frac{ax}{dt} = -Ax(t),$$

and discrete dynamical systems :

$$x(k+1) = Ax(k), \ k \in N,$$

are stable if A satisfies Schur stability ($|\lambda| < 1$).

Theorem 2.15. *[13] A is a square matrix of $C^{n\times n}$ and $Q \in H^n, \ Q \succ 0$ then*

$$|\lambda| < 1, \ \forall\lambda \in \sigma(A)$$

if and only if

$$\exists X \succeq 0, \ X \in H^n \ : \ X - AXA^* + Q \succeq 0 \ \textit{and} \ X[X - AXA^* + Q] = 0.$$

We deduce from this theorem that, $|\lambda| < 1$ if and only if

$$\forall Q \in H^n, \ \exists X \succeq 0 \ \textit{such that} \ X - AXA^* + Q \succeq 0 \ \textit{and} \ X[S_A(X) + Q] = 0.$$

Which represents a problem $(SDLCP)(S_A, Q)$.

Here H^n is the set of Hermitian matrices, and $S_A(X) = X - AXA^*$ is Stein transformation.

Chapter 3

SDLCP and primal-dual central path method

3.1 Presentation of Problem

Consider the following (SDLCP)

Find a pair $(Z,\ R) \in S^n \times S^n$ *that achieved this conditions*

$$Z,\ R \in S^n_+,\ R\ =\ \hbar(Z) + Q,\ \ and\ \ Z \bullet R\ =\ Tr(ZR)\ =\ 0, \tag{3.1}$$

where $\hbar$ is a linear transformation and $Q \in S^n$.

The solution set of the System (3.1), the strict feasible set and the feasible set are each subsets of $\mathbb{R}^{n\times n}$ indicated by

$$\begin{aligned} \mathcal{S} &= \{(Z,\ R) \in \mathcal{F}\ :\ Tr(ZR)\ =\ 0\}. \\ \mathcal{F}^0 &= \{\ (Z,\ R) \in \mathcal{F}\ :\ Z\ \succ\ 0, R\ \succ\ 0\}, \\ \mathcal{F} &= \{\ (Z,\ R) \in S^n \times S^n,\ R\ =\ \hbar(Z) + Q :\ Z\ \succeq\ 0,\ R\ \succeq\ 0\}, \end{aligned}$$

In this work, we suppose that the (SDLCP) satisfies all conditions :

(1) $\mathcal{F}^0$ is not empty,

(2) $\hbar$ is monotone ($< \ \hbar(Z), \ Z \ > \geq \ 0, \ \forall Z \in S^n$),

(3) $\hbar$ is self adjoint ($\hbar \ = \ \hbar^T$, such that : $< \ \hbar(Z), \ R \ > = < \ Z, \ \hbar^T(R) \ > \ \forall Z, \ R \in S^n$).

Since for $Z, \ R \in S^n_+, \ Z \bullet R \ = \ 0 \Longleftrightarrow ZR \ = \ 0$. Then, finding a solution of (3.1) is similar to solving this system :

$$\begin{cases} R \ = \ \hbar(Z) + Q, \\ ZR \ = \ 0, \\ Z \ \succeq \ 0, \ R \ \succeq \ 0. \end{cases} \tag{3.2}$$

Theorem 3.1. *[45]*
*Under conditions **(1)**, **(2)**, the set of solution of (SDLCP)*

$$\mathcal{S} \ = \ \{(\ Z, \ R \ ) \in \mathcal{F} : Tr(ZR) \ = \ 0\},$$

is nonempty and convex.

These results are similar to the ones obtained in [8].

Proposition 3.2. $f(Z) \ = \ Tr(Z(\hbar(Z) + Q))$ *is a convex function, if conditions **(2)**, **(3)** are satisfied.*

Theorem 3.3. *Assume that the conditions **(1)**, **(2)** and **(3)** are satisfied, then (SDLCP) is equivalent to the convex semi definite problem* (PO)

$$(PO) \begin{cases} min_Z \ Tr(Z(\hbar(Z) + Q)), \\ Z \ \succeq \ 0, \ R \ = \ \hbar(Z) + Q \ \succeq \ 0. \end{cases}$$

Therefore, finding the solution of (3.1) is similar to find the minimizer of (PO)

3.2 Central path method for SDLCP

3.2.1 Logarithmic barrier function

For solving the system (PO), we associate with it the nonlinear barrier problem :

$$(PO)_\mu \begin{cases} min[\ \chi_\mu(Z,\ R) \ = \ ZR - \mu \log\det(ZR)\ ], \quad \mu \ > \ 0, \\ R \ = \ \hbar(Z) + Q, \\ Z \ \succ \ 0,\ R \ \succ \ 0. \end{cases}$$

Where $\mu \ > \ 0$.

In the next theorems, we suppose that the conditions **(1), (2), (3)** are achieved.

Theorem 3.4. *The function* $\chi_\mu(Z,\ R),\ \mu \ > \ 0$ *is strictly convex.*

Theorem 3.5. *There exists* $(\ Z^0,\ R^0\ ) \in \mathcal{F}^0$*, the set*

$$\Omega_\mu \ = \ \left\{ Z \in S^n,\ \chi_\mu(Z) \ \leqslant \ \chi_\mu(Z^0) \right\}.$$

is compact.

The problem $(PO)_\mu$ have a unique solution, if the function $\chi_\mu(\ Z,\ R\ ),\ \mu \succ 0$ is strictly convex.

We know that the system $(PO)_\mu$ is a convex semi definite optimization problem, then the (**K. K. T**) conditions are necessary and sufficient and write as follow :

$$\begin{cases} ZR - \mu I \ = \ 0, \\ R \ = \ \hbar(Z) + Q, \\ Z \ \succ \ 0,\ R \ \succ \ 0. \end{cases} \tag{3.3}$$

where $\mu \ > \ 0$. Then the resolution of the sys (3.3) is equivalent to the resolution of $(PO)_\mu$.

We suppose that a strictly feasible pair $((Z^0, R^0) \in \mathcal{F}^0)$ exists, which is the (IPC) for (SDLCP), and L is monotone, the System (3.3) and $(PO)_\mu$ have a unique solution for a fixed $\mu > 0$ [19].

Definition 3.1. ($Z(\mu),\ R(\mu)$) *is the solution of* $(PO)_\mu$ *for* $\mu\ >\ 0$ *is and*

$$C\ =\ \{\ (Z(\mu),\ R(\mu))\ :\ \mu\ >\ 0\ \}\,.$$

The ***central path****(the set of all solutions of the system* $(PO)_\mu$*) .*

The binary $(Z(\mu),\ R(\mu))$ converges to the solution $(Z,\ R)$ of problem (SDLCP), as $\mu \to 0$,

Lemma 3.1. *Under the conditions* ***(1), (2)****, the set*

$$\{(Z(\mu),\ R(\mu)) : 0\ \leq\ \mu\ \leq\ \mu^-\}\,,$$

is bounded for all $\mu^-\ >\ 0$.

Theorem 3.6. $\lim_{\mu\to 0}\,(Z(\mu),\ R(\mu))\ =\ (Z^*, R^*)$, *where* $(Z^*,\ R^*)$ *is a solution of (SDLCP).*

3.3 Classical search direction

To solve the system (3.3), we use primal-dual path following algorithms.
The important idea of latter is to approximately follow the central path and reach the solution of (3.3), using Newton's step such that

$$Z^{(k+1)}\ =\ Z^{(k)} + \alpha \bigtriangleup Z,\quad R^{(k+1)}\ =\ R^{(k)} + \alpha \bigtriangleup R.$$

We note that $(Z,\ R)$ is a solution of (SDLCP) if and only if it is a solution of (3.3) for $\mu\ =\ 0$.
The first equation in system (3.3) is nonlinear, a direct resolution is generally impossible that is

$$F_\mu(Z,\ R)\ =\ 0\ \ \textit{and}\ \ (Z,\ R) \in S^n_{++} \times S^n_{++},\quad \mu \in \mathbb{R}^*_+, \tag{3.4}$$

which represents a system of nonlinear equation where $F_\mu : S^n \times S^n \longrightarrow S^n \times \mathbb{R}^{n\times n}$ is a function defined by :

$$F_\mu(Z, R)\ =\ \begin{pmatrix} ZR - \mu I \\ \hbar(Z) + Q - R \end{pmatrix}\ =\ 0,$$

we use Newton's method for solving nonlinear equation (3.4), we obtain

$$F_\mu(Z,\ R) + \bigtriangledown F_\mu(Z,\ R)(\triangle Z,\ \triangle R)^T \ = \ 0,$$

the system of linear equations :

$$\begin{cases} \hbar(\triangle Z) \ = \ \triangle R \\ Z \triangle R + \triangle Z R \ = \ \mu I - ZR. \end{cases} \tag{3.5}$$

Which is equivalent to

$$\begin{cases} \hbar(\triangle Z) \ = \ \triangle R \\ \triangle Z + Z \triangle R R^{-1} \ = \ \mu R^{-1} - Z. \end{cases} \tag{3.6}$$

Let $\mu \ > \ 0$ and $(Z,\ R)$ be a strictly primal-dual realizable point, the Newton direction $(\triangle Z,\ \triangle R)$ is the unique solution of (3.5).

3.3.1 primal dual algorithm for SDLCP

Algorithm strategy

(1) We start with (Z^0, R^0) a strictly primal dual realizable point and belonging to a neighborhood of central path, with known μ^0 $\left(\mu^0 = \dfrac{Tr(Z^0R^0)}{n}\right)$;

(2) We build the new pair where $\triangle Z,\ \triangle R$ are directions of Newton;

(3) We verify that $Z^{(k+1)},\ R^{(k+1)}$ are strictly realizable for all $\mu \ > \ 0$ and we distinguish two cases :

- If $n\mu^k < \epsilon$ where ϵ is a given precision, then $Z^{(k+1)},\ R^{(k+1)}$ are approximate solutions of the system (3.3).

- Otherwise $n\mu^k \geq \epsilon,\ \ Z^{(k+1)},\ R^{(k+1)}$ are not the approximate optimal solutions of (3.3), in this case the parameter μ reduces to $\mu^+ \ = \ (1-\theta)\mu$ where $0 \ < \ \theta \ < \ 1$ and we remain close to the central trajectory.

Remark 3.1. *In particular $\theta \ = \ \Theta(\frac{1}{\sqrt{n}})$, the algorithm is called at **Small step** and if $\theta \ = \ 0(1)$ then the algorithm is called at **large step***

Symmetrization technique

Under the previous conditions, this system admits a unique solution $(\triangle Z,\ \triangle R)$ which in general is not symmetrical. So to deal with this problem Y. Zhang [47] proposes to replace the last equation in (3.5) by the following equation :

$$H_P(Z \triangle R + \triangle ZR) = \mu I - H_P(ZR).$$

where H_P is a given linear transformation defined by :

$$H_P(M) = \frac{1}{2}(PMP^{-1} + P^{-T}M^TP^T), \tag{3.7}$$

with P is an invertible matrix and M is a real square matrix of order n.
Using this technique, Newton's second equation linearized in (3.5) will be :

$$\begin{cases} \hbar(\Delta Z) = \Delta R, \\ \Delta Z + P\Delta RP^T = \mu R^{-1} - Z, \end{cases} \tag{3.8}$$

there are several choices for the matrix P:

P	References
I	Alizadeh, Haeberly and Overton [5] ;
$R^{\frac{1}{2}} or Z^{\frac{1}{2}}$	Helmberg, Kojima and Monteiro (H K M) [23];
Z^{-1}	Kojima [11];
R	Kojima [11].

Table 3.1: Several choices of the matrix P

3.4 Nesterov-Todd direction

We use the symmetrization schema of N-T [42], where

$$\begin{aligned} P &= Z^{\frac{1}{2}}(Z^{\frac{1}{2}}RZ^{\frac{1}{2}})^{-\frac{1}{2}}Z^{\frac{1}{2}} \\ &= R^{-\frac{1}{2}}(R^{\frac{1}{2}}ZR^{\frac{1}{2}})^{\frac{1}{2}}R^{-\frac{1}{2}}. \end{aligned}$$

Let $B = P^{\frac{1}{2}}$, we define the matrix ω by

$$\omega = \frac{1}{\sqrt{\mu}}B^{-1}ZB^{-1} = \frac{1}{\sqrt{\mu}}BRB, \tag{3.9}$$

thus we have

$$\omega^{2} = \frac{1}{\mu}B^{-1}ZRB. \tag{3.10}$$

The matrices $\omega, B \in S^n_{++}$. When we use (3.9), the sys (3.6) becomes

$$\begin{cases} \tilde{\hbar}(D_Z) = D_R, \\ D_Z + D_R = \omega^{-1} - \omega, \end{cases} \tag{3.11}$$

with

$$D_Z = \frac{1}{\sqrt{\mu}}B^{-1}\Delta Z B^{-1},\ D_R = \frac{1}{\sqrt{\mu}}B\Delta R B,\ \ and\ \ \tilde{\hbar}(D_Z) = B\hbar(BD_ZB)B. \tag{3.12}$$

$\tilde{\hbar}$ is also monotone on S^n.

The new sys (3.11) has a unique symmetric solution $(D_Z,\ D_R)$.

The directions $(D_Z,\ D_R)$ do not achieve orthogonality, because

$$D_Z \bullet D_R = Trace(D_R D_Z) = \tfrac{1}{\mu}\Delta Z \bullet \hbar(\Delta Z) \geq 0.$$

3.5 Proximity measure

The iterations produced by the algorithm remain feasible and close to the central trajectory, we are obliged to introduce a proximity measure which is defined by :

$$\delta(\omega) = \tfrac{1}{2}\|\omega^{-1} - \omega\|. \tag{3.13}$$

Note that

$$\delta(\omega) \;=\; 0 \Leftrightarrow \omega \;=\; \omega^{-1} \Leftrightarrow ZR \;=\; \mu I.$$

Which justifies the belonging of the points to the central trajectory.

3.6 primal-dual IP algorithm

We describe the general outline of IP algorithm for SDLCP.

First, we use $\tau \;\geq\; 1$ and suppose that for certain known μ^0, and the existing strictly feasible point (Z^0, R^0) such that $\delta(Z^0,\; R^0,\; \mu^0) \;\leq\; \tau$.

Now, using the directions $(\Delta Z,\; \Delta R)$ and taking the step size $\alpha \in (0,1)$, the algorithm produces a new iterate $(Z^{(k+1)},\; R^{(k+1)}) = (Z^{(k)} + \alpha\Delta Z,\; R^{(k)} + \alpha\Delta R)$.

We note that the search for an optimal primal dual solution is equivalent to setting $Z \bullet R$ to zero, this is expressed by updating μ via :

$$\mu^{(k+1)} \;=\; (1-\theta)\mu^{(k)} \;\; with \;\; 0 \;<\; \theta \;<\; 1$$

The primal-dual algorithm for solving SDLCP is stated as follows :

Input:

A threshold parameter $\tau \geq 1$;

an accuracy parameter $\varepsilon \geq 0$;

barrier update parameter $0 < \theta < 1$;

A strictly feasible pair $Z^0 \succ 0,\ R^0 \succ 0$ and $\mu^0 = Trace(Z^0 R^0)/n$: $\delta(Z^0,\ R^0,\ \mu^0) \leq \tau$

begin

$Z := Z^0;\ R := R^0;\ \mu := \mu^0;$

while $n\mu \geq \varepsilon$ **do**

begin

$\mu^{k+1} = (1-\theta)\mu^k;$

while $\delta(Z,\ R,\ \mu) > \tau$ **do**

begin

-Solve System (3.11) and use (3.12) to obtain $(\Delta Z,\ \Delta R)$;

-determine a suitable step size α;

-update $(Z^{k+1},\ R^{k+1}) = (Z^k + \alpha\Delta Z,\ R^k + \alpha\Delta R)$.

end

end

end

Algorithm 1: Primal-dual IP algorithm for SDLCP

Chapter 4

Primal-dual IPMs for SDLCP based on new kernel functions

In this chapter, we develop an algorithm of a primal dual IPM based on a new kernel function to solve the SDLCP problem. First we start with the notion of a matrix function which will be used later in this chapter and the introduction of a new kernel function which must verify the conditions of eligibility . Then we establishes the convergence of the algorithm and the analysis of the complexity.

This results are similar to those obtained in [11], [6].

4.1 Kernel function and their properties

We begin by the matrix function from a differentiable function $g(t)$ and with real values . Then, we present the notion of a kernel function and their properties for SDLCP.

Now, we present Spectral decomposition for matrices belonging to S^n , it allows us to extend the definition of the real function g to $g : S^n \to S^n$

Theorem 4.1. *Let* $A \in \mathbb{R}^{n\times n}$

$$A \in S^n \iff \exists Q \in \mathbb{R}^{n\times n} \text{ where } Q^TQ = I,\ Q^TAQ = B, \text{ and } B \text{ is a diagonal matrix.}$$

Definition 4.1. *Let $\omega \in S^n$, and*

$$\omega \;=\; H^T diag(\gamma_1(\omega), \gamma_2(\omega), \ldots, \gamma_n(\omega))H,$$

is a spectral decomposition where H is any orthogonal matrix that diagonalizes ω, the matrix valued function $g : S^n \to S^n$ is given by

$$g(\omega) \;=\; H^T diag(g(\gamma_1(\omega)), g(\gamma_2(\omega)), \ldots, g(\gamma_n(\omega)))H,$$

We note that the function g is well defined when g is well defined at each eigenvalue of ω.

since g is differentiable, then $g'(x)$ is

$$g'(\omega) \;=\; H^T diag(g'(\gamma_1(\omega)), g'(\gamma_2(\omega)), \ldots, g'(\gamma_n(\omega)))H.$$

Now, we need to some notion connected to the matrix function. For more details, consult the works ([15], [36]).

Definition 4.2. *A matrix $M(x)$ is a function matrix if every element of $M(x)$ is a function of x, that is $M(x) = [M_{ij}(x)]$.*

Let $H(x)$ and $M(x)$ be two differentiable functions matrices. So we have:

$$\begin{cases} \frac{d}{dx}Trace(g(H(x))) &=& Trace(g(H^{'}(x))H^{'}(x)), \\ \frac{d}{dx}(H(x)M(x)) &=& H^{'}(x)M(x) + H(x)M^{'}(x). \\ \frac{d}{dx}Trace(H(x)) &=& Trace(H^{'}(x)), \\ \frac{d}{dx}H(x) &=& \left[\frac{d}{dx}H_{ij}(x)\right] = H^{'}(x), \end{cases}$$

4.1.1 A new set of kernel functions

Definition 4.3. *We said that $g(x) : [0, \infty) \to [0, \infty)$ is a kernel function if g is twice differentiable and satisfies*

(1) $g(1) = g'(1) = 0$;

(2) $g''(x) > 0 \ for \ all \ x \ > \ 0$;

(3) $\lim_{x\to 0} g(x) = \lim_{x\to +\infty} g(x) = +\infty$.

Clearly, the first and second properties say that $g(x)$ is a strictly convex function where $g(1) = 0$.

We can describe $g(x)$ by

$$g(x) = \int_1^x \int_1^\zeta g''(\xi) d\xi d\zeta.$$

where $g''()$ the second derivative of g. Now, we present the new function g

$$g(x) = \frac{1}{2}(2x^2 + \frac{1}{x^2} - 5) + e^{\frac{1}{x}-1}. \tag{4.1}$$

Now, we liste $g'(), g''() \ and \ g'''()$ as follows :

$$g'(x) = 2x - \frac{1}{x^3} - \frac{1}{x^2}e^{\frac{1}{x}-1}, \tag{4.2}$$

$$g''(x) = 2 + \frac{3}{x^4} + (\frac{2}{x^3} + \frac{1}{x^4})e^{\frac{1}{x}-1}, \tag{4.3}$$

$$g'''(x) = -\frac{12}{x^5} - (\frac{6}{x^5} + \frac{1}{x^6} + \frac{6}{x^4})e^{\frac{1}{x}-1}. \tag{4.4}$$

We have $\lim_{x\to 0} g(x) \ = \ \lim_{x\to +\infty}, g(x) \ = \ +\infty$ and $g(1) \ = \ g'(1) = 0$. from (4.3), (4.4) we get $g''(x) \ > \ 1, g'''(x) \ < \ 0$. Then $g(x)$ is a barrier function.

4.1.2 Qualification of a kernel function

We present the qualification conditions of a kernel barrier function $g(x)$ and some properties concerning these conditions which are crucial in the complexity . We begin by this technical lemma:

Lemma 4.1. *Consider the barrier function $g(x)$. If $g(x)$ verifies the following two properties:*

(1)- $xg''(x) - g'(x) > 0, \quad for \ x > 1$,

(2)- $g'''(x) < 0, \quad for\ x > 0,$

Then, it verifies

$$g''(x)g'(\beta x) - \beta g'(x)g''(\beta x) > 0,\ for\ x\ >\ 1,\ \beta > 1.$$

Lemma 4.2. *Consider the function $g(x)$ defined as in (4.1), then*

$$\begin{aligned} xg''(x) + g'(x) &> 0, && for\ x < 1, && (4.5)\\ xg''(x) - g'(x) &> 0, && for\ x > 1, && (4.6)\\ g'''(x) &< 0, && for\ x\ >\ 0, && (4.7)\\ 2g''(x)^2 - g'''(x)g'(x) &> 0, && for\ x <\ 1, && (4.8)\\ g''(x)g'(\beta x) - \beta g'(x)g''(\beta x) &> 0, && for\ x > 1,\ \beta\ >\ 1. && (4.9) \end{aligned}$$

Proof. For the two ineauqlity (4.5) and (4.6), using (4.2) and (4.3), we have

$$tg''(x) + g'(x) \ = \ 4x + \frac{2}{x^3} + (\frac{1}{x^2} + \frac{1}{x^3})e^{\frac{1}{x}-1} > 0,\ for\ x\ <\ 1,$$

and

$$tg''(x) - g'(x) \ = \ \frac{4}{x^3} + (\frac{1}{x^3} + \frac{3}{x^2})e^{\frac{1}{x}-1} > 0,\ for\ x\ >\ 1.$$

For $g'''(x) < 0\ for\ x > 0$, verified by the (4.4), .

For (4.8), we have

$$2g''(x)^2 - g'''(x)g'(x) = K(x) + W(x)e^{2(\frac{1}{x}-1)} + Q(x)e^{(\frac{1}{t}-1)} > 0\ for\ x < 1,$$

where $K(x) = 8 + \frac{48}{x^4} + \frac{6}{x^8},\ W(x) = (\frac{2}{x^6} + \frac{2}{x^7} + \frac{1}{x^8})$ and $Q(x) = (\frac{28}{x^3} + \frac{20}{x^4} + \frac{2}{x^5} + \frac{6}{x^7} + \frac{6}{x^8} - \frac{1}{x^9})$.

Finally, we use (4.6) and (4.7) to get (4.9), . □

The proximity function $G(\omega) : S^n_+ \to \mathbb{R}_+$ is

$$G(\omega) = Trace(g(\omega)) = \sum_{i=1}^{n} g(\gamma_i(\omega)), \forall \gamma_i(\omega) > 0.$$

Now, we show the important property of the barrier function according to the first two conditions of definition 4.3

Lemma 4.3. *Let $G(\omega)$ be the barrier function associated with a kernel function , $G(\omega)$ is strictly convex with $\omega \succ 0$ and verify $G(I) = G^{'}(I) = 0$.*

4.1.3 Properties of eligible kernel function

Lemma 4.4. *This properties are equivalent:*

(1) $g(\sqrt{x_1x_2}) \leq \frac{1}{2}(g(x_1) + g(x_2))$ *for* $x_1, x_2 > 0$,

(2) $xg''(x) + g'(x) > 0$ *for* $x > 0$,

(3) $g(e^{\xi})$ *is convex.*

Lemma 4.5. *For $g(x)$, the conditions holds*

(1) $\frac{1}{2}(x-1)^2 \leq g(x) \leq \frac{1}{2}(g'(x))^2$, *for* $x > 0$,

(2) $g(x) \leq 4(x-1)^2$, *for* $x \geq 1$.

Proof. in condition(1), using $(g''(x) > 1)$, we get the first inequality

$$g(x) = \int_1^x \int_1^\zeta g''(\xi)d\xi d\zeta \geq \int_1^t \int_1^\zeta d\xi d\zeta = \frac{1}{2}(x-1)^2.$$

For the second inequality we have

$$g(x) = \int_1^x \int_1^\zeta g''(\xi)d\xi d\zeta \leq \int_1^x \int_1^\zeta g''(\xi)g''(\zeta)d\xi d\zeta = \frac{1}{2}(g'(x))^2.$$

For item (2), $g(1) = g'(1) = 0$, $g''(x) = 8$ and $g'''(x) < 0$, we get

$$g(x) \leq \frac{1}{2}(g''(1))(x-1)^2 = 4(x-1)^2.$$

□

Theorem 4.2. *Suppose that ω_1 and $\omega_2 \in S^n_{++}$ and G is the matrix function, Then*

$$G\left(\left[(\omega_1^{\frac{1}{2}}\omega_2\omega_1^{\frac{1}{2}})^{\frac{1}{2}}\right]\right) \leq \frac{1}{2}(G(\omega_1)+G(\omega_2)).$$

4.2 New search directions

To define new search directions, we will use the system:

$$\begin{cases} \tilde{\hbar}(D_Z) = D_R, \\ D_Z + D_R = -g'(\omega). \end{cases} \tag{4.10}$$

The directions are not orthogonal i.e $D_Z \bullet D_R \geq 0$.
we can form a binary (Z_+, R_+) according to $Z_+ = Z + \alpha\Delta Z$ and $R_+ = R + \alpha\Delta R$.

To analyze an algorithm of interior points, the measure of proximity is
$\delta(\omega) : S^n_{++} \longrightarrow \mathbb{R}_+$, as follows :

$$\delta(\omega) \;\;=\;\; \tfrac{1}{2}\| - g'(\omega)\| = \tfrac{1}{2}\sqrt{Tr(g'(\omega)^2)} = \tfrac{1}{2}\|D_Z + D_R\|, \tag{4.11}$$

According to lemma (4.3), $G(\omega)$ is strictly convex and reaches its minimum value zero at $\omega = I$, then we have

$$\delta(\omega) = 0 \Leftrightarrow \omega = I \Leftrightarrow G(\omega) = 0.$$

4.2.1 The form of large-update primal-dual IP algorithm

Input:

A threshold parameter $\tau \geq 1$;

an accuracy parameter $\varepsilon \geq 0$;

barrier update parameter $0 < \theta < 1$;

A strictly feasible binary $Z^0 \succ 0$, $R^0 \succ 0$ and $\mu^0 = Trace(Z^0 R^0)/n$:

$G(Z^0, R^0, \mu^0) \leq \tau$

begin

$Z := Z^0$; $R := R^0$; $\mu := \mu^0$;

while $n\mu \geq \varepsilon$ **do**

begin

$\mu^{k+1} = (1-\theta)\mu^k$;

while $G(Z, R, \mu) > \tau$ **do**

begin

Solve Sys (4.10) and use Eq. (3.12) to obtain $(\Delta Z, \Delta R)$;

determine the step size α;

update $(Z^{k+1}, R^{k+1}) := (Z^k + \alpha\Delta Z, R^k + \alpha\Delta R)$

end

end

end

Algorithm 2: primal-dual IP algorithm for SDLCP

4.3 Upper bound of $G(W)$

Let $\Gamma : [0, \infty) \to [1, \infty)$ and $\rho : [0, \infty) \to (0, 1]$ the reverse function of $g(x)$, for $x \geq 1$, and the reverse of $-\frac{1}{2}g'(x)$ respectively,

$$s = g(x) \Leftrightarrow x = \Gamma(s), \; x \geq 1;$$

$$s = -\frac{1}{2}g^{'}(x) \Leftrightarrow x = \rho(s), \; x \leq 1.$$

Theorem 4.3. *[11]*

For all ω positive definite matrix, and for all $\beta > 1$ we have

$$G(\beta\omega) \leq ng\left(\beta\Gamma\left(\frac{G(\omega)}{n}\right)\right).$$

By consequence of the theorem, we get

if $G(\omega) \leq \tau$, $\beta = \frac{1}{\sqrt{1-\theta}}$, then after $\mu-$update, $G\left(\frac{1}{\sqrt{1-\theta}}\omega\right)$ is bounded by $ng\left(\beta\Gamma\left(\frac{G(\omega)}{n}\right)\right)$.

As Γ is increasing and $G(\omega) \leq \tau$ then $\Gamma\left(\frac{G(\omega)}{n}\right) \leq \Gamma\left(\frac{\tau}{n}\right)$ and so

$$G_0 \leq L_g(n, \theta, \tau) := ng\left(\beta\Gamma\left(\frac{\frac{\tau}{n}}{\sqrt{1-\theta}}\right)\right)$$

where G_0 denotes the value of $G(\omega)$ after updating μ

4.4 The decrease of the function barrier

we obtain new iterates,

$$Z_+ = \sqrt{\mu}B(\omega+\alpha D_Z)B \textit{ and } R_+ = \sqrt{\mu}B^{-1}(\omega+\alpha D_R)B^{-1},$$

so we have

$$\omega_+^2 = (\omega+\alpha D_Z)(\omega+\alpha D_R).$$

and

$$\omega_+^2 = (\omega+\alpha D_Z)^{\frac{1}{2}}(\omega+\alpha D_Z)^{\frac{1}{2}}(\omega+\alpha D_R)(\omega+\alpha D_Z)^{\frac{1}{2}}(\omega+\alpha D_Z)^{\frac{-1}{2}}.$$

We assume that $(\omega+\alpha D_Z)\succ 0$ and $(\omega+\alpha D_R)\succ 0$ for such α step size that ensures strict feasibility.

ω_+^2 is similar to

$$\omega_+^2 = (\omega+\alpha D_Z)^{\frac{1}{2}}(\omega+\alpha D_R)(\omega+\alpha D_Z)^{\frac{1}{2}}.$$

The eigenvalues of ω_+ are the same as

$$\left[(\omega+\alpha D_Z)^{\frac{1}{2}}(\omega+\alpha D_R)(\omega+\alpha D_Z)^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

We define the proximity after one step by

$$G(\omega_+) = G([(\omega+\alpha D_Z)^{\frac{1}{2}}(\omega+\alpha D_R)(\omega+\alpha D_Z)^{\frac{1}{2}}]^{\frac{1}{2}}).$$

By Theorem (4.2), we have $G(\omega_+) \leq \frac{1}{2}[G((\omega+\alpha D_Z)+G(\omega+\alpha D_R)]$.

Define for $\alpha > 0$,

$U(\alpha) = G(\omega_+) - G(\omega)$ and $U_1(\alpha) = \frac{1}{2}[G((\omega+\alpha D_Z)+G(\omega+\alpha D_R)] - G(\omega)$.

It is easily seen that, $U_1(0) = U(0) = 0$ and $U(\alpha) \leq U_1(\alpha)$. Furthermore, $U_1(\alpha)$ is a convex function.

$$U_1'(\alpha) = \tfrac{1}{2}Tr(g'((\omega + \alpha D_Z)D_Z + g'(\omega + \alpha D_R)D_R),$$

and

$$U_1''(\alpha) = \tfrac{1}{2}Tr(g''((\omega + \alpha D_Z)D_Z^2 + g''(\omega + \alpha D_R)D_R^2).$$

Hence, using (4.10) and (4.11), we obtain

$$U_1'(0) = \frac{1}{2}Tr(g'(\omega)(D_Z + D_R)) = \frac{1}{2}Tr(-g'(\omega)^2) = -2\delta^2(\omega). \tag{4.12}$$

Lemma 4.6. *[48] We have*

$$U_1''(\alpha) \leq 2\delta^2 U^{''}(\gamma_{min}(\omega) - 2\alpha\delta)$$

Lemma 4.7. *[6] We have $U_1'(\alpha) \leq 0$, If α verifies the inequality*

$$-g^{'}(\gamma_{min}(\omega) - 2\alpha\delta) + g^{'}(\gamma_{min}(\omega)) \leq 2\delta.$$

Lemma 4.8. *[6] Suppose that the value of α which satisfies lemma (4.7) is :*

$$\overline{\alpha} = \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)),$$

Lemma 4.9. *[6] Let ρ and $\overline{\alpha}$ then*

$$\overline{\alpha} \geq \tilde{\alpha} = \frac{1}{g''(\rho(2\delta))}.$$

Using $\tilde{\alpha}$ as the default step size. According to the previous lemma, we have $\tilde{\alpha} \leq \overline{\alpha}$.

Lemma 4.10. *[36] we have*

$$h(x) \ \leq \ \frac{th^{'}(0)}{2} \quad forall \ x \in [0, x^*].$$

If $h(0) = 0$, $h^{'}(0) < 0$ and realize its global minimum at $x^* > 0$, and $h^{''}(x)$ is increasing over the interval $[0, x^*]$, where $h(x)$be a convex function which is twice differentiable.

Lemma 4.11. *[6] If the deplacement step verifies $\alpha \leq \overline{\alpha}$, then we have*

$$U(\alpha) \leq -\alpha\delta^2.$$

According to the results of lemma (4.11) and $\tilde{\alpha} = \frac{1}{g''(\rho(2\delta))}$, *we obtain*

$$U(\tilde{\alpha}) \leq -\frac{\delta^2}{g''(\rho(2\delta))} \tag{4.13}$$

4.4.1 Bound of $\delta(\omega)$ in terms of $G(\omega)$

Lemma 4.12. *For any $\omega \succ 0$,*

$$\delta \geq \sqrt{\frac{G(\omega)}{2}}. \tag{4.14}$$

Proof. Using Lemma(4.5) and equation(4.11), we get

$$\delta^2 = \frac{1}{4}Tr(g'(\omega)^2) \geq \frac{2}{4}\sum_{i=1}^{n} g(\gamma_i(\omega)) \geq \frac{1}{2}G(\omega),$$

hence $\delta \geq \sqrt{\frac{G(\omega)}{2}}$. □

4.4.2 Iterations bound

Consider the total iterations' number, then

$$G_{K-1} > \tau, 0 \leq G_K \leq \tau.$$

Lemma 4.13. *[36] Let $x_0, x_1, ..., x_k$ be a sequence of positive numbers satisfying :*

$$x_{k+1} \leq x_k - \beta^- x_k^{1-\gamma}, k = 0, 1, ..., K-1,$$

such that :

$$\beta^- > 0, \gamma \in (0, 1]; \quad then \quad K \leq \left[\frac{x_0^\gamma}{\beta^- \gamma}\right].$$

Lemma 4.14. *Let K be the total number of inner iterations, then we have*

$$K \ \leq \frac{{G_0}^{\gamma}}{\beta^{-}\gamma}.$$

Lemma 4.15. *Let k be the number of outer iterations required for algorithm to converge to a primal-dual optimal solution, then we have*

$$k \ \leq \ \frac{1}{\theta} log\left(\frac{n}{\epsilon}\right).$$

The total iterations' number, namely

$$\frac{{G_0}^{\gamma}}{\theta\beta^{-}\gamma} log\left(\frac{n}{\epsilon}\right)$$

4.5 Complexity analysis

Lemma 4.16. *[1] For $G(x)$, we have*

$$\sqrt{1+s} \leq \Gamma(s) \leq 1+\sqrt{2s}, \ \textit{for } s \geq 0.$$

Theorem 4.4. *[1] One has*

$$G(\omega_+) \leq \frac{4}{1-\theta}(\sqrt{2\tau}+\sqrt{n}\theta)^2,$$

if $G(\omega) \leq \tau$, where $0 \leq \theta \leq 1$ and $\omega_+ = \frac{\omega}{\sqrt{1-\theta}}$.

Proof. From ([11], theorem 3.3.2) when $\beta = \frac{1}{\sqrt{1-\theta}}$, and According to Lemma (4.16), we get

$$\begin{aligned} G(\omega_{+)} \ &\leq \ ng\left(\tfrac{1}{\sqrt{1-\theta}}\Gamma(\tfrac{G(\omega)}{n}\right) \\ &\leq \ 4n\left(\tfrac{1+\sqrt{2(\frac{\tau}{n})}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \\ &\leq \ \tfrac{4}{1-\theta}\left(\sqrt{2\tau}+\sqrt{n}\theta\right)^2 = G_0, \end{aligned}$$

$1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, for $0 \leq \theta < 1$, G_0 is an upper bound of $G(\omega)$. □

Lemma 4.17. *[1] For any $\omega \succ 0$,*

$$\delta \geq \sqrt{\frac{1}{2}}. \tag{4.15}$$

Proof. we suppose that $\tau \geq 1$ and using $G(\omega) \geq \tau$, by Lemma(4.12), we obtain $\delta \geq \sqrt{\frac{1}{2}}$.

□

Lemma 4.18. *[1] One has*

$$\overline{\alpha} \geq \frac{1}{6 + 2(6\delta + 1)(1 + log(4\delta + 1))^2.} \tag{4.16}$$

Proof. We compute $\rho(2\delta) = s$, where ρ be the inverse of $-\frac{1}{2}g'(t)$ for $x \in [0, 1)$. This implies

$$-g'(x) = 4\delta \Leftrightarrow -2x + \frac{1}{x^3} + \frac{1}{x^2}e^{\frac{1}{x}-1} = 4\delta$$

which leads us to

$$e^{\frac{1}{x}-1} = x^2(4\delta + 2x - \frac{1}{x^3}), \tag{4.17}$$

this implies that

$$x \geq \frac{1}{1 + log(4\delta + 1)}$$

Using (4.17) and the definition of $g''(x)$.

If $s \leq 1$, we have $\frac{s-1}{s^5} \leq 0$ and

$$\frac{1}{s^2} \leq (1 + log(4\delta + 1))^2.$$

Then,

$$g''(x) = 2 + \frac{3}{x^4} + (\frac{2}{x^3} + \frac{1}{x^4})e^{\frac{1}{x}-1} \leq 6 + 2(6\delta + 1)(1 + log(4\delta + 1))^2$$

we obtain

$$\overline{\alpha} \geq \frac{1}{6 + 2(6\delta + 1)(1 + log(4\delta + 1))^2.}$$

As a default step size, we take

$$\tilde{\alpha} = \frac{1}{6 + 2(6\delta + 1)(1 + log(4\delta + 1))^2.} \tag{4.18}$$

□

Lemma 4.19. *[1] Let* $\tilde{\alpha}$ *and* $G(\omega) \geq 1$*, then*

$$U(\tilde{\alpha}) \leq -\frac{\sqrt{G_0}}{33(1 + log(2\sqrt{2G_0 + 1}))^2}. \tag{4.19}$$

Proof. By ([6], Lemma 4.5) and $\overline{\alpha} \geq \tilde{\alpha}$, we have

$$U(\tilde{\alpha}) \leq -\frac{\delta^2}{6 + 2\delta(6 + \sqrt{2})(1 + log(4\delta + 1))^2} \leq -\frac{1}{2}\left(\frac{G}{6 + 2\frac{\sqrt{G}}{\sqrt{2}}(6 + \sqrt{2})(1 + log(4\frac{\sqrt{G}}{\sqrt{2}} + 1))^2}\right),$$

therefore

$$U(\tilde{\alpha}) \leq -\frac{\sqrt{G_0}}{33(1 + log(2\sqrt{2G_0 + 1}))^2}.$$

□

Lemma 4.20. *[1] Let* K *be the total inner iterations' number in the outer iteration. Then,*

$$K \leq 66\left(1 + log(2\sqrt{2G_0} + 1))^2\right) G_0^{\frac{1}{2}}.$$

Proof. We utilize ([36], proposition 1.2.3), by taking $x_k = G_k$, $\beta^- = \frac{1}{33(1+log(2\sqrt{2G_0}+1))^2}$ and $\gamma = \frac{1}{2}$ we get the result. □

Theorem 4.5. *[1] If $\tau \geq 1$, the total number of iterations is not more than*

$$66(1 + log(2\sqrt{2G_0} + 1))^2)G_0^{\frac{1}{2}} \tfrac{1}{\theta} log \tfrac{n\mu^0}{\varepsilon}.$$

Proof. In the algorithm, $n\mu \leq \varepsilon$, $\mu^k = (1-\theta)^k \mu^0$ and $\mu^0 = \frac{Z_0^t R_0}{n}$. By easy computation, we get

$$k \leq \frac{1}{\theta} log \frac{n\mu^0}{\varepsilon}.$$

The upper bound of the total iterations, namely

$$\frac{K}{\theta} log \frac{n\mu^0}{\varepsilon} \leq \frac{66}{\theta}(1 + log(2\sqrt{2G_0} + 1))^2)G_0^{\frac{1}{2}} log \frac{n\mu^0}{\varepsilon}.$$

□

Remark 4.1. *We suppose that $\tau = O(n)$, $\theta = \Theta(1)$ and $G_0^{\frac{1}{2}} = O(\sqrt{n})$, we obtain the solution at most $O(\sqrt{n}(logn)^2 log\frac{n}{\varepsilon})$.*

Chapter 5

Numerical tests

The objective is to introduce three SDLCPs for testing the algorithm. The implementation is in ”Matlab”.

”inn” and ”out”: number of inner and outer number of iterations,
”T” means the time.

$\tau = 2$, $\varepsilon = 10^{-6}$, $\alpha \in \{\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1,\ \frac{1}{log(4\delta)},\ \frac{1}{log(4\delta+1)}\}$ and $\theta \in \{\ 0.95,\ 0.99\}$, $\mu_0 \in \{Tr(ZR)/n,\ 0.005,\ 0.0005\}$.

We supply a point $(Z_0,\ R_0)$ such that IPC and $G(Z_0,\ R_0,\ \mu_0)\ \leqslant\ \tau$ are satisfied.
We begin by the monotone (SDLCP) defined by the two sided transformation [2].

The next example present the monotone (SDLCP) the same as the (SDLS) problem and the last one is the reformulation of (NS-SDLS) problem [23].

Example 5.1. *[1] We consider the monotone (SDLCP) defined as follow [2],* $\hbar(Z) = AZA^T$, $n = m = 5$ *where*

$$A = \begin{pmatrix} 17.25 & -1.75 & -1.75 & -1.75 & -1.75 \\ -1.75 & 16.25 & -2 & 0 & 0 \\ -1.75 & -2 & 16.25 & -2 & 0 \\ -1.75 & 0 & -2 & 16.25 & -2 \\ -1.75 & 0 & 0 & -2 & 16.25 \end{pmatrix},$$

$$Q = \begin{pmatrix} -9.25 & 1.25 & 1.25 & 1.25 & 1.25 \\ 1.25 & -8.25 & 1.5 & 0 & 0 \\ 1.25 & 1.5 & -8.25 & 1.5 & 0 \\ 1.25 & 0 & 1.5 & -8.25 & 1.5 \\ 1.25 & 0 & 0 & 1.5 & -8.25 \end{pmatrix}.$$

The strictly feasible point $Z^0 \succ 0$ *is disposed by*

$$Z^0 = \begin{pmatrix} 0.0620 & 0 & 0 & 0 & 0 \\ 0 & 0.0620 & 0 & 0 & 0 \\ 0 & 0 & 0.0620 & 0 & 0 \\ 0 & 0 & 0 & 0.0620 & 0 \\ 0 & 0 & 0 & 0 & 0.0620 \end{pmatrix}.$$

The distinctive solution $Z^* \in S_+^5$ is given by

$$Z^* = \begin{pmatrix} 0.0313 & 0.0020 & 0.0020 & 0.0020 & 0.0020 \\ 0.0020 & 0.0313 & 0.0019 & 0 & 0 \\ 0.0020 & 0.0019 & 0.0312 & 0.0019 & 0 \\ 0.0020 & 0 & 0.0019 & 0.0312 & 0.0019 \\ 0.0020 & 0 & 0 & 0.0019 & 0.0313 \end{pmatrix}.$$

The both inner and outer iterations' number in addition to the time for diverse choices of α, θ and μ are presented in the next tables [1]

$\theta = 0.5$				
α/μ	$Tr(ZR)/n$	0.05	0.005	0.0005
	inn / out / T	inn / out / T	inn / out / T	inn / out / T
0,3	74 / 22 / 17 $\times 10^{-2}$	71 / 19 / 17$\times 10^{-2}$	65 / 15 / 15$\times 10^{-2}$	61 / 12 / 15$\times 10^{-2}$
0,5	43 / 22 / 12$\times 10^{-2}$	41 /19 / 12$\times 10^{-2}$	37 / 15 / 12$\times 10^{-2}$	35 / 12 /10$\times 10^{-2}$
0,7	23 / 22 / 9$\times 10^{-2}$	22 / 19 / 17$\times 10^{-2}$	20 / 15 / 7$\times 10^{-2}$	19 / 12 / 7$\times 10^{-2}$
0,9	22/ 22 / 10$\times 10^{-2}$	21 / 19 / 15$\times 10^{-2}$	18 / 15 / 7$\times 10^{-2}$	16 / 12 / 9$\times 10^{-2}$
1	22 / 22 / 10$\times 10^{-2}$	19 / 18 / 9$\times 10^{-2}$	17 / 15 / 7$\times 10^{-2}$	15 / 12 / 7$\times 10^{-2}$
1/log(4δ)	7 / 22 / 7$\times 10^{-2}$	16 / 19 / 9$\times 10^{-2}$	12 / 15 / 9$\times 10^{-2}$	25 / 12 /10$\times 10^{-2}$
1/log(4δ+1)	20 / 22 / 9$\times 10^{-2}$	21 / 19 / 6$\times 10^{-2}$	24 / 15 / 9$\times 10^{-2}$	30 / 12 /10$\times 10^{-2}$

Table 5.1: Number of inner, outer iterations and the time for several choices of α and μ, with $\theta = 0.5$.

$\theta\ =\ 0.89$				
α/μ	$Tr(ZR)/n$	0.05	0.005	0.0005
	inn / out / T	inn / out / T	inn /out / T	inn / out / T
0.3	58 / 7 / 14$\times10^{-2}$	57 / 6 / 14$\times10^{-2}$	55 / 5 / 12$\times10^{-2}$	53 / 4 / 14$\times10^{-2}$
0.5	31 / 7 / 10$\times10^{-2}$	31 /6 / 10$\times10^{-2}$	30 / 5 / 9$\times10^{-2}$	28 / 4 / 9 $\times10^{-2}$
0.7	21 / 7 / 9$\times10^{-2}$	20 / 6 / 7$\times10^{-2}$	19 / 5 / 9$\times10^{-2}$	18 / 4 / 9$\times10^{-2}$
0.9	14/ 7 / 7$\times10^{-2}$	14 / 6 / 7$\times10^{-2}$	13 / 5 / 6$\times10^{-2}$	12 / 4 / 6$\times10^{-2}$
1	9 / 7 / 6$\times10^{-2}$	8 / 6 / 6$\times10^{-2}$	8 / 5 / 6$\times10^{-2}$	7 / 4 / 6$\times10^{-2}$
1/log(4δ)	14 / 7 / 7$\times10^{-2}$	20 / 6 / 7$\times10^{-2}$	22 / 5 / 9$\times10^{-2}$	36 / 4 / 10$\times10^{-2}$
1/log(4δ+1)	22 / 7 / 7$\times10^{-2}$	24 / 6 / 9$\times10^{-2}$	30 / 5 / 9$\times10^{-2}$	36 / 4 /10$\times10^{-2}$

Table 5.2: Number of inner, outer iterations and the time for several choices of α and μ, with $\theta\ =\ 0.89$.

$\theta\ =\ 0.95$				
α/μ	$Tr(ZR)/n$	0.05	0.005	0.0005
	inn / out / T	inn / out / T	inn /out / T	inn / out / T
0.3	53 / 5 / 12$\times10^{-2}$	60 / 5 / 14$\times10^{-2}$	56 / 4 / 14$\times10^{-2}$	51 / 3 / 10$\times10^{-2}$
0.5	29 / 5 / 9$\times10^{-2}$	33 / 5 / 9$\times10^{-2}$	30 / 4 / 9$\times10^{-2}$	28 / 3 / 9$\times10^{-2}$
0.7	18 / 5 / 7$\times10^{-2}$	20 / 5 / 7$\times10^{-2}$	18 / 4 / 7$\times10^{-2}$	17 / 3 / 7$\times10^{-2}$
0.9	11/ 5 / 7$\times10^{-2}$	12 / 5 / 7$\times10^{-2}$	11 / 4 / 7$\times10^{-2}$	10 / 3 / 6$\times10^{-2}$
1	7 / 5 / 6$\times10^{-2}$	7 / 5 / 6$\times10^{-2}$	7 / 4 / 6$\times10^{-2}$	6 / 3 / 6$\times10^{-2}$
1/log(4δ)	20 / 5 / 7$\times10^{-2}$	23 / 5 / 7$\times10^{-2}$	32 / 4 / 10$\times10^{-2}$	38 / 3 / 10$\times10^{-2}$
1/log(4δ+1)	24 / 5 / 9$\times10^{-2}$	31 / 5 / 9$\times10^{-2}$	34 / 4 / 9$\times10^{-2}$	41 / 3 / 10$\times10^{-2}$

Table 5.3: Number of inner, outer iterations and the time for several choices of α and μ, with $\theta\ =\ 0.95$.

Example 5.2. *[1] The (SDLCP) which is equivalent to (SDLS) problem, is given by*

$\hbar(Z) = \frac{1}{2}(A^T AZ + ZA^T A)$ *and* $Q = -\frac{1}{2}(A^T B + B^T A)$,

where

$$A = \begin{pmatrix} 6 & -1 & 0 & 0 & 0 \\ -0.1 & 6 & -1 & 0 & 0 \\ 0 & -0.1 & 6 & -1 & 0 \\ 0 & 0 & -0.1 & 6 & -1 \\ 0 & 0 & 0 & -0.1 & 6 \\ 0 & 0 & 0 & 0 & -0.1 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.4 & 1 & 0 & 0 & 0 \\ -0.4 & -0.4 & 1 & 0 & 0 \\ -0.4 & 0 & -0.4 & 1 & 0 \\ -0.4 & 0 & 0 & -0.4 & 1 \\ -0.4 & 0 & 0 & 0 & -0.4 \end{pmatrix}.$$

The initial point $Z^0 \succ 0$ *is*

$$Z^0 = \begin{pmatrix} 0.2369 & 0 & 0 & 0 & 0 \\ 0 & 0.2369 & 0 & 0 & 0 \\ 0 & 0 & 0.2369 & 0 & 0 \\ 0 & 0 & 0 & 0.2369 & 0 \\ 0 & 0 & 0 & 0 & 0.2369 \end{pmatrix}.$$

The solution $Z^* \in S^5_+$ *is given by*

$$Z^* = \begin{pmatrix} 0.1639 & -0.0215 & -0.0342 & -0.0328 & -0.0300 \\ -0.0215 & 0.1553 & -0.0227 & -0.0019 & -0.0027 \\ -0.0342 & -0.0227 & 0.1558 & -0.0194 & 0.0014 \\ -0.0328 & -0.0019 & -0.0194 & 0.1564 & -0.0189 \\ -0.0300 & -0.0027 & 0.0014 & -0.0189 & 0.1598 \end{pmatrix}$$

In the next tables, we present the iterations' number and the time [1].

$\theta = 0.5$				
α/μ	$Tr(ZR)/n$	0.05	0.005	0.0005
	inn / out / T	inn / out / T	inn /out / T	inn / out / T
0.3	74 / 22 / 21×10^{-2}	69 / 18 / 20×10^{-2}	65 / 15 / 7×10^{-2}	61 / 12 / 18×10^{-2}
0.5	43 / 22 / 15×10^{-2}	40 /18 / 15×10^{-2}	37 / 15 / 17×10^{-2}	35 / 12 / 15×10^{-2}
0.7	23 / 22 / 11×10^{-2}	21 / 18 / 11×10^{-2}	20 / 15 / 11×10^{-2}	19 / 12 / 10×10^{-2}
0.9	22/ 22 / 10×10^{-2}	20 / 18 / 12×10^{-2}	18 / 15 / 9×10^{-2}	16 / 12 / 10×10^{-2}
1	22 / 22 / 12×10^{-2}	19 / 18 / 12×10^{-2}	17 / 15 / 11×10^{-2}	15 / 12 / 6×10^{-2}
1/log(4δ)	8 / 22 / 7×10^{-2}	12 / 18 / 11×10^{-2}	18 / 15 / 12×10^{-2}	26 / 12 / 14×10^{-2}
1/log(4δ+1)	20 / 22 / 10×10^{-2}	21 / 18 / 10×10^{-2}	25 / 15 / 10×10^{-2}	31 / 12 / 12×10^{-2}

Table 5.4: Number of inner, outer iterations and the time for several choices of α, and μ with $\theta = 0.5$.

$\theta\ =\ 0.89$				
α/μ	$Tr(ZR)/n$	0.05	0.005	0.0005
	inn / out / T	inn / out / T	inn /out / T	inn / out / T
0.3	58 / 7 / 20×10^{-2}	57 / 6 / 20×10^{-2}	55 / 5 / 17×10^{-2}	53 / 4 / 17×10^{-2}
0.5	31 / 7 / 12×10^{-2}	31 /6 / 12×10^{-2}	29 / 5 / 10×10^{-2}	28 / 4 / 12×10^{-2}
0.7	21 / 7 / 10×10^{-2}	20 / 6 / 10×10^{-2}	19 / 5 / 7×10^{-2}	18 / 4 / 10×10^{-2}
0.9	14/ 7 / 9×10^{-2}	14 / 6 / 9×10^{-2}	13 / 5 / 7×10^{-2}	12 / 4 / 7×10^{-2}
1	8 / 7 / 4×10^{-2}	8 / 6 / 6×10^{-2}	8 / 5 / 6×10^{-2}	8 / 4 / 7×10^{-2}
$1/\log(4\delta)$	19 / 7 / 7×10^{-2}	23 / 6 / 10×10^{-2}	29 / 5 / 12×10^{-2}	31 / 4 / 14×10^{-2}
$1/\log(4\delta{+}1)$	23 / 7 / 10×10^{-2}	26 / 6 / 10×10^{-2}	31 / 5 / 14×10^{-2}	38 / 4 / 15×10^{-2}

Table 5.5: Number of inner, outer iterations and the time for several choices of α, and μ with $\theta\ =\ 0.89$.

$\theta\ =\ 0.95$				
α/μ	$Tr(ZR)/n$	0.05	0.005	0.0005
	inn / out / T	inn / out / T	inn /out / T	inn / out / T
0.3	54 / 5 / 18×10^{-2}	61 / 5 / 20×10^{-2}	56 / 4 / 20×10^{-2}	52 / 3 / 18×10^{-2}
0.5	29 / 5 / 12×10^{-2}	33 /5 / 10×10^{-2}	30 / 4 / 12×10^{-2}	28 / 3 / 9×10^{-2}
0.7	18 / 5 / 9×10^{-2}	20 / 5 / 6×10^{-2}	18 / 4 / 7×10^{-2}	17 / 3 / 9×10^{-2}
0.9	11/ 5 / 7×10^{-2}	12 / 5 / 7×10^{-2}	11 / 4 / 7×10^{-2}	10 / 3 / 6×10^{-2}
1	7 / 5 / 7×10^{-2}	7 / 5 / 4×10^{-2}	7 / 4 / 6×10^{-2}	7 / 3 / 7×10^{-2}
$1/\log(4\delta)$	18 / 5 / 7×10^{-2}	21 / 5 / 7×10^{-2}	25 / 4 / 12×10^{-2}	41 / 3 / 17×10^{-2}
$1/\log(4\delta{+}1)$	25 / 5 / 10×10^{-2}	32 / 5 / 12×10^{-2}	36 / 4 / 12×10^{-2}	42 / 3 / 14×10^{-2}

Table 5.6: Number of inner, outer iterations and the time for several choices of α, and μ with $\theta = 0.95$.

Example 5.3. *[1]*

The matrices A and B of NS-SDLS are given by :

$$A = \begin{pmatrix} -0.3157 & 0.0330 & 0.0603 \\ -0.3274 & -0.0158 & 0.0625 \\ -0.3569 & 0.0787 & 0.0563 \\ -0.2994 & 0.0301 & 0.0496 \\ -0.3243 & -0.0048 & 0.0715 \\ -0.3447 & 0.0736 & 0.0545 \\ -0.2417 & 0.0709 & 0.0522 \\ -0.2063 & -0.0099 & 0.0233 \\ -0.3285 & 0.1585 & 0.0979 \\ -0.2484 & 0.0878 & 0.0622 \\ -0.2196 & 0.0023 & 0.0280 \\ -0.3148 & 0.1506 & 0.0922 \end{pmatrix},$$

and

$$B = \begin{pmatrix} -1.4257 & 0.1528 & 0.4398 \\ -1.4024 & -0.3092 & 0.4187 \\ -1.3766 & 0.4366 & 0.4197 \\ -1.4274 & 0.1424 & 0.4353 \\ -1.3994 & -0.3095 & 0.4206 \\ -1.3716 & 0.4285 & 0.4193 \\ -1.4269 & 0.1581 & 0.4335 \\ -1.4015 & 0.3229 & 0.4214 \\ -1.3767 & -0.4189 & 0.4333 \\ -1.4257 & 0.1515 & 0.4358 \\ -1.3989 & 0.3276 & 0.4217 \\ -1.3724 & 0.1454 & 0.4356 \end{pmatrix}.$$

Lyapunov transformation $\hbar(Z)$ is symmetric and strictly monotone given by

$$\hbar(Z) = \frac{1}{2}(G^{-1}Z + ZG^{-1}) \text{ and } Q = -\frac{1}{2}(G^{-1}A^TB + B^TAG^{-1}), \tag{5.1}$$

where $G = A^TA$.

SDLCP has the unique solution Z^* *is given by*

$$Z^* = \begin{pmatrix} 5.1571 & -0.2615 & 1.9198 \\ -0.2609 & 6.2655 & -4.0779 \\ 1.9199 & -4.0778 & 0.6117 \end{pmatrix}$$

The iterations' number and the time with α, μ *and* $\theta \in \{0.50, 0.80, 0.90\}$ *with* $Z^0 = I$ *are disposed in the next tables.[1]*

$\theta = 0.50$				
α/μ	$Tr(ZR)/n$	0.05	0.005	0.0005
	inn / out / T	inn / out / T	inn /out / T	inn / out / T
0.3	6 / 22 / 7×10^{-2}	6 / 18 / 7×10^{-2}	6 / 14 / 6×10^{-2}	6 / 11 / 6×10^{-2}
0.5	3 / 22 / 6×10^{-2}	3/ 18 / 6×10^{-2}	3 / 14 / 6×10^{-2}	3 / 11 / 6×10^{-2}
0.7	2 / 22 / 7×10^{-2}	2 / 18 / 6×10^{-2}	2 / 14 / 6×10^{-2}	2 / 11 / 4×10^{-2}
0.9	2 / 22 / 6×10^{-2}	2 / 18 / 4×10^{-2}	2 / 14 / 6×10^{-2}	2 / 11 / 4×10^{-2}
1	2 / 22 / 6×10^{-2}	2 / 18 / 6×10^{-2}	2 / 14 / 6×10^{-2}	2 / 11 / 6×10^{-2}
1/log(4δ)	15 / 22 / 9×10^{-2}	17 / 18 / 9×10^{-2}	15 / 14 / 6×10^{-2}	15 / 11 / 6×10^{-2}
1/log(4δ+1)	15 / 22 / 7×10^{-2}	17 / 18 / 9×10^{-2}	15 / 14 / 9×10^{-2}	15 / 11 / 9×10^{-2}

Table 5.7: Number of inner, outer iterations and the time for several choices of α, and μ with $\theta = 0.5$.

$\theta = 0.80$				
α/μ	$Tr(ZR)/n$	0.05	0.005	0.0005
	inn / out / T	inn / out / T	inn /out / T	inn / out / T
0.3	6 / 10 / 4×10^{-2}	6 / 8 / 0.06	6 / 6 / 4×10^{-2}	6 / 5 / 0.06
0.5	3 / 10 / 4×10^{-2}	3 / 8 / 4×10^{-2}	3 / 6 / 6×10^{-2}	3 / 5 / 6×10^{-2}
0.7	2 / 10 / 4×10^{-2}	2 / 8 / 6×10^{-2}	2 / 6 / 6×10^{-2}	2 / 5 / 4×10^{-2}
0.9	2 / 10 / 4×10^{-2}	2 / 8 / 6×10^{-2}	2 / 6 / 4×10^{-2}	2/ 5 / 6×10^{-2}
1	2 / 10 / 6×10^{-2}	2 / 8 / 6×10^{-2}	2 / 6 / 6×10^{-2}	2/ 5 / 4×10^{-2}
1/log(4δ)	16 / 10 / 7×10^{-2}	17 / 8 / 9×10^{-2}	15 / 6 / 7×10^{-2}	17 / 5 / 9×10^{-2}
1/log(4δ+1)	16 / 10 / 4×10^{-2}	17 / 8 / 7×10^{-2}	16 / 6 / 7×10^{-2}	17 / 5 / 7×10^{-2}

Table 5.8: Number of inner, outer iterations and the time for several choices of α, and μ with $\theta = 0.80$.

$\theta = 0.90$				
α/μ	$Tr(ZR)/n$	0.05	0.005	0.0005
	inn / out / T	inn / out / T	inn /out / T	inn / out / T
0.3	6 / 7 / 6×10^{-2}	6 / 6 / 6×10^{-2}	6 / 5 / 6×10^{-2}	6 / 4 / 6×10^{-2}
0.5	3 / 7 / 4×10^{-2}	3/ 6 / 4×10^{-2}	3 / 5 / 4×10^{-2}	3 / 4 / 4×10^{-2}
0.7	2 / 7 / 4×10^{-2}	2/ 6 / 6×10^{-2}	2 / 5 / 4×10^{-2}	2 / 4 / 6×10^{-2}
0.9	2/ 7 / 6×10^{-2}	2 / 6 / 6×10^{-2}	2 / 5 / 6×10^{-2}	2 / 4 /4×10^{-2}
1	2 / 7 / 4×10^{-2}	2 / 6 / 4×10^{-2}	2 / 5 / 6×10^{-2}	2 / 4 / 6×10^{-2}
1/log(4δ)	16 / 7 / 7×10^{-2}	17/ 6 / 7×10^{-2}	17 / 5 / 7×10^{-2}	17 / 4 / 7×10^{-2}
1/log(4δ+1)	16 / 7 / 6×10^{-2}	17 / 6 / 7×10^{-2}	17 / 5 / 7×10^{-2}	17 / 4 / 6×10^{-2}

Table 5.9: Number of inner, outer iterations and the time for several choices of α, and μ with $\theta = 0.90$.

The tables' results show the effectiveness of the algorithm through the role of our kernel *function $g(t)$, we achieved the better results than [2], [23] by iterations' number and the time produced that depends on the values of α, θ and μ.*

Conclusion and perspectives

The new class of search directions based on new kernel function to solve the (SDLCP) by primal-dual path following algorithm is proposed. The new directions are not orthogonal. The new kernel function has good properties. In particular, we have obtained a better existing complexity.

We conducted a numerical study on some existing examples, under a special choice of parameters $(\alpha,\ \theta,\ \mu)$, we obtain the best known iteration bound for the algorithm with large-update method, namely $O\left(\sqrt{n}(logn)^2 log\frac{n}{\epsilon}\right)$.

A great improvement concerning the time and the number of iterations was showed by the implementation of the algorithm.

We recommand for the future research to focus on the development of a primal-dual interior point algorithm based on a new kernel function with another symmetrization scheme, whose goal is to improve the complexity, and extended this methods to another problems.

Bibliography

[1] N. Abdessemed , R. benacer ,N. boudiaf .*A new kernel function generating the best complexity analysis for monotone SDLCP*. Advances in Mathematics: Scientific Journal (11) pp 925-941, (2022).

[2] M. Achache , N. Tabchouche .*A full Nesterov-Todd step primal-dual path- folowing interior point algorithm for semi definite linear complementarity problems*. Croation Operational Research Review (9) pp 37-50, (2018).

[3] M. Achache , N. Boudiaf and A. Keragel .*Le Problème de complémentarité linéaire semidéfini,* . Revue d'analyse numérique et de la théorie de l'approximation. Tome 38, No 2, pp 115-129, (2009).

[4] M. Achache .*Complexity Analysis of An interior point algorithm for semi definite optimization based on a kernel function with a double barrier term*. Acta Mathematica Sinica, English Series (3) pp 543-556, (2015).

[5] F. Alizadeh, J. A. Haeberly, and M. Overton .*Primal-dual interior point methods for semidefinite programming. convergence rates, stability and numerical results*. SIAM. J. optimization(8) pp 746-768, (1998).

[6] Y. Q. Bai , M. Elghami , C. Roos . *A comparative study of kernel functions for primal-dual interior point algorithms in linear optimization*. SIAM J. optim 15(1)pp 101-128 ,(2004).

[7] C. Besse. *Introduction a l'analyse matriciel*. Laboratoire MIP. Université Paul Sabatier, Septembre, (2003).

[8] N. Boudiaf .*Problème de complémentarité linéaire semidéfini, Etude théorique et algorithmique*, Thèse de doctorat en sciences mathématiques, université de Batna, Algerie , (2012).

[9] R. W. Cottle, J. S. Pang, R. Stone. *The linear complementarity problem*, Academic Press. Boston, (1997).

[10] E. Deklerk. *Interior point methods for semidefinite programming.* Thesis technische . University Delf. University of pretoria, (1997).

[11] M. Elghami *New primal-dual interior point methods based on kernel functions*, Ph. D. thesis, Delft University, Netherland ,(2005).

[12] M. S. Gowda, Y. song. *On semidefinite complementarity problem.* Methematical programming. series A, (88) pp 575-587, (2000).

[13] M. S. Gowda, Y. Song . *Some new results for the semidefinite linear complementarity problem.* SIAM journal on matrix Analysis a Applications, 24(1) pp 25-39, (2003).

[14] M. S. Gowda, T. Pathasaraty. *Complementarity forms of the theorems of lyapunov and stein and related results. linear algebras and its applications.* (320) pp 131-144 , (2000).

[15] R. A. Horn, R. J. charles. *Matrix Analysis*, Cambridge university press, Cambridge (1986).

[16] R. A. Horn, R. J. charles. *Topic in Matrix Analysis*, Cambridge university press, Cambridge (1991).

[17] S. Karamardian. *The complementarity problem.* Mathematical programming (2), (1972).

[18] A. Keraghel. *Analyse convexe, Theorie fondamentale et exercices.* Université de setif, Faculté des sciences. Laboratoire de Mathématique Fondamentale et Numérique, (2001).

[19] M. Kojima , M. Shindoh , S. Hara . *Interior point methods for monotone semidefinite linear complementarity in symmetric matrices.* SIAM J. Optimization (7)pp 86-125, (1997).

[20] M. Kojima , M. Shida , S. Shindoh . *Search direction in the SDP and the monotone SDLCP, generalization and inexact computation*. Resaerch Report B-327 dept of Mathematical and computing Sciences, Tokyo Institute of Technology,Oh-Okyama, Meguro, Tokyo 152, Japan, March, (1997).

[21] M. Kojima , S. Hara . *Linear algebra for semidefinite programming*. Resaerch Report B-290, dept of Mathematical and computing Sciences, Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo 152, Japan, RIMS Kokyuroko to appear revised (1996).

[22] M. Kojima , M. Shindoh, S. Hara . *Interior point methods for the monotone semidefinite linear complementarity in symetric matrices*. SIAM J. optimization, (7) pp 86-125, (1997).

[23] N. G. B. Krislock. *Numerical solution of semidefinite constrained least squares problems.* Master of science. The university of British colombia, Canada (2003).

[24] D. G. Luenberger. *Introduction to linear and nonlinear programming* pp 344-349 (1973).

[25] Z. Liu, W. Sun. *Polynomial complexity of primal-dual interior point methods for convex quadratic programming*, To appear in applied mathematics and computation (2008).

[26] Z. Liu, W. Sun. *An infeasible interior point algorithms with full Newton step for linear optimization*, Numer. Algor. (46) pp 173-188 (2007).

[27] M. Minoux. *Programmation mathématique, Theorie et algorithmes*, (Tome 1) Dunod, Paris (1983).

[28] R. D. C Menteiro. *Polynomial convergence of new family of primal-dual algorithms for semidefinite programming*, Research Memorandum No 647, The institute of statistical mathematics, Tokyo, 106, Japan, (to appear in SIAM J on optimization (1996).

[29] R. D. C Menteiro, T. Tsuchiya. *Polynomial convergence of primal-dual algorithms for semidefinite programming based on Monteiro and Zhang family of directions*, SIAM J on optimization (8) pp 797-812 (1998).

[30] R. D. C Menteiro, T. Tsuchiya. *Polynomiality of primal-dual algorithms for semidefinite linear complementarity problems based on Kojima- Shindoh- Hara family of directions*, Math. program (84) pp 39-53, Springer. Verlag (1999).

[31] R. D. C. Monteiro, *Primal-dual path-following algorithms for semidefinite programming*, SIAM J. Optim. (7) pp 663-678 (1997).

[32] Y. Nestrov, M. Todd. *Primal-dual interior point methods for self scaled cones.* SIAM. J. Optimization, (8) pp 324-364, (1998).

[33] J. Nocedal, S. J. Wright. *Numerical optimization*, Springer series in operations research (1999).

[34] M. R. peyghami , S. Fathi Hafshejani , S. Chen . *A primal-dual interior point method for semidefinite optimization based on a class of trigonometric barrier functions*. J. Oper. Res(44) pp 319-323 ,(2016).

[35] J. Peng , C. Roos , T. Terlaky. *New class of polynomial primal-dual methods for linear and semidefinite optimization*, European. J. Oper. Res 143(2)pp 234-256, (2002).

[36] J. Peng , C. Roos , T. Terlaky. *Self regular functions and new search directions for linear and semidefinite optimization*. Mathematical programming; (93) pp 129-171, (2002).

[37] J. Peng, C. Roos, T. Terlaky. *Self-regularity. A new paradigm for Primal-Dual Interior Point Algorithm*. Princeton University Press, (2002).

[38] J. Peng, C. Roos, T. Terlaky. *New complexity analysis of the primal-dual Newton method for semidefinite optimization based on the Nesterov-Todd direction*. Journal of optimization theory and application. Vol(109). Princeton University Press, (2002).

[39] J. Peng, C. Roos, T. Terlaky. *New complexity analysis of the primal-dual Newton method for $P_*(k)$ linear complementarity problem*. Manuscrit (1998).

[40] W.H. Press, B.P. Flannery, S.A. Tenkolsky, W.T. Vetterling. *Numerical Recipes in Pascal*. The Art of Scientific computing. Combridge University Prees, (1987).

[41] M. J. Todd . *Semidefinite optimization.* Cambridge University Press, acta Numerica (10) pp 515-560, (2001).

[42] M. J. Todd . *A study of search directions in primal-dual interior point methods for semidefinite programming.* Optim. Methods Softw (11) pp 1-46, (1999).

[43] G. Q. Wang , Y. Q. Bai , C. Roos. *Primal-dual interior point algorithm for semidefinite optimization based on a simple kernel function.* J. Math. Model Algorithms (4)pp 409-433, (2005).

[44] G. Q. Wang , Y. Q. Bai. *A new primal-dual path-following interior point algorithm for semidefinite optimization* . J. Math. Analysis and applications 353 pp 339-349, (2009).

[45] Y. Song. *The P and globally uniquely solvable properties in semidefinite linear complementarity problem.* Phd thesis, the university of maryland, USA,(2000).

[46] S. J. Wright. *Primal dual interior point methods*, SIAM, Philadelphia, USA (1997).

[47] Y. Zhang. *On extending some primal-dual interior-point algorithms from linear programming to semidefinite programming.* SIAM Journal on Optimization, (8) pp 365-386, (1998).

[48] M.W. Zhang. *A large-update interior-point algorithm for convex quadratic semidefinite optimization based on a new kernel function.* Acta Mathematica Sinica. English Series, 28 (11) pp 2313-2328, (2012).

[49] T. Parthasarathy, D. Sampangi Raman, B.Sriparna. " Relationship Between Strong Monotonicity Property, -Property, and the Gus-Property in Semidefinite Linear Complementarity Problems ", Mathematics of Operations Research, (2002)

[50] G. Wang, D. Zhu. "A unified kernel function approach to primal-dual interior-point algorithms for convex quadratic SDO", Numerical Algorithms, (2011).

[51] Y. Q. Bai. "A Class of Large-Update and Small Update Primal-Dual Interior-Point Algorithms for Linear Optimization", Journal of Optimization Theory and Applications,09 (2008).

[52] Gowda, M.. ”On some interconnections between strict monotonicity, globally uniquely solvable, and P properties in semidefinite linear complementarity problems”, Linear Algebra and Its Applications, (2003).

[53] Y. Q. Bai. ”A New Efficient Large-Update Primal-Dual Interior-Point Method Based on a finite Barrier”, SIAM Journal on Optimization (2002).

[54] B. Kheirfam. ”A primal-dual interior-point algorithm for symmetric optimization based on a new kernel function with trigonometric barrier term yielding the best known iteration bounds”, Afrika Matematika, (2016).

[55] E. de Klerk, C. Roos, T. Terlaky. ”Primal-dual potential reduction methods for semidefinite programming using affine-scaling directions”, Applied Numerical Mathematics, (1999).

ملخص

في هذه الاطروحة, نقترح وظيفة نواة جديدة تؤثر بشكل كبير على تحسين التعقيد, و تلعب دورا حاسما في اقتراح فئة جديدة من اتجاهات البحث لحل مسالة التكملة الخطية شبه المحددة الرتيبة باستخدام خوارزمية النقطة الداخلية التالية للنقطة الرئسية الثنائية .

هذه الاتجاهات ليست متعامدة مما يجعل الدراسة اكثر صعوبة. تم اجراء دراسة نظرية و خوارزمية و عددية، حصلنا على افضل تعقيد معروف حتى الان لهذه الخوارزميات ذات الخطوات الكبيرة و التي هي من الترتيب $.O\ (\sqrt{n}(\log n)^2\log(n/\varepsilon))$

اظهر تنفيذ الخوارزمية تحسنا كبيرا فيما يتعلق الوقت و عدد التكرارات.

الكلمات المفتاحية:

مسائل التكملة الخطية شبه المحدودة, طرق النقطة الداخلية , طريقة نيوتن الاولية المزدوجة, التعقيد متعدد الحدود, دالة النواة.

Abstract

In this thesis, we propose a new kernel function which has a great influence on the improvement of the complexity and a crucial role concerning the introduction of a new class of search directions to solve the monotone semidefinite linear complementarity problem (SDLCP) by primal-dual following path interior point algorithm. This directions are not orthogonal what makes the study more difficult.

A theoretical, algorithmic and numerical study was carried out. We obtain currently best known iteration bound for the algorithm with large-update method, namely $O\ (\sqrt{n}(\log n)^2\log(n/\varepsilon))$. The implementation of the algorithm showed a great improvement concerning the time and the number of iterations.

Keywords:

SDLCP, Interior Point Methods, method of Newton, polynomial complexity, kernel function.

Résumé

Dans cette thèse, nous proposons une nouvelle fonction noyau qui a une grande influence sur l'amélioration de la complexité et un rôle crucial concernant l'introduction d'une nouvelle classe des directions de recherche pour résoudre le problème de complémentarité linéaire semi défini monotone (SDLCP) par la méthode de trajectoire centrale primal-duale. Ces directions ne sont pas orthogonales ce qui rend l'étude plus difficile.

Une étude théorique, algorithmique et numérique a été réalisée. Nous obtenons actuellement la meilleure itération connue pour les algorithmes à grand pas, qui est de l'ordre $O(\sqrt{n}\ (\log n)^2\ \log(n/\varepsilon))$. L'implémentation de l'algorithme a montré une grande amélioration concernant le temps et le nombre des itérations.

Mots de passe:

SDLCP, MPI, méthode de Newton, complexité polynomiale, fonction noyau.