

Democratic and Popular Republic of Algeria
Ministry of Higher Education and Scientific Research
University of Batna 2, Mostefa Ben Boulaïd
Faculty of Mathematics and Computer Science
Department of Mathematics
Laboratory of Mathematical Techniques, LMT



THESIS

SUBMITTED FOR THE DEGREE OF
DOCTORATE IN MATHEMATICS

Speciality: Analysis

by

Haithem Benharzallah

**On discrete, differential and integral spline quasi-interpolants
and their applications**

Defended on November 18, 2023

Jury Members:

Saïd Guedjiba	Professor	University of Batna 2	Chair
Abdelaziz Mennouni	Professor	University of Batna 2	Supervisor
Domingo Barrera Rosillo	Professor	University of Granada	Co-supervisor
Maïssa Kada	Associate Professor	University of Batna 2	Examiner
Soheyb Milles	Associate Professor	University Center of Barika	Examiner
Abdelmouhcene Sengouga	Associate Professor	University of M'sila	Examiner

Contents

1	Preliminaries	3
1.1	Discrete quasi-interpolant	3
1.2	Quadrature rules	6
1.3	Evaluations for Tensor Product approximation	6
1.4	Continuous blending case	7
2	Spline QI for a model arising from biology	9
2.1	Approximating system	10
2.2	Continuous blending approximation	11
2.3	Error analysis	13
2.4	Numerical examples	15
2.4.1	Example 1	15
2.4.2	Example 2	17
2.4.3	Example 3	18
2.4.4	Example 4	20
3	Two schemes of QI for system of FIEs	21
3.1	Introduction	21
3.2	Discrete quasi-interpolant recall	22
3.3	Tensor product approximation	24
3.3.1	Quasi-interpolation to approximate the kernels	24
3.3.2	Approximating equation	25
3.4	Continuous blending approximation	27
3.5	Error Analysis	30
3.6	Computation of vectors and matrices	33
3.6.1	Evaluations for Tensor Product approximation	33
3.6.2	Continuous blending case	34
3.6.3	System of two integral equations	36
3.6.4	Approximating the kernel by quasi-interpolation	36
3.7	Three specific systems of integral equations	36
3.8	Numerical examples	43
4	Multi-quadratic trigonometric B-spline for IEs	44
4.1	Introduction	44
4.2	A generalized quasi-interpolation scheme	45
4.3	Solving periodic second kind integral equation of Fredholm type	47
4.3.1	First degenerate kernel method	47
4.3.2	Second degenerate kernel method	48
4.3.3	A tensor product scheme via the main quasi-interpolation	49
4.3.4	Kernel's blending approximation	51
4.4	Convergence analysis	53

4.5	Numerical examples and application	54
5	New construction of C^1-cubic QI splines	55
5.1	Introduction	55
5.2	Bernstein–Bézier Form of Cubic Splines on a Type-1 Triangulation	56
5.3	Clough–Tocher for Refinement C^1 Quasi-Interpolating Splines	58
5.4	Choosing Parameters	62
5.5	Numerical Tests	64
5.6	Conclusions	65
5.7	Masks	65
5.8	Masks That Do Not Depend on Parameters	65
	Bibliography	75

Introduction

Recently, spline quasi-interpolants have drawn increasing attention as a general approximation for more efficient calculation. Due to its stability, fast performance, low processing cost, and short linear systems, it has been claimed that its support is limited. The primary issue with quasi-interpolants is that they can be accomplished while having to solve any systems of linear equations. Spline quasi-interpolants are valuable for a variety of applications. Several fields, including approximation theory, machine learning, geometric design, the amusement business, etc., have used spline quasi-interpolants. The traditional option is smooth spline spaces with a lower degree, which is very challenging in arbitrary triangulations. Solving several kinds of integral equations using spline quasi-interpolants is a significant application.

In recent years, many authors have proposed various ways to solve Fredholm integral equations using spline kernel approximations in order to reduce the number of terms in the approximate kernel. The outcomes of this limitation enable great accuracy while avoiding needless processing expenses associated with large linear systems. In [2], the authors employed a couple of well-known approaches in the theory of integral equations (for example, see [1]) to solve an integral equation of the second kind of Fredholm type via the discrete quartic spline quasi-interpolant. Their effort yields two results: First, they accomplish an approximation order of $O(h^5)$ for the left technique; secondly, the authors achieve an approximation order of $O(h^6)$ for the right approach. Recently, the authors of [38] introduced new results concerning the septic polynomial B-splines. They developed the discrete septic spline quasi-interpolant, applying calculations to all coefficients. Also, the authors employed the results they discovered to solve the second-kind generalized integral equation of the Fredholm type and provide a convergence analysis. The principal purpose of [44] is to look into two building univariate and bivariate sums of C^1 quadratic spline quasi-interpolants in a bounded domain of \mathbb{R}^3 . The goal of this project is to find a solution to the functional boundary generator coefficient. [48] wants to make a group of quadrature rules that can be used to combine spline quasi-interpolants and fix problems at the ends of lines. The authors obtain correction weights by solving several linear algebraic equation systems. Their motivation for using such rules arises from their utility in solving Fredholm integral equations of the second class, as in [8]. In the same perspective, Nyström methods are used to solve a second kind of weakly singular integral equation using an approach based on appropriate smoothing changes of variables and product integration principles, in which related results of [49] are extended to a broader class of weakly singular integral equations.

The creation of a multivariate normalized B-spline basis for the quadratic C^1 -continuous spline space based on triangulation in \mathbb{R}^s $s \geq 1$ with a generalized Powell-Sabin refinement is considered by the author of [54]. Another paper uses a Nyström approach to solve a weakly singular integral equation, employing first smoothing change of variables and then product quasi-interpolation by smooth splines on a uniform grid (see [55]). As an extension of the previous Bernstein approximation for complex exponentials, the authors of [57] work on a multinomial spline approximation strategy based on spline quasi-interpolants on a restricted interval. It uses multinomial Lagrange-Bernstein approximants instead of Knockaert's technique (see [28]). In [4], an unconventional low-cost spline approximation technique is built to approximate bivariate functions. It is used for approximate digital elevation, and after that, the downscaling process's correctness is examined. The aim of [12] is to offer quasi-interpolation strategies that

are based on a type-1 uniform triangulation with a Powell-Sabin refinement. Contrary to how pseudo-interpolation splines are typically constructed on the 6-split, the method described does not call on a specific set of suitable basis functions. By setting the Bézier coordinates of the approximating splines to appropriate combinations of the provided data values, the splines are immediately defined.

Building cubic spline quasi-interpolation approximation presented on a refined partition was studied by the authors of [13]. Compared to those found in the literature, these schemes have fewer degrees of freedom. In particular, they offered a strategy for lowering them while maintaining perfect smoothness and cubic precision by imposing super-smoothing criteria. The resulting constraints are intended to express the B-spline coefficients linked to a finer partition compared to those linked to the earlier partition. The authors of [22] used spline Gauss quadrature rules to solve boundary value problems (BVPs) using the Nyström method. The applicable partial differential equation inside a domain is transformed into the Fredholm integral equation of the second order on the boundary (BIE) for solving BVPs. Then, using the Nyström method, the Fredholm integral equation is resolved by converting the BIE problem into a linear system using a special quadrature rule. The authors used the 2D Laplace problem with rounded edges and smooth bounds to illustrate this strategy.

The goal of this thesis is to present more numerical schemes via spline quasi-interpolants in order to solve various classes of functional problems. The remainder of this thesis is structured as follows: The first chapter examines the notion of spline functions and spline quasi-interpolants, including an analysis, definitions, and preliminary results. In the second chapter, we present a significant class of integral equations derived from biological models. The third chapter aims to approximate the kernels of a Fredholm second-kind integral equations system via two schemes of spline quasi-interpolants: the continuous blending sum scheme and the tensor product one. We look at how approximate solutions are built and present particular theoretical conclusions. We offer some specific theoretical results about the approximation error analysis of these methods. We give our findings by performing numerical tests. Numerical solutions to second-class Fredholm integral equations are obtained using two schemes of spline quasi-interpolants; the objective of the fourth chapter is to develop a generalised quasi-interpolation for periodic data in order to solve integral equations of the second kind in $L^2[0, 2]$ using periodic data. We use four degenerate kernel approaches via a new generalised quasi-interpolation to approximate the kernel. The initial type-1 triangulation is improved in the second chapter to determine the best approximation order. We focus on a Clough-Tocher (CT) improvement that creates six micro-triangles for every square made up of two macro-triangles that share an edge.

Chapter 1

Preliminaries

This chapter defines and offers basic results for the principles of spline quasi-interpolants and spline functions.

1.1 Discrete quasi-interpolant

Let us consider the nodes $\Psi = \Psi_n = \xi_0, \xi_1, \xi_2, \dots, \xi_n$ in the interval $\mathcal{I} := [a, b]$ with

$$\xi_w = a + w(h_w)_w \quad \text{and} \quad h_w = \xi_w - \xi_{w-1} \quad \text{for all} \quad 1 \leq w \leq n.$$

Let $\tau_n := \{\tau_w, 0 \leq w \leq n\}$ denote the uniform partition of \mathcal{I} into n subintervals with mesh length h .

Set $\mathcal{T}_m := S_n(\mathcal{I}, S_n)$, the space of splines of class C^{m-1} on this partition

Also,

$$\mathcal{J}_m := \{0, 1, \dots, n + m - 2\}.$$

The interval $[\tau_{j-m+1}, \tau_{j+1}]$ represents the support of $\varphi_{j,m}$, with multiple knots at the endpoints, i.e.

$$a = \tau_0 = \tau_{-1} = \dots = \tau_{-m+1} \quad \text{and} \quad b = \tau_n = \tau_{n+1} = \dots = \tau_{n+m-2}.$$

We note that the space $\mathcal{S}_m(\mathcal{T}_n)$ of splines of order m on \mathcal{T}_n admits a basis $\varphi_{j,m}, j \in \mathcal{J}_m$ composed by $n + m - 1$ B-splines. Moreover, the representation of monomials using symmetric functions of interior knots $\mathcal{N}_j := \{\tau_{j-m+2}, \dots, \tau_j\}$ in $\text{supp } \varphi_{j,m}$, defined by

$$\sigma_0(N_j) := 1 \quad \text{and} \quad \sigma_{r'}(\mathcal{N}_j) := \sum_{1 \leq \eta_1 < \dots < \eta_{r'} \leq m-1} \tau_{j+1-\eta_1} \cdots \tau_{j+1-\eta_{r'}}, \quad \text{for } 1 \leq r' \leq m-1.$$

Abbreviate by PQIOs, the discrete spline quasi-interpolant operator

$$\sigma g = \sum_{w \in \mathcal{J}_n} \gamma_w(g) \varphi_w,$$

whose coefficients are linear combinations of discrete values of g on the set of data points \mathcal{X}_n .

For $0 \leq r' \leq m-1$, we have the Mersden identity

$$\gamma_{r'}(\tau) := \tau^{r'} = \sum_{j \in \mathcal{J}_m} \theta_{j,m}^{(r')} \varphi_{j,m}(\tau),$$

where

$$\theta_{j,m}^{(r)} := \left(\binom{m-1}{r'} \right)^{-1} \sigma_{r'}(N_j).$$

Let $\mathcal{J}_3 := \{0, 1, \dots, n+1\}$. Define the Greville points :

$$\theta_0 := a, \theta_{n+1} := b, \theta_j := \frac{1}{2}(\tau_{j-1} + \tau_j), \quad 1 \leq j \leq n$$

and

$$\mathcal{S}_n := \{\theta_j, 0 \leq j \leq n+1\}.$$

As in [6, 45, 46], we investigate to the following pQIOs

$$\sigma_{m,n}(f) := \sum_{j \in J_m} \gamma_{j,m}(g) \varphi_{j,m}, \quad m \geq 3. \quad (1.1)$$

We recall that the coefficients $\gamma_{j,m}(g)$ are respectively $\gamma_{0,m}(g) = g(\tau_0), \gamma_{n+1,m}(g) = g(\tau_n)$

For $1 \leq j \leq n$

$$\gamma_{j,m}(g) = A_{j,m}g(\theta_{j-1}) + B_{j,m}g(\theta_j) + C_{j,m}g(\theta_{j+1}).$$

Using the B-spline expansion of $c_s(\tau) = \tau^s$ for $\tau \geq 0$, we get (see [19])

$$\gamma_0 = \sum_{j \in J_m} \varphi_j, \quad \gamma_1 = \sum_{j \in J_m} \theta_j \varphi_j, \quad \gamma_1 = \sum_{j \in J_m} \tau_{j-1} \tau_j \varphi_j \quad (1.2)$$

Let us consider the space P_{m-1} of polynomials up to degree $m-1$ and let

$$\begin{aligned} g(\mathcal{S}_n) &:= \{g_p := g(\theta_p), 0 \leq p \leq n+1\}, \text{ for } m \text{ odd,} \\ g(\mathcal{T}_n) &:= \{g_p := g(\tau_p), 0 \leq p \leq n\}, \text{ for } m \text{ even.} \end{aligned}$$

To be more precise, the PQIO σ_m is development to be exact on the space P_{m-1} .

$$\sigma_m c_s = c_s, \quad 0 \leq s \leq m-1 \quad (1.3)$$

and these equalities are satisfied if and only if

$$\gamma_{j,m}(c_s) = \theta_{j,m}^{(s)}, \quad j \in \mathcal{J}_m, \quad 0 \leq s \leq m-1. \quad (1.4)$$

For $m-1 \leq j \leq n-1$, following [6], we have

$$\mu_{j,m}(g) = \begin{cases} \sum_{s=0}^{m-1} \alpha_{j,s}^{(m)} g_s, & \text{for } 0 \leq j \leq m-3, \\ \sum_{s=0}^{m-1} \alpha_{j,s}^{(m)} g_{n-m+i+1}, & \text{for } n+1 \leq j \leq n+m-2, \end{cases} \quad (1.5)$$

For $0 \leq s \leq 2$, we obtain:

$$A_{j,m} = -\frac{\delta_j^2 \delta'_{j+1}}{\delta_j + \delta'_{j+1}}, \quad B_{j,m} = 1 + \delta_j \delta_{j+1}, \quad C_{j,m} = -\frac{\delta_j (\delta'_{j+1})^2}{\delta_j + \delta'_{j+1}}, \quad (1.6)$$

with

$$\delta_j = \frac{h_j}{h_{j-1} + h_j}, \quad \delta'_j = 1 - \delta_j = \frac{h_{j-1}}{h_{j-1} + h_j}. \quad (1.7)$$

Let g be a function with a bounded m' -th derivative in \mathcal{I} , we have

$$\|g - \sigma_{m',n}g\|_{\infty, \mathcal{I}} = O(h^{m'}).$$

In view of the importance of pQIOs, we recall some useful expression of cubic and quadratic spline quasi-interpolant.

The quadratic QI of a known function g is given by

$$\sigma_{3,n}(g) := \sum_{j=0}^{n+1} \gamma_{j,3}(g) \varphi_{j,3},$$

where the coefficients are defined by

$$\gamma_{j,3}(g) := \frac{1}{8}(-g_{j-1} + 10g_j - g_{j+1}), \quad 2 \leq j \leq n-1. \quad (1.8)$$

For $j = 0, 1, n, n+1$, we have

$$\begin{aligned} \gamma_{0,3}(g) &:= g_0, & \gamma_{1,3}(g) &:= -\frac{1}{3}g_0 + \frac{3}{2}g_1 - \frac{1}{6}g_2, \\ \gamma_{n+1,3}(g) &:= g_{n+1}, & \gamma_{n,3}(g) &:= -\frac{1}{6}g_{n-1} + \frac{3}{2}g_n - \frac{1}{3}g_{n+1} \quad i = 1 \cdots m'. \end{aligned} \quad (1.9)$$

The QI $\sigma_{3,n}(g)$ can be described as: (see [47])

$$\sigma_{3,n}(g) := \sum_{j=0}^{n+1} g(\theta_j) \vartheta_{j,3}, \quad (1.10)$$

with

$$\begin{aligned} \vartheta_{0,3} &:= \varphi_{0,3} + A_{1,3}\varphi_{1,3}, & \vartheta_{1,3} &:= B_{1,3}\varphi_{1,3} + C_{2,3}\varphi_{2,3}. \\ \vartheta_{j,3} &:= C_{j-1,3}\varphi_{j-1,3} + B_{j,3}\varphi_{j,3} + A_{j+1,3}\varphi_{j+1,3}, & 2 \leq j \leq n-1. \\ \vartheta_{n,3} &:= C_{n-1,3}\varphi_{n-1,3} + B_{n,3}\varphi_{n,3}, & \vartheta_{n+1,3} &:= A_{n,3}\varphi_{n,3} + \varphi_{n+1,3}, \end{aligned} \quad (1.11)$$

for some the specific linear functions $\tau_{j,3}$ which are combinations of the B-splines

The cubic spline quasi-interpolant is given by

$$\sigma_{4,n}(g) := \sum_{j=0}^{n+2} \gamma_{j,4}(g) \varphi_{j,4},$$

where the coefficients are defined by

$$\gamma_{j,4}(g) := \frac{1}{6}(-g_{j-2} + 8g_{j-1} - g_j), \quad 2 \leq j \leq n, \quad i = 1 \cdots m'.$$

Moreover,

$$\begin{aligned} \gamma_{0,4}(g) &:= g_0, & \gamma_{1,4}(g) &:= \frac{7}{18}g_0 + g_1 - \frac{1}{2}g_2 + \frac{1}{9}g_3, \\ \gamma_{n+2,4}(g) &:= g_n, & \gamma_{n+1,4}(g) &:= \frac{7}{18}g_n + g_{n-1} - \frac{1}{2}g_{n-2} + \frac{1}{9}g_{n-3}, \quad i = 1 \cdots m'. \end{aligned}$$

Finally, $\mathcal{Q}_{4,n}$ reads as

$$\sigma_{4,n}(g) := \sum_{j=0}^n g(\tau_j) \sigma_{j,4}, \quad i = 1 \cdots m',$$

for some the specific linear functions $\sigma_{j,4}$ which are combinations of the B-splines.

η	$\xi_{0,\eta}$	$\lambda_{0,\eta}$	$\xi_{1,\eta}$	$\lambda_{1,\eta}$	$\xi_{2,\eta}$	$\lambda_{2,\eta}$
1	1.23×10^{-2}	2.33×10^{-1}	4.41×10^{-1}	3.94×10^{-1}	1	5×10^{-1}
2	5.44×10^{-1}	1.01×10^{-1}	1.95×10^{-1}	2.73×10^{-1}	2	5×10^{-1}

Table 1.1: *Weights and points for $m = 2, 3$*

1.2 Quadrature rules

In order to evaluate the values of ρ_m , ϱ_m , ρ_m , $\widehat{\Phi}_m$, $\Phi_m^{(2)}$ and $\overline{\Phi}_m$, numerical quadrature rules are request. Popular quadrature rules in this situation are the the Gauss quadrature formulas (abbr. GQF) of order $2m' - 1$, with B-splines weights. We follow [7] and references therein to compute the weights and nodes of GQF.

1.3 Evaluations for Tensor Product approximation

Let

$$\rho_m := ((\rho_{1,i,m})_{i \in J_m}, (\rho_{2,i,m})_{i \in J_m}, \dots, (\rho_{m',i,m})_{i \in J_m})^T, \quad \text{with } \rho_{j,i,m} = \int_a^b \varphi_{i,m} g_j.$$

We evaluate the integrals $\rho_{j,i,m}$ via the GQF for quadratic and cubic B-spline weights with $m' = 2, 3$. Here, we give some approximate results.

$$\begin{aligned} \int_{\tau_0}^{\tau_1} \varphi_{0,3} g &\simeq h \sum_{\eta=0}^1 \lambda_{0,\eta} g(a + h_p \xi_{0,\eta}), \\ \int_{\tau_0}^{\tau_2} \varphi_{1,3} g &\simeq h \sum_{\eta=0}^1 \lambda_{1,\eta} g(a + h_p \xi_{1,\eta}), \\ \int_{\tau_{i-2}}^{\tau_{i+1}} \varphi_{i,3} g &\simeq h \sum_{\eta=0}^1 \lambda_{2,\eta} g(a + h_p (\xi_{2,\eta} + i - 2)), \quad 2 \leq i \leq n-1, \\ \int_{\tau_{n-2}}^{\tau_n} \varphi_{n,3} g &\simeq h \sum_{\eta=0}^1 \lambda_{1,\eta} g(b - h_p \xi_{1,\eta}), \\ \int_{\tau_{n-1}}^{\tau_n} \varphi_{n+1,3} g &\simeq h \sum_{\eta=0}^1 \lambda_{0,\eta} g(b - h_p \xi_{0,\eta}). \end{aligned}$$

Also, the following calculus are given

$$\begin{aligned} \int_{\tau_0}^{\tau_1} \varphi_{0,4} g &\simeq h \sum_{\eta=0}^2 \lambda_{0,\eta} g(a + h_p \xi_{0,\eta}), \\ \int_{\tau_0}^{\tau_2} \varphi_{1,4} g &\simeq h \sum_{\eta=0}^2 \lambda_{1,\eta} g(a + h_p \xi_{1,\eta}), \\ \int_{\tau_0}^{\tau_3} \varphi_{2,4} g &\simeq h \sum_{\eta=0}^2 \lambda_{2,\eta} g(a + h_p \xi_{2,\eta}), \\ \int_{\tau_{i-3}}^{\tau_{i+1}} \varphi_{i,4} g &\simeq h \sum_{\eta=0}^2 (\lambda_{3,\eta} g(a + h_p (\xi_{3,\eta} + i - 3))), \quad 3 \leq i \leq n-1, \end{aligned}$$

$$\begin{aligned}\int_{\tau_{n-3}}^{\tau_n} \varphi_{n,4} g &\simeq h \sum_{\eta=0}^2 \lambda_{2,\eta} g(b - h_p \xi_{2,\eta}), \\ \int_{\tau_{n-2}}^{\tau_n} \varphi_{n+1,4} g &\simeq h \sum_{\eta=0}^2 \lambda_{1,\eta} g(b - h_p \xi_{1,\eta}), \\ \int_{\tau_{n-1}}^{\tau_n} \varphi_{n+2,4} g &\simeq h \sum_{\eta=0}^2 \lambda_{0,\eta} g(b - h_p \xi_{0,\eta}).\end{aligned}$$

In the following Table 3.7, we present some values of the points $\xi_{i,\eta}$ and the weights $\lambda_{i,\eta}$.

η	$\xi_{0,\eta}$	$\lambda_{0,\eta}$	$\xi_{1,\eta}$	$\lambda_{1,\eta}$	$\xi_{2,\eta}$	$\lambda_{2,\eta}$	$\xi_{3,\eta}$	$\lambda_{3,\eta}$
0	$6, 3 \cdot 10^{-2}$	$1, 29 \cdot 10^{-1}$	$2, 37 \cdot 10^{-1}$	$1, 83 \cdot 10^{-1}$	$2, 37 \cdot 10^{-1}$	$1, 83 \cdot 10^{-1}$	1,052	$1, 86 \cdot 10^{-1}$
1	$3, 02 \cdot 10^{-1}$	$1, 05 \cdot 10^{-1}$	$7, 16 \cdot 10^{-1}$	$2, 69 \cdot 10^{-1}$	$7, 16 \cdot 10^{-1}$	$2, 69 \cdot 10^{-1}$	2	$6, 30 \cdot 10^{-1}$
2	$6, 37 \cdot 10^{-1}$	$1, 7 \cdot 10^{-2}$	1,326	$4, 9 \cdot 10^{-2}$	1,326	$4, 9 \cdot 10^{-2}$	2,949	$1, 86 \cdot 10^{-1}$

Table 1.2: The points $\xi_{i,\eta}$ and the weights $\lambda_{i,\eta}$ for $m' = 3$ and $m = 4$

1.4 Continuous blending case

Now, we use the GQF with B-spline weights with $m' = 4$ and $m = 5$ to approximate the integrals appearing in the matrices $\widehat{\Phi}_m$ and $\overline{\Phi}_m$, and in the vector ρ_m . We give the associate formulas of order $O(h^7)$

$$\begin{aligned}\int_{\tau_0}^{\tau_1} \varphi_{0,3} g &\simeq h \sum_{\eta=0}^3 \lambda_{0,\eta} g(a + h_p \xi_{0,\eta}), \\ \int_{\tau_0}^{\tau_2} \varphi_{1,3} g &\simeq h \sum_{\eta=0}^3 \lambda_{1,\eta} g(a + h_p \xi_{1,\eta}), \\ \int_{\tau_{i-2}}^{\tau_{i+1}} \varphi_{i,3} g &\simeq h \sum_{\eta=0}^3 \lambda_{2,k} g(a + h_p (\xi_{2,\eta} + i - 2)), \quad 2 \leq i \leq n-1, \\ \int_{\tau_{n-2}}^{\tau_n} \varphi_{n,3} g &\simeq h \sum_{\eta=0}^3 \lambda_{1,\eta} g(b - h_p \xi_{1,\eta}), \\ \int_{\tau_{n-1}}^{\tau_n} \varphi_{n+1,3} g &\simeq h \sum_{\eta=0}^3 \lambda_{0,\eta} g(b - h_p \xi_{0,\eta}).\end{aligned}$$

In table , we introduce the points $\xi_{i,\eta}$ and the weights $\lambda_{i,\eta}$.

η	$\xi_{0,\eta}$	$\lambda_{0,\eta}$	$\xi_{1,\eta}$	$\lambda_{1,\eta}$	$\xi_{2,\eta}$	$\lambda_{2,\eta}$
0	4.9×10^{-2}	1.11×10^{-1}	2.01×10^{-1}	1.11×10^{-1}	4.99×10^{-1}	7.6×10^{-2}
1	2.39×10^{-1}	1.44×10^{-1}	6.11×10^{-1}	3.15×10^{-1}	1.158	4.25×10^{-1}
2	5.18×10^{-1}	6.9×10^{-2}	1.113	2.10×10^{-1}	1.843	4.25×10^{-1}
3	7.96×10^{-1}	1.1×10^{-1}	6.24×10^{-1}	3.3×10^{-1}	2.502	7.6×10^{-1}

Table 1.3: *Points and weights for $m' = 4$ and $m = 3$*

η	$\xi_{0,\eta}$	$\lambda_{0,\eta}$	$\xi_{1,\eta}$	$\lambda_{1,\eta}$	$\xi_{2,\eta}$	$\lambda_{2,\eta}$	$\xi_{3,\eta}$	$\lambda_{3,\eta}$
0	$3, 1.10^{-2}$	$7, 1.10^{-2}$	$1, 28.10^{-1}$	$6, 6.10^{-1}$	$3, 11.10^{-1}$	$4, 6.10^{-1}$	$6, 19.10^{-1}$	$2, 5.10^{-2}$
1	$1, 55.10^{-1}$	$1, 01.10^{-1}$	$4, 04.10^{-1}$	$1, 99.10^{-1}$	$7, 71.10^{-1}$	$2, 45.10^{-1}$	1,294	$2, 39.10^{-1}$
2	$3, 52.10^{-1}$	$6, 1.10^{-2}$	$7, 80.10^{-1}$	$1, 77.10^{-1}$	1,323	$3, 22.10^{-1}$	2	$4, 73.10^{-1}$
3	$5, 83.10^{-1}$	$1, 7.10^{-1}$	1,211	$5, 5.10^{-1}$	1,913	$1, 27.10^{-1}$	2,707	$2, 39.10^{-1}$
4	$8, 04.10^{-1}$	2.10^{-2}	1,629	5.10^{-3}	2,485	$1, 3.10^{-2}$	3,382	$2, 5.10^{-1}$

Table 1.4: *for $m' = 5$ and $m = 4$*

$$\begin{aligned}
\int_{\tau_0}^{\tau_1} \varphi_{0,4} g &\simeq h \sum_{\eta=0}^4 \varpi_{0,\eta} g(a + h_p \xi_{0,\eta}), \\
\int_{\tau_0}^{\tau_2} \varphi_{1,4} g &\simeq h \sum_{\eta=0}^4 \varpi_{1,\eta} g(a + h_p \xi_{1,\eta}), \\
\int_{\tau_0}^{\tau_3} \varphi_{2,4} g &\simeq h \sum_{\eta=0}^4 \varpi_{2,\eta} g(a + h_p \xi_{2,\eta}), \\
\int_{\tau_{i-3}}^{\tau_{i+1}} \varphi_{i,4} g &\simeq h \sum_{\eta=0}^4 (\varpi_{3,\eta} g(a + h_p (\xi_{3,\eta} + i - 3))), \quad 3 \leq i \leq n-1, \\
\int_{\tau_{n-3}}^{\tau_n} \varphi_{n,4} g &\simeq h \sum_{\eta=0}^4 \varpi_{2,\eta} g(b - h_p \xi_{2,\eta}), \\
\int_{\tau_{n-2}}^{\tau_n} \varphi_{n+1,4} g &\simeq h \sum_{\eta=0}^4 \varpi_{1,\eta} g(b - h_p \xi_{1,\eta}), \\
\int_{\tau_{n-1}}^{\tau_n} \varphi_{n+2,4} g &\simeq h \sum_{\eta=0}^4 \varpi_{0,\eta} g(b - h_p \xi_{0,\eta}).
\end{aligned}$$

We compute the matrix $\Phi_m^{(2)}$ and the vector ϱ_m in both quadratic and cubic cases,

$$\begin{aligned}
\int_{\tau_0}^{\tau_n} f &\simeq h_p \sum_{m=0}^{\frac{n}{2}-1} \left(\frac{41}{420} f(\tau_{2m}) + \frac{18}{35} f\left(\tau_{2ml} + \frac{h}{3}\right) + \frac{9}{140} f\left(\tau_{2m} + \frac{2h_p}{3}\right) + \frac{68}{105} f(\tau_{2m} + h_p) \right. \\
&\quad \left. + \frac{9}{140} f\left(\tau_{2m} + \frac{4h_p}{3}\right) + \frac{18}{35} f\left(\tau_{2m} + \frac{5h_p}{3}\right) + \frac{41}{420} f(\tau_{2m} + 2h_p) \right).
\end{aligned}$$

Chapter 2

Spline QI for a model arising from biology

Integral equations are used in physics, hydrodynamics, electromagnetics, biology, and many other science and engineering domains. For this aim, the Kulkarni, Galerkin, Nyström, and degenerate kernel approaches are often used. They've been thoroughly researched in the scientific community. Mennouni has recently developed the Kulkarni and Galerkin methods for solving second-kind functional operator equations with bounded but noncompact operators. To solve second-kind integral equations with the Cauchy kernel, he used a series of orthogonal finite rank projections (see [37]). Then, for a class of the second kind, generalised singular integral equations with Cauchy kernel and constant coefficients in $L^2([0, 1], \mathbb{C})$, an extended version of the piecewise regular Galerkin method. Next, he used the Kulkarni approach for an integro-differential equation to expand and improve the results of previous work. An improvement via projection for solving the integrodifferential equations is introduced in [36]. The study of [38] presented three significant contributions. First, all of the septic quasi-interpolants' coefficients are determined. Then, the authors used the results to solve a class of second-kind Fredholm integral equations. Finally, the authors offered three degenerate kernel approaches. In recent years, many authors have presented various ways to solve integral equations using spline quasi-interpolants to reduce the number of terms in the kernel's approximation. As a result of the restrictions imposed, great accuracy is achieved while avoiding needless processing expenses associated with large linear systems (cf. [50]).

In this chapter, we describe the following significant class of integral equations of the second kind

$$u(\xi) \int_a^b \Phi(\xi - \tau) d\tau + \int_a^b \Phi(\tau - \xi) u(\tau) d\tau = f(\xi) \quad a \leq \xi \leq b. \quad (2.1)$$

This integral equation derived from biological models, (see [16, 18]). It is found in G. B. Ermentrout's research on a mathematical model for frog eyesight.

The above integral equation reads as follows:

$$u(\xi) + \frac{1}{\int_a^b \Phi(\xi - \tau) d\tau} \int_a^b \Phi(\tau - \xi) u(\tau) d\tau = \frac{f(\xi)}{\int_a^b \Phi(\xi - \tau) d\tau} \quad a \leq \xi \leq b. \quad (2.2)$$

Putting

$$h(\xi) = \frac{1}{\int_a^b \Phi(\xi - \tau) d\tau}, \quad \int_a^b \Phi(\xi - \tau) d\tau \neq 0 \quad (2.3)$$

and

$$g(\xi) = \frac{f(\xi)}{\int_a^b \Phi(\xi - \tau) d\tau}, \quad (2.4)$$

$$k(\xi, \tau) = -\Phi(\tau - \xi), \quad (2.5)$$

we obtain

$$u(\xi) = g(\xi) + h(\xi) \int_a^b \Phi(\xi, \tau) u(\tau) d\tau, \quad a \leq \xi \leq b. \quad (2.6)$$

2.1 Approximating system

We recall that the approximation $\Phi_{n,m}$ of the kernel Φ can be given as

$$\Phi_{n,m}(\xi, \tau) = \sum_{s,r \in J_m} \Phi_m(s, r) \varphi_{s,m}(\xi) \varphi_{r,m}(\tau), \quad (2.7)$$

where $J_m := \{0, \dots, n + m - 2\}$ and the values $\Phi_m(s, r)$ are the entries of $\Phi_{n,m}$ given by (2.7).

Replacing Φ by $\Phi_{n,m}$ in (2.6), we get

$$u_{n,m}(\xi) = g(\xi) + h(\xi) \int_a^b \Phi_{n,m}(\xi, \tau) u_{n,m}(\tau) d\tau, \quad (2.8)$$

System (2.9) gives

$$\begin{aligned} u_{n,m}(\xi) &= g(\xi) + h(\xi) \sum_{s,r \in J_m} \Phi_m(s, r) \varphi_{s,m}(\xi) \int_a^b \varphi_{r,m}(\tau) u_{n,m}(\tau) d\tau, \\ &= g(\xi) + h(\xi) \sum_{s \in J_m} \varphi_{s,m}(\xi) \sum_{r \in J_m} \Phi_m(s, r) \int_a^b \varphi_{r,m}(\tau) u_{n,m}(\tau) d\tau. \end{aligned}$$

Putting

$$\alpha_{s,m} = \sum_{r \in J_m} \Phi_m(s, r) \int_a^b \varphi_{r,m}(\tau) u_{n,m}(\tau) d\tau,$$

we obtain

$$u_{n,m}(\xi) = g(\xi) + h(\xi) \sum_{s \in J_m} \varphi_{s,m}(\xi) \alpha_{s,m}. \quad (2.9)$$

Or equivalently,

$$u_{n,m}(\tau) = g(\tau) + h(\tau) \sum_{i \in J_m} \varphi_{i,m}(\tau) \alpha_{i,m}.$$

Thus,

$$\alpha_{s,m} = \sum_{r \in J_m} \Phi_m(s, r) \int_a^b \varphi_{r,m}(\tau) g(\tau) d\tau + \sum_{i \in J_m} \alpha_{i,m} \sum_{r \in J_m} \Phi_m(s, r) \int_a^b h(\tau) \varphi_{s,m}(\tau) \varphi_{i,m}(\tau) d\tau.$$

Letting

$$\begin{aligned} \alpha &:= (\alpha_{i,m})_{i \in J_m}, \\ \rho_{i,m} &:= \int_a^b \varphi_{i,m}(\tau) g(\tau) d\tau, \\ \rho_m &:= (\rho_{i,m})_{i \in J_m}, \end{aligned}$$

Let us consider the Gram matrix $\Upsilon_m := (\Upsilon_m(i, j))_{i,j \in J_m}$ associated with the corresponding B-splines, i.e.

$$\Upsilon_m(i, j) := \int_a^b h(\tau) \varphi_{i,m}(\tau) \varphi_{j,m}(\tau) d\tau.$$

The following theorem has been demonstrated:

Theorem 2.1.1. *The coefficients α_m of the approach integral system's solution (2.9) fulfill the linear equations:*

$$(I - \Phi_m \Upsilon_m) \alpha_m = \Phi_m \rho_m.$$

By standard calculus, we get for $m = 3, 4$ where Υ_3 and Υ_4 are pentadiagonal and heptadiagonal symmetric matrices, respectively.

Moreover, the main diagonal of Υ_3 is $(0.2h, 0.2h, 0.55h, \dots, 0.55h, 0.2h, 0.2h)$ and the 1-diagonal and 2-diagonal of Υ_3 are respectively,

$$(0.1166h, 0.2083h, 0.2166h, \dots, 0.2166h, 0.2083h, 0.1166h) \quad \text{and} \quad (0.0166h, 0.0083h, \dots, 0.0083h, 0.0166h).$$

However, the main diagonal of Υ_4 is $(12h, 18.6h, 27.45h, 40.26h, \dots, 40.26h, 27.45h, 18.6h, 12h)$.

For $k = 1, 2, 3$, the k -diagonals of Υ_4 are respectively

$$\begin{aligned} &(7.35h, 13.125h, 18.866h, 19.85h, \dots, 19.85h, 18.866h, 13.125h, 7.35h), \\ &(1.55h, 2.9h, 1.99h, 2h, \dots, 2h, 1.99h, 2.9h, 1.55h), \\ &(0.1h, 0.025h, 0.0166h, \dots, 0.0166h, 0.025h, 0.1h). \end{aligned}$$

2.2 Continuous blending approximation

The goal of this section is to approximate the kernels Φ , via the quasi-interpolation. To this end, let us approximate the kernels $\Phi_{n,m}(\cdot, \cdot)$ as follows:

$$\Phi_{n,m}(\xi, \tau) = (\sigma_{n,m} \otimes \mathcal{I} + \mathcal{I} \otimes \sigma_{n,m} - \sigma_{n,m} \otimes \sigma_{n,m})\Phi(\xi, \tau),$$

where

$$\begin{aligned} (\sigma_{n,m} \otimes \mathcal{I})\Phi(\xi, \tau) &= \sum_{s \in J_m} \gamma_{s,m}(\Phi(\cdot, \tau))\varphi_{s,m}(\xi), \\ (\mathcal{I} \otimes \sigma_{n,m})\Phi(\xi, \tau) &= \sum_{r \in J_m} \gamma_{r,m}(\Phi(\xi, \cdot))\varphi_{r,m}(\tau), \\ (\sigma_{n,m} \otimes \sigma_{n,m})\Phi(\xi, \tau) &= \sum_{s,r \in J_m} \Phi_m(s, r)\varphi_{s,m}(\xi)\varphi_{r,m}(\tau). \end{aligned}$$

Letting

$$\widehat{\Phi}_{s,m} := \gamma_{s,m}(\Phi(\cdot, \tau)) \quad \text{and} \quad \overline{\Phi}_{r,m} := \gamma_{r,m}(\Phi(\xi, \cdot)),$$

we get

$$\Phi_{n,m}(\xi, \tau) = \sum_{s \in J_m} \widehat{\Phi}_{s,m}(\tau)\varphi_{s,m}(\xi) + \sum_{r \in J_m} \overline{\Phi}_{r,m}(\xi)\varphi_{r,m}(\tau) - \sum_{s,r \in J_m} \Phi_m(s, r)\varphi_{s,m}(\xi)\varphi_{r,m}(\tau),$$

Theorem 2.2.1. For some vectors α_m, β_m and ϱ_m, ρ_m , the vector $\chi^B := (\alpha, \beta)^T$ satisfies the following linear system

$$(I - \kappa)\chi = \Xi,$$

where

$$\Xi := (\varrho_m, \rho_m)^T,$$

and

$$\kappa^B := \begin{pmatrix} \widehat{\Phi}_m & \Phi_m^{(2)} - \Phi^m \\ \Upsilon_m & \overline{\Phi}_m \end{pmatrix}.$$

Proof. By replacing k to $k_{n,m}$ in the approximation integral equation, it is obvious that

$$u_{n,m}(\xi) = g(\xi) + h(\xi) \int_a^b \Phi_{n,m}(\xi, \tau)u_{n,m}(\tau)d\tau.$$

We have

$$\begin{aligned} u_{n,m}(\xi) &= g(\xi) + h(\xi) \left(\sum_{s \in J_m} \varphi_{s,m}(\xi) \int_a^b \widehat{\Phi}_{s,m}(\tau)u_{n,m}(\tau)d\tau + \sum_{r \in J_m} \overline{\Phi}_{r,m}(\xi) \int_a^b \varphi_{r,m}(\tau)u_{n,m}(\tau)d\tau \right. \\ &\quad \left. - \sum_{s,r \in J_m} \Phi_m(s, r)\varphi_{s,m}(\xi) \int_a^b \varphi_{r,m}(\tau)u_{n,m}(\tau)d\tau \right), \end{aligned}$$

which leads to

$$u_{n,m}(\xi) = g(\xi) + h(\xi) \sum_{s \in J_m} \alpha_{s,m} \varphi_{s,m}(\xi) + h(\xi) \sum_{r \in J_m} \beta_{r,m} \overline{\Phi_{r,m}}(\xi).$$

Or equivalently,

$$u_{n,m}(\tau) = g(\tau) + h(\tau) \sum_{w \in J_m} \alpha_{w,m} \varphi_{w,m}(\tau) + h(\tau) \sum_{v \in J_m} \beta_{v,m} \overline{\Phi_{v,m}}(\tau).$$

Hence

$$\begin{aligned} \sum_{s \in J_m} \alpha_{s,m} \varphi_{s,m}(\xi) + \sum_{r \in J_m} \beta_{r,m} \overline{\Phi_{r,m}}(\xi) &= \int_a^b \widehat{\Phi}_{n,m}(\xi, \tau) u_{n,m}(\tau) d\tau \\ &= \int_a^b \left(\sum_{s \in J_m} \widehat{\Phi}_{s,m}(\tau) \varphi_{s,m}(\xi) + \sum_{r \in J_m} \overline{\Phi_{r,m}}(\xi) \varphi_{r,m}(\tau) \right. \\ &\quad \left. - \sum_{s,r \in J_m} \Phi_m(s,r) \varphi_{s,m}(\xi) \varphi_{r,m}(\tau) \right) u_{n,m}(\tau) d\tau \\ &= \sum_{s \in J_m} \varphi_{s,m}(\xi) \left(\int_a^b \widehat{\Phi}_{s,m}(\tau) u_{n,m}(\tau) d\tau - \sum_{r \in J_m} \Phi_m(s,r) \int_a^b \varphi_{r,m}(\tau) u_{n,m}(\tau) d\tau \right) \\ &\quad + \sum_{r \in J_m} \overline{\Phi_{r,m}}(\xi) \int_a^b B_{r,m}(\tau) u_{n,m}(\tau) d\tau. \end{aligned}$$

This equality is satisfied with the following options for

$$\alpha_m := (\alpha_{s,m})_{s \in J_m} \quad \text{and} \quad \beta_m := (\beta_{r,m})_{r \in J_m}.$$

$$\begin{aligned} \alpha_{s,m} &= \left(\int_a^b \widehat{\Phi}_{s,m}(\tau) u_{n,m}(\tau) d\tau - \sum_{r \in J_m} \Phi_m(s,r) \int_a^b \varphi_{r,m}(\tau) u_{n,m}(\tau) d\tau \right) \\ \beta_{r,m} &= \int_a^b \varphi_{r,m}(\tau) u_{n,m}(\tau) d\tau. \end{aligned}$$

Let $\varrho_m := (\varrho_{s,m})_{s \in J_m}$ and $\rho_m := (\rho_{r,m})_{r \in J_m}$ be the vectors with entries

$$\varrho_{s,m} = \int_a^b \widehat{\Phi}_{s,m}(\tau) g(\tau) d\tau, \quad \rho_{r,m} := \int_a^b \varphi_{r,m}(\tau) g(\tau) d\tau,$$

Let us consider the matrices $\widehat{\Phi}_m$, $\overline{\Phi}_m$ and $\Phi_m^{(2)}$, with entries

$$\begin{aligned} \widehat{\Phi}_m(s,r) &:= \int_a^b h(\tau) \widehat{\Phi}_{s,m}(\tau) \varphi_{r,m}(\tau) d\tau, \quad \overline{\Phi}_m(s,r) := \int_a^b h(\tau) \varphi_{s,m}(\tau) \overline{\Phi_{r,m}}(\tau) d\tau, \\ \Phi_m^{(2)}(s,r) &:= \int_a^b h(\tau) \widehat{\Phi}_{s,m}(\tau) \overline{\Phi_{r,m}}(\tau) d\tau. \end{aligned}$$

We obtain

$$\begin{aligned} \beta_{r,m} &= \int_a^b \varphi_{r,m}(\tau) \left(g(\tau) + h(\tau) \sum_{w \in J_m} \alpha_{w,m} \varphi_{w,m}(\tau) + h(\tau) \sum_{v \in J_m} \beta_{v,m} \overline{\Phi_{v,m}}(\tau) \right) d\tau, \\ \beta_{r,m} &= \rho_{r,m} + \sum_{w \in J_m} \Upsilon_m(w,r) \alpha_{w,m} + \sum_{v \in J_m} \overline{\Phi}_m(r,v) \beta_{v,m}, \\ \alpha_{s,m} &= \left(\int_a^b \widehat{\Phi}_{s,m}(\tau) \left(g(\tau) + h(\tau) \sum_{w \in J_m} \alpha_{w,m} \varphi_{w,m}(\tau) + h(\tau) \sum_{v \in J_m} \beta_{v,m} \overline{\Phi_{v,m}}(\tau) \right) d\tau \right. \\ &\quad \left. - \sum_{r \in J_m} \Phi_m(s,r) \int_a^b \varphi_{r,m}(\tau) \left(g(\tau) + h(\tau) \sum_{w \in J_m} \alpha_{j_{w,m}} \varphi_{w,m}(\tau) + \sum_{q=1}^{m'} \sum_{v \in J_m} \beta_{q_{v,m}} \overline{\Phi_{j_{q_{v,m}}}}(\tau) \right) d\tau \right). \end{aligned}$$

Or equivalently,

$$\begin{aligned}\alpha_{s,m} &= \left(\varrho_{s,m} - \sum_{r \in J_m} \Phi_m(s,r) \rho_{r,m} \right) + \sum_{w \in J_m} \alpha_{w,m} \left(\widehat{\Phi}_m(s,w) - \sum_{r \in J_m} \Phi_m(s,r) \Upsilon_m(r,w) \right) \\ &+ \sum_{v \in J_m} \beta_{v,m} \left(\Phi_m^{(2)}(s,v) - \sum_{r \in J_m} \Phi_m(s,r) \overline{\Phi}_m(r,v) \right),\end{aligned}$$

we get the the matrix form

$$\begin{aligned}\alpha_m &= \varrho - \Phi^m \rho_m + (\widehat{\Phi}_m - \Phi^m \Upsilon_m^B) \alpha_m + (\Phi_m^{(2)} - \Phi^m \overline{\Phi}_m) \beta_m, \\ \beta_m &= \rho_m + \Upsilon_m^B \alpha_m + \overline{\Phi}_m \beta_m.\end{aligned}$$

Note that since

$$\overline{\Phi}_m \beta_m = \beta_m - \rho_m - \Upsilon_m^B \alpha_m,$$

we obtain

$$\alpha_m = \rho + \widehat{\Phi}_m \alpha_m + (\Phi_m^{(2)} - \Phi^m) \beta_m.$$

□

2.3 Error analysis

This section's goal is to review various findings and examine the error convergence. We provide this using the kernel's approximation error.

The approximate integral equations (2.9) reads as

$$u_{n,m}(\xi) = g(\xi) + h(\xi) \Phi_{n,m} u_{n,m}(\xi), \quad ,$$

where

$$\Phi_{n,m} u_{n,m}(\xi) = \int_a^b \Phi_{n,m}(\xi, \tau) u_{n,m}(\tau) d\tau, \quad .$$

Theorem 2.3.1. *Let Φ be an integrable derivative function. Assume that $b + a - s_j = s_{n+1-j}$. Then:*

$$\left| \int_a^b (k - \sigma_{m,n} k) \Phi \right| \leq C h^{m'+1} \left(\|\Phi^{(1)}\|_{\infty, I} \|k^{(m')}\|_{\infty, I} + \|k^{(m'+1)}\|_{\infty, I} \right)$$

for some constant C and for all $k \in C^{m'+1}(I)$.

Proof. See [7] and references therein. □

Theorem 2.3.2. *Assume that $\Lambda \in C^{m', m'+1}(I^2)$. Then, it holds*

$$\|u - u_{n,m'}\|_{\infty, I} = O(h_p^{m'}), \quad (\text{resp. } \|u - u_{n,m'}\|_{\infty, I} = O(h_p^{r_{m'}})),$$

where

$$r_{m'} := \begin{cases} 2m, & \text{if } m' \text{ is even,} \\ 2m + 1, & \text{if } m' \text{ is odd.} \end{cases}$$

Proof. We have

$$\begin{aligned}u - u_{n,m'} &= (I - h\Lambda)^{-1} k - (I - h\Lambda_{n,m'})^{-1} k \\ &= (I - h\Lambda_{n,m'})^{-1} [(I - h\Lambda_{n,m'}) - (I - h\Lambda)] (I - h\Lambda)^{-1} k \\ &= (I - h\Lambda_{n,m'})^{-1} h(\Lambda - \Lambda_{n,m'}) u.\end{aligned}$$

Hence

$$\|u - u_{n,m'}\|_{\infty} \leq \|(I - h\Lambda_{n,m'})^{-1}\|_{\infty} \|h\|_{\infty} \|(\Lambda - \Lambda_{n,m'})\|_{\infty} \|u\|_{\infty}.$$

As in [7], for each $i, j = 1, \dots, m'$ we have

$$\Lambda_{n,m} = \begin{cases} (\sigma_{n,m} \otimes \mathcal{I} + \mathcal{I} \otimes \sigma_{n,m} - \sigma_{n,m} \otimes \sigma_{n,m})\Phi & \text{in CB case,} \\ (\sigma_{n,m} \otimes \sigma_{n,m})\Phi, & \text{in TB one,} \end{cases}$$

and

$$\Phi - \Phi_{n,m} = \begin{cases} (\mathcal{I} - \sigma_{n,m}) \otimes (\mathcal{I} - \sigma_{n,m})\Phi, & \text{in CB case,} \\ ((\mathcal{I} - \sigma_{n,m}) \otimes \mathcal{I})\Phi + \mathcal{I} \otimes (\mathcal{I} - \sigma_{n,m})\Phi - (\mathcal{I} - \sigma_{n,m}) \otimes (\mathcal{I} - \sigma_{n,m})\Phi, & \text{in TB one.} \end{cases}$$

Again, we need the following results, which follows from [7],

$$\max_{\xi \in I} \left| \int_a^b ((\mathcal{I} - \sigma_{n,m}) \otimes \mathcal{I})\Phi(\xi, \tau)u(\tau)d\tau \right| \leq C_1 h_p^m \left\| \Phi^{(m,0)} \right\|_{\infty} \|u\|_{\infty},$$

and

$$\max_{\xi \in I} \left| \int_a^b (\mathcal{I} \otimes (\mathcal{I} - \sigma_{n,m}))\Phi(\xi, \tau)u_j(\tau)d\tau \right| \leq C_2 h_p^{m+1} \left(\left\| \Phi^{(0,m)} \right\|_{\infty} \|u^1\|_{\infty} + \left\| \Phi^{(0,m+1)} \right\|_{\infty} \right).$$

Hence

$$\begin{aligned} \max_{\xi \in I} \left| \int_a^b (\mathcal{I} - \sigma_{n,m}) \otimes (\mathcal{I} - \sigma_{n,m})\Phi(\xi, \tau)u(\tau)d\tau \right| &\leq C_2 h_p^{m+1} \left(\left\| (\mathcal{I} - \sigma_{n,m})K^{(0,m)} \right\|_{\infty} \|u^1\|_{\infty} \right. \\ &\quad \left. + \left\| (\mathcal{I} - \sigma_{n,m})\Phi^{(0,m+1)} \right\|_{\infty} \right) \\ &\leq C_2 h_p^{m+1} \left(C_1 h_p^m \left\| \Phi^{(m,m)} \right\|_{\infty} \|u^1\|_{\infty} \right. \\ &\quad \left. + C_1 h_p^m \left\| \Phi^{(m,m+1)} \right\|_{\infty} \right) \\ &= C_3 h_p^{2m+1} \left(\left\| \Phi^{(m,m)} \right\|_{\infty} \|u^1\|_{\infty} + \left\| \Phi^{(m,m+1)} \right\|_{\infty} \right), \end{aligned}$$

and hence

$$\left\| \Phi - \Phi_{n,m} \right\|_{\infty} = \begin{cases} O(h_p^m), & \text{in the TB case,} \\ O(h_p^{2m}), & \text{in the CB one.} \end{cases}$$

Since Φ and following [1], it follows that the operator $(I - h\Phi_{n,m})$ are invertible for n large enough, and their inverses are uniformly bounded with respect to n . Moreover,

$$\left\| (I - h\Phi_{n,m})^{-1} \right\|_{\infty} \leq \frac{\left\| (I - h\Phi)^{-1} \right\|_{\infty}}{1 - \left\| (I - h\Phi)^{-1} \right\|_{\infty} \left\| \Phi - \Phi_{n,m} \right\|_{\infty}}, \quad n \geq N_0.$$

Choose N_0 so that

$$\left\| \Phi - \Phi_{n,m} \right\|_{\infty} \leq \frac{1}{2 \left\| (I - h\Phi)^{-1} \right\|_{\infty}}, \quad n \geq N_0.$$

This implies that

$$\left\| (I - h\Phi_{n,m})^{-1} \right\|_{\infty} \leq 2 \left\| (I - h\Phi)^{-1} \right\|_{\infty}.$$

On the other hand,

$$\|u - u_{n,m}\|_{\infty} \leq \left\| (I - h\Phi_{n,m})^{-1} \right\|_{\infty} \|h\|_{\infty} \left\| \Phi - \Phi_{n,m} \right\|_{\infty} \|u\|_{\infty}.$$

Thus,

$$\|u - u_{n,m}\|_{\infty} \leq \begin{cases} 2Ch_p^m \left\| (I - h\Phi)^{-1} \right\|_{\infty} \|h\|_{\infty} \|u\|_{\infty}, & \text{in the TP case,} \\ 2Ch_p^{2m} \left\| (I - h\Phi)^{-1} \right\|_{\infty} \|h\|_{\infty} \|u\|_{\infty}, & \text{in the CB one.} \end{cases}$$

□

2.4 Numerical examples

2.4.1 Example 1

Consider the integral equation

$$u(\xi) \int_0^1 e^{\xi-\tau} d\tau + \int_0^1 e^{\tau-\xi} u(\tau) d\tau = \frac{e^\xi}{2} + (e-1)e^{\xi-1} \xi^2 \cos(\xi). \quad (2.10)$$

Or equivalently,

$$u(\xi) = g(x) + h(\xi) \int_0^1 \Phi(\xi, \tau) u(\tau) d\tau, \quad (2.11)$$

where,

$$h(\xi) = \frac{1}{(-1+e)e^{-1+\xi}}, \quad \Phi(\xi, \tau) = -e^{\tau-\xi} \quad \text{and} \quad g(\xi) = \frac{e^{1-2\xi}}{2(-1+e)} + \xi^2 \cos(\xi).$$

The exact solution is $u(\xi) = \xi^2 \cos(\xi) \in W_r, r \geq 1$. We compute the approximate solution using the continuous blending approach. We get

$$\Upsilon_{1_3} = \begin{pmatrix} \Upsilon_{1_3}^{1,1} & \Upsilon_{1_3}^{1,2} & \Upsilon_{1_3}^{1,3} & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \Upsilon_{1_3}^{2,1} & \Upsilon_{1_3}^{2,2} & \Upsilon_{1_3}^{2,3} & \Upsilon_{1_3}^{2,4} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \Upsilon_{1_3}^{3,1} & \Upsilon_{1_3}^{3,2} & \Upsilon_{1_3}^{3,3} & \Upsilon_{1_3}^{3,4} & \Upsilon_{1_3}^{3,5} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \Upsilon_{1_3}^{4,2} & \Upsilon_{1_3}^{4,3} & \Upsilon_{1_3}^{4,4} & \Upsilon_{1_3}^{4,5} & \Upsilon_{1_3}^{4,6} & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 & \Upsilon_{1_3}^{n+1,n-1} & \Upsilon_{1_3}^{n+1,n} & \Upsilon_{1_3}^{n+1,n+1} & \Upsilon_{1_3}^{n+1,n+2} \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \Upsilon_{1_3}^{n+2,n} & \Upsilon_{1_3}^{n+2,n+1} & \Upsilon_{1_3}^{n+2,n+2} \end{pmatrix},$$

where,

$$\begin{aligned} \Upsilon_{1_3}^{1,1} &= \frac{-98304e^{7/8} + 86753e}{-1+e}, \\ \Upsilon_{1_3}^{1,2} &= \frac{-16(-9404e^{7/8} + 8299e)}{-1+e}, \\ \Upsilon_{1_3}^{1,3} &= \frac{64(-817e^{7/8} + 721e)}{-1+e}, \\ \Upsilon_{1_3}^{2,1} &= \frac{-16(-9404e^{7/8} + 8299e)}{-1+e}, \end{aligned}$$

$$\begin{aligned} \Upsilon_{1_3}^{2,2} &= \frac{256(-96e^{3/4} - 815e^{7/8} + 794e)}{-1+e}, \\ \Upsilon_{1_3}^{2,3} &= \frac{-32(-1583e^{3/4} - 1105e^{7/8} + 2208e)}{-1+e}, \\ \Upsilon_{1_3}^{2,4} &= \frac{32(-817 + 721e^{1/8})e^{3/4}}{-1+e}, \\ \Upsilon_{1_3}^{3,1} &= \frac{64(-817e^{7/8} + 721e)}{-1+e}, \end{aligned}$$

$$\begin{aligned}
\Upsilon_{13}^{3,2} &= \frac{-32(-1583e^{3/4} - 1105e^{7/8} + 2208e)}{-1 + e}, \\
\Upsilon_{13}^{3,3} &= \frac{192(-128e^{5/8} - 431e^{3/4} + 335e^{7/8} + 128e)}{-1 + e}, \\
\Upsilon_{13}^{3,4} &= \frac{-32(-1583e^{5/8} - 288e^{3/4} + 1487e^{7/8})}{-1 + e}, \\
\Upsilon_{13}^{3,5} &= \frac{32(-817 + 721e^{1/8})e^{5/8}}{-1 + e}, \\
\Upsilon_{13}^{4,2} &= \frac{32(-817e^{3/4} + 721e^{7/8})}{-1 + e}, \\
\\
\Upsilon_{13}^{4,3} &= \frac{-32(-1583e^{5/8} - 288e^{3/4} + 1487e^{7/8})}{-1 + e}, \\
\Upsilon_{13}^{4,4} &= \frac{192(-128\sqrt{e} - 431e^{5/8} + 335e^{3/4} + 128e^{7/8})}{-1 + e}, \\
\Upsilon_{13}^{4,5} &= \frac{-32(-1583\sqrt{e} - 288e^{5/8} + 1487e^{3/4})}{-1 + e}, \\
\Upsilon_{13}^{4,6} &= \frac{32(-817\sqrt{e} + 721e^{5/8})}{-1 + e}, \\
\Upsilon_{13}^{n-1,n-3} &= \frac{32(-817e^{3/8} + 721\sqrt{e})}{-1 + e}, \\
\\
\Upsilon_{13}^{n-1,n-2} &= \frac{-32(-1583e^{1/4} - 288e^{3/8} + 1487\sqrt{e})}{-1 + e}, \\
\Upsilon_{13}^{n-1,n-1} &= \frac{192(-128e^{1/8} - 431e^{1/4} + 335e^{3/8} + 128\sqrt{e})}{-1 + e}, \\
\Upsilon_{13}^{n-1,n} &= \frac{-32(-1583e^{1/8} - 288e^{1/4} + 1487e^{3/8})}{-1 + e}, \\
\Upsilon_{13}^{n-1,n+1} &= \frac{32(-817e^{1/8} + 721e^{1/4})}{-1 + e}, \\
\Upsilon_{13}^{n,n-2} &= \frac{32(-817e^{3/8} + 721\sqrt{e})}{-1 + e}, \\
\\
\Upsilon_{13}^{n,n-1} &= \frac{-32(-1583e^{1/8} - 288e^{1/4} + 1487e^{3/8})}{-1 + e}, \\
\Upsilon_{13}^{n,n} &= \frac{192(-128 - 431e^{1/8} + 335e^{1/4} + 128e^{3/8})}{-1 + e}, \\
\Upsilon_{13}^{n,n+1} &= \frac{-32(-2400 + 433e^{1/8} + 1487e^{1/4})}{-1 + e}, \\
\Upsilon_{13}^{n,n+2} &= \frac{64(-817 + 721e^{1/8})}{-1 + e}, \\
\Upsilon_{13}^{n+1,n-1} &= \frac{32(-817 + 721e^{1/8})e^{1/8}}{-1 + e}, \\
\Upsilon_{13}^{n+1,n} &= \frac{-32(-2400 - 433e^{1/8} + 1487e^{1/4})}{-1 + e},
\end{aligned}$$

$$\begin{aligned}
\Upsilon_{13}^{n+1,n+1} &= \frac{256(-938 + 719e^{1/8} + 96e^{1/4})}{-1 + e}, \\
\Upsilon_{13}^{n+1,n+2} &= \frac{-16(-10221 + 9020e^{1/8})}{-1 + e}, \\
\Upsilon_{13}^{n+2,n} &= \frac{64(-817 + 721e^{1/8})}{-1 + e}, \\
\Upsilon_{13}^{n+2,n+1} &= \frac{-16(-10221 + 9020e^{1/8})}{-1 + e}, \\
\Upsilon_{13}^{n+2,n+2} &= \frac{3(-37131 + 32768e^{1/8})}{-1 + e}.
\end{aligned}$$

n	$\ u - u_{3,n}\ _\infty$
8	1.2e-10
16	1e-12
32	7.02e-13
64	8.5e-15
128	4.44e-16

Table 2.1: *Example 1*

2.4.2 Example 2

Consider the integral equation

$$u(\xi) \int_0^1 e^{\xi-\tau} d\tau + \int_0^1 e^{\tau-\xi} u(\tau) d\tau = \frac{1}{2} e^{-1-\xi} (e + e^2(-\cos(1) + \sin(1)) + 2(-1 + e)e^{2\xi} \sin(\xi)), \quad (2.12)$$

or equivalently,

$$u(\xi) = g(\xi) + h(\xi) \int_0^1 \Phi(\xi, \tau) u(\tau) d\tau. \quad (2.13)$$

where,

$$h(\xi) = \frac{1}{(-1 + e)e^{-1+\xi}}, \quad \Phi(\xi, \tau) = -e^{\tau-\xi} \quad \text{and} \quad g(\xi) = \frac{e^{1-2\xi}(1 + e(-\cos(1) + \sin(1)))}{2(-1 + e)} + \sin(\xi).$$

The exact solution is $u(\xi) = \sin(\xi) \in W_r, r \geq 1$. We compute the approximate solution using the continuous blending method, we get

n	$\ u - u_{3,n}\ _\infty$
8	1.3e-10
16	1.35e-12
32	1.02e-15
64	1.5e-15
128	3.5e-16

Table 2.2: *Example 2*

2.4.3 Example 3

Consider the integral equation

$$u(\xi) \int_0^1 e^{\pi(\xi-\tau)} d\tau + \int_0^1 e^{\pi(\tau-\xi)} u(\tau) d\tau = \frac{e^{\pi(-1+\xi)}(-1+e^\pi)(e^\xi + \frac{e^{\pi-2\pi\xi}(-1+e^{1+\pi})\pi}{(-1+e^\pi)(1+\pi)})}{\pi}. \quad (2.14)$$

Or equivalently,

$$u(\xi) = g(\xi) + h(\xi) \int_0^1 \Phi(\xi, \tau) u(\tau) d\tau, \quad (2.15)$$

where,

$$h(\xi) = \frac{\pi}{(-1+e^\pi)(-1+\xi)e^\pi}, \quad \Phi(\xi, \tau) = -e^{\pi(\tau-\xi)} \quad \text{and} \quad g(\xi) = e^\xi + \frac{e^{\pi-2\pi\xi}(-1+e^{1+\pi})\pi}{(-1+e^\pi)(1+\pi)}.$$

The exact solution is $u(\xi) = e^\xi \in W_r, r \geq 1$. We compute the approximate solution using the continuous blending method, we obtain

$$\Upsilon_{3_3} = \begin{pmatrix} \Upsilon_{3_3}^{1,1} & \Upsilon_{3_3}^{1,2} & \Upsilon_{3_3}^{1,3} & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \Upsilon_{3_3}^{2,1} & \Upsilon_{3_3}^{2,2} & \Upsilon_{3_3}^{2,3} & \Upsilon_{3_3}^{2,4} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \Upsilon_{3_3}^{3,1} & \Upsilon_{3_3}^{3,2} & \Upsilon_{3_3}^{3,3} & \Upsilon_{3_3}^{3,4} & \Upsilon_{3_3}^{3,5} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \Upsilon_{3_3}^{4,2} & \Upsilon_{3_3}^{4,3} & \Upsilon_{3_3}^{4,4} & \Upsilon_{3_3}^{4,5} & \Upsilon_{3_3}^{4,6} & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \ddots \\ 0 & 0 & \cdots & \cdots & 0 & 0 & \Upsilon_{3_3}^{n+1,n-1} & \Upsilon_{3_3}^{n+1,n} & \Upsilon_{3_3}^{n+1,n+1} & \Upsilon_{3_3}^{n+1,n+2} \\ 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \Upsilon_{3_3}^{n+2,n} & \Upsilon_{3_3}^{n+2,n+1} & \Upsilon_{3_3}^{n+2,n+2} \end{pmatrix},$$

where,

$$\begin{aligned} \Upsilon_{3_3}^{1,1} &= \frac{e^{7/8}(-98304 + 98304e^{\pi/8} - 12288e^{\pi/8}\pi + 768e^{\pi/8}\pi^2 - 32e^{\pi/8}\pi^3 + e^{\pi/8}\pi^4)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{1,2} &= \frac{16e^{7/8}(9216 - 9216e^{\pi/8} + 192\pi + 960e^{\pi/8}\pi - 4\pi^2 - 44e^{\pi/8}\pi^2 + e^{\pi/8}\pi^3)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{1,3} &= \frac{64e^{7/8}(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{2,1} &= \frac{16e^{7/8}(9216 - 9216e^{\pi/8} + 192\pi + 960e^{\pi/8}\pi - 4\pi^2 - 44e^{\pi/8}\pi^2 + e^{\pi/8}\pi^3)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{2,2} &= \frac{256e^{3/4}(-96 - 768e^{\pi/8} + 864e^{\pi/4} - 48e^{\pi/8}\pi - 72e^{\pi/4}\pi + e^{\pi/8}\pi^2 + 2e^{\pi/4}\pi^2)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{2,3} &= \frac{32e^{3\pi/4}(1536 + 768e^{\pi/8} - 2304e^{\pi/4} + 48\pi + 336e^{\pi/8}\pi + 96e^{\pi/4}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{2,4} &= \frac{32e^{3\pi/4}(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{3,1} &= \frac{64e^{7/8}(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{3,2} &= \frac{32e^{3\pi/4}(1536 + 768e^{\pi/8} - 2304e^{\pi/4} + 48\pi + 336e^{\pi/8}\pi + 96e^{\pi/4}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{3,3} &= \frac{192e^{5\pi/8}(-128 - 384e^{\pi/8} + 384e^{\pi/4} + 128e^{3\pi/8} - 48e^{\pi/8}\pi - 48e^{\pi/4}\pi + e^{\pi/8}\pi^2 - e^{\pi/4}\pi^2)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{3,4} &= \frac{32e^{5\pi/8}(1536 - 1536e^{\pi/4} + 48\pi + 288e^{\pi/8}\pi + 48e^{\pi/4}\pi - \pi^2 + e^{\pi/4}\pi^2)}{(-1+e^\pi)\pi^4}, \\ \Upsilon_{3_3}^{3,5} &= \frac{32e^{5\pi/8}(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1+e^\pi)\pi^4} \end{aligned}$$

$$\begin{aligned}
\Upsilon_{3_3}^{4,2} &= \frac{32e^{3\pi/4}(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{4,3} &= \frac{32e^{5\pi/8}(1536 - 1536e^{\pi/4} + 48\pi + 288e^{\pi/8}\pi + 48e^{\pi/4}\pi - \pi^2 + e^{\pi/4}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{4,4} &= \frac{192e^{5\pi/8}(-128 - 384e^{\pi/8} + 384e^{\pi/4} + 128e^{3\pi/8} - 48e^{\pi/8}\pi - 48e^{\pi/4}\pi + e^{\pi/8}\pi^2 - e^{\pi/4}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{4,5} &= \frac{32e^{\pi/2}(1536 - 1536e^{\pi/4} + 48\pi + 288e^{\pi/8}\pi + 48e^{\pi/4}\pi - \pi^2 + e^{\pi/4}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{4,6} &= \frac{32e^{\pi/2}(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n,n-2} &= \frac{32e^{\pi/4}(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n,n-1} &= \frac{32e^{\pi/8}(1536 - 1536e^{\pi/4} + 48\pi + 288e^{\pi/8}\pi + 48e^{\pi/4}\pi - \pi^2 + e^{\pi/4}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n,n} &= \frac{192(-128 - 384e^{\pi/8} + 384e^{\pi/4} + 128e^{3\pi/8} - 48e^{\pi/8}\pi - 48e^{\pi/4}\pi + e^{\pi/8}\pi^2 - e^{\pi/4}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n,n+1} &= \frac{32(2304 - 768e^{\pi/8} - 1536e^{\pi/4} + 96\pi + 336e^{\pi/8}\pi + 48e^{\pi/4}\pi - e^{\pi/8}\pi^2 + e^{\pi/4}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n,n+2} &= \frac{64(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n+1,n-2} &= \frac{32(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n+1,n-1} &= \frac{32(2304 - 768e^{\pi/8} - 1536e^{\pi/4} + 96\pi + 336e^{\pi/8}\pi + 48e^{\pi/4}\pi - e^{\pi/8}\pi^2 + e^{\pi/4}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n+1,n} &= \frac{256(-864 + 768e^{\pi/8} + 96e^{\pi/4} - 72\pi - 48e^{\pi/8}\pi - 2\pi^2 - e^{\pi/8}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n+1,n+1} &= \frac{256(-864 + 768e^{\pi/8} + 96e^{\pi/4} - 72\pi - 48e^{\pi/8}\pi - 2\pi^2 - e^{\pi/8}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n+1,n+2} &= \frac{16(9216 - 9216e^{\pi/8} + 960\pi + 192e^{\pi/8}\pi + 44e^{\pi/8}\pi^2 + \pi^3)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n+2,n} &= \frac{64(-768 + 768e^{\pi/8} - 48\pi - 48e^{\pi/8}\pi - \pi^2 + e^{\pi/8}\pi^2)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n+2,n+1} &= \frac{16(9216 - 9216e^{\pi/8} + 960\pi + 192e^{\pi/8}\pi + 44e^{\pi/8}\pi^2 + \pi^3)}{(-1 + e^\pi)\pi^4}, \\
\Upsilon_{3_3}^{n+2,n+2} &= \frac{-98304 + 98304e^{\pi/8} - 12288\pi - 768\pi^2 - 32\pi^3 - \pi^4}{(-1 + e^\pi)\pi^4}
\end{aligned}$$

n	$\ u - u_{3,n}\ _\infty$
8	4e-7
16	1e-8
32	8e-12
64	5.1e-14
128	4.01e-16

Table 2.3: *Example 3*

2.4.4 Example 4

Consider the integral equation

$$u(\xi) \int_0^1 e^{\pi(\xi-\tau)} d\tau + \int_0^1 e^{\pi(\tau-\xi)} u(\tau) d\tau = \frac{e^{-\pi\xi}(6 + e^{1+\pi}(-2 + 3\pi + \pi^3))}{(1 + \pi)^4} + e^\xi \xi^3. \quad (2.16)$$

Or equivalently,

$$u(\xi) + h(\xi) \int_0^1 \Phi(\xi, \tau) u(\tau) d\tau = g(\xi), \quad (2.17)$$

where,

$$h(\xi) = \frac{\pi}{(-1 + e^\pi)(-1 + \xi)e^\pi}, \quad \Phi(\xi, \tau) = -e^{\pi(\tau-\xi)},$$

and

$$g(\xi) = \frac{e^{\pi-2\pi\xi}\pi(6 + e^{1+\pi}(-2 + 3\pi + \pi^3))}{(-1 + e^\pi)(1 + \pi)^4} + e^\xi \xi^3.$$

The exact solution is $u(\xi) = \xi^3 e^\xi \in W_r, r \geq 1$.

n	$\ u - u_{3,n}\ _\infty$
8	3e-6
16	1.2e-7
32	8e-11
64	5.1e-14
128	4.9e-14

Table 2.4: *Example 4*

Chapter 3

Two schemes of spline quasi-interpolants to system of Fredholm integral equations

3.1 Introduction

Quadratic spline quasi-interpolants subject to bounded domains were studied recently by Sablonnière and show how they can be used in various approximation theory areas (see [19]). In the same direction, another paper uses multivariate and univariate quadratic spline quasi-interpolants to derive several essential approximation formulas for approximated functions' derivatives. It makes use of these operators' superconvergence features to obtain exceedingly precise derivatives of approximated functions at certain locations (cf. [20]). In [20], the authors show how to use three separate variables in a quasi-interpolation approach for quadratic piecewise polynomials to enable more successful reconstruction and display of gridded volume data. The authors use local averaging to obtain the Bernstein-B'ezier coefficients of the splines from the available data. The authors of [7] considered the Fredholm second-kind integral equation of the form:

$$u(\xi) = g(\xi) + \int_a^b \Phi(\xi, \tau)u(\tau)d\tau, \quad a \leq \xi \leq b. \quad (3.1)$$

They used two alternative spline quasi-interpolation approaches to approximate the kernel $\Phi(., .)$: the continuous blending approximation based on the sum of two univariate spline quasi-interpolants tensor product one. Furthermore, the approximation operators in question are not projectors. The authors' tensor product approximation is unique, resulting in a more accessible and less expensive computation. The continuous blending approximation's performance reviews are also crucial.

Motivated by the above considerations, the goal of this work is to consider on the space of all continuous functions $\mathcal{X} := C^0([a, b], \mathbb{R})$, equipped with the uniform-norm $\|\cdot\|_\infty$, the system of linear Fredholm integral equations of the form

$$u_i(\xi) = g_i(\xi) + \sum_{j=1}^m \int_a^b \Phi_{ij}(\xi, \tau)u_j(\tau)d\tau, \quad 1 \leq i \leq m, \quad (3.2)$$

where $g_i \in \mathcal{X}$, $i = 1 \cdots m$, $\Phi_{ij}(., .) \in \mathcal{X} \times \mathcal{X}$, $i, j = 1 \cdots m$ are known functions and u_i , $i = 1 \cdots m$ are the unknown functions to be determined.

The request coefficients in these approximate formulas solve a specific linear equations systems. In addition, when the system (3.2) is combined with two degenerate integral equations systems, we get two degenerate integral equations systems. We approach the kernels of the given integral equations system via two methods. First, we develop the tensor product, obtain new results, and prove that the order of convergence for this method is $O(h^l)$, where h is the mesh length of the partition, and l is the order of the spline. Then, by the sum of two univariate spline quasi-interpolants, we apply continuous blending to approximate the solution of the present problem. Also, we analyze the convergence of this second approximation method, and we prove that the order of the convergence is $O(h^{2l})$ when l is even and $O(h^{2l+1})$ where l is odd. The matrix systems are different from these obtained in [7].

3.2 Discrete quasi-interpolant recall

Let us consider the nodes $\xi_0, \xi_1, x_2, \dots, \xi_n$ in the interval $\mathcal{I} := [a, b]$ with

$$\xi_\eta = a + kh_\eta \quad \text{and} \quad h = \xi_\eta - \xi_{\eta-1} \quad \text{for all} \quad 1 \leq \eta \leq n.$$

Let $T_n := \{\tau_\eta, 0 \leq \eta \leq n\}$ denote the uniform partition of the interval \mathcal{I} into n subintervals with mesh length h .

Define $\mathcal{T}_m := S_n(\mathcal{I}, \mathcal{S}_n)$ to be the space of splines of class C^{m-1} on this partition. Consider the set $\mathcal{J}_m := \{0, 1, \dots, n+m-2\}$. the support of $\varphi_{j,m}$ is the interval $[\tau_{j-m+1}, \tau_{j+1}]$, with multiple knots at the endpoints, i.e. $a = \tau_0 = \tau_{-1} = \dots = \tau_{-m+1}$ and $b = \tau_n = \tau_{n+1} = \dots = \tau_{n+m-2}$.

We recall that the space $\mathcal{S}_m(T_n)$ of splines of order m on T_n admits a basis $\varphi_{j,m}, j \in \mathcal{J}_m$ composed by $n+m-1$ B-splines. Moreover, the representation of monomials using symmetric functions of interior knots $\mathcal{N}_j := \{\tau_{j-m+2}, \dots, \tau_j\}$ in supp $\varphi_{j,m}$, defined by

$$\sigma_0(N_j) := 1 \quad \text{and} \quad \sigma_{r'}(\mathcal{N}_j) := \sum_{1 \leq \eta_1 < \dots < \eta_{r'} \leq m-1} \tau_{j+1-\eta_1} \cdots \tau_{j+1-\eta_{r'}}, \quad \text{for } 1 \leq r' \leq m-1.$$

Abbreviating by PQIOs, the discrete spline quasi-interpolant operator

$$\lambda f = \sum_{k \in \mathcal{J}_n} \gamma_k(g) \varphi_k,$$

whose coefficients are linear combinations of discrete values of g on the set of data points xi_n .

For $0 \leq r' \leq m-1$, we have the Mersden identity

$$\gamma_{r'}(\tau) := \tau^{r'} = \sum_{j \in \mathcal{J}_m} \theta_{j,m}^{(r')} \varphi_{j,m}(\tau),$$

where

$$\theta_{j,m}^{(r')} := \left(\binom{m-1}{r'} \right)^{-1} \sigma_{r'}(N_j).$$

Letting

$$s_0 := a, \quad s_{n+1} := b, \quad s_j := \frac{1}{2}(\tau_{j-1} + \tau_j), \quad 1 \leq j \leq n$$

and

$$\mathcal{S}_n := \{s_j, 0 \leq j \leq n+1\}.$$

Through the chapter, we shall investigate to the following pQIOs

$$\sigma_{m,n}(g) := \sum_{j \in \mathcal{J}_m} \gamma_{j,m}(g) \varphi_{j,m}, \quad m \geq 3. \quad (3.3)$$

We recall that the coefficients $\gamma_{j,m}(g)$ are linear combinations of values of g on the set T_n for m even or the set \mathcal{S}_n for m odd

Let us consider the space P_{m-1} of polynomials up to degree $m-1$. Letting

$$\begin{aligned} g_i(\mathcal{S}_n) &:= \{g_{ij} := g_i(s_j), 0 \leq j \leq n+1\}, \quad \text{for } m \text{ odd,} \\ g_i(\mathcal{T}_n) &:= \{g_{ij} := g_i(\tau_j), 0 \leq j \leq n\}, \quad \text{for } m \text{ even.} \end{aligned}$$

and these equalities are satisfied if and only if

$$\gamma_{j,m}(c_{r'}) = \theta_{j,m}^{(r')}, \quad j \in \mathcal{J}_m, \quad 0 \leq r' \leq m-1. \quad (3.4)$$

For $m-1 \leq j \leq n-1$, we choose

$$\gamma_{j,m}(g_i) = \begin{cases} \sum_{s=0}^{m-1} \alpha_{j,s}^{(m)} g_{i,j-m+i+2}, & \text{for } m \text{ odd,} \\ \sum_{s=0}^{m-2} \alpha_{j,s}^{(m)} g_{i,j-m+i+2}, & \text{for } m \text{ even.} \end{cases} \quad (3.5)$$

We note that the pQIO $\sigma_{m,n}$ is constructed to be exact on P_{m-1} , that is

$$\sigma_{m,n}c_{r'} = c_{r'}, \quad 0 \leq r' \leq m-1.$$

That is to say

$$\sum_{j \in J_m} \gamma_{j,m}(c_{r'})\varphi_{j,m} = \sum_{j \in J_m} \theta_{j,m}^{(r')}\varphi_{j,m}(\tau), \quad 0 \leq r' \leq m-1.$$

For m odd, we use as functionals associated with boundary B-splines

$$\gamma_{j,m}(g_i) = \begin{cases} \sum_{s=0}^{m-1} \alpha_{j,s}^{(m)} g_{i,s}, & \text{for } 0 \leq j \leq m-2, \\ \sum_{s=0}^{m-1} \alpha_{j,s}^{(m)} g_{i,n-m+i+2}, & \text{for } n \leq j \leq n+m-2. \end{cases} \quad (3.6)$$

For otherwise, we have

$$\gamma_{j,m}(g_i) = \begin{cases} \sum_{s=0}^{m-1} \alpha_{j,s}^{(m)} g_{i,s}, & \text{for } 0 \leq j \leq m-3, \\ \sum_{s=0}^{m-1} \alpha_{j,s}^{(m)} g_{i,n-m+i+1}, & \text{for } n+1 \leq j \leq n+m-2, \end{cases} \quad (3.7)$$

We recall that for any function f with a bounded m -th derivative in \mathcal{I} ,

$$\|g - \sigma_{m,n}f\|_{\infty, \mathcal{I}} = O(h^m).$$

In view of the importance of pQIOs, we recall some useful expression of cubic and quadratic quadratic spline quasi-interpolant.

The quadratic QI of a given function f is given by

$$\sigma_{3,n}(g) := \sum_{j=0}^{n+1} \gamma_{j,3}(g)\varphi_{j,3},$$

where the coefficients are defined by

$$\gamma_{j,3}(g) := \frac{1}{8}(-g_{j-1} + 10g_j - g_{j+1}), \quad 2 \leq j \leq n-1.$$

For $j = 0, 1, n, n+1$, we have

$$\begin{aligned} \gamma_{0,3}(g) &:= g_0, & \gamma_{1,3}(g) &:= -\frac{1}{3}g_0 + \frac{3}{2}g_1 - \frac{1}{6}g_2, \\ \gamma_{n+1,3}(g) &:= g_{n+1}, & \gamma_{n,3}(g) &:= -\frac{1}{6}g_{n-1} + \frac{3}{2}g_n - \frac{1}{3}g_{n+1}. \end{aligned}$$

The QI $\sigma_{3,n}(g)$ can be described as

$$\sigma_{3,n}(g) := \sum_{j=0}^{n+1} g(s_j)L_{j,3},$$

for some the specific linear functions $L_{j,3}$ which are combinations of the B-splines. C^2 cubic spline quasi-interpolant

The cubic QI is defined by

$$\sigma_{4,n}(g) := \sum_{j=0}^{n+2} \gamma_{j,4}(g)\varphi_{j,4},$$

where the coefficients are defined by

$$\gamma_{j,4}(g) := \frac{1}{6}(-g_{j-2} + 8g_{j-1} - g_j), \quad 2 \leq j \leq n.$$

Moreover,

$$\begin{aligned}\gamma_{0,4}(g) &:= g_0, & \gamma_{1,4}(g) &:= \frac{7}{18}g_0 + g_1 - \frac{1}{2}g_2 + \frac{1}{9}g_3, \\ \gamma_{n+2,4}(g) &:= g_n, & \gamma_{n+1,4}(g) &:= \frac{7}{18}g_n + g_{n-1} - \frac{1}{2}g_{n-2} + \frac{1}{9}g_{n-3}.\end{aligned}$$

Finally, $\sigma_{4,n}$ reads as

$$\sigma_{4,n}(g) := \sum_{j=0}^n g(\tau_j) L_{j,4},$$

for some the specific linear functions $L_{j,4}$ which are combinations of the B-splines.

3.3 Tensor product approximation

The first aim of this chapter is to approximate the kernels φ_{ij} by the tensor product (TP) of two univariate PQIOs σ_1 and σ_2 in the variables ξ and τ , i.e.

$$\Phi_{ij}(\xi, \tau) \approx (\sigma_1 \oplus \sigma_2)\Phi_{ij}(\xi, \tau) = \sum_{s,r \in J} \Phi_{ij}(s, r)\varphi_s(\xi)\varphi_r(\tau). \quad (3.8)$$

This first approach conduce to a degenerate kernel in the space $\mathcal{S}(\mathcal{T}_n)$, which is used to calculate the approximate solution by solving a system of equations. In addition, the coefficients $K_{ij}(s, r)$ are appropriate linear combinations of values $\Phi_{ij}(\theta_s, \theta_r)$, for $(s, r) \in \mathcal{J} \times \mathcal{J}$.

The second aim of this chapter is to approximate the kernels Φ_{ij} by the continuous blending (CB) sum of the two univariate pQIOs σ_1 and σ_2 i.e.

$$\begin{aligned}\Phi_{ij}(\xi, \tau) &\approx (\sigma_1 \otimes \sigma_2)\Phi_{ij}(\xi, \tau) \\ &= (\sigma_1 \otimes \mathcal{I} + \mathcal{I} \otimes \sigma_2 - \sigma_1 \otimes \sigma_2)\Phi_{ij}(\xi, \tau) \\ &= \sum_{s \in J} \widehat{\Phi}_{ij}^s(\tau)\varphi_s(\xi) + \sum_{r \in J} \overline{\Phi}_{ij}^r(\xi)\varphi_r(\tau) - \sum_{s,r \in J} \Phi_{ij}(s, r)\varphi_s(\xi)\varphi_r(\tau),\end{aligned}$$

where

$$\widehat{\varphi}_{ij_s}(\tau) := \gamma_s(\Phi_{ij}(\cdot, t)), \quad \overline{\Phi}_{ij_r}(\xi) := \gamma_r(\Phi_{ij}(\xi, \cdot)), \quad i, j = 1 \cdots m',$$

and \mathcal{I} denotes the identity operator. We will show that the corresponding solutions read as follows:

1. In the TP case,

$$u_i^{TP}(\xi) = g_i(\xi) + \sum_{s \in J} \varphi_s(\xi)\alpha_{i_s}, \quad i = 1 \cdots m'.$$

2. In the CB case,

$$u_i^{CB}(\xi) = g_i(\xi) + \sum_{s \in J} \alpha_{i_s}\varphi_s(\xi) + \sum_{j=1}^{m'} \sum_{r \in J} \beta_{j_r}\overline{\Phi}_{ij_r}(\xi), \quad i = 1 \cdots m'.$$

3.3.1 Quasi-interpolation to approximate the kernels

The purpose of this section is to demonstrate how to use the tensor product approach. First, we approximate the kernels Φ_{ij} $i, j = 1, 2, \dots, m'$ using quasi-interpolation. The approximate solution in the space $S_m(\mathcal{T}_n)$ is then calculated using the obtained degenerate kernels and system of equations.

Letting

$$\begin{aligned}\Phi_{ij}^{m,n}(\xi, \tau) &:= (\sigma_{m,n} \otimes \sigma_{m,n})\Phi_{ij}(\xi, \tau), \\ \varphi_m(\tau) &:= (\varphi_{0,m}(\tau), \dots, \varphi_{n+m-2,m}(\tau)),\end{aligned}$$

and

$$\gamma_m := (\gamma_{0,m}, \gamma_{1,m}, \dots, \gamma_{n+m-2,m})^T.$$

Hence,

$$\begin{aligned}\Phi_{ij}^{m,n}(\xi, \tau) &:= \sigma_{m,n}(\sigma_{m,n}(\Phi_{ij}(\xi, \cdot))(\tau)) \\ &= \sigma_{m,n} \left(\sum_{s \in J_m} \gamma_{s,m}(\Phi_{ij}(\xi, \cdot)) \varphi_{s,m}(\tau) \right) \\ &= \sigma_{m,n}(\varphi_m(t) \gamma_m(\Phi_{ij}(\xi, \cdot))).\end{aligned}$$

Let us defined the matrix P_m as follows

$$P_m(s, r) = \begin{cases} \alpha_{s,r}^m, & \text{for } 0 \leq s, r \leq n + m - 2, & \text{if } m \text{ is odd,} \\ \alpha_{s,r}^m, & \text{for } 0 \leq s \leq n + m - 2, \quad 0 \leq r \leq n, & \text{if } m \text{ is even,} \end{cases}$$

and letting

$$\theta_{r,m} = \begin{cases} s_r, & \text{for } 0 \leq r \leq m' = n + m - 2, & \text{if } m \text{ is odd,} \\ \tau_r, & \text{for } 0 \leq r \leq m' = n, & \text{if } \ell \text{ is even,} \end{cases}$$

we get

$$\gamma_{s,m}(\Phi_{ij}(\xi, \cdot)) = \sum_{r=0}^{n+m-2} P_m(s, r) \varphi_{ij}(\xi, \theta_{r,m}), \quad 0 \leq s \leq n + m - 2.$$

Therefore,

$$\begin{aligned}\Phi_{ij}^{m,n}(\xi, \tau) &= \varphi_m(\tau) P_m \begin{pmatrix} \sigma_{m,n}(\Phi_{ij}(\cdot, \theta_{0,m}))(\xi) \\ \sigma_{m,n}(\Phi_{ij}(\cdot, \theta_{1,m}))(\xi) \\ \vdots \\ \sigma_{m,n}(\Phi_{ij}(\cdot, \theta_{m',m}))(\xi) \end{pmatrix} \\ &= \varphi_{m'}(t) P_m \begin{pmatrix} \sum_{s \in J_m} \gamma_{s,m}(\Phi_{ij}(\cdot, \theta_{0,m})) \varphi_{s,m}(\xi) \\ \sum_{s \in J_m} \gamma_{s,m}(\Phi_{ij}(\cdot, \theta_{1,m})) \varphi_{s,m}(\xi) \\ \vdots \\ \sum_{s \in J_m} \gamma_{s,m}(\Phi_{ij}(\cdot, \theta_{m',m})) \varphi_{s,m}(\xi) \end{pmatrix} \\ &= \varphi_m(\tau) \Phi_{ij}^m \varphi_m(\xi)^T\end{aligned}$$

where

$$\Phi_{ij}^m := P_m \begin{pmatrix} \Phi_{ij}(\theta_{0,m}, \theta_{0,m}) & \Phi_{ij}(\theta_{1,m}, \theta_{0,m}) & \cdots & \Phi_{ij}(\theta_{m',m}, \theta_{0,m}) \\ \Phi_{ij}(\theta_{0,m}, \theta_{1,m}) & \Phi_{ij}(\theta_{1,m}, \theta_{1,m}) & \cdots & \Phi_{ij}(\theta_{m',m}, \theta_{1,m}) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{ij}(\theta_{0,m}, \theta_{m',m}) & \Phi_{ij}(\theta_{1,m}, \theta_{m',m}) & \cdots & \Phi_{ij}(\theta_{m',m}, \theta_{m',m}) \end{pmatrix} P_m^T. \quad (3.9)$$

3.3.2 Approximating equation

We recall that the approximation $\Phi_{ij}^{n,m}$ of the kernel Φ can be written as

$$\Phi_{ij}^{n,m}(\xi, \tau) = \sum_{s,r \in J_m} \Phi_{ij}^m(s, r) \varphi_{s,m}(\xi) \varphi_{r,m}(\tau), \quad (3.10)$$

where $J_m := \{0, \dots, n + m - 2\}$, and the values $\Phi_{ij}^m(s, r)$ are the entries of the matrix Φ_{ij}^m given by (3.9). Replacing Φ_{ij} by $\Phi_{ij}^{n,m}$ in (1.1), we get

$$u_i^{n,m}(\xi) = g_i(\xi) + \sum_{j=1}^{m'} \int_a^b \Phi_{ij}^{n,m}(\xi, \tau) u_j^{n,m}(\tau) d\tau, \quad (3.11)$$

Equation (3.10) gives

$$\begin{aligned} u_i^{n,m}(\xi) &= g_i(\xi) + \sum_{j=1}^{m'} \sum_{s,r \in J_m} \Phi_{ij}^m(s,r) \varphi_{s,m}(\xi) \int_a^b \varphi_{r,m}(\tau) u_j^{n,m}(\tau) d\tau, \\ &= g_i(\xi) + \sum_{s \in J_m} \varphi_{s,m}(\xi) \sum_{j=1}^{m'} \sum_{r \in J_m} \Phi_{ij}^m(s,r) \int_a^b \varphi_{r,m}(\tau) u_j^{n,m}(\tau) d\tau. \end{aligned}$$

Putting

$$\alpha_{s,m}^i = \sum_{j=1}^{m'} \sum_{r \in J_m} \Phi_{ij}^m(s,r) \int_a^b \varphi_{r,m}(\tau) u_j^{n,m}(\tau) d\tau,$$

we obtain

$$u_i^{n,m}(\xi) = g_i(\xi) + \sum_{s \in J_m} \varphi_{s,m}(\xi) \alpha_{s,m}^i. \quad (3.12)$$

Or equivalently,

$$u_i^{n,m}(\tau) = g_i(\tau) + \sum_{k \in J_m} \varphi_{k,m}(\tau) \alpha_{k,m}^i.$$

Thus,

$$\alpha_{s,m}^i = \sum_{j=1}^{m'} \left(\sum_{r \in J_m} \Phi_{ij}^m(s,r) \int_a^b \varphi_{r,m}(\tau) g_j(\tau) d\tau + \sum_{k \in J_m} \alpha_{k,m}^j \sum_{r \in J_m} \Phi_{ij}^m(s,r) \int_a^b \varphi_{s,m}(\tau) \varphi_{k,m}(\tau) d\tau \right).$$

Letting

$$\begin{aligned} \alpha &:= ((\alpha_{i,m}^1)_{i \in J_m}, (\alpha_{i,m}^2)_{i \in J_m}, \dots, (\alpha_{i,m}^{m'})_{i \in J_m})^T, \\ \rho_{i,m}^j &:= \int_a^b \varphi_{i,m}(\tau) g_j(\tau) d\tau, \\ \rho_m &:= ((\rho_{i,m}^1)_{i \in J_m}, (\rho_{i,m}^2)_{i \in J_m}, \dots, (\rho_{i,m}^{m'})_{i \in J_m})^T, \end{aligned}$$

and

$$\Phi_m := \begin{pmatrix} \Phi_{11}^m & \Phi_{12}^m & \dots & \Phi_{1m'}^m \\ \Phi_{21}^m & \Phi_{22}^m & \dots & \Phi_{2m'}^m \\ \dots & \dots & \ddots & \dots \\ \Phi_{m'1}^m & \Phi_{m'2}^m & \dots & \Phi_{m'm'}^m \end{pmatrix}_{m \times m}.$$

Let us consider the Gram matrix $\Upsilon_m := (\Upsilon_m(i,j))_{i,j \in J_m}$ associated with the corresponding B-splines, i.e.

$$\Upsilon_m(i,j) := \int_a^b \varphi_{i,m}(\tau) \varphi_{j,m}(\tau) d\tau.$$

We have proved the following theorem:

Theorem 3.3.1. *The coefficients α_m of the solution (3.11) of the approximate integral equation satisfy the linear equations*

$$(I - \Phi_m \Upsilon_m) \alpha_m = \Phi_m \rho_m.$$

By standard calculus, we get for $m = 3, 4$ where Υ_3 and Υ_4 are pentadiagonal and heptadiagonal symmetric matrices, respectively.

Moreover, the main diagonal of Υ_3 is $(0.2h, 0.2h, 0.55h, \dots, 0.55h, 0.2h, 0.2h)$ and the 1-diagonal and 2-diagonal of Υ_3 are respectively,

$$(0.1166h, 0.2083h, 0.2166h, \dots, 0.2166h, 0.2083h, 0.1166h)$$

and

$$(0.0166h, 0.0083h, \dots, 0.0083h, 0.0166h).$$

However, the main diagonal of Υ_4 is $(12h, 18.6h, 27.45h, 40.26h, \dots, 40.26h, 27.45h, 18.6h, 12h)$. For $k = 1, 2, 3$, the k -diagonals of Υ_4 are respectively

$$\begin{aligned} &(7.35h, 13.125h, 18.866h, 19.85h, \dots, 19.85h, 18.866h, 13.125h, 7.35h), \\ &(1.55h, 2.9h, 1.99h, 2h, \dots, 2h, 1.99h, 2.9h, 1.55h), \\ &(0.1h, 0.025h, 0.0166h, \dots, 0.0166h, 0.025h, 0.1h). \end{aligned}$$

3.4 Continuous blending approximation

The goal of this section is to approximate the kernels $\Phi_{ij, i, j \in J_m}$ via the quasi-interpolation. To this end, let us approximate the kernels $\Phi_{ij}^{n, m}(\cdot, \cdot)$ as follows

$$\Phi_{ij}^{n, m}(\xi, \tau) = (\sigma_{n, m} \otimes \mathcal{I} + \mathcal{I} \otimes \sigma_{n, m} - \sigma_{n, m} \otimes \sigma_{n, m}) \Phi_{ij}(\xi, \tau),$$

where

$$\begin{aligned} (\sigma_{n, m} \otimes \mathcal{I}) \Phi_{ij}(\xi, \tau) &= \sum_{s \in J_m} \gamma_{s, m}(\Phi_{ij}(\cdot, \tau)) \varphi_{s, m}(\xi), \\ (\mathcal{I} \otimes \sigma_{n, m}) \Phi_{ij}(\xi, \tau) &= \sum_{r \in J_m} \gamma_{r, m}(\Phi_{ij}(\xi, \cdot)) \varphi_{r, m}(\tau), \\ (\sigma_{n, m} \otimes \sigma_{n, m}) \Phi_{ij}(\xi, \tau) &= \sum_{s, r \in J_m} \Phi_{ij}^m(s, r) \varphi_{s, m}(\xi) \varphi_{r, m}(\tau). \end{aligned}$$

Letting

$$\widehat{\Phi_{ij, s, m}} := \gamma_{s, m}(\Phi_{ij}(\cdot, \tau)) \quad \text{and} \quad \overline{\Phi_{ij, r, m}} := \gamma_{r, m}(\Phi_{ij}(\xi, \cdot)),$$

we get

$$\begin{aligned} \Phi_{ij}^{n, m}(\xi, \tau) &= \sum_{s \in J_m} \widehat{\Phi_{ij, s, m}}(\tau) \varphi_{s, m}(\xi) + \sum_{r \in J_m} \overline{\Phi_{ij, r, m}}(\xi) \varphi_{r, m}(\tau) \\ &\quad - \sum_{s, r \in J_m} \Phi_{ij}^m(s, r) \varphi_{s, m}(\xi) \varphi_{r, m}(\tau), \end{aligned}$$

Theorem 3.4.1. For some vectors α_m, β_m and ϱ_m, ρ_m where

$$\begin{aligned} \alpha_m &:= ((\alpha_{1, m})_{i \in J_m}, (\alpha_{2, m})_{i \in J_m}, \dots, (\alpha_{m', m})_{i \in J_m})^T, \\ \beta_m &:= ((\beta_{1, m})_{i \in J_m}, (\beta_{2, m})_{i \in J_m}, \dots, (\beta_{m', m})_{i \in J_m})^T, \\ \varrho_m &:= ((\varrho_{1, m})_{i \in J_m}, (\varrho_{2, m})_{i \in J_m}, \dots, (\varrho_{m', m})_{i \in J_m})^T, \\ \rho_m &:= ((\rho_{1, m})_{i \in J_m}, (\rho_{2, m})_{i \in J_m}, \dots, (\rho_{m', m})_{i \in J_m})^T, \end{aligned}$$

such that, the vector $\chi^B := (\alpha, \beta)^T$ satisfies the following linear system

$$(I - \kappa^B) \chi^B = b^B,$$

with

$$b^B := (\varrho_m, \rho_m)^T,$$

and

$$\kappa^B := \begin{pmatrix} \widehat{\Phi_m} & \Phi_m^{(2)} - \Phi^m \\ \Upsilon_m^B & \overline{\Phi_m} \end{pmatrix},$$

with

$$\widehat{\Phi_m} := \begin{pmatrix} \widehat{\Phi_{11}} & \widehat{\Phi_{12}} & \cdots & \widehat{\Phi_{1m'}} \\ \widehat{\Phi_{21}} & \widehat{\Phi_{22}} & \cdots & \widehat{\Phi_{2m'}} \\ \cdots & \cdots & \ddots & \cdots \\ \widehat{\Phi_{m'1}} & \widehat{\Phi_{m'2}} & \cdots & \widehat{\Phi_{m'm'}} \end{pmatrix},$$

$$\begin{aligned}\Phi_m^{(2)} &:= \begin{pmatrix} \Phi_{11_m}^{(2)} & \Phi_{12_m}^{(2)} & \cdots & \Phi_{1m'_m}^{(2)} \\ \Phi_{21_m}^{(2)} & \Phi_{22_m}^{(2)} & \cdots & \Phi_{2m'_m}^{(2)} \\ \cdots & \cdots & \ddots & \cdots \\ \Phi_{m'1_m}^{(2)} & \Phi_{m'2_m}^{(2)} & \cdots & \Phi_{m'm'_m}^{(2)} \end{pmatrix}, \\ \Phi^m &:= \begin{pmatrix} \Phi_{11}^m & \Phi_{12}^m & \cdots & \Phi_{1m'}^m \\ \Phi_{21}^m & \Phi_{22}^m & \cdots & \Phi_{2m'}^m \\ \cdots & \cdots & \ddots & \cdots \\ \Phi_{m'1}^m & \Phi_{m'2}^m & \cdots & \Phi_{m'm'}^m \end{pmatrix}, \\ \widehat{\Phi_{m'm'}} &:= \begin{pmatrix} \overline{\Phi_{11}} & \overline{\Phi_{12}} & \cdots & \overline{\Phi_{1m'}} \\ \overline{\Phi_{21}} & \overline{\Phi_{22}} & \cdots & \overline{\Phi_{2m'}} \\ \cdots & \cdots & \ddots & \cdots \\ \overline{\Phi_{m'1}} & \overline{\Phi_{m'2}} & \cdots & \overline{\Phi_{m'm'}} \end{pmatrix},\end{aligned}$$

and

$$\Upsilon_m^B := \begin{pmatrix} \Upsilon_m & 0 & 0 & 0 \\ 0 & \Upsilon_m & 0 & 0 \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \Upsilon_m \end{pmatrix}.$$

Proof. It follows from the approximate integral equation by replacing Φ_{ij} by $\Phi_{ij}^{n,m}$ that

$$u_i^{n,m}(\xi) = g_i(\xi) + \sum_{j=1}^{m'} \int_a^b \Phi_{ij}^{n,m}(\xi, \tau) u_j^{n,m}(\tau) d\tau,$$

We have

$$\begin{aligned}u_i^{n,m}(\xi) &= g_i(\xi) + \sum_{j=1}^{m'} \left(\sum_{s \in J_m} \varphi_{s,m}(\xi) \int_a^b \widehat{\Phi_{ij_{s,m}}}(\tau) u_j^{n,m}(\tau) d\tau \right. \\ &\quad + \sum_{r \in J_m} \overline{\Phi_{ij_{r,m}}}(\xi) \int_a^b \varphi_{r,m}(\tau) u_j^{n,m}(\tau) d\tau \\ &\quad \left. - \sum_{s,r \in J_m} \Phi_{ij}^m(s, r) \varphi_{s,m}(\xi) \int_a^b \varphi_{r,m}(\tau) u_j^{n,m}(\tau) d\tau \right),\end{aligned}$$

which leads to

$$u_i^{n,m}(\xi) = g_i(\xi) + \sum_{s \in J_m} \alpha_{i_s,m} \varphi_{s,m}(\xi) + \sum_{j=1}^{m'} \sum_{r \in J_m} \beta_{j_r,m} \overline{\Phi_{ij_{r,m}}}(\xi).$$

Or equivalently,

$$u_j^{n,m}(\tau) = g_j(\tau) + \sum_{w \in J_m} \alpha_{j_w,m} \varphi_{w,m}(\tau) + \sum_{q=1}^{m'} \sum_{v \in J_m} \beta_{q_v,\ell} \overline{\Phi_{jq_{v,m}}}(\tau).$$

Hence

$$\begin{aligned}
\sum_{s \in J_m} \alpha_{i_{s,m}} \varphi_{s,m}(\xi) &+ \sum_{j=1}^{m'} \sum_{r \in J_m} \beta_{j_{r,m}} \overline{\Phi_{ij_{r,m}}}(\xi) = \sum_{j=1}^{m'} \int_a^b \Phi_{ij}^{n,m}(\xi, \tau) u_j^{n,m}(\tau) d\tau \\
&= \sum_{j=1}^{m'} \int_a^b \left(\sum_{s \in J_m} \widehat{\Phi_{ij_{s,m}}}(\tau) \varphi_{s,m}(\xi) + \sum_{r \in J_m} \overline{\Phi_{ij_{r,m}}}(\xi) \varphi_{r,m}(\tau) \right. \\
&\quad \left. - \sum_{s,r \in J_m} \Phi_{ij}^m(s,r) \varphi_{s,m}(\xi) \varphi_{r,m}(\tau) \right) u_j^{n,m}(\tau) d\tau \\
&= \sum_{s \in J_m} \varphi_{s,m}(\xi) \sum_{j=1}^{m'} \left(\int_a^b \widehat{\Phi_{ij_{s,m}}}(\tau) u_j^{n,m}(\tau) d\tau \right. \\
&\quad \left. - \sum_{r \in J_m} \Phi_{ij}^m(s,r) \int_a^b \varphi_{r,m}(\tau) u_j^{n,m}(\tau) d\tau \right) \\
&\quad + \sum_{j=1}^{m'} \sum_{r \in J_m} \overline{\Phi_{ij_{r,m}}}(\xi) \int_a^b \varphi_{r,m}(\tau) u_j^{n,m}(\tau) d\tau.
\end{aligned}$$

This equality is satisfied with the following choices for $X_{i_m} := (X_{i_{s,m}})_{s \in J_m}$ and $\beta_{j_m} := (\beta_{j_{r,m}})_{r \in J_m}$:

$$\begin{aligned}
\alpha_{i_{s,m}} &= \sum_{j=1}^{m'} \left(\int_a^b \widehat{\varphi_{ij_{s,m}}}(\tau) u_j^{n,m}(\tau) d\tau - \sum_{r \in J_m} \Phi_{ij}^m(s,r) \int_a^b \varphi_{r,m}(\tau) u_j^{n,m}(\tau) d\tau \right) \\
\beta_{j_{r,m}} &= \int_a^b \varphi_{r,m}(\tau) u_j^{n,m}(\tau) d\tau.
\end{aligned}$$

Let $\varrho_{i_m} := (\varrho_{i_{s,m}})_{s \in J_m}$ and $\rho_{j_m} := (\rho_{j_{r,m}})_{r \in J_m}$ be the vectors with entries

$$\varrho_{i_{s,m}} = \sum_{j=1}^{m'} \int_a^b \widehat{\Phi_{ij_{s,m}}}(\tau) g_j(\tau) d\tau, \quad \rho_{j_{r,m}} := \int_a^b \varphi_{r,m}(\tau) g_j^{n,m}(\tau) d\tau,$$

Let us consider the matrices

$$\widehat{\Phi}_m := (\widehat{\Phi_{ij_m}})_{i,j=1,\dots,m'}, \quad \overline{\Phi}_m := (\overline{\Phi_{ij_m}})_{i,j=1,\dots,m'} \quad \text{and} \quad \Phi_m^{(2)} := (\Phi_{ij_m}^{(2)})_{i,j=1,\dots,m'},$$

with entries

$$\begin{aligned}
\widehat{\Phi_{ij_m}}(s,r) &:= \int_a^b \widehat{\Phi_{ij_{s,m}}}(\tau) \varphi_{r,m}(\tau) d\tau, \quad \overline{\Phi_{ij_m}}(s,r) := \int_a^b \varphi_{s,m}(\tau) \overline{\Phi_{ij_{r,m}}}(\tau) d\tau, \\
\Phi_{ij_m}^{(2)}(s,r) &:= \sum_{q=1}^{m'} \int_a^b \widehat{\Phi_{iq_{s,m}}}(\tau) \overline{\Phi_{jq_{r,m}}}(\tau) d\tau.
\end{aligned}$$

We obtain

$$\begin{aligned}
\beta_{j_{r,m}} &= \int_a^b \varphi_{r,m}(\tau) \left(g_j(\tau) + \sum_{w \in J_m} \alpha_{j_{w,m}} \varphi_{w,m}(\tau) + \sum_{q=1}^{m'} \sum_{v \in J_m} \beta_{q_{v,m}} \overline{\Phi_{jq_{v,m}}}(\tau) \right) d\tau, \quad i, j = 1, \dots, m', \\
\beta_{j_{r,m}} &= \rho_{j_{r,m}} + \sum_{w \in J_m} \Upsilon_m(w,r) \alpha_{j_{w,m}} + \sum_{q=1}^{m'} \sum_{v \in J_m} \overline{\Phi_{jq_m}}(r,v) \beta_{q_{v,m}} d\tau, \\
\alpha_{i_{s,m}} &= \sum_{j=1}^{m'} \left(\int_a^b \widehat{\Phi_{ij_{s,m}}}(\tau) \left(g_j(\tau) + \sum_{w \in J_m} \alpha_{j_{w,m}} \varphi_{w,m}(\tau) + \sum_{q=1}^{m'} \sum_{v \in J_m} \beta_{q_{v,m}} \overline{\Phi_{jq_{v,m}}}(\tau) \right) d\tau \right. \\
&\quad \left. - \sum_{r \in J_m} \Phi_{ij}^m(s,r) \int_a^b \varphi_{r,m}(\tau) \left(g_j(\tau) + \sum_{w \in J_m} \alpha_{j_{w,m}} \varphi_{w,m}(\tau) + \sum_{q=1}^{m'} \sum_{v \in J_m} \beta_{q_{v,m}} \overline{\Phi_{jq_{v,m}}}(\tau) \right) d\tau \right).
\end{aligned}$$

Or equivalently,

$$\begin{aligned} \alpha_{i_s, m} &= \left(\varrho_{i_s, m} - \sum_{j=1}^{m'} \sum_{r \in J_m} \Phi_{ij}^m(s, r) \rho_{j_r, m} \right) + \sum_{j=1}^{m'} \sum_{w \in J_m} \alpha_{j_w, m} \left(\widehat{\Phi}_{ij_m}(s, w) - \sum_{r \in J_m} \Phi_{ij}^m(s, r) \Upsilon_m(r, w) \right) \\ &+ \sum_{q=1}^{m'} \sum_{v \in J_m} \beta_{q_v, m} \left(\Phi_{iq_m^{(2)}}(s, v) - \sum_{j=1}^{m'} \sum_{r \in J_m} \Phi_{ij}^m(s, r) \overline{\Phi}_{jq_m}(r, v) \right), \end{aligned}$$

we get the the matrix form

$$\begin{aligned} \alpha_m &= \varrho - \Phi^m \rho_m + (\widehat{\Phi}_m - \Phi^m \Phi_m^B) \alpha_m + (\Phi_m^{(2)} - \Phi^m \overline{\Phi}_m) \beta_m, \\ \beta_m &= \rho_m + \Upsilon_m^B \alpha_m + \overline{\Phi}_m \beta_m. \end{aligned}$$

Note that since

$$\overline{\Phi}_m \beta_m = \beta_m - \rho_m - \Upsilon_m^B \alpha_m,$$

we obtain

$$\alpha_m = \varrho + \widehat{\Phi}_m \alpha_m + (\Phi_m^{(2)} - \Phi^m) \beta_m.$$

□

3.5 Error Analysis

The purpose of this section is to analysis the error convergence, we recall some results . We give via the approximation error of the kernel.

We need the following Theorem, which follows from [6].

Define the operators Φ_{ij} as follows

$$\Phi_{ij} u_j(\xi) = \int_a^b \Phi_{ij}(\xi, \tau) u_j(\tau) d\tau, \quad i, j = 1, 2, \dots, m'.$$

Then the system (1.1) can be written in operator form as

$$u_i(\xi) = g_i(\xi) + \sum_{j=1}^{m'} \Phi_{ij} u_j(\xi), \quad 1 \leq i \leq m',$$

which equivalent to the following matrix form

$$\mathcal{I}_{m'} U = G + \Phi U,$$

where

$$\mathcal{I}_{m'} := \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & I \end{pmatrix}_{m' \times m'}, \quad \Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \dots & \Phi_{1m} \\ \Phi_{21} & \Phi_{22} & \dots & \Phi_{2m} \\ \dots & \dots & \ddots & \dots \\ \Phi_{m'1} & \Phi_{m'2} & \dots & \Phi_{m'm'} \end{pmatrix}_{m' \times m'},$$

and

$$U := (u_1 \ u_2 \ \dots \ u_{m'})_{m' \times 1}^T, \quad G := (g_1 \ g_2 \ \dots \ g_{m'})_{m' \times 1}^T.$$

The approximate integral equations (3.11) reads as

$$u_i^{n, m}(\xi) = g_i(\xi) + \sum_{j=1}^{m'} \Phi_{ij}^{n, m} u_j^{n, m}(\xi), \quad 1 \leq i \leq m',$$

where

$$\Phi_{ij}^{n,m} u_j^{n,m}(\xi) = \int_a^b \Phi_{ij}^{n,m}(\xi, \tau) u_j^{n,m}(\tau) d\tau, \quad i, j = 1, 2, \dots, m'.$$

Then can be written in matrix form as

$$\mathcal{I}_m U^{n,m} = G + \Phi^{n,M} U^{n,m},$$

where

$$\Phi^{n,m} = \begin{pmatrix} \Phi_{11}^{n,m} & \Phi_{12}^{n,m} & \cdots & \Phi_{1m'}^{n,m} \\ \Phi_{21}^{n,m} & \mathcal{K}_{22}^{n,m} & \cdots & \Phi_{2m'}^{n,m} \\ \cdots & \cdots & \ddots & \cdots \\ \Phi_{m'1}^{n,m} & \Phi_{m'2}^{n,m} & \cdots & \Phi_{m'm'}^{n,m} \end{pmatrix}, \quad U^{n,m} = \begin{pmatrix} u_1^{n,m} \\ u_2^{n,m} \\ \vdots \\ u_{m'}^{n,m} \end{pmatrix}_{m' \times 1}.$$

Theorem 3.5.1. *Let f be a function of integrable derivative. It is assumed that the points of evaluation $S_n := s_j, 0 \leq j \leq n+1$ are symmetric in the interval I , i.e. $b+a-s_j = s_{n+1-j}$. Then there exists a constant C independent of h such that for $f \in C^{m'+1}(I)$, we have*

$$\left| \int_a^b (g - \sigma_{m,n} g) f \right| \leq Ch^{m'+1} \left(\|f^{(1)}\|_{\infty, I} \|g^{(m')}\|_{\infty, I} + \|g^{(m'+1)}\|_{\infty, I} \right).$$

Proof. See [7] and references therein. \square

Let U be the solution of the systems of linear Fredholm integral equations and $U^{n,m}$ be the solution of the approximate systems of linear Fredholm integral equations.

Theorem 3.5.2. *Assume that $\Phi_{ij} \in C^{m', m'+1}(I^2)$. Then, it holds*

$$\|U - U^{n,m'}\|_{\infty, I} = O(h^{m'}), \quad (\text{resp. } \|U - U_{m',n}\|_{\infty, I} = O(h^{r_{m'}})),$$

where

$$r_{m'} := \begin{cases} 2m', & \text{if } m' \text{ is even,} \\ 2m+1, & \text{if } m' \text{ is odd.} \end{cases}$$

Proof. We have

$$\begin{aligned} U - U^{n,m} &= (I_m - \Phi)^{-1} F - (I_m - \Phi^{n,m})^{-1} G \\ &= (I_m - \Phi^{n,m})^{-1} [(I_m - \Phi^{n,m}) - (I_m - \Phi)] (I_m - \Phi)^{-1} G \\ &= (I_m - \Phi^{n,m})^{-1} (\Phi - \Phi^{n,m}) U. \end{aligned}$$

Hence

$$\begin{aligned} \|U - U^{n,m}\|_{\infty} &\leq \|(I_m - \Phi^{n,m})^{-1}\|_{\infty} \|(\Phi - \Phi^{n,m})U\|_{\infty} \\ &= \|(I_{m'} - \Phi^{n,m})^{-1}\|_{\infty} \left\| \begin{pmatrix} \sum_{j=1}^{m'} [\Phi_{1j} - \Phi_{1j}^{n,m}] u_j \\ \sum_{j=1}^{m'} [\Phi_{2j} - \Phi_{2j}^{n,m}] u_j \\ \vdots \\ \sum_{j=1}^{m'} [\Phi_{1j} - \Phi_{1j}^{n,m}] u_j \end{pmatrix} \right\|_{\infty} \\ &= \max_{1 \leq i \leq m'} \left\{ \left\| \sum_{j=1}^{m'} (I - \Phi_{ij}^{n,m})^{-1} \right\|_{\infty} \right\} \max_{1 \leq i \leq m'} \left\{ \left\| \sum_{j=1}^{m'} [\Phi_{ij} - \Phi_{ij}^{n,m}] u_j \right\|_{\infty} \right\} \\ &\leq \max_{1 \leq i \leq m'} \left\{ \sum_{j=1}^{m'} \|(I - \Phi_{ij}^{n,m})^{-1}\|_{\infty} \right\} \max_{1 \leq i \leq m'} \left\{ \sum_{j=1}^{m'} \left\| [\Phi_{ij} - \Phi_{ij}^{n,m}] u_j \right\|_{\infty} \right\} \\ &\leq \max_{1 \leq i \leq m'} \left\{ \sum_{j=1}^{m'} \|(I - \Phi_{ij}^{n,m})^{-1}\|_{\infty} \right\} \max_{1 \leq i \leq m'} \left\{ \sum_{j=1}^{m'} \left\| \Phi_{ij} - \Phi_{ij}^{n,m} \right\|_{\infty} \|u_j\|_{\infty} \right\}. \end{aligned}$$

As in [7], for each $i, j = 1, \dots, m'$ we have

$$\Phi_{ij}^{n,m} = \begin{cases} (\sigma_{n,m} \otimes \mathcal{I} + \mathcal{I} \otimes \sigma_{n,m} - \sigma_{n,m} \otimes \sigma_{n,m}) \Phi_{ij} & \text{in CB case,} \\ (\sigma_{n,m} \otimes \sigma_{n,m}) \Phi_{ij}, & \text{in TB one,} \end{cases}$$

and

$$\Phi_{ij} - \Phi_{ij}^{n,m} = \begin{cases} (\mathcal{I} - \sigma_{n,m}) \otimes (\mathcal{I} - \sigma_{n,m}) \Phi_{ij}, & \text{in CB case,} \\ ((\mathcal{I} - \sigma_{n,m}) \otimes \mathcal{I}) \Phi_{ij} + \mathcal{I} \otimes (\mathcal{I} - \sigma_{n,m}) \Phi_{ij} - (\mathcal{I} - \sigma_{n,m}) \otimes (\mathcal{I} - \sigma_{n,m}) \Phi_{ij}, & \text{in TB one.} \end{cases}$$

Again, we need the following results, which follows from [7],

$$\max_{i \in I} \left| \int_a^b ((\mathcal{I} - \sigma_{n,m}) \otimes \mathcal{I}) \Phi_{ij}(\xi, \tau) u_j(\tau) d\tau \right| \leq C_1 h^{m'} \left\| \Phi_{ij}^{(m',0)} \right\|_{\infty} \|u_j\|_{\infty},$$

and

$$\max_{i \in I} \left| \int_a^b (\mathcal{I} \otimes (\mathcal{I} - \sigma_{n,m})) \Phi_{ij}(\xi, \tau) u_j(\tau) d\tau \right| \leq C_2 h^{m+1} \left(\left\| \Phi_{ij}^{(0,m)} \right\|_{\infty} \|u_j^1\|_{\infty} + \left\| \Phi_{ij}^{(0,m+1)} \right\|_{\infty} \right).$$

Hence

$$\begin{aligned} \max_{i \in I} \left| \int_a^b (\mathcal{I} - \sigma_{n,m}) \otimes (\mathcal{I} - \sigma_{n,m}) \Phi_{ij}(\xi, \tau) u_j(\tau) d\tau \right| &\leq C_2 h^{m+1} \left(\left\| (\mathcal{I} - \sigma_{n,m}) \Phi_{ij}^{(0,m)} \right\|_{\infty} \|u_j^1\|_{\infty} \right. \\ &\quad \left. + \left\| (\mathcal{I} - \sigma_{n,m}) \Phi_{ij}^{(0,m+1)} \right\|_{\infty} \right) \\ &\leq C_2 h^{m+1} \left(C_1 h^m \left\| \Phi_{ij}^{(m,m)} \right\|_{\infty} \|u_j^1\|_{\infty} \right. \\ &\quad \left. + C_1 h^m \left\| \Phi_{ij}^{(m,m+1)} \right\|_{\infty} \right) \\ &= C_3 h^{2m+1} \left(\left\| \Phi_{ij}^{(m,m)} \right\|_{\infty} \|u_j^1\|_{\infty} + \left\| \Phi_{ij}^{(m,m+1)} \right\|_{\infty} \right), \end{aligned}$$

and hence

$$\left\| \Phi_{ij} - \Phi_{ij}^{n,m} \right\|_{\infty} = \begin{cases} O(h^m), & \text{in the TB case,} \\ O(h^{2m}), & \text{in the CB one.} \end{cases}$$

Since Φ_{ij} are compact for all $i, j = 1 \dots m$, the theory developed in [1], assures that for n large enough, the operator $(I - \Phi_{ij}^{n,m})^{-1}$ are invertible, and their inverses are uniformly bounded with respect to n . Moreover,

$$\left\| (I - \Phi_{ij}^{n,m})^{-1} \right\|_{\infty} \leq \frac{\left\| (I - \Phi_{ij})^{-1} \right\|_{\infty}}{1 - \left\| (I - \Phi_{ij})^{-1} \right\|_{\infty} \left\| \Phi_{ij} - \Phi_{ij}^{n,m} \right\|_{\infty}}, \quad n \geq N_0.$$

Choose N_0 so that

$$\left\| \Phi_{ij} - \Phi_{ij}^{n,m} \right\|_{\infty} \leq \frac{1}{2 \left\| (I - \Phi_{ij})^{-1} \right\|_{\infty}}, \quad n \geq N_0.$$

This implies that

$$\left\| (I - \Phi_{ij}^{n,m})^{-1} \right\|_{\infty} \leq 2 \left\| (I - \Phi_{ij})^{-1} \right\|_{\infty}.$$

On the other hand,

$$\|U - U^{n,m}\|_{\infty} \leq \max_{1 \leq i \leq m'} \left\{ \sum_{j=1}^{m'} \left\| (I - \Phi_{ij}^{n,m})^{-1} \right\|_{\infty} \right\} \max_{1 \leq i \leq m'} \left\{ \sum_{j=1}^{m'} \left\| \Phi_{ij} - \Phi_{ij}^{n,m} \right\|_{\infty} \|u_j\|_{\infty} \right\}.$$

Thus,

$$\|U - U^{n,m}\|_\infty \leq \begin{cases} 2Ch^m \max_{1 \leq i \leq m'} \left\{ \sum_{j=1}^{m'} \|(I - \Phi_{ij})^{-1}\|_\infty \right\} \|U\|_\infty, & \text{in the TP case,} \\ 2Ch^{r_m} \max_{1 \leq i \leq m'} \left\{ \sum_{j=1}^{m'} \|(I - \Phi_{ij})^{-1}\|_\infty \right\} \|U\|_\infty, & \text{in the CB one.} \end{cases}$$

□

3.6 Computation of vectors and matrices

In order to evaluate the values of ρ_m , ϱ_m , ρ_m , $\widehat{\Phi}_m$, $\Phi_m^{(2)}$ and $\overline{\Phi}_{m'}$, numerical quadrature rules are request. Popular quadrature rules in this situation are the the Gauss quadrature formulas (abbr. GQF) of order $2m' - 1$, with B-splines weights. We follow [7] and references therein to compute the nodes and weights of GQF.

η	$\xi_{0,\eta}$	$\lambda_{0,\eta}$	$\xi_{1,\eta}$	$\lambda_{1,\eta}$	$\xi_{2,\eta}$	$\lambda_{2,\eta}$
1	1.23×10^{-2}	2.33×10^{-1}	4.41×10^{-1}	3.94×10^{-1}	1	5×10^{-1}
2	5.44×10^{-1}	1.01×10^{-1}	1.95×10^{-1}	2.73×10^{-1}	2	5×10^{-1}

Table 3.1: for $m' = 2$ and $m = 3$

3.6.1 Evaluations for Tensor Product approximation

Letting

$$\rho_m := ((\rho_{1,i,m})_{i \in J_m}, (\rho_{2,i,\ell})_{i \in J_m}, \dots, (\rho_{m',i,m})_{i \in J_m})^T, \quad \text{with } \rho_{j,i,m} = \int_a^b \varphi_{i,m} g_j.$$

We evaluate the integrals $\rho_{j,i,m}$ via the GQF for quadratic and cubic B-spline weights with $m' = 2$ and $m' = 3$. Here, we give some approximate results.

$$\begin{aligned} \int_{\tau_0}^{\tau_1} \varphi_{0,3} \Psi_j &\simeq h \sum_{\eta=0}^1 \lambda_{0,\eta} \Psi_j(a + h\xi_{0,\eta}), \\ \int_{\tau_0}^{\tau_2} \varphi_{1,3} \Psi_j &\simeq h \sum_{\eta=0}^1 \lambda_{1,\eta} \Psi_j(a + h\xi_{1,\eta}), \\ \int_{\tau_{i-2}}^{\tau_{i+1}} \varphi_{i,3} \Psi_j &\simeq h \sum_{\eta=0}^1 \lambda_{2,\eta} \Psi_j(a + h(\xi_{2,\eta} + i - 2)), \quad 2 \leq i \leq n-1, \\ \int_{\tau_{n-2}}^{\tau_n} \varphi_{n,3} \Psi_j &\simeq h \sum_{\eta=0}^1 \lambda_{1,\eta} \Psi_j(b - h\xi_{1,\eta}), \\ \int_{\tau_{n-1}}^{\tau_n} \varphi_{n+1,3} \Psi_j &\simeq h \sum_{\eta=0}^1 \lambda_{0,\eta} \Psi_j(b - h\xi_{0,\eta}). \end{aligned}$$

Also, the following calculus are given

$$\begin{aligned}
\int_{\tau_0}^{\tau_1} \varphi_{0,4} \Psi_j &\simeq h \sum_{\eta=0}^2 \lambda_{0,\eta} \Psi_j(a + h\xi_{0,\eta}), \\
\int_{\tau_0}^{\tau_2} \varphi_{1,4} \Psi_j &\simeq h \sum_{\eta=0}^2 \lambda_{1,\eta} \Psi_j(a + h\xi_{1,\eta}), \\
\int_{\tau_0}^{\tau_3} \varphi_{2,4} \Psi_j &\simeq h \sum_{\eta=0}^2 \lambda_{2,\eta} \Psi_j(a + h\xi_{2,\eta}), \\
\int_{\tau_{i-3}}^{\tau_{i+1}} \varphi_{i,4} \Psi_j &\simeq h \sum_{\eta=0}^2 (\lambda_{3,\eta} \Psi_j(a + h(\xi_{3,\eta} + i - 3))), \quad 3 \leq i \leq n-1, \\
\int_{\tau_{n-3}}^{\tau_n} \varphi_{n,4} \Psi_j &\simeq h \sum_{\eta=0}^2 \lambda_{2,\eta} \Psi_j(b - h\xi_{2,\eta}), \\
\int_{\tau_{n-2}}^{\tau_n} \varphi_{n+1,4} \Psi_j &\simeq h \sum_{\eta=0}^2 \lambda_{1,\eta} \Psi_j(b - h\xi_{1,\eta}), \\
\int_{\tau_{n-1}}^{\tau_n} \varphi_{n+2,4} \Psi_j &\simeq h \sum_{\eta=0}^2 \lambda_{0,\eta} \Psi_j(b - h\xi_{0,\eta}).
\end{aligned}$$

In Table 3.7, we present some values of the points $\xi_{i,\eta}$ and the weights $\lambda_{i,\eta}$.

η	$\xi_{0,\eta}$	$\lambda_{0,\eta}$	$\xi_{1,\eta}$	$\lambda_{1,\eta}$	$\xi_{2,\eta}$	$\lambda_{2,\eta}$	$\xi_{3,\eta}$	$\lambda_{3,\eta}$
0	$6, 3 \cdot 10^{-2}$	$1, 29 \cdot 10^{-1}$	$2, 37 \cdot 10^{-1}$	$1, 83 \cdot 10^{-1}$	$2, 37 \cdot 10^{-1}$	$1, 83 \cdot 10^{-1}$	1,052	$1, 86 \cdot 10^{-1}$
1	$3, 02 \cdot 10^{-1}$	$1, 05 \cdot 10^{-1}$	$7, 16 \cdot 10^{-1}$	$2, 69 \cdot 10^{-1}$	$7, 16 \cdot 10^{-1}$	$2, 69 \cdot 10^{-1}$	2	$6, 30 \cdot 10^{-1}$
2	$6, 37 \cdot 10^{-1}$	$1, 7 \cdot 10^{-2}$	1,326	$4, 9 \cdot 10^{-2}$	1,326	$4, 9 \cdot 10^{-2}$	2,949	$1, 86 \cdot 10^{-1}$

Table 3.2: The points $\lambda_{i,\eta}$ and the weights $\lambda_{i,\eta}$ for $m' = 3$ and $m = 4$

3.6.2 Continuous blending case

Now, we use the GQF with B-spline weights with $m = 4$ and $m = 5$ to approximate the integrals appearing in the matrices $\widehat{\Phi}_m$ and $\overline{\Phi}_m$, and in the vector ρ_m . We give the associate formulats of order $O(h^7)$

$$\begin{aligned}
\int_{\tau_0}^{\tau_1} \varphi_{0,3} \Psi_j &\simeq h \sum_{k=0}^3 \lambda_{0,k} \Psi_j(a + h\xi_{0,k}), \\
\int_{\tau_0}^{\tau_2} \varphi_{1,3} \Psi_j &\simeq h \sum_{k=0}^3 \lambda_{1,k} \Psi_j(a + h\xi_{1,k}), \\
\int_{\tau_{i-2}}^{\tau_{i+1}} \varphi_{i,3} \Psi_j &\simeq h \sum_{k=0}^3 \lambda_{2,k} \Psi_j(a + h(\xi_{2,k} + i - 2)), \quad 2 \leq i \leq n-1, \\
\int_{\tau_{n-2}}^{\tau_n} \varphi_{n,3} \Psi_j &\simeq h \sum_{k=0}^3 \lambda_{1,k} \Psi_j(b - h\xi_{1,k}), \\
\int_{\tau_{n-1}}^{\tau_n} \varphi_{n+1,3} \Psi_j &\simeq h \sum_{k=0}^3 \lambda_{0,k} \Psi_j(b - h\xi_{0,k}).
\end{aligned}$$

In table , we introduce the points $\xi_{i,k}$ and the weights $\lambda_{i,k}$.

η	$\xi_{0,\eta}$	$\lambda_{0,\eta}$	$\xi_{1,\eta}$	$\lambda_{1,\eta}$	$\xi_{2,\eta}$	$\lambda_{2,\eta}$
0	4.9×10^{-2}	1.11×10^{-1}	2.01×10^{-1}	1.11×10^{-1}	4.99×10^{-1}	7.6×10^{-2}
1	2.39×10^{-1}	1.44×10^{-1}	6.11×10^{-1}	3.15×10^{-1}	1.158	4.25×10^{-1}
2	5.18×10^{-1}	6.9×10^{-2}	1.113	2.10×10^{-1}	1.843	4.25×10^{-1}
3	7.96×10^{-1}	1.1×10^{-1}	6.24×10^{-1}	3.3×10^{-1}	2.502	7.6×10^{-1}

Table 3.3: Points and weights for $m' = 4$ and $m = 3$

η	$\xi_{0,\kappa}$	$\lambda_{0,\eta}$	$\xi_{1,\eta}$	$\lambda_{1,\eta}$	$\xi_{2,\eta}$	$\lambda_{2,\eta}$	$\xi_{3,\eta}$	$\lambda_{3,\eta}$
0	$3, 1.10^{-2}$	$7, 1.10^{-2}$	$1, 28.10^{-1}$	$6, 6.10^{-1}$	$3, 11.10^{-1}$	$4, 6.10^{-1}$	$6, 19.10^{-1}$	$2, 5.10^{-2}$
1	$1, 55.10^{-1}$	$1, 01.10^{-1}$	$4, 04.10^{-1}$	$1, 99.10^{-1}$	$7, 71.10^{-1}$	$2, 45.10^{-1}$	1,294	$2, 39.10^{-1}$
2	$3, 52.10^{-1}$	$6, 1.10^{-2}$	$7, 80.10^{-1}$	$1, 77.10^{-1}$	1,323	$3, 22.10^{-1}$	2	$4, 73.10^{-1}$
3	$5, 83.10^{-1}$	$1, 7.10^{-1}$	1,211	$5, 5.10^{-1}$	1,913	$1, 27.10^{-1}$	2,707	$2, 39.10^{-1}$
4	$8, 04.10^{-1}$	2.10^{-2}	1,629	5.10^{-3}	2,485	$1, 3.10^{-2}$	3,382	$2, 5.10^{-1}$

Table 3.4: Points and weights for $m' = 5$ and $m = 4$

$$\begin{aligned}
\int_{\tau_0}^{\tau_1} \varphi_{0,4} \Psi_j &\simeq h \sum_{k=0}^4 \lambda_{0,k} \Psi_j(a + h\xi_{0,k}), \\
\int_{\tau_0}^{\tau_2} \varphi_{1,4} \Psi_j &\simeq h \sum_{k=0}^4 \lambda_{1,k} \Psi_j(a + h\xi_{1,k}), \\
\int_{\tau_0}^{\tau_3} \varphi_{2,4} \Psi_j &\simeq h \sum_{k=0}^4 \lambda_{2,k} \Psi_j(a + h\xi_{2,k}), \\
\int_{\tau_{i-3}}^{\tau_{i+1}} \varphi_{i,4} \Psi_j &\simeq h \sum_{k=0}^4 (\lambda_{3,k} \Psi_j(a + h(\xi_{3,k} + i - 3))), \quad 3 \leq i \leq n-1, \\
\int_{\tau_{n-3}}^{\tau_n} \varphi_{n,4} \Psi_j &\simeq h \sum_{k=0}^4 \lambda_{2,k} \Psi_j(b - h\xi_{2,k}), \\
\int_{\tau_{n-2}}^{\tau_n} \varphi_{n+1,4} \Psi_j &\simeq h \sum_{k=0}^4 \lambda_{1,k} \Psi_j(b - h\xi_{1,k}), \\
\int_{\tau_{n-1}}^{\tau_n} \varphi_{n+2,4} \Psi_j &\simeq h \sum_{k=0}^4 \lambda_{0,k} \Psi_j(b - h\xi_{0,k}).
\end{aligned}$$

We compute the matrix $\Phi_m^{(2)}$ and the vector ϱ_m in both quadratic and cubic cases,

$$\begin{aligned}
\int_{\tau_0}^{\tau_n} \Psi_j &\simeq h \sum_{m=0}^{\frac{n}{2}-1} \left(\frac{41}{420} \Psi_j(\tau_{2m}) + \frac{18}{35} \Phi_j \left(\tau_{2m} + \frac{h}{3} \right) + \frac{9}{140} \Psi_j \left(\tau_{2m} + \frac{2h}{3} \right) + \frac{68}{105} \Psi_j(\tau_{2m} + h) \right. \\
&\quad \left. + \frac{9}{140} \Psi_j \left(\tau_{2m} + \frac{4h}{3} \right) + \frac{18}{35} \Psi_j \left(\tau_{2m} + \frac{5h}{3} \right) + \frac{41}{420} \Psi_j(\tau_{2m} + 2h) \right).
\end{aligned}$$

3.6.3 System of two integral equations

Let us consider the system of two integral equations of the form

$$\begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} = \begin{pmatrix} g_1(\xi) \\ g_2(\xi) \end{pmatrix} + \int_a^b \begin{pmatrix} \Phi_{11}(\xi, \tau) & \Phi_{12}(\xi, \tau) \\ \Phi_{21}(\xi, \tau) & \Phi_{22}(\xi, \tau) \end{pmatrix} \begin{pmatrix} u_1(\tau) \\ u_2(\tau) \end{pmatrix} dt.$$

Where

$$\begin{cases} u_1(\xi) = g_1(\xi) + \int_a^b \Phi_{11}(\xi, \tau)u_1(\tau)d\tau + \int_a^b \Phi_{12}(\xi, \tau)u_2(\tau)d\tau \\ u_2(\xi) = g_2(\xi) + \int_a^b \Phi_{21}(\xi, \tau)u_1(\tau)d\tau + \int_a^b \Phi_{22}(\xi, \tau)u_2(\tau)d\tau \end{cases}.$$

Proposition 3.6.1. For some vectors α, β and ϱ, ρ where $\alpha := (\alpha_{1_m}, \alpha_{2_m})^T$, $\beta := (\beta_{1_m}, \beta_{2_m})^T$ and $\varrho := (\varrho_{1_m}, \varrho_{2_m})^T$, $\rho := (\rho_{1_m}, \rho_{2_m})^T$. Then, the vector $\chi^B := (\alpha, \beta)^T$ satisfies the following system of linear equations

$$(\mathcal{I} - \kappa^B)\chi^B = b^B,$$

where

$$b^B := (\varrho_{m'}, \rho_m)^T,$$

and

$$\kappa^B := \begin{pmatrix} \begin{pmatrix} \widehat{\Phi_{11_m}} & \widehat{\Phi_{12_m}} \\ \widehat{\Phi_{21_m}} & \widehat{\Phi_{22_m}} \end{pmatrix} & \begin{pmatrix} \Phi_{11_m}^{(2)} - \Phi_{11_m} & \Phi_{12_m}^{(2)} - \Phi_{12_m} \\ \Phi_{21_m}^{(2)} - \Phi_{21_m} & \Phi_{22_m}^{(2)} - \Phi_{22_m} \end{pmatrix} \\ \begin{pmatrix} \Upsilon_m & 0 \\ 0 & \Upsilon_m \end{pmatrix} & \begin{pmatrix} \overline{\Upsilon_{11_m}} & \overline{\Upsilon_{12_m}} \\ \overline{\Upsilon_{21_m}} & \overline{\Upsilon_{22_m}} \end{pmatrix} \end{pmatrix}.$$

3.6.4 Approximating the kernel by quasi-interpolation

In this case, the approximate kernel is

$$\begin{aligned} \Phi_{1_{m,n}}(\xi, \tau) &= (\mathcal{Q}_{m,n} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{Q}_{m,n} - \mathcal{Q}_{m,n} \otimes \mathcal{Q}_{m,n})\Phi_1(s, t), \\ \Phi_{2_{m,n}}(\xi, \tau) &= (\mathcal{Q}_{m,n} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{Q}_{m,n} - \mathcal{Q}_{m,n} \otimes \mathcal{Q}_{m,n})\Phi_2(s, t), \end{aligned}$$

where

$$\begin{aligned} (\sigma_{m,n} \otimes \mathcal{I})\Phi_1(\xi, \tau) &= \sum_{i \in J_m} \gamma_{i,m}(\Phi_1(\cdot, \tau))\varphi_{i,m}(\xi), \\ (\sigma_{m,n} \otimes \mathcal{I})\Phi_2(\xi, \tau) &= \sum_{i \in J_m} \gamma_{i,m}(\Phi_2(\cdot, \tau))\varphi_{i,m}(\xi), \end{aligned}$$

and

$$\begin{aligned} (\mathcal{I} \otimes \sigma_{m,n})\Phi_1(\xi, \tau) &= \sum_{j \in J_m} \gamma_{j,m}(\Phi_1(\xi, \cdot))\varphi_{j,m}(\tau), \\ (\mathcal{I} \otimes \sigma_{m,n})\Phi_2(\xi, \tau) &= \sum_{j \in J_m} \gamma_{j,m}(\Phi_2(\xi, \cdot))\varphi_{j,m}(\tau). \end{aligned}$$

Also,

$$\begin{aligned} (\sigma_{m,n} \otimes \sigma_{m,n})\Phi_1(\xi, \tau) &= \sum_{i,j \in J_m} \Phi_{1_m}(i, j)\varphi_{i,m}(\xi)\varphi_{j,m}(\tau), \\ (\sigma_{m,n} \otimes \sigma_{m,n})\Phi_2(\xi, \tau) &= \sum_{i,j \in J_m} \Phi_{2_m}(i, j)\varphi_{i,m}(\xi)\varphi_{j,m}(\tau). \end{aligned}$$

3.7 Three specific systems of integral equations

In this section, we look at three specific systems of integral equations and provide results that describe each one.

Firstly, let us consider the following system of integral equations

$$\begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} = \begin{pmatrix} g_1(\xi) \\ g_2(\xi) \end{pmatrix} + \int_a^b \begin{pmatrix} 0 & \Phi_1(\xi, \tau) \\ \Phi_2(\xi, \tau) & 0 \end{pmatrix} \begin{pmatrix} u_1(\tau) \\ u_2(\tau) \end{pmatrix} dt,$$

where

$$\begin{aligned} u_1(\xi) &= f_1(\tau) + \int_a^b \Phi_1(\xi, \tau) u_2(\tau) d\tau, \\ u_2(\xi) &= f_2(\tau) + \int_a^b \Phi_2(\xi, \tau) u_1(t) d\tau. \end{aligned}$$

For this scenario, we provide some specific outcomes.

Proposition 3.7.1. *For some vectors α, β and ϱ, ρ where $\alpha := (\alpha_{1_m}, \alpha_{2_m})^T$, $\beta := (\beta_{1_m}, \beta_{2_m})^T$ and $\varrho := (\varrho_{1_m}, \varrho_{2_m})^T$, $\rho := (\rho_{1_m}, \rho_{2_m})^T$, the following system of linear equations*

$$(\mathcal{I} - \kappa^B)\chi^B = b^B$$

is satisfied by the vector $\chi^B := (\alpha, \beta)^T$, where

$$\kappa^B := \begin{pmatrix} \begin{pmatrix} 0 & \widehat{\Phi}_{2_m} \\ \widehat{\Phi}_{1_m} & 0 \end{pmatrix} & \begin{pmatrix} -\Phi_{2_m} & \Phi_{12_m}^{(2)} \\ \Phi_{21_m}^{(2)} & -\Phi_{1_m} \end{pmatrix} \\ \begin{pmatrix} 0 & \Upsilon_m \\ \Upsilon_m & 0 \end{pmatrix} & \begin{pmatrix} 0 & \overline{\Phi}_{1_m} \\ \overline{\Phi}_{2_m} & 0 \end{pmatrix} \end{pmatrix}, \quad \text{and } b^B := (\varrho_m, \rho_m)^T.$$

Proof. We have

$$\begin{aligned} \Phi_{1_{m,n}}(\xi, \tau) &= \sum_{i \in J_m} \widehat{\Phi}_{1_{i,m}}(\tau) \varphi_{i,m}(\xi) + \sum_{j \in J_m} \overline{\Phi}_{1_{j,m}}(\xi) \varphi_{j,m}(\tau) - \sum_{i,j \in J_m} \Phi_{1_m}(i, j) \varphi_{i,m}(\xi) \varphi_{j,m}(\tau), \\ \Phi_{2_{m,n}}(\xi, \tau) &= \sum_{i \in J_m} \widehat{\Phi}_{2_{i,m}}(\tau) \varphi_{i,m}(\xi) + \sum_{j \in J_m} \overline{\Phi}_{2_{j,m}}(\xi) \varphi_{j,m}(\tau) - \sum_{i,j \in J_m} \Phi_{2_m}(i, j) \varphi_{i,m}(\xi) \varphi_{j,m}(\tau), \end{aligned}$$

where

$$\begin{aligned} \widehat{\Phi}_{1_{i,m}}(\tau) &:= \gamma_{i,m}(\Phi_1(\cdot, \tau)), \quad \widehat{\Phi}_{2_{i,m}}(\tau) = \gamma_{i,m}(\Phi_2(\cdot, \tau)) \\ \overline{\Phi}_{1_{j,m}}(\xi) &:= \gamma_{j,m}(\Phi_1(\xi, \cdot)), \quad \overline{\Phi}_{2_{j,m}}(\xi) = \gamma_{j,m}(\Phi_2(\xi, \cdot)). \end{aligned}$$

Hence

$$\begin{aligned} u_{1_{m,n}}(\xi) &= g_1(\xi) + \sum_{i \in J_m} \alpha_{2_{i,m}} \varphi_{i,m}(\xi) + \sum_{j \in J_m} \beta_{2_{j,m}} \overline{\Phi}_{1_{j,m}}(\xi), \\ u_{2_{m,n}}(\xi) &= g_2(\xi) + \sum_{i \in J_m} \alpha_{1_{i,m}} \varphi_{i,m}(\xi) + \sum_{j \in J_m} \beta_{1_{j,m}} \overline{\Phi}_{2_{j,m}}(\xi). \end{aligned}$$

The following options for equality satisfy this requirement

$$\begin{aligned} \alpha_{1_m} &= (\alpha_{1_{i,m}})_{i \in J_m}, \quad \alpha_{2_m} := (\alpha_{2_{i,m}})_{i \in J_m} \\ \beta_{1_m} &= (\beta_{1_{i,m}})_{i \in J_m}, \quad \beta_{2_m} := (\beta_{2_{i,m}})_{i \in J_m}, \end{aligned}$$

where

$$\begin{aligned} \alpha_{1_{i,m}} &= \int_a^b \widehat{\Phi}_{2_{i,m}}(\tau) u_{1_{m,n}}(\tau) d\tau - \sum_{j \in J_m} \Phi_{2_m}(i, j) \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau, \\ \alpha_{2_{i,m}} &= \int_a^b \widehat{\Phi}_{1_{i,m}}(\tau) u_{2_{m,n}}(\tau) d\tau - \sum_{j \in J_m} \Phi_{1_m}(i, j) \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau. \end{aligned}$$

As a result, we get the following approximate integral equations.

$$\begin{aligned}
u_{1_{m,n}}(\xi) &= g_1(\xi) + \sum_{i \in J_m} \varphi_{i,m}(\xi) \int_a^b \widehat{\Phi_{1_{i,m}}}(\tau) u_{2_{m,n}}(\tau) d\tau + \sum_{j \in J_m} \overline{\Phi_{1_{j,m}}}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau \\
&\quad - \sum_{i,j \in J_m} \Phi_{1_m}(i,j) \varphi_{i,m}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau, \\
u_{2_{m,n}}(\xi) &= g_2(\xi) + \sum_{i \in J_m} \varphi_{i,m}(\xi) \int_a^b \widehat{\Phi_{2_{i,m}}}(t) u_{1_{m,n}}(\tau) d\tau + \sum_{j \in J_m} \overline{\Phi_{2_{j,m}}}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau \\
&\quad - \sum_{i,j \in J_m} \Phi_{2_m}(i,j) \varphi_{i,m}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau.
\end{aligned}$$

and

$$\begin{aligned}
\beta_{1_{j,m}} &= \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau, \\
\beta_{2_{j,m}} &= \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau.
\end{aligned}$$

Alternatively,

$$\begin{aligned}
\alpha_{1_{i,m}} &= \int_a^b \widehat{\Phi_{2_{i,m}}}(\tau) [g_1(\tau) + \sum_{r \in J_m} \alpha_{2_{r,m}} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{2_{s,m}} \overline{\Phi_{1_{s,m}}}(\tau)] d\tau \\
&\quad - \sum_{j \in J_m} \Phi_{2_m}(i,j) \int_a^b \varphi_{j,m}(\tau) [g_1(\tau) + \sum_{r \in J_m} \alpha_{2_{r,m}} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{2_{s,m}} \overline{\Phi_{1_{s,m}}}(\tau)] d\tau, \\
\alpha_{2_{i,m}} &= \int_a^b \widehat{k_{1_{i,m}}}(\tau) [g_2(\tau) + \sum_{r \in J_m} \alpha_{1_{r,m}} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{1_{s,m}} \overline{\Phi_{2_{s,m}}}(\tau)] d\tau \\
&\quad - \sum_{j \in J_m} \Phi_{1_m}(i,j) \int_a^b \varphi_{j,m}(\tau) [g_2(\tau) + \sum_{r \in J_m} \alpha_{1_{r,m}} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{1_{s,m}} \overline{\Phi_{2_{s,m}}}(\tau)] d\tau.
\end{aligned}$$

Let $\varrho_{1_m} := (\varrho_{1_{i,m}})_{i \in J_m}$, $\varrho_{2_m} := (\varrho_{2_{i,m}})_{i \in J_m}$ and $\rho_{1_m} := (\rho_{1_{i,m}})_{i \in J_m}$, $\rho_{2_m} := (\rho_{2_{i,m}})_{i \in J_m}$ be the vectors with entries

$$\begin{aligned}
\varrho_{1_{i,m}} &= \int_a^b \widehat{\Phi_{2_{i,m}}}(\tau) g_1(\tau) d\tau, & \varrho_{2_{i,m}} &= \int_a^b \widehat{\Phi_{1_{i,m}}}(\tau) g_2(\tau) d\tau, \\
\rho_{1_{i,m}} &= \int_a^b \varphi_{i,m}(\tau) g_1(\tau) d\tau, & \rho_{2_{i,m}} &= \int_a^b \varphi_{i,m}(\tau) g_2(\tau) d\tau,
\end{aligned}$$

and $(\widehat{\Phi_{1_m}}, \widehat{\Phi_{2_m}})$, $(\overline{\Phi_{1_m}}, \overline{\Phi_{2_m}})$ and $(\Phi_{1_m}^{(2)}, \Phi_{2_m}^{(2)})$ be the matrices with entries

$$\begin{aligned}
\widehat{\Phi_{1_m}}(i,j) &= \int_a^b \widehat{\Phi_{1_{i,m}}}(\tau) \varphi_{j,m}(\tau) d\tau, & \widehat{\Phi_{2_m}}(i,j) &= \int_a^b \widehat{\Phi_{2_{i,m}}}(\tau) \varphi_{j,m}(\tau) g_2(\tau) d\tau, \\
\overline{\Phi_{1_m}}(i,j) &= \int_a^b \varphi_{i,m}(\tau) \overline{\Phi_{1_{j,m}}}(\tau) d\tau, & \overline{\Phi_{2_m}}(i,j) &= \int_a^b \varphi_{i,m}(t) \overline{\Phi_{2_{j,m}}}(\tau) d\tau.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\Phi_{11_m}^{(2)}(i,j) &= \int_a^b \widehat{\Phi_{2_{i,m}}}(\tau) \overline{\Phi_{2_{j,m}}}(\tau) d\tau, \\
\Phi_{12_m}^{(2)}(i,j) &= \int_a^b \widehat{\Phi_{2_{i,m}}}(\tau) \overline{\Phi_{1_{j,m}}}(\tau) d\tau, \\
\Phi_{21_m}^{(2)}(i,j) &= \int_a^b \widehat{\Phi_{1_{i,m}}}(\tau) \overline{\Phi_{2_{j,m}}}(\tau) d\tau, \\
\Phi_{22_m}^{(2)}(i,j) &= \int_a^b \widehat{\Phi_{1_{i,m}}}(\tau) \overline{\Phi_{1_{j,m}}}(\tau) d\tau,
\end{aligned}$$

. Hence

$$\begin{aligned}
\alpha_{1_i,m} &= (\varrho_{1_i,m} - \sum_{j \in J_m} \Phi_{2_m}(i,j)\rho_1) + \sum_{r \in J_m} \alpha_{2_{r,m}}(\widehat{\Phi_{2_m}}(i,r) - \sum_{j \in J_m} \Phi_{2_m}(i,j)\Upsilon_m(j,r)) \\
&+ \sum_{s \in J_m} \beta_{2_{s,m}}(\Phi_{2_m}^{(2)}(i,s) - \sum_{j \in J_m} \Phi_{2_m}(i,j)\overline{\Phi_{1_{s,m}}}(j,s))\alpha_{2_{i,m}} \\
&= (\varrho_{2_{i,m}} - \sum_{j \in J_m} \Phi_{1_m}(i,j)\rho_2) + \sum_{r \in J_m} \alpha_{1_{r,m}}(\widehat{\Phi_{1_m}}(i,r) - \sum_{j \in J_m} \Phi_{1_m}(i,j)\Upsilon_m(j,r)) \\
&+ \sum_{s \in J_m} \beta_{1_{s,m}}(\Phi_{1_m}^{(2)}(i,s) - \sum_{j \in J_m} \Phi_{1_m}(i,j)\overline{\Phi_{2_m}}(j,s)).
\end{aligned}$$

and hence

$$\begin{aligned}
\beta_{1_j,m} &= \int_a^b \varphi_{j,m}(\tau)[g_1(\tau) + \sum_{r \in J_m} \alpha_{2_{r,m}}\varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{2_{s,m}}\overline{\Phi_{1_{s,m}}}(\tau)]d\tau, \\
\beta_{2_j,m} &= \int_a^b \varphi_{j,m}(\tau)[g_2(\tau) + \sum_{r \in J_m} \alpha_{1_{r,m}}\varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{1_{s,m}}\overline{\Phi_{2_{s,m}}}(\tau)]d\tau,
\end{aligned}$$

which eventually leads to

$$\beta_{1_m} = \rho_1 + \Upsilon_m \alpha_{2_m} + \overline{\Phi_{1_m}} \beta_{2_m}, \quad \text{and} \quad \beta_{2_m} = \rho_2 + \Upsilon_m \alpha_{1_m} + \overline{\Phi_{2_m}} \beta_{1_m}.$$

So,

$$\begin{pmatrix} \beta_{1_m} \\ \beta_{2_m} \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} 0 & \Upsilon_m \\ \Upsilon_m & 0 \end{pmatrix} \begin{pmatrix} \alpha_{1_m} \\ \alpha_{2_m} \end{pmatrix} + \begin{pmatrix} 0 & \overline{\Phi_{1_m}} \\ \overline{\Phi_{2_m}} & 0 \end{pmatrix} \begin{pmatrix} \beta_{1_m} \\ \beta_{2_m} \end{pmatrix}.$$

Thus, we obtain

$$\begin{pmatrix} \alpha_{1_m} \\ \alpha_{2_m} \end{pmatrix} = \begin{pmatrix} \varrho_{1_m} \\ \varrho_{2_m} \end{pmatrix} + \begin{pmatrix} 0 & \widehat{\Phi_{2_m}} \\ \widehat{\Phi_{1_m}} & 0 \end{pmatrix} \begin{pmatrix} \alpha_{1_m} \\ \alpha_{2_m} \end{pmatrix} + \begin{pmatrix} -\Phi_{2_m} & \Phi_{12_m}^{(2)} \\ \Phi_{21_m}^{(2)} & -\Phi_{1_m} \end{pmatrix} \begin{pmatrix} \beta_{1_m} \\ \beta_{2_m} \end{pmatrix}.$$

□

Finally, we consider the following integral system

$$\begin{pmatrix} u_1(\xi) \\ u_2(\xi) \end{pmatrix} = \begin{pmatrix} g_1(\xi) \\ g_2(\xi) \end{pmatrix} + \int_a^b \begin{pmatrix} \Phi_2(\xi, \tau) & \Phi_1(\xi, \tau) \\ \Phi_2(\xi, \tau) & \Phi_1(\xi, \tau) \end{pmatrix} \begin{pmatrix} u_1(\tau) \\ u_2(\tau) \end{pmatrix} d\tau,$$

where

$$\begin{aligned}
u_1(\xi) &= g_1(\xi) + \int_a^b \Phi_1(\xi, \tau)u_2(\tau)d\tau + \int_a^b \Phi_2(\xi, \tau)u_1(\tau)d\tau, \\
u_2(\xi) &= g_2(\xi) + \int_a^b \Phi_2(\xi, \tau)u_1(\tau)d\tau + \int_a^b \Phi_1(\xi, \tau)u_2(\tau)d\tau.
\end{aligned}$$

In the following statement, we provide a concrete result for this third case.

Proposition 3.7.2. For some vectors α, β and ϱ, ρ where $\alpha := (\alpha_{1_m}, \alpha_{2_m})^T$, $\beta := (\beta_{1_m}, \beta_{2_m})^T$ and $\varrho := (\varrho_{1_m}, \varrho_{2_m})^T$, $\rho := (\rho_{1_m}, \rho_{2_m})^T$, the following system of linear equations

$$(I - \kappa^B)\chi^B = b^B$$

is satisfied by the vector $\chi^B := (\alpha, \beta)^T$, where

$$\kappa^B := \begin{pmatrix} \begin{pmatrix} \widehat{\Phi_{2_m}} & \widehat{\Phi_{2_m}} \\ \widehat{\Phi_{1_m}} & \widehat{\Phi_{1_m}} \end{pmatrix} & \begin{pmatrix} \Phi_{11_m}^{(2)} - \Phi_{2_m} & \Phi_{12_m}^{(2)} \\ \Phi_{21_m}^{(2)} & \Phi_{22_m}^{(2)} - \Phi_{1_m} \end{pmatrix} \\ \begin{pmatrix} \Upsilon_m & \Upsilon_m \\ \Upsilon_{ml} & \Upsilon_m \end{pmatrix} & \begin{pmatrix} \overline{\Phi_{2_m}} & \overline{\Phi_{1_m}} \\ \overline{\Phi_{2_m}} & \overline{\Phi_{1_m}} \end{pmatrix} \end{pmatrix}, \quad \text{and} \quad b^B := (\varrho_m, \rho_m)^T.$$

Proof. We have

$$\begin{aligned}\Phi_{1_{m,n}}(\xi, \tau) &= \sum_{i \in J_m} \widehat{\Phi_{1_{i,m}}}(\tau) \varphi_{i,m}(\xi) + \sum_{j \in J_m} \overline{\Phi_{1_{j,m}}}(\xi) \varphi_{j,m}(\tau) - \sum_{i,j \in J_m} \Phi_{1_m}(i,j) \varphi_{i,m}(\xi) \varphi_{j,m}(\tau), \\ \Phi_{2_{m,n}}(\xi, \tau) &= \sum_{i \in J_m} \widehat{\Phi_{2_{i,m}}}(\tau) \varphi_{i,m}(\xi) + \sum_{j \in J_m} \overline{\Phi_{2_{j,m}}}(\xi) \varphi_{j,m}(\tau) - \sum_{i,j \in J_m} \Phi_{2_m}(i,j) \varphi_{i,m}(\xi) \varphi_{j,m}(\tau),\end{aligned}$$

where

$$\widehat{\Phi_{1_{i,m}}}(\tau) = \gamma_{i,m}(\Phi_1(\cdot, \tau)), \quad \widehat{\Phi_{2_{i,m}}}(\tau) := \gamma_{i,m}(\Phi_2(\cdot, \tau)),$$

and

$$\overline{\Phi_{1_{j,m}}}(\xi) = \gamma_{j,m}(\Phi_1(\xi, \cdot)), \quad \overline{\Phi_{2_{j,m}}}(\xi) := \gamma_{j,m}(\Phi_2(\xi, \cdot)).$$

Hence

$$\begin{aligned}u_{1_{m,n}}(\xi) &= g_1(\xi) + \sum_{i \in J_m} \varphi_{i,m}(\xi) \int_a^b \widehat{\Phi_{1_{i,m}}}(\tau) u_{2_{m,n}}(\tau) d\tau + \sum_{j \in J_m} \overline{\Phi_{1_{j,m}}}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau \\ &\quad - \sum_{i,j \in J_m} \Phi_{1_m}(i,j) \varphi_{i,m}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau \\ &\quad + \sum_{i \in J_m} \varphi_{i,m}(\xi) \int_a^b \widehat{\Phi_{2_{i,m}}}(\tau) u_{1_{m,n}}(\tau) d\tau + \sum_{j \in J_m} \overline{\Phi_{2_{j,m}}}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau \\ &\quad - \sum_{i,j \in J_m} \Phi_{2_m}(i,j) \varphi_{i,m}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau, \\ u_{2_{m,n}}(\xi) &= g_2(\xi) + \sum_{i \in J_m} \varphi_{i,m}(\xi) \int_a^b \widehat{\Phi_{2_{i,m}}}(\tau) u_{1_{m,n}}(\tau) d\tau + \sum_{j \in J_m} \overline{\Phi_{2_{j,m}}}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau \\ &\quad - \sum_{i,j \in J_m} \Phi_{2_m}(i,j) \varphi_{i,m}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau \\ &\quad + \sum_{i \in J_m} \varphi_{i,m}(\xi) \int_a^b \widehat{\Phi_{1_{i,m}}}(\tau) u_{2_{m,n}}(\tau) d\tau + \sum_{j \in J_m} \overline{\Phi_{1_{j,m}}}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau \\ &\quad - \sum_{i,j \in J_m} \Phi_{1_m}(i,j) \varphi_{i,m}(\xi) \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau.\end{aligned}$$

Therefore

$$\begin{aligned}u_{1_{m,n}}(\xi) &= g_1(\xi) + \sum_{i \in J_m} \alpha_{2_{i,m}} \varphi_{i,m}(\xi) + \sum_{j \in J_m} \beta_{2_{j,m}} \overline{\Phi_{1_{j,m}}}(\xi) + \sum_{i \in J_m} \alpha_{1_{i,m}} \varphi_{i,m}(\xi) + \sum_{j \in J_m} \beta_{1_{j,m}} \overline{\Phi_{2_{j,m}}}(\xi), \\ u_{2_{m,n}}(\xi) &= g_2(\xi) + \sum_{i \in J_m} \alpha_{1_{i,m}} \varphi_{i,m}(\xi) + \sum_{j \in J_m} \beta_{1_{j,m}} \overline{\Phi_{2_{j,m}}}(\xi) + \sum_{i \in J_m} \alpha_{2_{i,m}} \varphi_{i,m}(\xi) + \sum_{j \in J_m} \beta_{2_{j,m}} \overline{\Phi_{1_{j,m}}}(\xi).\end{aligned}$$

This equality is satisfied with the following choices for

$$\alpha_{1_m} := (\alpha_{1_{i,m}})_{i \in J_m}, \quad \alpha_{2_m} := (\alpha_{2_{i,m}})_{i \in J_m} \quad \text{and} \quad \beta_{1_m} := (\beta_{1_{i,m}})_{i \in J_m}, \quad \beta_{2_m} := (\beta_{2_{i,m}})_{i \in J_m},$$

where

$$\left\{ \begin{aligned} \alpha_{1_{i,m}} &= \int_a^b \widehat{\Phi_{2_{i,m}}}(\tau) u_{1_{m,n}}(\tau) d\tau - \sum_{j \in J_m} \Phi_{2_m}(i,j) \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau, \\ \alpha_{2_{i,m}} &= \int_a^b \widehat{\Phi_{1_{i,m}}}(\tau) u_{2_{m,n}}(\tau) d\tau - \sum_{j \in J_m} \Phi_{1_m}(i,j) \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau, \end{aligned} \right.$$

and

$$\beta_{1_{j,m}} = \int_a^b \varphi_{j,m}(\tau) u_{1_{m,n}}(\tau) d\tau, \quad \text{and} \quad \beta_{2_{j,m}} = \int_a^b \varphi_{j,m}(\tau) u_{2_{m,n}}(\tau) d\tau.$$

Or, equivalently

$$\begin{aligned}
\alpha_{1_i,m} &= \int_a^b \widehat{\Phi}_{2_i,m}(\tau) [g_1(\tau) + \sum_{r \in J_m} \alpha_{2_r,m} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{2_s,m} \overline{\varphi_{1_s,m}}(\tau) \\
&+ \sum_{r \in J_m} \alpha_{1_r,m} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{1_s,m} \overline{\Phi_{2_s,m}}(\tau)] d\tau \\
&- \sum_{j \in J_m} \Phi_{2_m}(i,j) \int_a^b \varphi_{j,m}(\tau) [g_1(\tau) + \sum_{r \in J_m} \alpha_{2_r,m} \varphi_{r,m}(\tau) \\
&+ \sum_{s \in J_m} \beta_{2_s,m} \overline{\Phi_{1_s,m}}(\tau) + \sum_{r \in J_m} \alpha_{1_r,m} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{1_s,m} \overline{\varphi_{2_s,m}}(\tau)] d\tau, \\
\alpha_{2_i,m} &= \int_a^b \widehat{\Phi}_{1_i,m}(\tau) [g_2(\tau) + \sum_{r \in J_m} \alpha_{1_r,m} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{1_s,m} \overline{\Phi_{2_s,m}}(\tau) \\
&+ \sum_{r \in J_m} \alpha_{2_r,m} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{2_s,m} \overline{\Phi_{1_s,m}}(\tau)] d\tau \\
&- \sum_{j \in J_m} \Phi_{1_m}(i,j) \int_a^b \varphi_{j,m}(\tau) [g_2(\tau) + \sum_{r \in J_m} \alpha_{1_r,m} \varphi_{r,m}(\tau) \\
&+ \sum_{s \in J_m} \beta_{1_s,m} \overline{\Phi_{2_s,m}}(\tau) + \sum_{r \in J_m} \alpha_{2_r,m} \varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{2_s,m} \overline{\Phi_{1_s,m}}(\tau)] d\tau.
\end{aligned}$$

Assume that

$$\varrho_{1_m} := (\varrho_{1_i,m})_{i \in J_m}, \quad \varrho_{2_m} := (\varrho_{2_i,m})_{i \in J_m} \quad \text{and} \quad \rho_{1_m} := (\rho_{1_i,m})_{i \in J_m}, \quad \rho_{2_m} := (\rho_{2_i,m})_{i \in J_m}$$

are vectors with entries

$$\begin{aligned}
\varrho_{1_i,m} &= \int_a^b \widehat{\Phi}_{2_i,m}(\tau) g_1(\tau) d\tau, & \varrho_{2_i,m} &= \int_a^b \widehat{\Phi}_{1_i,m}(\tau) g_2(\tau) d\tau, \\
\rho_{1_i,m} &= \int_a^b \varphi_{i,m}(\tau) g_1(\tau) d\tau, & \rho_{2_i,m} &= \int_a^b \varphi_{i,m}(\tau) g_2(\tau) d\tau,
\end{aligned}$$

respectively. Moreover, the matrices

$$(\widehat{\Phi}_{1_m}, \widehat{\Phi}_{2_m}), \quad (\overline{\Phi}_{1_m}, \overline{\Phi}_{2_m}), \quad (\Phi_{1_m}^{(2)}, \Phi_{2_m}^{(2)}),$$

respectively, have entries

$$\begin{aligned}
\widehat{\Phi}_{1_m}(i,j) &= \int_a^b \widehat{\Phi}_{1_i,m}(\tau) \varphi_{j,m}(\tau) d\tau, & \widehat{\Phi}_{2_m}(i,j) &= \int_a^b \widehat{\Phi}_{2_i,m}(\tau) \varphi_{j,m}(\tau) g_2(\tau) d\tau, \\
\overline{\Phi}_{1_m}(i,j) &:= \int_a^b \varphi_{i,m}(\tau) \overline{\Phi_{1_j,m}}(\tau) d\tau, & \overline{\Phi}_{2_m}(i,j) &= \int_a^b \varphi_{i,m}(\tau) \overline{\Phi_{2_j,m}}(\tau) d\tau, \\
\Phi_{1_m}^{(2)}(i,j) &:= \int_a^b \widehat{\Phi}_{1_i,m}(\tau) \overline{\Phi_{1_j,m}}(\tau) d\tau, & \Phi_{2_m}^{(2)}(i,j) &:= \int_a^b \widehat{\Phi}_{2_i,m}(\tau) \overline{\Phi_{2_j,m}}(\tau) d\tau.
\end{aligned}$$

So that

$$\begin{aligned}
\alpha_{1_i,m} &= (\varrho_{1_i,m} - \sum_{j \in J_m} \Phi_{2_m}(i,j)\rho_1) + \sum_{r \in J_m} \alpha_{2_{r,m}}(\widehat{\Phi}_{2_m}(i,r) - \sum_{j \in J_m} \Phi_{2_m}(i,j)\Upsilon_m(j,r)) \\
&+ \sum_{s \in J_m} \beta_{2_{s,m}}(\Phi_{12_m}^{(2)}(i,s) - \sum_{j \in J_m} \Phi_{2_m}(i,j)\overline{\Phi}_{1_{s,m}}(j,s)) + \sum_{r \in J_m} \alpha_{1_{r,m}}(\widehat{\Phi}_{2_m}(i,r) \\
&- \sum_{j \in J_m} \Phi_{2_m}(i,j)\Upsilon_m(j,r)) \\
&+ \sum_{s \in J_m} \beta_{1_{s,m}}(\Phi_{11_m}^{(2)}(i,s) - \sum_{j \in J_m} \Phi_{1_m}(i,j)\overline{\Phi}_{2_m}(j,s)), \alpha_{2_{i,m}} \\
&= (\varrho_{2_{i,m}} - \sum_{j \in J_m} \Phi_{1_m}(i,j)\rho_2) + \sum_{r \in J_m} \alpha_{1_{r,m}}(\widehat{\Phi}_{1_m}(i,r) - \sum_{j \in J_m} \Phi_{1_m}(i,j)\Upsilon_m(j,r)) \\
&+ \sum_{s \in J_m} \beta_{1_{s,m}}(\Phi_{21_m}^{(2)}(i,s) - \sum_{j \in J_m} \Phi_{1_m}(i,j)\overline{\Phi}_{2_m}(j,s)) + \sum_{r \in J_m} \alpha_{2_{r,m}}(\widehat{\Phi}_{1_m}(i,r) \\
&- \sum_{j \in J_m} \Phi_{1_m}(i,j)\Upsilon_m(j,r)) \\
&+ \sum_{s \in J_m} \beta_{2_{s,m}}(\Phi_{22_m}^{(2)}(i,s) - \sum_{j \in J_m} \Phi_{2_m}(i,j)\overline{\Phi}_{1_{s,m}}(j,s)).
\end{aligned}$$

In matricial form,

$$\begin{aligned}
\begin{pmatrix} \alpha_{1_m} \\ \alpha_{2_m} \end{pmatrix} &= \begin{pmatrix} \varrho_{1_m} \\ \varrho_{2_m} \end{pmatrix} - \begin{pmatrix} \Phi_{2_m} & 0 \\ 0 & \Phi_{1_m} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} \widehat{\Phi}_{2_m} - \Phi_{2_m}\Upsilon_m & \widehat{\Phi}_{2_m} - \Phi_{2_m}\Upsilon_m \\ \widehat{\Phi}_{1_m} - \Phi_{1_m}\Upsilon_m & \widehat{\Phi}_{1_m} - \Phi_{1_m}\Upsilon_m \end{pmatrix} \begin{pmatrix} \alpha_{1_m} \\ \alpha_{2_m} \end{pmatrix} \\
&+ \begin{pmatrix} \Phi_{11_m}^{(2)} - \Phi_{1_m}\overline{\Phi}_{2_m} & \Phi_{12_m}^{(2)} - \Phi_{2_m}\overline{\Phi}_{1_m} \\ \Phi_{21_m}^{(2)} - \Phi_{1_m}\overline{\Phi}_{2_m} & \Phi_{22_m}^{(2)} - \Phi_{2_m}\overline{\Phi}_{1_m} \end{pmatrix} \begin{pmatrix} \beta_{1_m} \\ \beta_{2_m} \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\beta_{1_{j,m}} &= \int_a^b \varphi_{j,m}(\tau)[g_1(\tau) + \sum_{r \in J_m} \alpha_{2_{r,m}}\varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{2_{s,m}}\overline{\Phi}_{1_{s,m}}(\tau) \\
&+ \sum_{r \in J_m} \alpha_{1_{r,m}}\varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{1_{s,m}}\overline{\Phi}_{2_{s,m}}(\tau)]d\tau, \\
\beta_{2_{j,m}} &= \int_a^b \varphi_{j,m}(\tau)[g_2(\tau) + \sum_{r \in J_m} \alpha_{1_{r,m}}\varphi_{r,m}(\tau) \\
&+ \sum_{s \in J_m} \beta_{1_{s,m}}\overline{\Phi}_{2_{s,m}}(\tau) + \sum_{r \in J_m} \alpha_{2_{r,m}}\varphi_{r,m}(\tau) + \sum_{s \in J_m} \beta_{2_{s,m}}\overline{\Phi}_{1_{s,m}}(\tau)]d\tau.
\end{aligned}$$

Thus,

$$\beta_{1_m} = \rho_1 + \Upsilon_m\alpha_{2_m} + \overline{\Phi}_{1_m}\beta_{2_m} + \Upsilon_m\alpha_{1_m} + \overline{\Phi}_{2_m}\beta_{1_m},$$

and

$$\beta_{2_m} = \rho_2 + \Upsilon_m\alpha_{1_m} + \overline{\Phi}_{2_m}\beta_{1_m} + \Upsilon_m\alpha_{2_m} + \overline{\Phi}_{1_m}\beta_{2_m}.$$

$$\begin{pmatrix} \beta_{1_m} \\ \beta_{2_m} \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} \Upsilon_m & \Upsilon_m \\ \Upsilon_m & \Upsilon_m \end{pmatrix} \begin{pmatrix} \alpha_{1_m} \\ \alpha_{2_m} \end{pmatrix} + \begin{pmatrix} \overline{\Phi}_{2_m} & \overline{\Phi}_{1_m} \\ \overline{\Phi}_{2_m} & \overline{\Phi}_{1_m} \end{pmatrix} \begin{pmatrix} \beta_{1_m} \\ \beta_{2_m} \end{pmatrix}.$$

sinse,

$$\begin{pmatrix} \alpha_{1_m} \\ \alpha_{2_m} \end{pmatrix} = \begin{pmatrix} \varrho_{1_m} \\ \varrho_{2_m} \end{pmatrix} + \begin{pmatrix} \widehat{\Phi}_{1_m} & \widehat{\Phi}_{2_m} \\ \widehat{\Phi}_{1_m} & \widehat{\Phi}_{2_m} \end{pmatrix} \begin{pmatrix} \alpha_{1_m} \\ \alpha_{2_m} \end{pmatrix} + \begin{pmatrix} \Phi_{11_m}^{(2)} - \Phi_{2_m} & \Phi_{12_m}^{(2)} \\ \Phi_{21_m}^{(2)} & \Phi_{22_m}^{(2)} - \Phi_{1_m} \end{pmatrix} \begin{pmatrix} \beta_{1_m} \\ \beta_{2_m} \end{pmatrix}$$

□

3.8 Numerical examples

In this section, we solved some cases using the methods described in this chapter for solving the given integral equation.

i	$[a_i, b_i]$	$\Phi_1(\xi, \tau)$	$\Phi_2(\xi, \tau)$	$u_1(\xi)$	$u_2(\xi)$
1	$[0, 1]$	$\exp(\xi - \tau)$	$\cos(\pi\xi\tau)$	$\exp(\xi)$	$\exp(-\xi)$
2	$[0, 1]$	$\exp(\xi\tau)$	$\cos(\pi\xi\tau)$	$\exp(\xi)$	$\sin(\xi)$
3	$[0, 1]$	$\exp(\xi\tau)$	$\xi^{10}\tau^2$	$\exp(\xi)$	$\cos(\xi)$

Table 3.5:

n	$\ u_1 - u_{1,3,n}\ _\infty$	$\ u_2 - u_{2,3,n}\ _\infty$
8	1.88e-8	9.47e-9
16	5.31e-11	4.84e-10
32	1.45e-13	5.09e-12
64	1.77e-15	5.58e-14
128	1.77e-15	9.99e-16

Table 3.6:

n	$\ u_1 - u_{1,3,n}\ _\infty$	$\ u_2 - u_{2,3,n}\ _\infty$
8	1.49e-8	1.33e-8
16	9.25e-11	4.99e-10
32	4.66e-13	5.96e-12
64	3.10e-15	5.59e-14
128	2.22e-15	9.99e-16

Table 3.7:

n	$\ u_1 - u_{1,3,n}\ _\infty$	$\ u_2 - u_{2,3,n}\ _\infty$
8	1.16e-9	1.50e-10
16	3.03e-11	7.65e-13
32	2.63e-13	3.33e-15
64	2.66e-15	4.44e-16
128	1.77e-15	4.44e-16

Table 3.8:

Chapter 4

Multi-quadratic trigonometric B-spline for solving integral equations

4.1 Introduction

Quasi-interpolations play an important role in the approximation theory and its applications. Quasi-interpolations have been examined thoroughly in the literature. In view of this, large research on quasi-interpolation is usually only for discrete function values. A lot of papers have been published on multi-quadratic quasi-interpolation, which needs to be better defined for periodic data since its kernel itself is not periodic. Lyche et al. (cf. [32]) developed a quasi-interpolation scheme via the trigonometric B-splines. They discussed its approximation orders for high-order derivatives. The goal of [25] is to construct a quasi-interpolant for periodic data by combining some techniques of the multi-quadratic quasi-interpolant and the trigonometric B-spline quasi-interpolant. This trigonometric B-spline quasi-interpolant can be considered as a special case of the new quasi-interpolant. In other work, the authors proposed a quasi-interpolation scheme for the linear functional data. This quasi-interpolant provided an optimal approximation order with respect to the smoothness of the right-hand function of the differential equation. The authors used the scheme to approximate the solution of the differential equation. Next, the authors used a meshless method for solving some partial differential equations whose solutions are periodic with respect to the spatial variable. This method is based on the multi-quadratic trigonometric B-spline quasi-interpolant. Then, the authors solved a Hamiltonian wave equation with periodic boundary conditions by using a meshless symplectic scheme based on multi-quadratic trigonometric quasi-interpolation. The authors presented the equation in space using an iterated derivative approximation method based on multi-quadratic trigonometric quasi-interpolation, and for the time, they used an appropriate symplectic scheme. A new meshless conservative or dissipative method for nonlinear time-dependent partial differential equations is proposed by Sun. In the same perspective, the authors solved multi-symplectic Hamiltonian partial differential equations on multi-symplectic formulation via a multi-symplectic quasi-interpolation method. The goal of [14] is to construct a new local spline quasi-interpolant for fitting $3D$ data defined on a sphere-like surface S . The authors investigated in this construction the tensor product of cubic polynomial B-splines and 2π -periodic uniform algebraic trigonometric B-splines of order four. Moreover, Gao and Zhou provided a general and an efficient scheme for explicitly constructing multiscale radial kernels with high-order generalized Strang-Fix conditions from a given univariate generator. The authors constructed the resulting kernels by taking a linear functional to the scaled of the generator with respect to the scale variable. Moreover, Gao and Zhou provided a general and efficient scheme for explicitly constructing multiscale radial kernels with high-order generalized Strang-Fix conditions from a given univariate generator. The authors constructed the resulting kernels by taking a linear function to the scale of the generator with respect to the scale variable. Moreover, reference [24] established three new multi-quadratic quasi-interpolation schemes for integral functionals without solving any minimization problem motivated by fair properties and wide applications of multi-quadratic quasi-interpolation. An iterated quasi-interpolation approach for approximating the high-order derivatives has been introduced only from given discrete function values. More recently, the solution of generalized integral equations of Fredholm type is addressed in [38], where septic spline quasi-interpolants are used. In the same perspective,

a general framework using spline quasi-interpolants to numerically solve generalized Fredholm integral equations of the second kind by two Nyström methods is considered in [5]. In addition, explicit results for octic spline quasi-interpolants are provided. Two efficient methods for solving a Fredholm integral equation of the second kind by two types of bivariate spline quasi-interpolants are introduced in [6].

This present chapter has two goals. On the one hand, we generalize the construction of the quasi-interpolant for periodic data given in [25]. In our construction, we use the n th-order trigonometric divided differences and a new kernel. This generalized scheme is also characterized by the periodicity and the smoothness as in [25]. On the other hand, we explore the new results to approximate the solution of the periodic second kind Fredholm integral equation by using four degenerate methods, namely the tensor product and the continuous blending sum of the present generalized trigonometric B-splines quasi-interpolants. We present the development of the approximate solutions, and we prove some theoretical results related to the error analysis of these methods. Numerical results are given in this paper to illustrate the obtained results.

4.2 A generalized quasi-interpolation scheme

In [25], the authors constructed a new quasi-interpolant, which is characterized by the periodicity as the trigonometric B-spline quasi-interpolant and the smoothness as the MQ quasi-interpolant. In their construction, the authors used the second-order trigonometric divided differences. We follow this scheme, and we generalize this new quasi-interpolant via the n th-order trigonometric divided differences where we use a new kernel.

For this aim, we recall some background on trigonometric spline presented in [25, 32].

Let $\mathcal{D} := \frac{d}{d\xi}$, consider the n th-order differential operator

$$\mathcal{D}_n = \begin{cases} \mathcal{D} (\mathcal{D}^2 + 1) (\mathcal{D}^2 + 4) \cdots \left(\mathcal{D}^2 + \left(\frac{n-1}{2} \right)^2 \right), & \text{for } n \text{ odd,} \\ \left(\mathcal{D}^2 + \frac{1}{4} \right) \left(\mathcal{D}^2 + \frac{9}{4} \right) \cdots \left(\mathcal{D}^2 + \left(\frac{n-1}{2} \right)^2 \right), & \text{for } n \text{ even.} \end{cases}$$

we define $\Theta(s) = \sin(s)$ and $\chi(s) = \cos(s)$. The null space of the operator \mathbb{T}_n is the space of trigonometric polynomials of order n which is given by

$$\mathbb{T}_n = \begin{cases} \text{span} \left\{ 1, \Theta(s), \chi(s), \dots, \Theta\left(\frac{(n-1)s}{2}\right), \chi\left(\frac{(n-1)s}{2}\right) \right\}, & \text{for } n \text{ odd,} \\ \text{span} \left\{ 1, \Theta\left(\frac{s}{2}\right), \chi\left(\frac{s}{2}\right), \dots, \Theta\left(\frac{(n-1)s}{2}\right), \chi\left(\frac{(n-1)s}{2}\right) \right\}, & \text{for } n \text{ even.} \end{cases}$$

We note that $\mathbb{T}_1 \subset \mathbb{T}_n$ if $n-1 \geq 0$ is even, but not if it is odd (see [32]).

Let $\{s_j\}_{0 \leq j \leq N}$ be a partition of the interval $\mathcal{J} := [0, 2\pi]$ such that $0 = s_0 < s_1 < \dots < s_j < s_{j+1} < \dots < s_n = 2\pi$. It is extended to the real line by defining $s_{kn+j} = 2k\pi + s_j$, $k \in \mathbb{Z}$.

We define the n th-order trigonometric divided difference of a function φ with respect to the space \mathbb{T}_n

as follows

$$[s_j, s_{j+1}, \dots, s_{j+n}]_{\mathbb{T}_n} \Phi = 2^n \begin{array}{c} \left| \begin{array}{cccc} \Theta(\frac{s_j}{2}) & \dots & \dots & \dots & \Theta(\frac{s_{j+n}}{2}) \\ \chi(\frac{s_j}{2}) & \dots & \dots & \dots & \chi(\frac{s_{j+n}}{2}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta(\frac{(n-1)s_j}{2}) & \dots & \dots & \dots & \Theta(\frac{(n-1)s_{j+n}}{2}) \\ \chi(\frac{(n-1)s_j}{2}) & \dots & \dots & \dots & \chi(\frac{(n-1)s_{j+n}}{2}) \\ \Phi(s_j) & \dots & \dots & \dots & \Phi(s_{j+n}) \end{array} \right| \\ \left| \begin{array}{cccc} 1 & \dots & \dots & \dots & 1 \\ \Theta(s_j) & \dots & \dots & \dots & \Theta(s_{j+n}) \\ \chi(s_j) & \dots & \dots & \dots & \chi(s_{j+n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta(\frac{ns_j}{2}) & \dots & \dots & \dots & \Theta(\frac{ns_{j+n}}{2}) \\ \chi(\frac{ns_j}{2}) & \dots & \dots & \dots & \chi(\frac{ns_{j+n}}{2}) \end{array} \right| \end{array}.$$

As in [25, 27], for some shape parameter c let us consider the periodic kernel

$$\Phi(s) = \sqrt{c^2 + \Theta^2(\frac{s}{2})}.$$

Taking the n th-order trigonometric divided differences to the kernel Φ , we get the n th-order trigonometric B-splines

$$\varphi_{j,n}(s) := [s_j, s_{j+1}, \dots, s_{j+n}]_{\mathbb{T}_n} \Phi(s - t).$$

Hence, the the main generalized quasi-interpolation scheme can be defined via these MQ trigonometric B-splines $\Phi_{j,n}$ as follows

$$\sigma_{N,l}^G u(s) = \sum_{j=0}^{n-1} \Theta(\frac{s_{j+N} - s_j}{2}) \sum_{i=1}^l \alpha_{j,i} \lambda_{j,i}(u) \Phi_{j,N}(s), \quad 1 \leq l \leq N, \quad \text{and } N - l \text{ is an even integer,}$$

where the coefficients $\alpha_{j,i}$ are given in [32] and $\lambda_{j,i}$ are some prescribed linear functionals.

Applying the second-order trigonometric divided differences to the kernel gives the MQ trigonometric B-splines

$$\varphi_j(s) = \frac{1}{2} [s_{j-1}, s_j, s_{j+1}]_{\mathbb{T}_2} \Phi = 2 \frac{\left| \begin{array}{ccc} \Theta(\frac{s_{j-1}}{2}) & \Theta(\frac{s_j}{2}) & \Theta(\frac{s_{j+1}}{2}) \\ \chi(\frac{s_{j-1}}{2}) & \chi(\frac{s_j}{2}) & \chi(\frac{s_{j+1}}{2}) \\ \Phi(s - s_{j-1}) & \Phi(s - s_j) & \Phi(s - s_{j+1}) \end{array} \right|}{\left| \begin{array}{ccc} 1 & 1 & 1 \\ \Theta(s_{j-1}) & \Theta(s_j) & \Theta(s_{j+1}) \\ \chi(s_{j-1}) & \chi(s_j) & \chi(s_{j+1}) \end{array} \right|}.$$

Taking these MQ trigonometric B-splines $\varphi_j(\cdot)$ as basis functions, we get

$$\sigma f(s) = \sum_{j=1}^n \Theta(\frac{s_{j+1} - s_{j-1}}{2}) f(s_j) \varphi_j(s),$$

with

$$s_0 = 0 < \dots < s_n = 2\pi.$$

and

$$\Theta(\frac{s_{j+1} - s_{j-1}}{2}) \varphi_j(s) = \frac{\Phi(s - s_{j+1}) - \chi(\frac{s_{j+1} - s_j}{2}) \Phi(s - s_j)}{2\Theta(\frac{s_{j+1} - s_j}{2})} - \frac{\chi(\frac{s_j - s_{j-1}}{2}) \Phi(s - s_j) - \Phi(s - s_{j-1})}{2\Theta(\frac{s_j - s_{j-1}}{2})}.$$

With some simple calculations and the periodicity of $f(s)$, we have

$$\sigma f(s) = \sum_{j=1}^n \gamma_j(f) \Phi_j(s),$$

Where the coefficients are defined by

$$\gamma_j(f) := \frac{f(s_{j-1})}{2\Theta(\frac{s_j-s_{j-1}}{2})} - \left(\frac{\chi(\frac{s_{j+1}-s_j}{2})}{2\Theta(\frac{s_{j+1}-s_j}{2})} + \frac{\chi(\frac{s_j-s_{j-1}}{2})}{2\Theta(\frac{s_j-s_{j-1}}{2})} \right) f(s_j) + \frac{f(s_{j+1})}{2\Theta(\frac{s_{j+1}-s_j}{2})}, \quad 2 \leq j \leq n-1,$$

For $j = 1, n$:

$$\begin{aligned} \gamma_1(f) &:= \frac{f(s_n)}{2\Theta(\frac{s_1-s_0}{2})} - \left(\frac{\chi(\frac{s_2-s_1}{2})}{2\Theta(\frac{s_2-s_1}{2})} + \frac{\chi(\frac{s_1-s_0}{2})}{2\Theta(\frac{s_1-s_0}{2})} \right) f(s_1) + \frac{f(s_2)}{2\Theta(\frac{s_2-s_1}{2})} \\ \gamma_n(f) &:= \frac{f(s_{n-1})}{2\Theta(\frac{s_n-s_{n-1}}{2})} - \left(\frac{\chi(\frac{s_2-s_1}{2})}{2\Theta(\frac{s_2-s_1}{2})} + \frac{\chi(\frac{s_n-s_{n-1}}{2})}{2\Theta(\frac{s_n-s_{n-1}}{2})} \right) f(s_n) + \frac{f(s_1)}{2\Theta(\frac{s_1-s_0}{2})} \end{aligned}$$

4.3 Solving periodic second kind integral equation of Fredholm type

For a given continuous function f , we consider the following Fredholm integral equation of the second kind

$$u(s) - \int_0^{2\pi} \psi(s, \tau) u(\tau) d\tau = f(s), \quad s \in \mathcal{I}. \quad (4.1)$$

We assume that $\psi(\cdot, \cdot) \in C(\mathcal{I} \times \mathcal{I}, \mathbb{C})$, and consider the uniform partition with mesh length $h := \frac{2\pi}{n}$, i.e $\mathcal{T}_n := \{s_i = ih, \quad 0 \leq i \leq n+1\}$, with $s_i : ih$

4.3.1 First degenerate kernel method

The first method consists of approximating by

$$\psi_n^F(s, t) := \sum_{j=1}^n \gamma_j(\psi(\cdot, \tau)) \Phi_j(s).$$

But,

$$\begin{aligned} \begin{bmatrix} \gamma_1(\psi(\cdot, \tau)) \\ \gamma_2(\psi(\cdot, \tau)) \\ \vdots \\ \gamma_n(\psi(\cdot, \tau)) \end{bmatrix} &= \begin{bmatrix} \left(-\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} \right) \psi(s_1, \tau) + \left(\frac{1}{2\sin \frac{h}{2}} \right) \psi(s_2, \tau) + \left(\frac{1}{2\sin \frac{h}{2}} \right) \psi(s_n, \tau) \\ \left(\frac{1}{2\sin \frac{h}{2}} \right) \psi(s_1, \tau) - \left(\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} \right) \psi(s_2, \tau) + \left(\frac{1}{2\sin \frac{h}{2}} \right) \psi(s_3, \tau) \\ \vdots \\ \left(\frac{1}{2\sin \frac{h}{2}} \right) \psi(s_1, \tau) + \left(\frac{1}{2\sin \frac{h}{2}} \right) \psi(s_{n-1}, \tau) - \left(\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} \right) \psi(s_n, \tau) \end{bmatrix} \\ &= \Lambda \begin{bmatrix} \psi(s_1, \tau) \\ \psi(s_2, \tau) \\ \vdots \\ \psi(s_n, \tau) \end{bmatrix}, \end{aligned}$$

Where

$$\Lambda := \begin{bmatrix} -\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} & \frac{1}{2\sin \frac{h}{2}} & 0 & 0 & \cdots & \frac{1}{2\sin \frac{h}{2}} \\ \frac{1}{2\sin \frac{h}{2}} & -\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} & \frac{1}{2\sin \frac{h}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2\sin \frac{h}{2}} & -\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} & \frac{1}{2\sin \frac{h}{2}} \\ \frac{1}{2\sin \frac{h}{2}} & 0 & \cdots & 0 & \frac{1}{2\sin \frac{h}{2}} & -\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} \end{bmatrix}$$

Then, replacing $\psi(.,.)$ by $\psi_n(.,.)$ in (4.1), we get the integral equation

$$u_n^F(s) - \int_0^{2\pi} \psi_n^F(s, \tau) u_n^F(\tau) d\tau = f(s) \quad (4.2)$$

for an approximate solution to u .

The approximate equation (4.2) reads as

$$u_n^F(s) - \sum_{j=1}^n \Phi_j(s) \int_0^{2\pi} \gamma_j(\psi(., \tau)) u_n^F(\tau) d\tau = f(s).$$

Putting

$$x_j^F := \int_0^{2\pi} \gamma_j(\psi(., \tau)) u_n^F(\tau) d\tau,$$

we get

$$u_n^F(s) = \sum_{j=1}^n x_j^F \Phi_j(s) + f(s),$$

so that

$$\sum_{j=1}^n x_j^F \Phi_j(s) + f(s) - \sum_{j=1}^n \Phi_j(s) \int_0^{2\pi} \gamma_j(\psi(., \tau)) \left[\sum_{m=1}^n x_m^F \Phi_m(\tau) + f(\tau) \right] d\tau = f(s).$$

Hence

$$\sum_{j=1}^n \Phi_j(s) \left[x_j^F - \int_0^{2\pi} \gamma_j(\psi(., \tau)) f(\tau) d\tau - \sum_{m=1}^n x_m^F \int_0^{2\pi} \gamma_j(\psi(., \tau)) \Phi_m(\tau) d\tau \right] = 0,$$

and hence

$$x_j^F - \int_0^{2\pi} \gamma_j(\psi(., \tau)) f(\tau) d\tau - \sum_{m=1}^n x_m^F \int_0^{2\pi} \gamma_j(\psi(., \tau)) \Phi_m(\tau) d\tau = 0.$$

Thus,

$$x_j^F - \sum_{m=1}^n x_m^F \int_0^{2\pi} \gamma_j(\psi(., \tau)) \Phi_m(\tau) d\tau = \int_0^{2\pi} \gamma_j(\psi(., \tau)) f(\tau) d\tau.$$

We have proved the following theorem:

Theorem 4.3.1. *The coefficients $X^F := x_j^F$ of the solution (4.2) satisfy the system of linear equations*

$$(I - A^F)X^F = b^F,$$

where

$$b_j^F = \int_0^{2\pi} \gamma_j(\psi(., \tau)) f(\tau) d\tau, \quad \text{and} \quad A_{mj}^F = \int_0^{2\pi} \gamma_j(\psi(., \tau)) \Phi_m(\tau) d\tau.$$

4.3.2 Second degenerate kernel method

The second kernel method is obtained by approximating the kernel at right to get

$$\psi_n^R(s, t) := \sum_{j=1}^n \gamma_j(\psi(s, .)) \Phi_j(\tau).$$

Where

$$\begin{bmatrix} \gamma_1(\psi(s, .)) \\ \gamma_2(\psi(s, .)) \\ \vdots \\ \gamma_n(\psi(s, .)) \end{bmatrix} = \begin{bmatrix} \left(-\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} \right) \psi(s, \tau_1) + \left(\frac{1}{2 \sin \frac{h}{2}} \right) \psi(s, \tau_2) + \left(\frac{1}{2 \sin \frac{h}{2}} \right) \psi(s, \tau_n) \\ \left(\frac{1}{2 \sin \frac{h}{2}} \right) \psi(s, \tau_1) - \left(\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} \right) \psi(s, \tau_2) + \left(\frac{1}{2 \sin \frac{h}{2}} \right) \psi(s, \tau_3) \\ \vdots \\ \left(\frac{1}{2 \sin \frac{h}{2}} \right) \psi(s, \tau_1) + \left(\frac{1}{2 \sin \frac{h}{2}} \right) \psi(s, \tau_{n-1}) - \left(\frac{\cos \frac{h}{2}}{\sin \frac{h}{2}} \right) \psi(s, \tau_n) \end{bmatrix},$$

Then, by replacing $\psi(.,.)$ by $\psi_n^R(.,.)$ the following approximate integral equation is obtained for an approximated solution u_n^R to u :

$$u_n^R(s) - \int_0^{2\pi} \psi_n^R(s, \tau) u_n^R(\tau) d\tau = f(s). \quad (4.3)$$

Following (4.3) we have

$$u_n^R(s) - \sum_{j=1}^n \gamma_j(\psi(s, .)) \int_0^{2\pi} \Phi_j(\tau) u_n^R(\tau) d\tau = f(s).$$

We obtain

$$u_n^R(s) = f(s) + \sum_{j=1}^n \gamma_j(\psi(s, .)) x_j^R,$$

with

$$x_j^R := \int_0^{2\pi} \Phi_j(\tau) u_n^R(\tau) d\tau,$$

so that

$$u_n^R(\tau) = f(\tau) + \sum_{m=1}^n \gamma_m(\psi(\tau, .)) x_m^R.$$

Hence

$$\sum_{j=1}^n \gamma_j(\psi(s, .)) \left[x_j^R - \int_0^{2\pi} \Phi_j(\tau) f(\tau) d\tau - \sum_{m=1}^n x_m^R \int_0^{2\pi} \gamma_m(\psi(\tau, .)) \Phi_j(\tau) d\tau \right] = 0.$$

Consequently,

$$x_j^R - \sum_{m=1}^n x_m^R \int_0^{2\pi} \gamma_m(\psi(\tau, .)) \Phi_j(\tau) d\tau = \int_0^{2\pi} \Phi_j(\tau) f(\tau) d\tau.$$

We have proved the following result:

Theorem 4.3.2. *The coefficients $X^R := x_j^R$ of the solution of (4.3) satisfy the system of linear equations*

$$(I - A^R)X^R = b^R,$$

where

$$b_j^R = \int_0^{2\pi} \Phi_j(\tau) f(\tau) d\tau, \quad \text{and} \quad A_{mj}^R = \int_0^{2\pi} \gamma_m(\psi(\tau, .)) \Phi_j(\tau) d\tau.$$

4.3.3 A tensor product scheme via the main quasi-interpolation

In this subsection, we introduce a tensor product scheme based on the quasi-interpolation operator to approximate the kernel $\psi(.,.)$ of 4.1 as follows:

$$\psi_n^P(s, \tau) := (\sigma_n^G \otimes \sigma_n^G) \psi(s, \tau).$$

Letting

$$\Psi = \Lambda \begin{bmatrix} \psi(s_1, \tau_1) & \psi(s_1, \tau_2) & \cdots & \psi(s_1, \tau_n) \\ \psi(s_2, \tau_1) & \psi(s_2, \tau_2) & \cdots & \psi(s_2, \tau_n) \\ \vdots & \vdots & & \vdots \\ \psi(s_n, \tau_1) & \psi(s_n, \tau_2) & \cdots & \psi(s_n, \tau_n) \end{bmatrix} \Lambda^T,$$

$$\Phi(t) := (\Phi_1(t), \Phi_2(t), \dots, \Phi_n(t)).$$

First, we will prove in the following Proposition that the approximate kernel $\psi_n^P(.,.)$ can be written in an important form as follows

$$\psi_n^P(s, t) = \sum_{i=1}^n \sum_{j=1}^n \Psi(i, j) \Phi_i(s) \Phi_j(\tau).$$

We have

$$\begin{aligned}\psi_n^P(s, \tau) &:= \sigma_n^G\left(\sigma_n^G(\psi(s, \cdot))(\tau)\right) \\ &= \sigma_n^G\left(\sum_{j=1}^n \gamma_j(\psi(s, \cdot))\Phi_j(\tau)\right) \\ &= \sigma_n^G\left(\Phi(\tau) \begin{bmatrix} \gamma_1(\psi(s, \cdot)) \\ \gamma_2(\psi(s, \cdot)) \\ \vdots \\ \gamma_n(\psi(s, \cdot)) \end{bmatrix}\right).\end{aligned}$$

But,

$$\begin{bmatrix} \gamma_1(\psi(s, \cdot)) \\ \gamma_2(\psi(s, \cdot)) \\ \vdots \\ \gamma_n(\psi(s, \cdot)) \end{bmatrix} = \Lambda \begin{bmatrix} \psi(s, \tau_1) \\ \psi(s, \tau_2) \\ \vdots \\ \psi(s, \tau_n) \end{bmatrix},$$

so that

$$\psi_n^P(s, \tau) = \Phi(\tau)\Lambda \begin{bmatrix} \sigma_n^G(\psi(\cdot, \tau_1))(s) \\ \sigma_n^G(\psi(\cdot, \tau_2))(s) \\ \vdots \\ \sigma_n^G(\psi(\cdot, \tau_n))(s) \end{bmatrix}.$$

Moreover,

$$\begin{aligned}\begin{bmatrix} \sigma_n^G(\psi(\cdot, \tau_1))(s) \\ \sigma_n^G(\psi(\cdot, \tau_2))(s) \\ \vdots \\ \sigma_n^G(\psi(\cdot, \tau_n))(s) \end{bmatrix} &= \begin{bmatrix} \sum_{j=1}^n \gamma_j(\psi(\cdot, \tau_1))\Phi_j(s) \\ \sum_{j=1}^n \gamma_j(\psi(\cdot, \tau_2))\Phi_j(s) \\ \vdots \\ \sum_{j=1}^n \gamma_j(\psi(\cdot, \tau_n))\Phi_j(s) \end{bmatrix} \\ &= \begin{bmatrix} \gamma_1(\psi(\cdot, \tau_1)) & \gamma_1(\psi(\cdot, \tau_2)) & \cdots & \gamma_1(\psi(\cdot, \tau_n)) \\ \gamma_2(\psi(\cdot, \tau_1)) & \gamma_2(\psi(\cdot, \tau_2)) & \cdots & \gamma_2(\psi(\cdot, \tau_n)) \\ \vdots & \vdots & & \vdots \\ \gamma_n(\psi(\cdot, \tau_1)) & \gamma_n(\psi(\cdot, \tau_2)) & \cdots & \gamma_n(\psi(\cdot, \tau_n)) \end{bmatrix} \Phi(s)^T \\ &= \begin{bmatrix} \psi(s_1, \tau_1) & \psi(s_1, \tau_2) & \cdots & \psi(s_1, \tau_n) \\ \psi(s_2, \tau_1) & \psi(s_2, \tau_2) & \cdots & \psi(s_2, \tau_n) \\ \vdots & \vdots & & \vdots \\ \psi(s_n, \tau_1) & \psi(s_n, \tau_2) & \cdots & \psi(s_n, \tau_n) \end{bmatrix} \Lambda^T \Phi(s)^T.\end{aligned}$$

Consequently,

$$\begin{aligned}\psi_n^P(s, \tau) &= \Phi(\tau)\Lambda \begin{bmatrix} \psi(s_1, \tau_1) & \psi(s_1, \tau_2) & \cdots & \psi(s_1, \tau_n) \\ \psi(s_2, \tau_1) & \psi(s_2, \tau_2) & \cdots & \psi(s_2, \tau_n) \\ \vdots & \vdots & & \vdots \\ \psi(s_n, \tau_1) & \psi(s_n, \tau_2) & \cdots & \psi(s_n, \tau_n) \end{bmatrix} \Lambda^T \Phi(s)^T \\ &= \Phi(\tau)\Psi\Phi(s)^T,\end{aligned}$$

and the proof the following result is complete.

Proposition 4.3.1. *The approximate kernel $\psi_n^P(\cdot, \cdot)$ satisfies the following matrix representation*

$$\psi_n^P(s, \tau) = \Phi(\tau)\Psi\Phi(s)^T.$$

Replacing ψ by $\psi_n^P(\cdot, \cdot)$ in (4.1), we get the following approximate equation

$$u_n^P(s) = f(s) + \int_0^{2\pi} \psi_n^P(s, \tau) u_n^P(\tau) d\tau. \quad (4.4)$$

Theorem 4.3.3. *The coefficients $X^P := X_m^P$ of the solution (4.4) satisfy the system of linear equations*

$$(I - \Psi A^P) X^P = \Psi b^P,$$

where

$$b_i^P = \int_0^{2\pi} \Phi_i(\tau) f(\tau) d\tau, \quad \text{and} \quad A(i, j)^P := \int_0^{2\pi} \Phi_i \Phi_j.$$

Proof. We have

$$\begin{aligned} u_n^P(s) &= f(s) + \sum_{j=1}^n \sum_{m=1}^n \Psi(m, j) \Phi_j(s) \int_0^{2\pi} \Phi_m(\tau) u_n^P(\tau) d\tau \\ &= f(s) + \sum_{j=1}^n \Phi_j(s) \sum_{m=1}^n \Psi(m, j) \int_0^{2\pi} \Phi_m(\tau) u_n^P(\tau) d\tau, \end{aligned}$$

so that

$$u_n^P(s) = f(s) + \sum_{j=1}^n X_j^P \Phi_j(s),$$

with

$$X_j^P := \sum_{m=1}^n \Psi(m, j) \int_0^{2\pi} \Phi_m(\tau) u_n^P(\tau) d\tau.$$

Hence

$$X_j^P = \sum_{m=1}^n \Psi(m, j) \int_0^{2\pi} \Phi_m(\tau) f(\tau) d\tau + \sum_{k=1}^n X_k \sum_{m=1}^n \Psi(m, j) \int_0^{2\pi} \Phi_m(\tau) \Phi_k(\tau) d\tau.$$

□

4.3.4 Kernel's blending approximation

Now, we propose to approximate the kernel $\psi(\cdot, \cdot)$ by

$$\psi_n^B(s, \tau) := (\sigma_n^G \otimes \mathcal{I} + \mathcal{I} \otimes \sigma_n^G - \sigma_n^G \otimes \sigma_n^G) \psi(s, \tau).$$

Next, we look for an appropriate expression for $\psi_n^B(s, \tau)$.

Lemma 4.3.1. *It holds*

$$\psi_n^B(s, \tau) := \sum_{j=1}^n \psi_j^{(1)}(\tau) \Phi_j(s) + \sum_{m=1}^n \psi_m^{(2)}(s) \Phi_m(\tau) - \sum_{j=1}^n \sum_{m=1}^n \Psi(m, j) \Phi_j(s) \Phi_m(\tau),$$

where

$$\psi_j^{(1)} = \gamma_j(\psi(\cdot, \tau)) \quad \text{and} \quad \psi_m^{(2)} = \gamma_j(\psi(s, \cdot)).$$

Proof. We have

$$(\sigma_n^G \otimes \mathcal{I}) \psi(s, \tau) = \sum_{j=1}^n \gamma_j(\psi(\cdot, \tau)) \Phi_j(s),$$

and

$$(\mathcal{I} \otimes \sigma_n^G) \psi(s, \tau) = \sum_{m=1}^n \gamma_j(\psi(s, \cdot)) \Phi_m(\tau).$$

By using

$$(\sigma_n^G \otimes \sigma_n^G) \psi(s, \tau) = \sum_{j=1}^n \sum_{m=1}^n \Psi(m, j) \Phi_j(s) \Phi_m(\tau)$$

we get the desired result. □

Replacing $\psi(.,.)$ by $\psi_n^B(.,.)$ in (4.1), we get an integral equation of the approximate solution u_n^B :

$$u_n^B(s) = f(s) + \int_0^{2\pi} \psi_n^B(s, \tau) u_n^B(\tau) d\tau.$$

Proposition 4.3.2. *For some vectors X and Y , the vector $\chi^B := (X, Y)^T$ satisfies the following system of linear equations*

$$(I - \kappa^B)\chi^B = b^B,$$

where

$$b^B := (M, N)^T \quad \text{and} \quad \kappa^B := \begin{bmatrix} \Psi^{(1)} & \Psi^{(3)} - \Psi \\ A^p & \Psi^{(1)} \end{bmatrix}.$$

Proof. We have

$$\begin{aligned} u_n^B(s) &= f(s) + \sum_{j=1}^n \Phi_j(s) \int_0^{2\pi} \psi_j^{(1)}(\tau) u_n^B(\tau) d\tau + \sum_{m=1}^n \psi_m^{(2)}(s) \int_0^{2\pi} \Phi_m(\tau) u_n^B(\tau) d\tau \\ &\quad - \sum_{j=1}^n \sum_{m=1}^n \Psi(m, j) \Phi_j(s) \int_0^{2\pi} \Phi_m(\tau) u_n^B(\tau) d\tau. \end{aligned}$$

The approximate solution has the form

$$u_n^B(s) = f(s) + \sum_{j=1}^n X_j \Phi_j(s) + \sum_{m=1}^n Y_m \psi_m^{(2)}(s).$$

Hence

$$\begin{aligned} \sum_{j=1}^n X_j \Phi_j(s) + \sum_{m=1}^n Y_m \psi_m^{(2)}(s) &= \int_0^{2\pi} \Psi_n(s, \tau) u_n^B(\tau) d\tau \\ &= \sum_{j=1}^n \Phi_j(s) \left(\int_0^{2\pi} \psi_j^{(1)}(\tau) u_n^B(\tau) d\tau - \sum_{m=1}^n \Psi(m, j) \int_0^{2\pi} \Phi_m(\tau) u_n^B(\tau) d\tau \right) \\ &\quad + \sum_{m=1}^n \psi_m^{(2)}(s) \int_0^{2\pi} \Phi_m(\tau) u_n^B(\tau) d\tau. \end{aligned}$$

It is sufficient to choose $X := (X_j)$ and $Y := (Y_m)$, where

$$\begin{aligned} X_j &:= \int_0^{2\pi} \psi_j^{(1)}(\tau) u_n^B(\tau) d\tau - \sum_{m=1}^n \Psi(m, j) \int_0^{2\pi} \Phi_m(\tau) u_n^B(\tau) d\tau, \\ Y_m &= \int_0^{2\pi} \Phi_m(\tau) u_n^B(\tau) d\tau. \end{aligned}$$

Letting

$$M_j := \int_0^{2\pi} \psi_j^{(1)}(\tau) f(\tau) d\tau, \quad N_j := \int_0^{2\pi} \Phi_j(\tau) f(\tau) d\tau,$$

and

$$\Psi^{(1)}(m, j) := \int_0^{2\pi} \psi_j^{(1)}(\tau) \Phi_m(\tau) d\tau, \quad \Psi^{(2)}(m, j) := \int_0^{2\pi} \Phi_j(\tau) \psi_m^{(2)}(\tau) d\tau, \quad \Psi^{(3)} := \int_0^{2\pi} \psi_j^{(1)}(\tau) \psi_m^{(2)}(\tau) d\tau.$$

We obtain

$$Y_m = N_m + \sum_{r=1}^n A^p(r, s) X_r + \sum_{s=1}^n \Psi^{(2)}(r, s) Y_s,$$

and

$$\begin{aligned} X_j &:= (M_j - \sum_{m=1}^n \Psi(m, j)N_m) + \sum_{r=1}^n X_r(\Psi^{(1)}(j, r) - \sum_{m=1}^n \Psi(m, j)A^p(m, r)) \\ &+ \sum_{s=1}^n Y_s(\Psi^{(3)}(j, s) - \sum_{m=1}^n \Psi(m, j)\Psi^{(2)}(m, s)), \end{aligned}$$

so that

$$X = M - \Psi N + (\Psi^{(1)} - \Psi A^p)X + (\Psi^{(3)} - \Psi\Psi^{(2)})Y,$$

and

$$Y = N + A^p X + \Psi^{(2)}Y.$$

By using

$$\Psi^{(2)}Y = Y - A^p X - N$$

we get

$$X = M + \Psi^{(1)}X + (\Psi^{(3)} - \Psi)Y,$$

and the proof is complete. \square

4.4 Convergence analysis

Let σ_n be the trigonometric B-spline quasi-interpolation scheme. Following [32], we have

$$\|\sigma_n f - f\|_\infty \leq O(h^2).$$

In order to prove the main result concerning the convergence analysis, we follow [32, 25] to prove the following theorem.

Theorem 4.4.1. *The following result holds*

$$\|(\sigma_n^G - \sigma_n)f\|_\infty = O(ch) + O(c^2(\pi - h)).$$

Proof. We have

$$2(\sigma_n^G f(s) - \sigma_n f(s)) = \sum_{j=1}^n \sin \frac{s_{j+1} - s_{j-1}}{2} (\Phi(s - s_j) - |\sin \frac{s - s_j}{2}|) [s_{j-1}, s_j, s_{j+1}]_{Tr_2} f.$$

Letting

$$I_N(s) := \sum_{j=1}^n \sin \frac{s_{j+1} - s_{j-1}}{2} (\Phi(s - s_j) - |\sin \frac{s - s_j}{2}|).$$

Since

$$\begin{aligned} (\sqrt{c^2 + \sin^2(\frac{s - s_j}{2})} - |\sin \frac{s - s_j}{2}|) &\leq c, \\ (\sqrt{c^2 + \sin^2(\frac{s - s_j}{2})} - |\sin \frac{s - s_j}{2}|) &\leq \frac{c^2}{2|\sin \frac{s - s_j}{2}|}, \quad |\sin \frac{s - s_j}{2}| \neq 0, \end{aligned}$$

we get

$$\begin{aligned} I_N(s) &\leq c \sum_{|s - s_j| \leq h} \sin \frac{s_{j+1} - s_{j-1}}{2} + c \sum_{2\pi - h \leq |s - s_j| \leq 2\pi} \sin \frac{s_{j+1} - s_{j-1}}{2} + \frac{c^2}{2} \sum_{h \leq |s - s_j| \leq 2\pi - h} \frac{\sin \frac{s_{j+1} - s_{j-1}}{2}}{|\sin \frac{s - s_j}{2}|} \\ &\leq 2c \sum_{|s - s_j| \leq h} \sin \frac{s_{j+1} - s_{j-1}}{2} + \frac{c^2}{2} \sum_{h \leq |s - s_j| \leq 2\pi - h} \frac{\sin \frac{s_{j+1} - s_{j-1}}{2}}{|\sin \frac{s - s_j}{2}|}. \end{aligned}$$

But

$$\begin{aligned} s_{j+1} - s_{j-1} &\leq |s_{j+1} - s| + |s - s_{j-1}| \\ &\leq |s_{j+1} - s_j| + 2|s - s_j| + |s_j - s_{j-1}| \\ &\leq 2h + 2|s - s_j|, \end{aligned}$$

hence

$$\sin \frac{s_{j+1} - s_{j-1}}{2} \leq \frac{s_{j+1} - s_{j-1}}{2} \leq h + |s - s_j|,$$

and hence

$$I_N(s) \leq O(ch) + \frac{c^2}{2} \sum_{h \leq |s - s_j| \leq 2\pi - h} \frac{\sin \frac{s_{j+1} - s_{j-1}}{2}}{|\sin \frac{s - s_j}{2}|}.$$

Moreover

$$\frac{c^2}{2} \sum_{h \leq |s - s_j| \leq 2\pi - h} \frac{\sin \frac{s_{j+1} - s_{j-1}}{2}}{|\sin \frac{s - s_j}{2}|} \leq c^2 \int_{h \leq |s - t| \leq 2\pi - h} \frac{1}{|\sin \frac{s - t}{2}|} dt + O(c^2 h).$$

Thus,

$$c^2 \int_{h \leq |s - t| \leq 2\pi - h} \frac{1}{|\sin \frac{s - t}{2}|} dt = 4c^2 \int_{\frac{h}{2} \leq |s| \leq \pi - \frac{h}{2}} \frac{1}{|\sin s|} ds \leq O(c^2(\pi - h)).$$

Consequently,

$$I_N(s) = O(ch) + O(c^2(\pi - h)) + O(c^2 h),$$

and the claim follows. \square

Corollary 4.4.2. *It holds*

$$\|\sigma_n^G f - f\|_\infty = O(h^2).$$

Proof. We note that

$$\|\sigma_n^G f - f\|_\infty \leq \|\sigma_n^G f - \sigma_n f\|_\infty + \|\sigma_n f - f\|_\infty.$$

Following [32],

$$\|\sigma_n f - f\|_\infty = O(h^2),$$

hence, we get the desired result. \square

4.5 Numerical examples and application

Loves equation is the Fredholm integral equation given by

$$u(s) - \int_0^{2\pi} \psi(s, \tau) u(\tau) d\tau = f(s), \quad (4.5)$$

where the kernel $\psi(s, t) = \cos[s + t]$. Table 1 shows that the results of the present method are better than those obtained.

$\ u - u_{1,3,n}\ _{\infty, [3.79, 3.394]}$		
n	Right 3	CB
8	4.5×10^{-1}	3.14×10^{-1}
16	5.41×10^{-2}	4.01×10^{-2}
32	1.33×10^{-2}	3.37×10^{-4}

Table 4.1:

Chapter 5

New Construction of C^1 -Cubic Quasi-Interpolation Splines

5.1 Introduction

A novel, non-standard technique for constructing bivariate quasi-interpolating splines over uniform partitions was proposed by T. Sorokina and F. Zeilfelder in [51, 53] (see also [40, 52]). The essential idea of this methodology is to define the quasi-interpolant by directly providing the coefficients of the Bernstein–Bézier (BB-) form of its restriction to each of the subsets forming the partition.

In [53], the construction of C^1 quartic quasi-interpolants over a type-1 triangulation is addressed, so that the largest polynomial space is reproduced, namely the space \mathbb{P}_3 of polynomials of total degree less than or equal to three (see Figure 5.1). The coefficients of the quasi-interpolant on each triangle are linear combinations of function values at vertices and midpoints in a neighborhood of the triangle. The quasi-interpolant is constructed from them.

In [51], the same strategy is applied to construct C^1 quadratic quasi-interpolants on a triangulation which the authors called of type-2. Starting from a decomposition of the plane into squares, each of them is divided into eight micro-triangles by means of its diagonals and the straight lines parallel to the coordinate axes passing through the center of the square (see Figure 5.1).

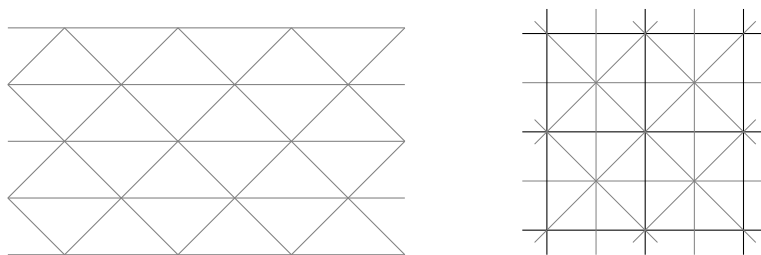


Figure 5.1: From left to right, type-1 and type-2 triangulations on which C^1 -continuous quasi-interpolants are constructed in [53, 51]: quartic and exact on \mathbb{P}_3 , and quadratic and exact on \mathbb{P}_2 , respectively.

The problem addressed in [53] is studied in detail in [9], proving that the approximation scheme proposed in [53] is a particular choice in a 19-parametric family of schemes. Moreover, different strategies for assigning values to the parameters are provided. In both [53] and [9], the quasi-interpolating splines interpolate the values at the vertices and the masks associated with the domain points that are key to the construction are applied taking into account the symmetries of the triangulation involved. Since the triangulation is uniform, these masks are independent of the specific triangle on which the quasi-interpolant is calculated (see also [10]).

Later, the cubic case was dealt with in [11], on the same triangulation used to construct quartic quasi-interpolants. The aim was to construct a C^1 cubic one, exact on \mathbb{P}_2 , from the values at vertices and midpoints. Since it is not possible to define a quasi-interpolant that interpolates values at vertices, the authors opted to find specific masks for key domain points, including vertices, without imposing any

symmetry. It was proved that there are unique masks that satisfy the required properties. Not being possible to achieve exactness on \mathbb{P}_3 , this paper presents a construction on a refinement of the initial type-1 triangulation in order to achieve the optimal approximation order. Specifically, we work on a Clough–Tocher (CT-) refinement [17], which produces a subdivision into six micro-triangles of each square formed by two macro-triangles sharing an edge.

The rest of the chapter is structured as follows. In Section 5.2, a type-1 triangulation endowed with a Clough–Tocher refinement is introduced, as well as the space of C^1 cubic splines defined over it. Further, a partition of the domain points associated with the micro-triangles is provided. In Section 5.3, the construction of quasi-interpolating splines is given and the general solution of the resulting problem. In Section 5.4, a method for selecting parameters based on the minimization of an upper bound of the quasi-interpolation error associated with the quartic monomials is proposed. In Section 5.5, the results of some numerical tests are given to illustrate the performance of the quasi-interpolation operator relative to the selected parameters. Finally, some details are included in an appendix.

5.2 Bernstein–Bézier Form of Cubic Splines on a Type-1 Triangulation

Let us suppose that the triangulation is spanned by the vectors $e_1 := (h, h)$ and $e_2 := (h, -h)$, with $h > 0$. Its vertices are $v_{i,j} := ie_1 + je_2$, which define the lattice $\mathcal{V} := \{v_{i,j}, i, j \in \mathbb{Z}\}$. These vertices define squares which can be decomposed into the triangles $\mathbb{T}_{i,j} \langle v_{i,j}, v_{i+1,j+1}, v_{i+1,j} \rangle$ and $\mathbb{B}_{i,j} \langle v_{i,j}, v_{i+1,j+1}, v_{i,j+1} \rangle$ (see Figure 5.2). Therefore, a type-1 triangulation results:

$$\Delta := \bigcup_{i,j \in \mathbb{Z}} (\mathbb{T}_{i,j} \cup \mathbb{B}_{i,j}).$$

When there is no need to distinguish between the types of triangles in Δ , we denote by \mathbb{T} any one of them.

To define the refinement of Δ to be used, let

$$t_{i,j} := \frac{1}{3}(v_{i,j} + v_{i+1,j+1} + v_{i+1,j}) \quad \text{and} \quad b_{i,j} := \frac{1}{3}(v_{i,j} + v_{i+1,j+1} + v_{i,j+1})$$

be the barycenters of $\mathbb{T}_{i,j}$ and $\mathbb{B}_{i,j}$, respectively. Then, the CT-refinement of each triangle is obtained by joining its vertices with its barycenter [17]. Each macro-triangle $\mathbb{T}_{i,j}$ and $\mathbb{B}_{i,j}$ is, respectively, divided into the following micro-triangles:

$$\begin{aligned} t_1^+ &= \langle v_{i,j}, v_{i+1,j+1}, t_{i,j} \rangle, & t_2^+ &= \langle v_{i+1,j+1}, v_{i+1,j}, t_{i,j} \rangle, & t_3^+ &= \langle v_{i+1,j}, v_{i,j}, t_{i,j} \rangle, \\ t_1^- &= \langle v_{i,j}, v_{i,j+1}, b_{i,j} \rangle, & t_2^- &= \langle v_{i,j+1}, v_{i+1,j+1}, b_{i,j} \rangle, & t_3^- &= \langle v_{i+1,j+1}, v_{i,j}, b_{i,j} \rangle. \end{aligned} \quad (5.1)$$

They are shown in Figure 5.2, bottom, where any reference to the subscripts of the micro-triangles has been avoided. As in the case of macro-triangles, the lower case letter t will be used to represent any of the micro-triangles of Δ_{CT} .

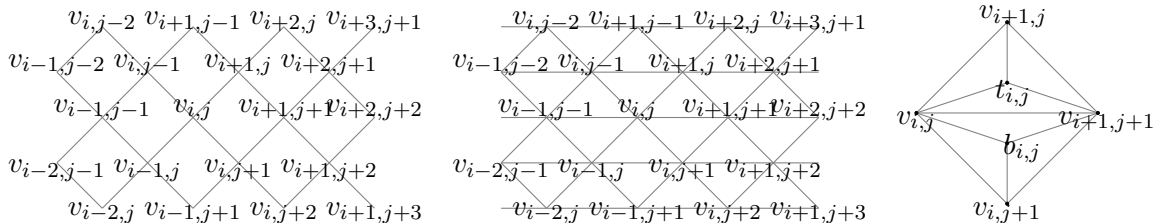


Figure 5.2: Top, from left to right, decomposition into squares induced by the vertices of Δ , type-1 triangulation. Bottom, CT-refinements of macro-triangles $\mathbb{T}_{i,j}$ and $\mathbb{B}_{i,j}$.

In this paper, we consider the space of C^1 cubic splines on Δ_{CT} defined by

$$S_3^1(\Delta_{\text{CT}}) := \{s \in C^1(\mathbb{R}^2) : s|_t \in \mathbb{P}_3 \text{ for all } t \in \Delta_{\text{CT}}\}.$$

where the restriction is $s|_t$ of $s \in S_3^1(\Delta_{CT})$ to a micro-triangle $t = \langle V_1, V_2, V_3 \rangle \in \Delta_{CT}$ a cubic polynomial, it can be represented using the cubic Bernstein polynomials

$$B_{\beta,t}(p) := \frac{3!}{\beta!} \tau^\beta = \frac{6}{\beta_1! \beta_2! \beta_3!} \tau_1^{\beta_1} \tau_2^{\beta_2} \tau_3^{\beta_3},$$

where the multi-index notations $\beta := (\beta_1, \beta_2, \beta_3) \in \mathbb{N}_0^3$, $|\beta| := \beta_1 + \beta_2 + \beta_3$ and $\beta! := \beta_1! \beta_2! \beta_3!$ have been used, and $\tau := (\tau_1, \tau_2, \tau_3)$ provides the barycentric coordinates of point $p \in \mathbb{R}^2$ with respect to t , i.e., $p = \sum_{i=1}^3 \tau_i V_i$ and $\sum_{i=1}^3 \tau_i = 1$. The coordinates τ_1, τ_2 and τ_3 are non-negative whenever p belongs to t .

Every polynomial $q \in \mathbb{P}_3$ can be expressed on t in terms of the cubic Bernstein basis polynomials $B_{\beta,t}$, $|\beta| = 3$, i.e., there exist values b_{β} such that

$$q(x, y) = q(\tau) = \sum_{|\beta|=3} b_{\beta,t} B_{\beta,t}(\tau).$$

Coefficients in $D_t := \{b_{\beta,t}, |\beta| = 3\}$ are said to be the Bernstein–Bézier (BB-) coefficients of q . They are linked to the domain points $\xi_{\beta,t}$ determined by the barycentric coordinates $\left(\frac{\beta_{1,t}}{3}, \frac{\beta_{2,t}}{3}, \frac{\beta_{3,t}}{3}\right)$ with respect to t . They determine the lattice $L_3(t)$. The graph of q on t is included in the convex hull of $\{(\xi_{\beta,t}, b_{\beta,t}), |\beta| = 3\}$.

On each micro-triangle, an element $s \in S_3^1(\Delta_{CT})$ is uniquely determined by ten BB-coefficients, associated with the corresponding domain points. When all macro-triangles are taken into account, a subset of domain points is obtained, which we note $\mathcal{D}_3(\Delta_{CT})$, i.e., $\mathcal{D}_3(\Delta_{CT}) = \bigcup_{t \in \Delta_{CT}} L_3(t)$, where the

union is formed without taking repetitions into account. To determine s , it is necessary to give the BB-coefficients associated with all the points of $\mathcal{D}_3(\Delta_{CT})$. As the triangulation is uniform, following the approach in [9, 10, 52, 53], it is sufficient to establish a partition $\{D_{i,j}, i, j \in \mathbb{Z}\}$ of $\mathcal{D}_3(\Delta_{CT})$ and define the BB-coefficients linked to the domain points in $D_{i,j}$.

Figure 5.3 shows the twenty-seven domain points forming $D_{i,j}$, which are linked to vertex $v_{i,j}$. Each of them has the subscripts of $v_{i,j}$. The vertices and barycenter have already been defined. They remaining domain points in $D_{i,j}$ are given next:

$$\begin{aligned} u_{i,j}^{1,1} &:= \frac{1}{3}(2v_{i,j} + v_{i+1,j+1}), & u_{i,j}^{1,0} &:= \frac{1}{3}(2v_{i,j} + v_{i+1,j}), \\ u_{i,j}^{2,1} &:= \frac{1}{3}(2v_{i,j} + t_{i,j}), & u_{i,j}^{-1,-1} &:= \frac{1}{3}(2v_{i,j} + v_{i-1,j-1}), \\ u_{i,j}^{1,-1} &:= \frac{1}{3}(2v_{i,j} + b_{i,j-1}), & u_{i,j}^{0,-1} &:= \frac{1}{3}(2v_{i,j} + v_{i,j-1}), \\ u_{i,j}^{-1,-2} &:= \frac{1}{3}(2v_{i,j} + t_{i-1,j-1}), & u_{i,j}^{-2,-1} &:= \frac{1}{3}(2v_{i,j} + b_{i-1,j-1}), \\ u_{i,j}^{-1,0} &:= \frac{1}{3}(2v_{i,j} + v_{i-1,j}), & u_{i,j}^{-1,1} &:= \frac{1}{3}(2v_{i,j} + t_{i-1,j}), \\ u_{i,j}^{0,1} &:= \frac{1}{3}(2v_{i,j} + v_{i,j+1}), & u_{i,j}^{1,2} &:= \frac{1}{3}(2v_{i,j} + b_{i,j}), \\ x_{i,j}^{1,1} &:= \frac{1}{3}(v_{i,j} + v_{i+1,j+1} + b_{i,j}), & x_{i,j}^{1,0} &:= \frac{1}{3}(v_{i,j} + v_{i+1,j} + b_{i,j-1}), \\ x_{i,j}^{0,1} &:= \frac{1}{3}(v_{i,j} + v_{i,j+1} + b_{i,j}), & y_{i,j}^{2,1} &:= \frac{1}{3}(v_{i,j} + 2t_{i,j}), \\ y_{i,j}^{1,-1} &:= \frac{1}{3}(v_{i,j} + 2b_{i,j-1}), & y_{i,j}^{-1,-2} &:= \frac{1}{3}(v_{i,j} + 2t_{i-1,j-1}), \\ y_{i,j}^{-2,-1} &:= \frac{1}{3}(v_{i,j} + 2b_{i-1,j-1}), & y_{i,j}^{-1,1} &:= \frac{1}{3}(v_{i,j} + 2t_{i-1,j}), \\ y_{i,j}^{1,2} &:= \frac{1}{3}(v_{i,j} + 2b_{i,j}), & z_{i,j}^{1,1} &:= \frac{1}{3}(v_{i,j} + v_{i+1,j+1} + t_{i,j}), \\ z_{i,j}^{1,0} &:= \frac{1}{3}(v_{i,j} + v_{i+1,j} + t_{i,j}), & z_{i,j}^{0,1} &:= \frac{1}{3}(v_{i,j} + v_{i,j+1} + t_{i-1,j}). \end{aligned}$$

For instance, with regard to the micro-triangle t_1^+ of $\mathbb{T}_{i,j}$ (see Figure 5.5) we write

$$\begin{aligned} \mathcal{Q}f|_{t_1^+} &= V_{i,j}B_{(3,0,0),t_1^+} + U_{i,j}^{1,1}B_{(2,1,0),t_1^+} + U_{i,j}^{2,1}B_{(2,0,1),t_1^+} + U_{i+1,j+1}^{-1,-1}B_{(1,2,0),t_1^+} \\ &+ Z_{i,j}^{1,1}B_{(1,1,1),t_1^+} + Y_{i,j}^{2,1}B_{(1,0,2),t_1^+} + V_{i+1,j+1}B_{(0,3,0),t_1^+} \\ &+ U_{i+1,j+1}^{-1,-2}B_{(0,2,1),t_1^+} + Y_{i+1,j+1}^{-1,-2}B_{(0,1,2),t_1^+} + T_{i,j}B_{(0,0,3),t_1^+}. \end{aligned}$$

Similar expressions are obtained for the restrictions of $\mathcal{Q}f$ to the other two micro-triangles of $\mathbb{T}_{i,j}$ and those three into which $\mathbb{B}_{i,j}$ is divided.

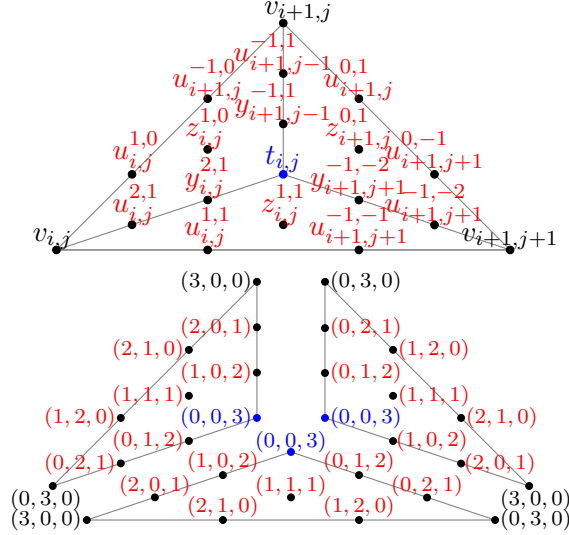


Figure 5.5: Top, the domain points associated with the three micro-triangles of the macro-triangle $\mathbb{T}_{i,j}$. They are denoted as shown in Figure 5.4 bottom; the indices corresponding to each micro-triangle, whose orientation is determined by the vertex ordering given by (5.1).

The BB-coefficients involved in the definition of $\mathcal{Q}f$ on each micro-triangle of $\mathbb{T}_{i,j}$ and $\mathbb{B}_{i,j}$ will be linear combinations of f at the specific domain points for cubic polynomials lying in the hexagon defined by the triangles sharing vertex $v_{i,j}$. Specifically, the union without repetitions $\mathcal{D}_3(\Delta) := \bigcup_{\mathbb{T} \in \Delta} L_3(\mathbb{T})$ is formed and decomposed as

$$\mathcal{D}_3(\Delta) = \bigcup_{i,j \in \mathbb{Z}} S_{i,j},$$

where the ordered subset $S_{i,j}$ consists of the thirty-seven domain points given below:

$$\begin{aligned} S_{i,j} := & \left\{ v_{i,j}, u_{i,j}^{1,1}, u_{i,j}^{1,0}, u_{i,j}^{0,-1}, u_{i,j}^{-1,-1}, u_{i,j}^{-1,0}, u_{i,j}^{0,1}, u_{i+1,j+1}^{-1,-1}, t_{i,j}, u_{i+1,j}^{-1,0}, b_{i,j-1}, \right. \\ & u_{i,j-1}^{0,-1}, t_{i-1,j-1}, u_{i-1,j-1}^{-1,-1}, b_{i-1,j-1}, u_{i-1,j}^{1,0}, t_{i-1,j}, u_{i,j+1}^{0,-1}, b_{i,j}, v_{i+1,j+1}, u_{i+1,j+1}^{0,-1}, \\ & u_{i+1,j}^{0,1}, v_{i+1,j}, u_{i+1,j}^{-1,-1}, u_{i,j-1}^{1,1}, v_{i,j-1}, u_{i,j-1}^{-1,0}, u_{i-1,j-1}^{1,0}, v_{i-1,j-1}, u_{i-1,j-1}^{0,1}, u_{i-1,j}^{0,-1}, \\ & \left. v_{i-1,j}, u_{i-1,j}^{1,1}, u_{i,j+1}^{-1,-1}, v_{i,j+1}, u_{i,j+1}^{1,0}, u_{i+1,j+1}^{-1,0} \right\}. \end{aligned}$$

The BB-coefficient P of a domain point p is a linear combination of values of f at points in $S_{i,j}$; its coefficients give rise to a vector $M(p)$, ordered as $S_{i,j}$, which is said to be the mask of p . If $f(S_{i,j}) := \{f(p), p \in S_{i,j}\}$ is also ordered as $S_{i,j}$, then

$$P = M(p) \cdot f(S_{i,j}) := \sum_{\ell=1}^{37} M(p)_\ell f(S_{i,j})_\ell,$$

where $M(p)_\ell$ and $f(S_{i,j})_\ell$ stand for the ℓ -th entries of $M(p)$ and $f(S_{i,j})$, respectively.

In the following, we state the problem that is the object of this work.

Proposition 5.3.1. *Find masks for the domain points in $D_{i,j}$ such that the associated quasi-interpolation operator \mathcal{Q} is exact on \mathbb{P}_3 and produces C^1 quasi-interpolating splines.*

The following result holds.

Proposition 5.3.2. *Problem 5.3.1 has a 17-parametric family of solutions.*

Proof. Given an arbitrary function f , C^1 continuity of $\mathcal{Q}f$ across segment $[v_{i,j}, v_{i+1,j}]$ is equivalent to the following conditions [29, Thm. 2.28] (see Figure 5.6 and the notations used for the domain points in Figures 5.3 and 5.4):

$$\begin{aligned} V_{i,j} + U_{i,j}^{1,0} - U_{i,j}^{1,-1} - U_{i,j}^{2,1} &= 0, \\ U_{i,j}^{1,0} + U_{i+1,j}^{-1,0} - X_{i,j}^{1,0} - Z_{i,j}^{1,0} &= 0, \\ U_{i+1,j}^{-1,0} + V_{i+1,j} - U_{i+1,j}^{2,1} - U_{i+1,j}^{-1,1} &= 0. \end{aligned}$$

For $[v_{i,j}, v_{i+1,j+1}]$,

$$\begin{aligned} V_{i,j} + U_{i,j}^{1,1} - U_{i,j}^{2,1} - U_{i,j}^{1,2} &= 0, \\ U_{i,j}^{1,1} + U_{i+1,j+1}^{-1,-1} - X_{i,j}^{1,1} - Z_{i,j}^{1,1} &= 0, \\ U_{i+1,j+1}^{-1,-1} + V_{i+1,j+1} - U_{i+1,j+1}^{-1,-2} - U_{i+1,j+1}^{2,1} &= 0. \end{aligned}$$

And for $[v_{i,j}, v_{i,j+1}]$,

$$\begin{aligned} V_{i,j} + U_{i,j}^{0,1} - U_{i,j}^{-1,1} - U_{i,j}^{1,2} &= 0, \\ U_{i,j}^{0,1} + U_{i,j+1}^{0,-1} - Z_{i,j}^{0,1} - X_{i,j}^{0,1} &= 0, \\ U_{i,j+1}^{0,-1} + V_{i,j+1} - U_{i,j+1}^{-1,-2} - U_{i,j+1}^{1,-1} &= 0. \end{aligned}$$

Regarding micro-edges, C^1 continuity across $[v_{i,j}, t_{i,j}]$ is equivalent to conditions

$$U_{i,j}^{2,1} - \frac{1}{3} (V_{i,j} + U_{i,j}^{1,0} + U_{i,j}^{1,1}) = 0, \quad Y_{i,j}^{2,1} - \frac{1}{3} (U_{i,j}^{2,1} + Z_{i,j}^{1,0} + Z_{i,j}^{1,1}) = 0.$$

Similarly, it is satisfied across $[v_{i+1,j}, t_{i,j}]$ and $[v_{i+1,j+1}, t_{i,j}]$, respectively, if and only if

$$\begin{aligned} U_{i+1,j}^{-1,1} - \frac{1}{3} (V_{i+1,j} + U_{i+1,j}^{-1,0} + U_{i+1,j}^{0,1}) &= 0, \\ Y_{i+1,j+1}^{-1,-1} - \frac{1}{3} (U_{i+1,j}^{-1,1} + Z_{i,j}^{1,0} + Z_{i+1,j}^{0,1}) &= 0, \end{aligned}$$

and

$$\begin{aligned} U_{i+1,j+1}^{-1,-2} - \frac{1}{3} (V_{i+1,j+1} + U_{i+1,j+1}^{-1,-1} + U_{i+1,j+1}^{0,-1}) &= 0, \\ Y_{i+1,j+1}^{-1,-2} - \frac{1}{3} (U_{i+1,j+1}^{-1,-2} + Z_{i,j}^{1,1} + Z_{i+1,j}^{0,1}) &= 0. \end{aligned}$$

For the micro-sides of macro-triangle $\mathbb{B}_{i,j}$, six new conditions are involved. For $[v_{i,j}, b_{i,j}]$, $[v_{i,j+1}, b_{i,j}]$ and $[v_{i+1,j+1}, b_{i,j}]$, C^1 regularity is equivalent to

$$U_{i,j}^{1,2} - \frac{1}{3} (V_{i,j} + U_{i,j}^{0,1} + U_{i,j}^{1,1}) = 0, \quad Y_{i,j}^{1,2} - \frac{1}{3} (U_{i,j}^{1,2} + X_{i,j}^{1,1} + X_{i,j}^{0,1}) = 0,$$

$$\begin{aligned} U_{i,j+1}^{1,-1} - \frac{1}{3} (V_{i,j+1} + U_{i,j+1}^{0,-1} + U_{i,j+1}^{1,0}) &= 0, \\ Y_{i,j+1}^{1,-1} - \frac{1}{3} (U_{i,j+1}^{1,-1} + X_{i,j}^{0,1} + X_{i,j+1}^{1,0}) &= 0, \end{aligned}$$

and

$$\begin{aligned} U_{i+1,j+1}^{2,-1} - \frac{1}{3} (V_{i+1,j+1} + U_{i+1,j+1}^{-1,-1} + U_{i+1,j+1}^{-1,0}) &= 0, \\ Y_{i+1,j+1}^{-2,-1} - \frac{1}{3} (U_{i+1,j+1}^{2,1} + X_{i,j}^{1,1} + X_{i,j+1}^{1,0}) &= 0, \end{aligned}$$

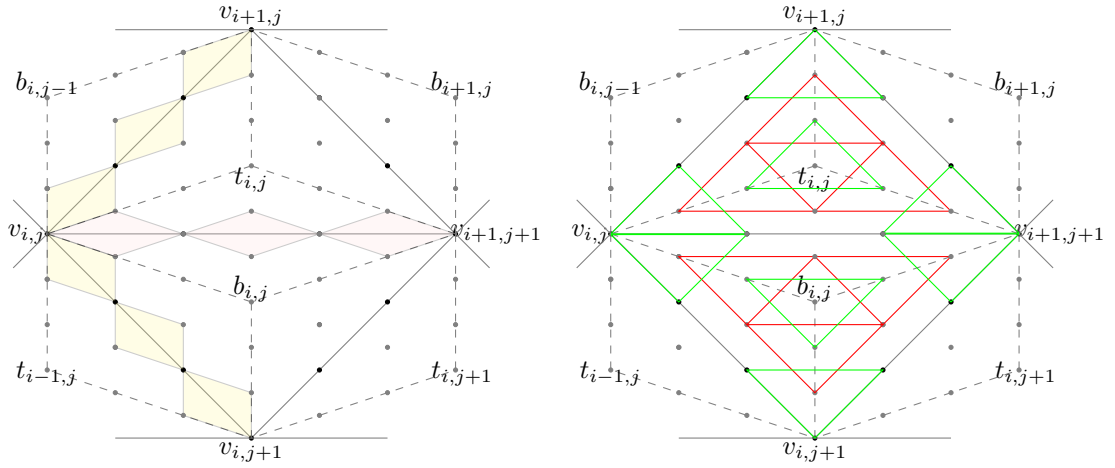


Figure 5.6: Schematic representation of the conditions to be imposed to achieve C^1 continuity on the macro-interval edges (top) and on the micro-edges and at barycenters (bottom). In each of the shaded parallelograms in the figure on the left, it must be fulfilled that the sum of the BB-coefficients of two opposite domain points must be equal to that of the other two. The C^1 continuity across the micro-edges of the triangle $\mathbb{T}_{i,j}$ is obtained if, in each of the two green and red \triangle -triangles closest to each vertex, it is satisfied that the BB-coefficient corresponding to the interior domain point is equal to one-third of the sum of those of the three vertices of the triangle. The same condition must be fulfilled for the ∇ -triangles of $\mathbb{B}_{i,j}$.

respectively. Finally, C^1 continuity at the barycenters of $\mathbb{T}_{i,j}$ and $\mathbb{B}_{i,j}$ is obtained if and only if

$$\begin{aligned} T_{i,j} - \frac{1}{3} \left(Y_{i,j}^{2,1} + Y_{i+1,j+1}^{-1,-2} + Y_{i+1,j-1}^{-1,1} \right) &= 0, \\ B_{i,j} - \frac{1}{3} \left(Y_{i,j}^{1,2} + Y_{i+1,j+1}^{-2,-1} + Y_{i,j+1}^{-1,-1} \right) &= 0. \end{aligned}$$

These are all equalities involving the values $f(p)$, $p \in S_{i,j}$, so $\mathcal{Q}f$ is C^1 continuous if and only if all the coefficients of the f -values in these equalities are zero. Therefore, the requirements on the C^1 continuity are equivalent to a system of equations having a 122-parametric family of solutions. To these equations must be added those related to the exactness of the operator on \mathbb{P}_3 . They are obtained by imposing that the BB-coefficients on each microtriangle of the monomials of degree less than or equal to three and those of their quasi-interpolants are equal. The resulting system can be solved with a Computer Algebra System, namely, Mathematica, obtaining the existence of a 17-parametric family of solutions. The free parameters are entries with indices 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 18, 19, 20, 21, and 22 of the mask $M(b_{i,j})$. \square

Remark 5.3.1. *The programme allowing the results to be checked is given in an appendix at the end of the manuscript.*

Figure 5.7 shows the mask relative to vertex $v_{i,j}$. The entries of the masks for $u_{i,j}^{0,-1}$, $u_{i,j}^{-1,-2}$, $u_{i,j}^{-2,-1}$

$\frac{4484399224513}{6514447513140}$		$\frac{2552223421511}{1085741252190}$		$\frac{5169718992073}{1085741252190}$		$\frac{8031525079}{232658839755}$	
$\frac{64504080325}{868593001752}$		$\frac{3505392583699}{4342965008760}$		$\frac{27952071335903}{2895310005840}$		$\frac{3505392583699}{4342965008760}$	
$\frac{5169718992073}{868593001752}$		$\frac{5169718992073}{868593001752}$		$\frac{5169718992073}{868593001752}$		$\frac{5169718992073}{868593001752}$	
$\frac{36832622942}{542870626095}$		$\frac{120096488429}{482551667640}$		$\frac{519419554543}{310211786340}$		$\frac{533680265081}{1447655002920}$	
$\frac{120096488429}{482551667640}$		$\frac{120096488429}{482551667640}$		$\frac{120096488429}{482551667640}$		$\frac{120096488429}{482551667640}$	
$\frac{6955013524199}{13028895026280}$		$\frac{144225313868}{1628611878285}$		$\frac{382180214323}{100531597425}$		$\frac{176728557928}{232658839755}$	
$\frac{144225313868}{1628611878285}$		$\frac{144225313868}{1628611878285}$		$\frac{144225313868}{1628611878285}$		$\frac{144225313868}{1628611878285}$	
$\frac{2095907565527}{2171482504380}$		$\frac{2725154777111}{4342965008760}$		$\frac{519419554543}{310211786340}$		$\frac{533680265081}{1447655002920}$	
$\frac{2725154777111}{4342965008760}$		$\frac{2725154777111}{4342965008760}$		$\frac{2725154777111}{4342965008760}$		$\frac{2725154777111}{4342965008760}$	
$\frac{2725154777111}{4342965008760}$		$\frac{2095907565527}{2171482504380}$		$\frac{6955013524199}{13028895026280}$		$\frac{36832622942}{542870626095}$	
$\frac{2095907565527}{2171482504380}$		$\frac{2095907565527}{2171482504380}$		$\frac{2095907565527}{2171482504380}$		$\frac{2095907565527}{2171482504380}$	
$\frac{31932271589}{120637916910}$		$\frac{31932271589}{120637916910}$		$\frac{73887390529}{5211558010512}$		$\frac{25733502500393}{13028895026280}$	
$\frac{31932271589}{120637916910}$		$\frac{31932271589}{120637916910}$		$\frac{31932271589}{120637916910}$		$\frac{31932271589}{120637916910}$	

Figure 5.7: Mask $M(v_{i,j})$.

and $u_{i,j}^{-1,0}$ are almost all zero, and the following expressions for their BB-coefficients are found:

$$\begin{aligned}
U_{i,j}^{-1,0} &= -\frac{5}{6}f(v_{i,j}) + 3f(u_{i,j}^{-1,0}) - \frac{3}{2}f(u_{i,j-1}^{0,-1}) + \frac{1}{3}f(v_{i,j-1}), \\
U_{i,j}^{0,-1} &= -\frac{5}{6}f(v_{i,j}) - \frac{3}{2}f(u_{i,j}^{-1,0}) + 3f(u_{i,j-1}^{0,-1}) + \frac{1}{3}f(v_{i+1,j+1}), \\
U_{i,j}^{-1,-2} &= -\frac{5}{6}f(v_{i,j}) + f(u_{i,j}^{-1,0}) + 2f(u_{i,j}^{0,-1}) - \frac{1}{2}f(u_{i,j-1}^{0,-1}) \\
&\quad - f(u_{i+1,j}^{-1,0}) + \frac{2}{9}f(v_{i+1,j+1}) + \frac{1}{9}f(v_{i,j-1}), \\
U_{i,j}^{-2,-1} &= -\frac{5}{6}f(v_{i,j}) + 2f(u_{i,j}^{-1,0}) + f(u_{i,j}^{0,-1}) - f(u_{i,j-1}^{0,-1}) \\
&\quad - \frac{1}{2}f(u_{i+1,j}^{-1,0}) + \frac{1}{9}f(v_{i+1,j+1}) + \frac{2}{9}f(v_{i,j-1}).
\end{aligned}$$

Fourteen of the masks relative to the remaining twenty-two domain points in $D_{i,j}$ do not depend on any parameters and appear in an appendix. Those of $t_{i,j}$, $b_{i,j}$, $y_{i,j}^{1,2}$, $y_{i,j}^{2,1}$, $y_{i,j}^{-1,-2}$, $y_{i,j}^{-2,-1}$, $x_{i,j}^{1,1}$, and $z_{i,j}^{1,1}$ have very long entries and will not be given.

Remark 5.3.2. *It can be proved that it is not possible to obtain quasi-interpolants with the required characteristics if the BB-coefficients are linear combinations of function values at the vertices lying in the hexagon $H_{i,j}$ determined by the six triangles sharing vertex $v_{i,j}$, and the midpoints of the edges of $H_{i,j}$. Neither is it possible to construct C^1 cubic quasi-interpolants exact on \mathbb{P}_3 in this way if function values at $v_{i,j}$ and at the eighteen vertices closest to it are used.*

Moreover, quasi-interpolation error estimates are found using a standard procedure [53].

Proposition 5.3.3. *There exists an absolute constant K such that for every $f \in C^{m+1}(\mathbb{R}^2)$, $0 \leq m \leq 2$,*

$$\|D^\gamma(f - \mathcal{Q}f)\|_{\infty, \mathbb{T}} \leq Kh^{m+1-|\gamma|} \|D^{m+1}f\|_{\infty, \Omega_{\mathbb{T}}}, \quad (5.2)$$

for all $0 \leq |\gamma| \leq 1$, $\gamma = (\gamma_1, \gamma_2)$, with $\Omega_{\mathbb{T}}$ denoting the union of the triangles in Δ having a non-empty intersection with T .

5.4 Choosing Parameters

An obvious choice is to make all parameters equal to zero. However, a reasonable strategy is to minimize an upper bound of the quasi-interpolation error for monomials of smaller degree non reproduced by the quasi-interpolation operator, namely $m_{k,4-k}(x, y) := x^k y^{4-k}$, $k = 0, 1, 2, 3, 4$. Let us suppose that the BB-coefficients of $m_{k,4-k}$ relative to each micro-triangle t_ℓ^+ , $\ell = 1, 2, 3$, of $\mathbb{T}_{i,j}$ are μ_{k,β,t_ℓ^+} , $|\beta| = 4$, and that those of the cubic quasi-interpolant $Qm_{k,4-k}$ are b_{k,γ,t_ℓ^+} , $|\gamma| = 4$. By degree elevation, $Qm_{k,4-k}|_{t_\ell^+}$ can be represented as a quartic polynomial having BB-coefficients b_{k,β,t_ℓ^+} , $|\beta| = 4$, which depend on

parameters $z_r := M(b_{i,j})_r$, $1 \leq r \leq 12$, and $z_r := M(b_{i,j})_{r+5}$, $13 \leq r \leq 17$. Therefore, the BB-coefficients of the restriction of $m_{k,4-k} - Qm_{k,4-k}$ to t_ℓ^+ have the form

$$\sigma_{k,t_\ell^+}(z) = c_{k,t_\ell^+} + \sum_{r=1}^{17} c_{k,t_\ell^+}^{(r)} z_r$$

for real values c_{k,t_ℓ^+} and $c_{k,t_\ell^+}^{(r)}$, where $z := (z_1, \dots, z_{17})$. Since the Bernstein polynomials relative to t_ℓ^+ form a partition of unity, then the infinity norm of $m_{k,4-k} - Qm_{k,4-k}$ is bounded by

$$\max \left\{ \left| \sigma_{k,t_\ell^+}(z) \right|, \ell = 1, 2, 3 \right\}.$$

Consequently, an upper bound for the quasi-interpolation errors for quartic monomials in the macro-triangle $\mathbb{T}_{i,j}$ is

$$U_+(z) := \max \left\{ \left| \sigma_{k,t_\ell^+}(z) \right|, \ell = 1, 2, 3; k = 0, 1, 2, 3, 4 \right\}.$$

Analogously, an upper bound of such errors in the macro-triangle $\mathbb{B}_{i,j}$ is written as

$$U_-(z) := \max \left\{ \left| \sigma_{k,t_\ell^-}(z) \right|, \ell = 1, 2, 3; k = 0, 1, 2, 3, 4 \right\},$$

where

$$\sigma_{k,t_\ell^-}(z) = c_{k,t_\ell^-} + \sum_{r=1}^{17} c_{k,t_\ell^-}^{(r)} z_r,$$

for real values c_{k,t_ℓ^-} and $c_{k,t_\ell^-}^{(r)}$. In short, the function

$$U(z) := \max \{U_+(z), U_-(z)\}$$

is an upper bound for the quasi-interpolation errors for quartic monomials in the square $\mathbb{T}_{i,j} \cup \mathbb{B}_{i,j}$.

Function U can be rewritten as

$$U(z) = \max_{1 \leq \alpha \leq 30} \frac{1}{c_\alpha} \left(d_\alpha + \sum_{\beta=1}^{17} e_{\alpha,\beta} |f_{\alpha,\beta} \cdot z| \right),$$

where $c_\alpha, d_\alpha, e_{\alpha,\beta} \in \mathbb{N}$, $f_{\alpha,\beta} \in \mathbb{Z}^{17}$ and $A \cdot B := \sum_{s=1}^{17} A_s B_s$. The number of terms involved in each sum depends on α , because some of them will be zero. Therefore, the minimization of U is equivalent to the following linear programming problem:

$$\begin{array}{l} \text{Minimize } \mu \\ \text{such that } \left\{ \begin{array}{l} d_\alpha + \sum_{\beta=1}^{17} e_{\alpha,\beta} (u_{\alpha,\beta} + v_{\alpha,\beta}) - c_\alpha \mu \leq 0, \quad 1 \leq \alpha \leq 30, \\ f_{\alpha,\beta} \cdot (Z^+ - Z^-) - u_{\alpha,\beta} + v_{\alpha,\beta} = 0, \quad 1 \leq \alpha \leq 30, 1 \leq \beta \leq 17, \\ u_{p,n}, v_{p,n}, X_1, X_2, Y_1, Y_2, Z_1, Z_2, \mu \geq 0, \end{array} \right. \end{array}$$

where it has been used that each variable z_r can be written as $z_r = z_r^+ - z_r^-$, $z_r^+, z_r^- \geq 0$, therefore $Z = Z^+ - Z^-$, with $Z^+ := (z_1^+, \dots, z_{17}^+)$ and $Z^- := (z_1^-, \dots, z_{17}^-)$. The solution of this problem has been

exactly determined by using Mathematica, and the minimum value $\mu = \frac{35971348390906381}{87945041427390}$ is reached at

$$\begin{array}{ll} Z_3^+ = \frac{33654106472661220639}{24647711830550794440}, & Z_6^- = \frac{28931119278287059059781}{79874153306877553434720}, \\ Z_7^- = \frac{147713415264798351289}{49295423661101588880}, & Z_9^- = \frac{71687410464642966611}{49295423661101588880}, \\ Z_{10}^- = \frac{3723562194545339719095199}{1118238146296285748086080}, & Z_{12}^- = \frac{3459921708110971652593}{12288331277981162066880}, \\ Z_{13}^- = \frac{143791532332245022121277}{1863730243827142913476800}, & Z_{15}^+ = \frac{9334610941403380115035381}{10064143316666571732774720}, \end{array}$$

being equal to zero all the remaining values. Therefore, the minimum is attained at point z^* with components $z_r^* = 0$ for $r \in \{1, 2, 4, 5, 8, 11, 14, 16, 17\}$, and

$$\begin{array}{ll} z_3^* = \frac{33654106472661220639}{24647711830550794440}, & z_6^* = -\frac{28931119278287059059781}{79874153306877553434720}, \\ z_7^* = -\frac{147713415264798351289}{49295423661101588880}, & z_9^* = -\frac{71687410464642966611}{49295423661101588880}, \\ z_{10}^* = -\frac{3723562194545339719095199}{1118238146296285748086080}, & z_{12}^* = -\frac{3459921708110971652593}{12288331277981162066880}, \\ z_{13}^* = -\frac{143791532332245022121277}{1863730243827142913476800}, & z_{15}^* = \frac{9334610941403380115035381}{10064143316666571732774720}. \end{array}$$

5.5 Numerical Tests

In this section, the performance of the quasi-interpolation operator Q^* defined by the masks provided by the solution above is tested. To perform this, we consider Franke's function

$$f_1(x_1, x_2) = \frac{3}{4} \exp\left(-\frac{(9x_1 - 2)^2}{4} - \frac{(9x_2 - 2)^2}{4}\right) + \frac{3}{4} \exp\left(-\frac{(9x_1 + 1)^2}{49} - \frac{9x_2 + 1}{10}\right) \\ + \frac{1}{2} \exp\left(-\frac{(9x_1 - 7)^2}{4} - \frac{(9x_2 - 3)^2}{4}\right) - \frac{1}{5} \exp\left(-\frac{(9x_1 - 4)^2}{4} - \frac{(9x_2 - 7)^2}{4}\right)$$

and Nielson's function

$$f_2(x_1, x_2) = \frac{x_2}{2} \cos^4(4(x_1^2 + x_2 - 1))$$

to produce quasi-interpolants on the unit square [21, 39]. The plots of f_1 and f_2 are shown in Figure ??, together with those of their quasi-interpolants obtained by dividing the unit interval into 256 equal parts.

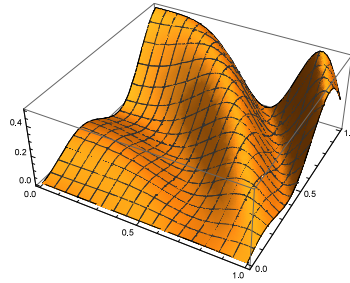


Figure 5.8: Top, from left to right, plots of the test functions. Bottom, the ones of their respective quasi-interpolants Q^*f_1 and Q^*f_2 with $h = 1/256$.

The quasi-interpolation error is estimated as

$$\max_{k,\ell=1,\dots,400} |Q^*f(x_k, y_\ell) - f(x_k, y_\ell)|,$$

x_k and y_ℓ being equally spaced points in $[0, 1]$. The numerical convergence order (NCO) is given by the rate

$$\text{NCO} := \log\left(\frac{E(h_2)}{E(h_1)}\right) / \log\left(\frac{h_2}{h_1}\right),$$

where $E(h)$ stands for the estimated error associated with the step length h .

The quasi-interpolation errors are estimated for different values of the step length h and the NCO are calculated. The results are shown in Table ?. They confirm the theoretical ones.

–	f_1	–	f_2	–
n	Estimated Error	NCO	Estimated Error	NCO
16	7.07377×10^{-1}	–	1.47146×10^{-1}	–
32	4.49051×10^{-2}	3.97753	1.44799×10^{-2}	3.34512
64	3.14830×10^{-3}	3.83423	8.62813×10^{-4}	4.06886
128	1.76965×10^{-4}	4.15304	5.36388×10^{-5}	4.00770
256	1.07615×10^{-5}	4.03951	3.48823×10^{-6}	3.94271

Table 5.1: Errors and NCOs for functions f_1 and f_2 with $h = 1/n$, $n = 20, 40, 80, 160$.

5.6 Conclusions

In this work, C^1 cubic quasi-interpolants have been defined on a Clough–Tocher refinement of a type-1 triangulation, providing directly their BB-coefficients on each of the micro-triangles of the sub-triangulation, which are linear combinations of the values taken by the approximated function at specific points in a neighborhood of each macro-triangle. Cubic polynomials are reproduced. The general problem has a 17-parametric family of solutions and a specific solution has been chosen, which minimizes an upper bound of the quasi-interpolation errors associated with the quartic monomials.

The results improve on those available for cubic quasi-interpolation over a type-1 triangulation since the quasi-interpolation operator is now exact on \mathbb{P}_3 instead of \mathbb{P}_2 .

5.7 Masks

This appendix includes the masks provided by Proposition 5.3.2 and which do not depend on the parameters indicated. Further, the remaining ones corresponding to the parameters values z_r^* were performed. They have been obtained by minimizing the considered upper bound of the quasi-interpolation errors of the quartic monomials. They are very lengthy expressions, but are included to provide the reader with as much information as possible.

5.8 Masks That Do Not Depend on Parameters

Mask of $u_{i,j}^{1,1}$:

$$\left(\begin{array}{cccccc} -\frac{47473646953}{77552946585}, & -\frac{285439445629}{361913750730}, & -\frac{533680265081}{482551667640}, & \frac{209207768203}{103403928780}, & \frac{382180214323}{33510532475}, & \\ \frac{209207768203}{103403928780}, & -\frac{533680265081}{482551667640}, & \frac{6856655665381}{2895310005840}, & \frac{120096488429}{160850555880}, & -\frac{1333910079319}{1447655002920}, & \\ -\frac{27952071335903}{965103335280}, & -\frac{1333910079319}{1447655002920}, & \frac{120096488429}{160850555880}, & -\frac{144225313868}{542870626095}, & -\frac{2725154777111}{1447655002920}, & \\ \frac{2095907565527}{723827501460}, & -\frac{6955013524199}{4342965008760}, & \frac{36832622942}{180956875365}, & -\frac{64504080325}{289531000584}, & \frac{3760571723053}{2171482504380}, & \\ -\frac{2552223421511}{361913750730}, & \frac{5169718992073}{361913750730}, & -\frac{8031525079}{77552946585}, & \frac{5169718992073}{361913750730}, & -\frac{2552223421511}{361913750730}, & \\ \frac{3760571723053}{2171482504380}, & -\frac{64504080325}{289531000584}, & \frac{36832622942}{180956875365}, & -\frac{6955013524199}{4342965008760}, & \frac{2095907565527}{723827501460}, & \\ -\frac{2725154777111}{1447655002920}, & -\frac{31932271589}{40212638970}, & -\frac{31932271589}{40212638970}, & \frac{73887390529}{1737186003504}, & -\frac{25733502500393}{43429650087600}, & \\ 343799911081 & 343799911081 & & & & \\ 361913750730 & 361913750730 & & & & \end{array} \right)$$

Mask of $u_{i,j}^{-1,-1}$:

$$\left(\begin{array}{cccccc} -\frac{211036174997}{232658839755}, & \frac{285439445629}{1085741252190}, & \frac{533680265081}{1447655002920}, & \frac{411215804477}{310211786340}, & -\frac{382180214323}{100531597425}, & \\ \frac{411215804477}{310211786340}, & \frac{533680265081}{1447655002920}, & -\frac{6856655665381}{8685930017520}, & -\frac{120096488429}{482551667640}, & -\frac{3009054929441}{4342965008760}, & \\ \frac{27952071335903}{2895310005840}, & -\frac{3009054929441}{4342965008760}, & -\frac{120096488429}{482551667640}, & \frac{144225313868}{1628611878285}, & \frac{2725154777111}{4342965008760}, & \\ -\frac{2095907565527}{2171482504380}, & \frac{6955013524199}{13028895026280}, & -\frac{36832622942}{542870626095}, & \frac{64504080325}{868593001752}, & -\frac{2312916720133}{6514447513140}, & \\ \frac{2552223421511}{1085741252190}, & -\frac{5169718992073}{1085741252190}, & \frac{8031525079}{232658839755}, & -\frac{5169718992073}{1085741252190}, & \frac{2552223421511}{1085741252190}, & \\ -\frac{2312916720133}{6514447513140}, & \frac{64504080325}{868593001752}, & -\frac{36832622942}{542870626095}, & \frac{6955013524199}{13028895026280}, & \frac{2095907565527}{13028895026280}, & \\ \frac{2725154777111}{4342965008760}, & \frac{31932271589}{120637916910}, & \frac{31932271589}{120637916910}, & -\frac{73887390529}{5211558010512}, & \frac{25733502500393}{130288950262800}, & \\ -\frac{343799911081}{1085741252190}, & -\frac{343799911081}{1085741252190}, & & & & \end{array} \right)$$

Mask of $u_{i,j}^{2,1}$:

$$\begin{pmatrix} -319149498787 & -285439445629 & -533680265081 & 364313661373 & 764360428646 \\ -465317679510 & 542870626095 & -723827501460 & 155105893170 & 100531597425 \\ 209207768203 & -533680265081 & 6856655665381 & 120096488429 & -2419651331509 \\ 155105893170 & 723827501460 & 4342965008760 & 241275833820 & 2171482504380 \\ -27952071335903 & -1333910079319 & 120096488429 & -288450627736 & -2725154777111 \\ -1447655002920 & 2171482504380 & 241275833820 & 1628611878285 & -2171482504380 \\ 2095907565527 & -6955013524199 & 73665245884 & -64504080325 & 4122485473783 \\ 1085741252190 & 6514447513140 & 542870626095 & -434296500876 & 3257223756570 \\ -2552223421511 & 5169718992073 & -16063050158 & 5169718992073 & -2552223421511 \\ 542870626095 & 542870626095 & 232658839755 & 542870626095 & 542870626095 \\ 3760571723053 & 64504080325 & 73665245884 & -6955013524199 & 2095907565527 \\ 3257223756570 & -434296500876 & 542870626095 & 6514447513140 & 1085741252190 \\ -2725154777111 & 31932271589 & -31932271589 & 73887390529 & 25733502500393 \\ -2171482504380 & 60318958455 & -60318958455 & 2605779005256 & 65144475131400 \\ 343799911081 & 343799911081 \\ 542870626095 & 542870626095 \end{pmatrix}$$

Mask of $u_{i,j}^{1,0}$:

$$\begin{pmatrix} -319149498787 & -285439445629 & -533680265081 & 519419554543 & 764360428646 \\ -465317679510 & 542870626095 & -723827501460 & 155105893170 & 100531597425 \\ 54101875033 & -533680265081 & 6856655665381 & 120096488429 & -3505392583699 \\ 155105893170 & 723827501460 & 4342965008760 & 241275833820 & 2171482504380 \\ -27952071335903 & -248168827129 & 120096488429 & -288450627736 & -2725154777111 \\ -1447655002920 & 2171482504380 & 241275833820 & 1628611878285 & 2171482504380 \\ 2095907565527 & -6955013524199 & 73665245884 & -64504080325 & 4484399224513 \\ 1085741252190 & 6514447513140 & 542870626095 & -434296500876 & 3257223756570 \\ -2552223421511 & 5169718992073 & -16063050158 & 5169718992073 & -2552223421511 \\ 542870626095 & 542870626095 & 232658839755 & 542870626095 & 542870626095 \\ 3398657972323 & -64504080325 & 73665245884 & -6955013524199 & 2095907565527 \\ 3257223756570 & 434296500876 & 542870626095 & 6514447513140 & 1085741252190 \\ -2725154777111 & 31932271589 & -31932271589 & 73887390529 & 25733502500393 \\ -2171482504380 & 60318958455 & -60318958455 & 2605779005256 & 65144475131400 \\ 343799911081 & 343799911081 \\ 542870626095 & 542870626095 \end{pmatrix}$$

Mask of $u_{i,j}^{1,-1}$:

$$\begin{pmatrix} -176728557928 & -285439445629 & -533680265081 & 829631340883 & 382180214323 \\ 232658839755 & 1085741252190 & 1447655002920 & 310211786340 & 100531597425 \\ 209207768203 & -533680265081 & 6856655665381 & 120096488429 & -5676875088079 \\ 310211786340 & 1447655002920 & 8685930017520 & 482551667640 & -4342965008760 \\ -27952071335903 & -1333910079319 & 120096488429 & -144225313868 & -2725154777111 \\ -2895310005840 & 4342965008760 & 482551667640 & 1628611878285 & 4342965008760 \\ 2095907565527 & -6955013524199 & 36832622942 & -64504080325 & 5208226725973 \\ 2171482504380 & 13028895026280 & 542870626095 & -868593001752 & 6514447513140 \\ -2552223421511 & 5169718992073 & -8031525079 & 5169718992073 & -2552223421511 \\ 1085741252190 & 1085741252190 & -232658839755 & 1085741252190 & 1085741252190 \\ 3760571723053 & -64504080325 & 36832622942 & -6955013524199 & 2095907565527 \\ 6514447513140 & 868593001752 & 542870626095 & 13028895026280 & 2171482504380 \\ -2725154777111 & -31932271589 & -31932271589 & 73887390529 & -25733502500393 \\ -4342965008760 & 120637916910 & -120637916910 & 5211558010512 & 130288950262800 \\ 343799911081 & 343799911081 \\ 1085741252190 & 1085741252190 \end{pmatrix}$$

Mask of $u_{i,j}^{-1,1}$:

$$\begin{pmatrix} -176728557928 & -285439445629 & -533680265081 & 209207768203 & 382180214323 \\ 232658839755 & 1085741252190 & 1447655002920 & 310211786340 & 100531597425 \\ 829631340883 & -533680265081 & 6856655665381 & 120096488429 & -1333910079319 \\ 310211786340 & 1447655002920 & 8685930017520 & 482551667640 & -4342965008760 \\ -27952071335903 & 5676875088079 & 120096488429 & -144225313868 & -2725154777111 \\ -2895310005840 & -4342965008760 & 482551667640 & 1628611878285 & -4342965008760 \\ 2095907565527 & -6955013524199 & 36832622942 & -64504080325 & 3760571723053 \\ 2171482504380 & 13028895026280 & 542870626095 & -868593001752 & 6514447513140 \\ -2552223421511 & 5169718992073 & -8031525079 & 5169718992073 & -2552223421511 \\ 1085741252190 & 1085741252190 & -232658839755 & 1085741252190 & 1085741252190 \\ 5208226725973 & -64504080325 & 36832622942 & -6955013524199 & 2095907565527 \\ 6514447513140 & 868593001752 & 542870626095 & 13028895026280 & 2171482504380 \\ -2725154777111 & -31932271589 & -31932271589 & 73887390529 & -25733502500393 \\ -4342965008760 & 120637916910 & -120637916910 & 5211558010512 & 130288950262800 \\ 343799911081 & 343799911081 \\ 1085741252190 & 1085741252190 \end{pmatrix}$$

Mask of $u_{i,j}^{0,1}$:

$$\left(\begin{array}{l} -\frac{319149498787}{465317679510}, -\frac{285439445629}{542870626095}, -\frac{533680265081}{723827501460}, \frac{54101875033}{155105893170}, \frac{764360428646}{100531597425}, \\ \frac{519419554543}{155105893170}, -\frac{533680265081}{723827501460}, \frac{6856655665381}{4342965008760}, \frac{120096488429}{241275833820}, -\frac{248168827129}{2171482504380}, \\ -\frac{27952071335903}{1447655002920}, \frac{3505392583699}{2171482504380}, \frac{120096488429}{241275833820}, -\frac{288450627736}{1628611878285}, -\frac{2725154777111}{2171482504380}, \\ 2095907565527, -\frac{6955013524199}{6514447513140}, \frac{73665245884}{542870626095}, -\frac{64504080325}{434296500876}, \frac{3398657972323}{3257223756570}, \\ 1085741252190, -\frac{2552223421511}{542870626095}, \frac{5169718992073}{542870626095}, -\frac{16063050158}{232658839755}, \frac{5169718992073}{542870626095}, -\frac{2552223421511}{542870626095}, \\ \frac{4484399224513}{3257223756570}, -\frac{64504080325}{434296500876}, \frac{73665245884}{542870626095}, -\frac{6955013524199}{6514447513140}, \frac{2095907565527}{1085741252190}, \\ \frac{3257223756570}{1085741252190}, -\frac{2725154777111}{2171482504380}, -\frac{31932271589}{60318958455}, -\frac{31932271589}{60318958455}, \frac{73887390529}{2605779005256}, -\frac{25733502500393}{65144475131400}, \\ \frac{343799911081}{542870626095}, \frac{343799911081}{542870626095} \end{array} \right)$$

Mask $u_{i,j}^{1,2}$:

$$\left(\begin{array}{l} -\frac{319149498787}{465317679510}, -\frac{285439445629}{542870626095}, -\frac{533680265081}{723827501460}, \frac{209207768203}{155105893170}, \frac{764360428646}{100531597425}, \\ \frac{364313661373}{155105893170}, -\frac{533680265081}{723827501460}, \frac{6856655665381}{4342965008760}, \frac{120096488429}{241275833820}, -\frac{1333910079319}{2171482504380}, \\ -\frac{27952071335903}{1447655002920}, -\frac{2419651331509}{2171482504380}, \frac{120096488429}{241275833820}, -\frac{288450627736}{1628611878285}, -\frac{2725154777111}{2171482504380}, \\ 2095907565527, -\frac{6955013524199}{6514447513140}, \frac{73665245884}{542870626095}, -\frac{64504080325}{434296500876}, \frac{3760571723053}{3257223756570}, \\ 1085741252190, -\frac{2552223421511}{542870626095}, \frac{5169718992073}{542870626095}, -\frac{16063050158}{232658839755}, \frac{5169718992073}{542870626095}, -\frac{2552223421511}{542870626095}, \\ \frac{4122485473783}{3257223756570}, -\frac{64504080325}{434296500876}, \frac{73665245884}{542870626095}, -\frac{6955013524199}{6514447513140}, \frac{2095907565527}{1085741252190}, \\ \frac{3257223756570}{1085741252190}, -\frac{2725154777111}{2171482504380}, -\frac{31932271589}{60318958455}, -\frac{31932271589}{60318958455}, \frac{73887390529}{2605779005256}, -\frac{25733502500393}{65144475131400}, \\ \frac{343799911081}{542870626095}, \frac{343799911081}{542870626095} \end{array} \right)$$

Mask of $y_{i,j}^{1,-1}$:

$$\left(\begin{array}{l} -\frac{176522697979}{827231430240}, -\frac{285439445629}{6514447513140}, \frac{1755007242871}{13028895026280}, \frac{43682365976659}{52115580105120}, -\frac{3914887523587}{26057790052560}, 0, \\ 0, \frac{6856655665381}{52115580105120}, \frac{23609319453817}{52115580105120}, -\frac{4979513337263}{52115580105120}, \frac{8038657526531}{26057790052560}, 0, 0, \\ -\frac{72112656934}{4885835634855}, -\frac{2725154777111}{26057790052560}, \frac{2095907565527}{13028895026280}, -\frac{41401785965929}{78173370157680}, \frac{48774214262167}{52115580105120}, \\ -\frac{231734735789}{3257223756570}, \frac{12969882253997}{156346740315360}, \frac{7618392504781}{26057790052560}, -\frac{8209241769433}{52115580105120}, -\frac{2520438176729}{19543342539420}, \\ 0, 0, 0, 0, 0, 0, 0, 0, -\frac{31932271589}{723827501460}, 0, 0, 0, -\frac{26064240688}{180956875365}, 0 \end{array} \right)$$

Mask of $y_{i,j}^{-1,1}$:

$$\left(\begin{array}{l} -\frac{176522697979}{827231430240}, -\frac{285439445629}{6514447513140}, 0, 0, -\frac{3914887523587}{26057790052560}, \frac{43682365976659}{52115580105120}, \frac{1755007242871}{13028895026280}, \\ \frac{6856655665381}{52115580105120}, 0, 0, \frac{8038657526531}{26057790052560}, -\frac{4979513337263}{52115580105120}, \frac{23609319453817}{52115580105120}, -\frac{72112656934}{4885835634855}, \\ 0, 0, 0, 0, 0, 0, 0, 0, -\frac{2520438176729}{19543342539420}, -\frac{8209241769433}{52115580105120}, \frac{7618392504781}{26057790052560}, \frac{12969882253997}{156346740315360}, \\ -\frac{231734735789}{3257223756570}, \frac{48774214262167}{52115580105120}, -\frac{41401785965929}{78173370157680}, \frac{2095907565527}{13028895026280}, \\ \frac{3257223756570}{1085741252190}, -\frac{2725154777111}{26057790052560}, -\frac{31932271589}{723827501460}, 0, 0, 0, -\frac{26064240688}{180956875365} \end{array} \right)$$

Mask of $z_{i,j}^{1,0}$:

$$\left(\begin{array}{l} -\frac{16044311288503}{10423116021024}, -\frac{285439445629}{434296500876}, -\frac{19207698623}{32170111176}, \frac{7517711285213}{1737186003504}, \frac{13541457918013}{1206379169100}, \\ \frac{105803839423}{103403928780}, -\frac{533680265081}{482551667640}, \frac{6856655665381}{3474372007008}, \frac{948342242035}{1158124002336}, -\frac{1604093855459}{496338858144}, \\ -\frac{2302892888971}{80425277940}, -\frac{610082577859}{1447655002920}, \frac{120096488429}{160850555880}, -\frac{72112656934}{3257223756570}, -\frac{2725154777111}{1737186003504}, \\ 2095907565527, -\frac{5562426555557}{52115580105120}, -\frac{1095500413283}{1737186003504}, \frac{1089486754079}{868593001752}, \frac{505619067587}{13028895026280}, \\ \frac{868593001752}{52115580105120}, -\frac{521386636583}{1447655002920}, \frac{2085546546332}{180956875365}, \frac{364490450669}{542870626095}, \frac{5169718992073}{361913750730}, -\frac{2552223421511}{361913750730}, \\ \frac{3519295889233}{2171482504380}, -\frac{64504080325}{289531000584}, \frac{36832622942}{180956875365}, -\frac{6955013524199}{4342965008760}, \frac{2095907565527}{723827501460}, \\ -\frac{2725154777111}{1447655002920}, -\frac{31932271589}{48255166764}, -\frac{31932271589}{40212638970}, \frac{73887390529}{1737186003504}, -\frac{25733502500393}{43429650087600}, \\ \frac{521386636583}{723827501460}, \frac{343799911081}{361913750730} \end{array} \right)$$

Mask of $z_{i,j}^{0,1}$:

$$\begin{aligned} & \left(\begin{array}{l} 61848672613411 \quad 285439445629 \quad 533680265081 \quad -209207768203 \quad -4369132774261 \\ 52115580105120 \quad 2171482504380 \quad 1447655002920 \quad -310211786340 \quad -1206379169100 \quad , \\ - \quad 8501061371657 \quad 339213799501 \quad -6856655665381 \quad -120096488429 \quad 1333910079319 \\ 8685930017520 \quad 206807857560 \quad 17371860035040 \quad 482551667640 \quad 4342965008760 \quad , \\ 2700000133115 \quad 28100144271473 \quad 15512464547161 \quad 72112656934 \quad 2725154777111 \\ 289531000584 \quad , 17371860035040 \quad , 5790620011680 \quad , 1628611878285 \quad , 4342965008760 \quad , \\ -2095907565527 \quad 6955013524199 \quad -36832622942 \quad 64504080325 \quad -3760571723053 \\ 2171482504380 \quad , 13028895026280 \quad , 542870626095 \quad , 868593001752 \quad , 6514447513140 \quad , \\ 2552223421511 \quad -5169718992073 \quad -1205912703113 \quad -1086920646923 \quad -2388094137299 \\ 1085741252190 \quad , 1085741252190 \quad , 1628611878285 \quad , 542870626095 \quad , 2171482504380 \quad , \\ 6440703111091 \quad -1218494914729 \quad 6656146000559 \quad -21722746362811 \quad -2095907565527 \\ 6514447513140 \quad , -868593001752 \quad , 8685930017520 \quad , 26057790052560 \quad , 4342965008760 \quad , \\ 2725154777111 \quad 31932271589 \quad 31932271589 \quad -73887390529 \quad 25733502500393 \\ 8685930017520 \quad , 120637916910 \quad , 241275833820 \quad , 5211558010512 \quad , 130288950262800 \quad , \\ -343799911081 \quad -37792053085 \quad) \\ -1085741252190 \quad , -434296500876 \end{array} \right) \end{aligned}$$

Mask of $x_{i,j}^{1,0}$:

$$\begin{aligned} & \left(\begin{array}{l} 61848672613411 \quad 285439445629 \quad -339213799501 \quad -8501061371657 \quad -4369132774261 \\ 52115580105120 \quad 2171482504380 \quad 206807857560 \quad -8685930017520 \quad 1206379169100 \quad , \\ -209207768203 \quad 533680265081 \quad -6856655665381 \quad 15512464547161 \quad 28100144271473 \\ 310211786340 \quad , 1447655002920 \quad , -17371860035040 \quad , 5790620011680 \quad , 17371860035040 \quad , \\ 2700000133115 \quad 1333910079319 \quad -120096488429 \quad 72112656934 \quad 2725154777111 \\ 289531000584 \quad , 4342965008760 \quad , 482551667640 \quad , 1628611878285 \quad , 8685930017520 \quad , \\ -2095907565527 \quad -21722746362811 \quad 6656146000559 \quad -1218494914729 \quad 6440703111091 \\ 4342965008760 \quad -26057790052560 \quad , 8685930017520 \quad -868593001752 \quad , 6514447513140 \quad , \\ -2388094137299 \quad -1086920646923 \quad 1205912703113 \quad 5169718992073 \quad 2552223421511 \\ 2171482504380 \quad , 542870626095 \quad , -1628611878285 \quad , -1085741252190 \quad , 1085741252190 \quad , \\ -3760571723053 \quad 64504080325 \quad -36832622942 \quad 6955013524199 \quad -2095907565527 \\ 6514447513140 \quad , 868593001752 \quad , 542870626095 \quad , 13028895026280 \quad , 2171482504380 \quad , \\ 2725154777111 \quad 31932271589 \quad 31932271589 \quad -73887390529 \quad 25733502500393 \\ 4342965008760 \quad , 241275833820 \quad , 120637916910 \quad , 5211558010512 \quad , 130288950262800 \quad , \\ -37792053085 \quad -343799911081 \quad) \\ -434296500876 \quad , -1085741252190 \end{array} \right) \end{aligned}$$

Mask of $x_{i,j}^{0,1}$:

$$\begin{aligned} & \left(\begin{array}{l} -16044311288503 \quad -285439445629 \quad -533680265081 \quad 105803839423 \quad 13541457918013 \\ 10423116021024 \quad , 434296500876 \quad , 482551667640 \quad , 103403928780 \quad , 1206379169100 \quad , \\ 7517711285213 \quad -19207698623 \quad 6856655665381 \quad 120096488429 \quad -610082577859 \\ 1737186003504 \quad -32170111176 \quad 3474372007008 \quad 160850555880 \quad -1447655002920 \quad , \\ -2302892888971 \quad -1604093855459 \quad 948342242035 \quad -72112656934 \quad -2725154777111 \\ 80425277940 \quad , 496338858144 \quad , 1158124002336 \quad , 325722375657 \quad , 1447655002920 \quad , \\ 2095907565527 \quad -6955013524199 \quad 36832622942 \quad -64504080325 \quad 3519295889233 \\ 723827501460 \quad , 4342965008760 \quad , 180956875365 \quad , 289531000584 \quad , 2171482504380 \quad , \\ -2552223421511 \quad 5169718992073 \quad 364490450669 \quad 2085546546332 \quad -521386636583 \\ 361913750730 \quad , 361913750730 \quad , 542870626095 \quad , 180956875365 \quad , 144765500292 \quad , \\ 505619067587 \quad 1089486754079 \quad -1095500413283 \quad -556242655557 \quad 2095907565527 \\ 1302889502628 \quad , 868593001752 \quad , 1737186003504 \quad , 5211558010512 \quad , 868593001752 \quad , \\ -2725154777111 \quad -31932271589 \quad -31932271589 \quad 73887390529 \quad -25733502500393 \\ 1737186003504 \quad , 40212638970 \quad , 48255166764 \quad , 1737186003504 \quad , 43429650087600 \quad , \\ 343799911081 \quad 521386636583 \quad) \\ 361913750730 \quad , 723827501460 \end{array} \right) \end{aligned}$$

Masks associated with the parameter values z_r^*

Mask of $t_{i,j}$:

$$\left(\begin{array}{l} -4531890127703 \quad 36006119259559 \quad -20242568383524532937 \quad 17297520671281 \\ 13028895026280 \quad 39086685078840 \quad -7042203380157369840 \quad 13028895026280 \quad , \\ 29593026919579 \quad 26994849992255621402765 \quad 9165456190681132751 \quad 30904875065308 \\ 5428706260950 \quad , 15974830661375510686944 \quad , 6161927957637698610 \quad , 24429178174275 \quad , \\ 64028235288551979169 \quad 576334620107504500896935 \quad -361472787357233 \\ 24647711830550794440 \quad , 223647629259257149617216 \quad , -26057790052560 \quad , \\ -5791606797469437299171 \quad -96451246858750373479 \quad 9488658832619183459 \\ 12288331277981162066880 \quad , -49295423661101588880 \quad , 7783487946489724560 \quad , \\ 399428393604596767 \quad -21206215098462805471 \quad -888284192303734585 \\ 3791955666238583760 \quad , 49295423661101588880 \quad , 1556697589297949412 \quad , \\ 1354491661355237551055957 \quad 1387013003869 \quad -2878083358530960513494389 \\ 1863730243827142913476800 \quad , 8685930017520 \quad , -10064143316666571732774720 \quad , \\ -1681864799399 \quad 4586462085623 \quad 179554805254886612513 \\ 558381215412 \quad , 697976519265 \quad , 2218294064749571499600 \quad , \\ 567425049055209047084219 \quad 726921707599278586955167 \quad -1087229746071719305197127 \\ 149098419506171433078144 \quad , 798741533068775534347200 \quad , -629008957291660733298420 \quad , \\ 70130888658695527738129 \quad -55746022850959428730643 \quad -292198896309538327 \\ 31949661322751021373888 \quad -279559536574071437021520 \quad -972935993311215570 \quad , \\ 148884558838306413569 \quad -7135795471566067417 \quad -1112520265706788528800107 \\ 49295423661101588880 \quad , 1895977833119291880 \quad , 3354714438888857244258240 \quad , \\ 1767938963792804738010533 \quad -232977362971875229 \quad -48551848730147429474011 \\ 670942887777714488516480 \quad , 34127600996147253840 \quad , 276487453754576146504800 \quad , \\ -2318866060366412313392359 \quad -7657959982383155961476047 \\ -838678609722214311064560 \quad , 3354714438888857244258240 \end{array} \right)$$

Mask of $b_{i,j}$:

$$\left(\begin{array}{l} (0, 0, 33654106472661220639, 0, 0, -28931119278287059059781, -147713415264798351289 \\ 24647711830550794440, -79874153306877553434720, 49295423661101588880, \\ 0, -71687410464642966611, -3723562194545339719095199, -3459921708110971652593 \\ 49295423661101588880, 1118238146296285748086080, 0, -12288331277981162066880, \\ 10915736212229383229, -17837846075447753051, -2964129356331035 \\ 3521101690078684920, -7783487946489724560, -27085397615989884, \\ 80572693526252875501, -584083821479971933, -143791532332245022121277, 0, \\ 49295423661101588880, 1945871986622431140, 1863730243827142913476800, 0, \\ 9334610941403380115035381, 6547984526167163665, 2061565679251584881570009 \\ 10064143316666571732774720, 0, 0, 88731762589982859984, 745492097530857165390720, \\ -3132760642018231502419567, 1490762720001245530293439, 325145030004122832756977 \\ 79874153306877553434720, 629008957291660733298420, -159748306613755106869440, \\ 8646494711181661614169, -4440165076962254041, -89518080410516343539 \\ 55911907314814287404304, -7783487946489724560, 49295423661101588880, \\ 14256041226850386701, -281846783691478699397989, 911334612517867838881345 \\ 3791955666238583760, -3354714438888857244258240, -134188577555542897703296, \\ 352952140065170249, -6065655156261675300631, 2681065109642364400574251 \\ 6825520199229450768, -55297490750915229300960, 838678609722214311064560, \\ 1821351235897392862040723) \\ 67094288777771448851648 \end{array} \right)$$

Mask of $y_{i,j}^{2,1}$:

$$\left(\begin{array}{l} -12271780430239 \quad 23115036621721 \quad -697752284589655474 \quad 192333463040581 \\ 11167624308240 \quad , 13028895026280 \quad , -146712570419945205 \quad , 5211558010512 \quad , \\ 29593026919579 \quad 26994849992255621402765 \quad 31613815182159137939 \quad 123384121964447 \\ 1809568753650 \quad , 5324943553791836895648 \quad , 16431807887033862960 \quad , 65144475131400 \quad , \\ 7928485881111597527 \quad 2940388026606543576781849 \quad -361472787357233 \\ 2053975985879232870 \quad , 372746048765428582695360 \quad , 8685930017520 \quad , \\ -5791606797469437299171 \quad -89635388032606534969 \quad 2391950060126985955 \\ -4096110425993720688960 \quad -32863615774067725920 \quad , 1037798392865296608 \quad , \\ -833558368032964139 \quad 2966315162024509075 \quad 4586462085623 \quad -869086434956909083 \\ 631992611039763960 \quad , 6572723154813545184 \quad , 232658839755 \quad , 648623995540810380 \quad , \\ 1447826441014520643970577 \quad 444445950829 \quad -2445437288558816620140901 \\ 621243414609047637825600 \quad , 2605779005256 \quad , 3354714438888857244258240 \quad , \\ -1681864799399 \quad 328025690686445610353 \quad 567425049055209047084219 \\ -186127071804 \quad , 1478862709833047666400 \quad , 49699473168723811026048 \quad , \\ 726921707599278586955167 \quad -1087229746071719305197127 \quad 70130888658695527738129 \\ 266247177689591844782400 \quad , 209669652430553577766140 \quad , 10649887107583673791296 \quad , \\ -55746022850959428730643 \quad -2370991526403268243 \quad 178056402439210119869 \\ 93186512191357145673840 \quad -2594495982163241520 \quad , 32863615774067725920 \quad , \\ -4003702544556302561 \quad -951454110398899835368811 \quad 1767938963792804738010533 \\ -631992611039763960 \quad , 1118238146296285748086080 \quad , 2236476292592571496172160 \quad , \\ 8956702739347499 \quad -48551848730147429474011 \quad -2318866060366412313392359 \\ 5687933499357875640 \quad , 92162484584858715501600 \quad , 279559536574071437021520 \quad , \\ -7657959982383155961476047 \\ -1118238146296285748086080 \end{array} \right)$$

Mask of $y_{i,j}^{-1,-2}$:

$$\left(\begin{array}{l} 279021003241424923, 0, -2095907565527, 2987873000594748871, 251548442921233 \\ 148256913266470944, 0, -13028895026280, 2527970444159055840, 260577900525600, \\ -391511615876220607, 0, 0, 2725154777111, -61670773096205223361, 11080236034699 \\ 78999076379970495, 0, 0, 26057790052560, 32863615774067725920, 6514447513140, \\ 119712715237402707269, -35841538350081461, 72112656934, -6856655665381 \\ 32863615774067725920, 0, -1625123856959393040, 0, 0, 4885835634855, -52115580105120, \\ 285439445629, -822872808792542327, 20342292042566996155, -123959409326402622509 \\ 6514447513140, 5188991964326483040, 6572723154813545184, 32863615774067725920, \\ -1437666942653, 41709834343289924069, -103267105684894211989, 33400355926961627, \\ 52115580105120, 16431807887033862960, 32863615774067725920, 2594495982163241520, \\ 0, 0, 0, 0, 0, 0, 0, 31083919823327614673, 31932271589 \\ 0, 1478862709833047666400, 723827501460, 0 \end{array} \right)$$

Mask of $y_{i,j}^{-2,-1}$:

$$\left(\begin{array}{l} -3512153958923412841, 0, 0, 132824112520673051, -5398434624971, 3368897166768107761 \\ 1037798392865296608, 0, 0, 252797044415905584, 5790620011680, 505594088831811168, \\ -2095907565527, 0, 0, 51400849925349169201, 2771608293079, -129982638408258761429 \\ 13028895026280, 0, 0, 32863615774067725920, 8685930017520, -32863615774067725920, \\ 2725154777111, 74797826319829301, 0, 0, 0, 0, 0, 34028298205171069 \\ 26057790052560, 1137586699871575128, 0, 0, 0, 0, 0, 2594495982163241520, \\ -78503269290786805121, 4837764506653326713, 5150734764713, -129974845906529243239 \\ 32863615774067725920, 2053975985879232870, 10423116021024, -32863615774067725920, \\ 18067899515277483949, -821616924236123443, 285439445629, -6856655665381 \\ 4694802253438246560, -5188991964326483040, 6514447513140, -52115580105120, \\ 72112656934, 0, 0, 0, 0, 0, 3045745544524018757, 0, 31932271589 \\ 4885835634855, 0, 0, 0, 0, 0, 295772541966609533280, 0, 723827501460 \end{array} \right)$$

Mask of $y_{i,j}^{1,2}$:

$$\left(\begin{array}{l} -22557725931173, 10223953983883, 14303048446047913787, 0, 0, -7343052350198304428891 \\ 39086685078840, 26057790052560, 8215903943516931480, 0, 0, 5324943553791836895648, \\ -81155974164104723453, -47075659357, -64871551638499128101 \\ 16431807887033862960, -13028895026280, 32863615774067725920, \\ -3723562194545339719095199, 0, -2814702740319531297679, 7580980524594648365 \\ 372746048765428582695360, 0, -4096110425993720688960, 1643180788703386296, \\ -3073350921486401347, -215819734497942031, 109744537127156581801 \\ 1037798392865296608, 252797044415905584, 32863615774067725920, \\ -2370363584125058801, -1437915323322245022121277, 0, 9334610941403380115035381 \\ 2594495982163241520, 621243414609047637825600, 0, 3354714438888857244258240, \\ 0, 0, 8919157102449659699, 2061565679251584881570009, -3132760642018231502419567 \\ 42253220280944219040, 248497365843619055130240, -266247177689591844782400, \\ 379450774843626130907008, -68314426406062357805065, 28616345252400386004019 \\ 52417413107638394441535, 10649887107583673791296, 46593256095678572836920, \\ -347571779754942689, -53480289501930992693, 5261189680556966689 \\ 259449598216324152, 32863615774067725920, 1263985222079527920, \\ -281846783691478699397989, -4234540751973561807544133, 203356487425511647 \\ 1118238146296285748086080, -2236476292592571496172160, 2275173399743150256, \\ -6065655156261675300631, 2681065109642364400574251, 1821351235897392862040723 \\ 18432496916971743100320, 279559536574071437021520, 223647629259257149617216 \end{array} \right)$$

Mask of $z_{i,j}^{1,1}$:

$$\left(\begin{array}{l} -55838625716687, 28252946643043, -35419776850673763263, 340862589821 \\ 52115580105120, 4342965008760, -2738634647838977160, 77552946585, \\ 36472270777393, 22784567744197314899045, 41709834343289924069, 184986982915643 \\ 1206379169100, 1774981184597278965216, 5477269295677954320, 86859300175200, \\ 16061835113666299907, 696077982004858901394227, -222664500682471 \\ 1564934084479415520, 24849736584361905513024, -2895310005840, \\ -4377474382328624429291, -103267105684894211989, 12649034228277897263 \\ 1365370141997906896320, -10954538591355908640, 1729663988108827680, \\ -95482998221203757, -32748200305393967077, -1629969089647167523 \\ 84265681471968528, 10954538591355908640, 864831994054413840, \\ 1550315458679682817567037, -64504080325, -4294689070651846383297973 \\ 207081138203015879275200, -108574125219, 1118238146296285748086080, \\ -10208893686044, 20678875968292, 31083919823327614673, 173021210789374032949883 \\ 542870626095, 542870626095, 492954236611015888800, 16566491056241270342016, \\ 1770022110879746120192767, -320297403910492795197703, 71449037995331751739129 \\ 88749059229863948260800, -17472471035879464813845, 3549962369194557930432, \\ -66283535540719222053683, -62692087856862133, 125189984533080661589 \\ 31062170730452381891280, -864831994054413840, 10954538591355908640, \\ -417844641336210772, -507465788526363562852763, 2754579679065107565823973 \\ -26333025459990165, 372746048765428582695360, 7454920975308571655390720, \\ -35841538350081461, -18213339302664427175101, -2445005075701749729458923 \\ 541707952319797680, 30720828194952905167200, 93186512191357145673840, \\ -8248110067234692639264367 \\ 372746048765428582695360 \end{array} \right)$$

Mask of $x_{i,j}^{1,1}$:

$$\begin{aligned}
& \left(\frac{25735396581967}{52115580105120}, \frac{20499774026527}{8685930017520}, \frac{4837764506653326713}{684658661959744290}, -\frac{147165410935}{62042357268}, -\frac{4542756612353}{241275833820}, \right. \\
& -\frac{95967048423019818088417}{8874905922986394826080}, -\frac{10549610284487262413}{782467042239707760}, -\frac{61898203625857}{17371860035040}, \\
& -\frac{78503269290786805121}{10954538591355908640}, -\frac{3594876144767525747936119}{124248682921809527565120}, \frac{69404143337381}{1447655002920}, \\
& \frac{3119383890644764638467}{1365370141997906896320}, \frac{137196682189771506217}{10954538591355908640}, -\frac{14677470679789039247}{1729663988108827680}, \\
& \frac{48558431048704523}{84265681471968528}, \frac{56878119221027123521}{10954538591355908640}, -\frac{62064145578652691}{864831994054413840}, \\
& -\frac{1508165407920643644274877}{207081138203015879275200}, \frac{322520401625}{868593001752}, \frac{6231253028279350105536421}{1118238146296285748086080}, \frac{2552223421511}{217148250438}, \\
& -\frac{5169718992073}{217148250438}, \frac{3045745544524018757}{98590847322203177760}, \frac{318105460850634877653593}{82832455281206351710080}, \\
& -\frac{2395882352848026640135327}{88749059229863948260800}, \frac{280444972598738032686143}{13977976828703571851076}, \\
& -\frac{72239927597313486139729}{3549962369194557930432}, \frac{72606043154575098047507}{31062170730452381891280}, -\frac{1629341147368958081}{864831994054413840}, \\
& -\frac{20212013123489501029}{2190907718271181728}, \frac{3225445712758438091}{210664203679921320}, \frac{211473573944672714508731}{372746048765428582695360}, \\
& -\frac{3346564108228489262512037}{745492097530857165390720}, \frac{74797826319829301}{379195566623858376}, \frac{2046729234925159151}{6144165638990581033440}, \\
& \left. \frac{2533527588429480231127171}{93186512191357145673840}, \frac{8602200118145614645937359}{372746048765428582695360} \right)
\end{aligned}$$

Conclusion and perspectives

To determine, as a future project, the conditions under which the previous methods could be applied to Volterra integral equations of the third kind. These techniques can also be used with nonlinear integrals and integro-differential equations, but some modifications are required.

Precisely, we aim to approximate the solution of integral equations of the type :

$$\alpha\varphi(s) - \beta \sum_{k=1}^m \int_a^s H_k(s, t, \psi(t))\varphi(t)dt = g(s), \quad m \in \mathbb{N}^*, \quad a \leq s \leq b,$$

$$\alpha\varphi(s) - \beta \sum_{k=1}^m \int_a^s H_k(s, t, \psi(t)) \ln|s-t| h(s, t)\varphi(t)dt = g(s), \quad m \in \mathbb{N}^*,$$

$$\alpha\varphi(s) - \frac{\beta}{\pi} \int_0^1 \frac{h(s, t)k(s, t, \psi(t))}{s-t} \varphi(t)dt = g(s), \quad , \quad 0 \leq s \leq 1.$$

Bibliography

- [1] M. Ahues, A. Largillier, B. V. Limaye, *Spectral Computations for Bounded Operators*, CRC, Boca Raton, 2001.
- [2] C. Allouch, P. Sablonnière, D. Sbibih, *Solving Fredholm integral equations by approximating kernels by spline quasi-interpolants*, Numer. Algor., 56 (2011) 437–453.
- [3] J. Appell, A. Kalitvin, P. Zabrejko, *Partial integral operators and integro-differential equations*, M. Dekker, 2000.
- [4] F.J. Ariza-Lopez, D. Barrera, S. Eddargani, M.J. Ibáñez and J.F. Reinoso, *Spline quasi-interpolation in the Bernstein basis and its application to digital elevation models*, Math. Meth. Appl. Sci., 46 (2023) 1687- 1698.
- [5] D. Barrera, H. Chebbah and A. Mennouni, *Two spline quasi-interpolation based Nyström methods for solving generalized Fredholm integral equations*, Submitted.
- [6] D. Barrera, F. El Mokhtari, M. J. Ibáñez , D. Sbibih, *Non uniform quasi-interpolation for solving Hammerstein integral equations*, Int. J. Comput. Math., 97 (2020) 72-84.
- [7] D. Barrera. F. Elmokhtari and D.Sbibih, *Two methods based on bivariate spline quasi-interpolants for solving Fredholm integral equations*, Appl. Numer. Math., 127 (2018) 78–94.
- [8] D. Barrera, M.J., Ibáñez, P., Sablonnière, D., Sbibih, *Near best univariate spline discrete quasi-interpolants on nonuniform partitions*, Constr. Approx., 28 (2008) 237-251.
- [9] D. Barrera, C. Dagnino, M. J. Ibáñez, S. Remogna, *Quasi-interpolation by C^1 quartic splines on type-1 triangulations*, J. of Comput. and Appl. Maths., 349 (2019) 225–238.
- [10] D. Barrera, C. Conti, C. Dagnino, M. J. Ibáñez, S. Remogna, *C^1 -Quartic Butterfly-spline interpolation on type-1 triangulations*. In: G. E. Fasshauer, M. Neamtu, L. L. Schumaker (Eds.), Approximation Theory XVI, Nashville, TN, USA, May 19–22, 2019. Springer Proceedings in Mathematics & Statistics, 336 (2021) 11–26.
- [11] D. Barrera, C. Dagnino, M. J. Ibáñez, S. Remogna, *Point and differential C^1 quasi-interpolation on three direction meshes*, J. of Comput. and Appl. Maths., 354 (2019) 373–389.
- [12] D. Barrera, S. Eddargani, M.J. Ibáñez, S. Remogna, *Spline quasi-interpolation in the Bernstein basis on the Powell–Sabin 6-split of a type-1 triangulation*, J. Comput. Appl. Math., 424 (2023) 115011.
- [13] D. Barrera, S. Eddargani, A. Lamnii, *On C^2 cubic quasi-interpolating splines and their computation by subdivision via blossoming*, J. Comput. Appl. Math., 420 (2023) 114834.
- [14] S. Bouhiri, A. Lamnii, M. Lamnii, A. Zidna, *A C^2 spline quasi-interpolant for fitting 3D data on the sphere and applications*, Math. Comput. Simulation, 164 (2019) 46–62.
- [15] L. Bougoffa, R.C. Rach, A. Mennouni, *An approximate method for solving a class of weakly-singular Volterra integro-differential equations*, Appl. Math. Comput., 217 (2011) 8907-8913.
- [16] H. Chebbah, A. Mennouni, K. Zennir, *Three methods to solve two classes of integral equations of the second kind*, Bol. Soc. Paran. Mat., 223 (2022) 2855-2866.
- [17] R.W. Clough, J.L.Tocher, *Finite element stiffness matrices for analysis of plates in bending*, in: Conference on Matrix Methods in Structural Mechanics, Wright Patterson Air Force Base, Ohio, (1965), 515–545.

- [18] M.C. De Bonis, M.P. Stanić, T.V. Tomović Mladenović, *Nystorm methods for approximating the solutions of an integral equation arising from a problem in mathematical biology*, Appl. Numer. Math., 171 (2022) 193–211.
- [19] F. Foucher P. Sablonnière, *Quadratic spline quasi-interpolants on bounded domains of \mathbb{R}^d , $d = 1; 2; 3$* , Rend. Sem. Mat. Univ. Pol. Torino, 61 (2003) 229-246.
- [20] F. Foucher P. Sablonnière, *Quadratic spline quasi-interpolants and collocation methods*, Math. Comput. Simulation 79 (2009), 3455-3465.
- [21] R. Franke, *Scattered data interpolation: tests of some methods*, Math. Comp., 38 (1982) 181–200.
- [22] A. Hashemian, H. Sliusarenko, S. Remogna, D. Barrera, M. Barton, *Solving boundary value problems via the Nyström method using spline Gauss rules*, Comput. Math. Appl., 143 (2023) 33-47.
- [23] W. Gao, Z. Sun, *High-order numerical solution of time-dependent differential equations with quasi-interpolation*, Appl. Numer. Math., 146 (2019) 276–290.
- [24] W. Gao X. Zhang, X. Zhou, *Multiquadric quasi-interpolation for integral functionals*, Math. Comput. Simulation, 177 (2020) 316–328.
- [25] W. Gao, Z. Wu, *A quasi-interpolation scheme for periodic data based on multiquadric trigonometric B-splines*, J. Comput. Appl. Math., 271 (2014) 20-30.
- [26] Y. N. Grigoriev et al., *Symmetries of integro-Differential Equations: With Applications in Mechanics and Plasma Physics*, Springer Netherlands, 2010.
- [27] R. Hardy, *Multiquadric equations of topography and other irregular surfaces*, J. Geophys. Res., 76 (1971) 1905–1915.
- [28] L. Knockaert, *Multinomial Lagrange–Bernstein approximants*, IEEE Signal Process. Lett., 13 (2006) 333–336.
- [29] M.-J. Lai, L.L. Schumaker, *Spline functions on triangulations*. Encycl. Math. Appl., 110. Cambridge University Press, Cambridge, 2007.
- [30] V. Lakshmikantham, *Theory of integro-differential Equations*, CRC Press, 1995.
- [31] P. Linz, *A Method for the Approximate Solution of Linear Integro-Differential Equations*. SIAM J. Numer. Anal., 11 (1974) 137–44.
- [32] T. Lyche, L. Schumaker, S. Stanley, *Quasi-interpolants based on trigonometric splines*, J. Approx. Theory, 95 (1998) 280–309.
- [33] A. Mennouni, *Two projection methods for skew-Hermitian operator equations*, Math. Comput. Modelling, 55 (2012) 1649–1654.
- [34] A. Mennouni, *The iterated projection method for integro-differential equations with Cauchy kernel*, J. Appl. Math. Inf. Sci., 31 (2013) 661–667.
- [35] A. Mennouni, *Piecewise constant Galerkin method for a class of Cauchy singular integral equations of the second kind in L^2* , J. Comput. Appl. Math. , 326 (2017) 268–272.
- [36] A. Mennouni, *Improvement by projection for integro-differential equations*, Math. Methods Appl. Sci., 2020. <https://doi.org/10.1002/mma.6318>.
- [37] A. Mennouni, *Two projection methods for skew-Hermitian operator equations*, Math. Comput. Modelling, 55 (2012) 1649–1654.
- [38] A. Mennouni, S. Zaouia, *Discrete septic spline quasi-interpolants for solving generalized Fredholm integral equation of the second kind*, Math. Sci., 11 (2017) 345–357.
- [39] G. M. Nielson, *A first order blending method for triangles based upon cubic interpolation*, Int. J. Numer. Meth. Engng. 15 (1978) 308–318.
- [40] G. Nürnberger, C. Rössl, H.-P. Seidel, F. Zeilfelder, *Quasi-Interpolation by quadratic piecewise polynomials in three variables*, Comput. Aided Geom. Des., 22 (2005) 221–249.
- [41] A. Pedas, E. Tamme, *Discrete Galerkin method for Fredholm integro-differential equations with weakly singular kernels*, J. Comput. Appl. Math., 213 (2008) 111–126.

- [42] I. Parts, A. Pedas, E. Tamme, *Piecewise polynomial collocation for Fredholm integro-differential equations with weakly singular kernels*. SIAM J. Numer. Anal., 43 (2005) 1897–1911.
- [43] M. Rahman, *Integral equations and their applications*, WIT Press, (2007)
- [44] S. Remogna, P. Sablonnière, *On trivariate blending sums of univariate and bivariate quadratic spline quasi-interpolants on bounded domains*, Comput. Aided Geom. Des., 28 (2011) 89–101.
- [45] P. Sablonnière, *Quadratic spline quasi-interpolants on bounded domains of \mathbb{R}^d , $d = 1; 2; 3$* , Rend. Sem. Mat. Univ. Pol. Torino 61 (2003) 263–278.
- [46] P. Sablonnière, *On some multivariate quadratic spline quasi-interpolants on bounded domains*. In Modern Developments in Multivariate Approximation, W. Haussmann et al. (eds), ISNM 145 BV (2003) 263–278.
- [47] P. Sablonnière, *Univariate spline quasi-interpolants and applications to numerical analysis*. Rend. Sem. Mat. Univ. Pol. Torino, 63 (2005) 107–118.
- [48] P. Sablonnière, D. Sbibih, M. Tahrichi, *High-order quadrature rules based on spline quasi-interpolants and application to integral equations*, Appl. Numer. Math., 62 (2012) 507–520.
- [49] C. Schneider, *Product integration for weakly singular integral equations*, Math. Comp., 36 (1981) 207–213.
- [50] L. L. Schumaker, *Spline Functions: Basic Theory*, Wiley-Interscience, New York, 1981.
- [51] T. Sorokina, F. Zeilfelder, *Optimal quasi-interpolation by quadratic C^1 splines on four-directional meshes*, in: Chui, C., et al. (Eds.), Approximation Theory, vol. XI. Gatlinburg 2004. Nashboro Press, Brentwood, TN, (2005) 423–438.
- [52] T. Sorokina, F. Zeilfelder, *Local Quasi-Interpolation by cubic C^1 splines on type-6 tetrahedral partitions*, IMA J. Numerical Analysis, 27 (2007) 74–101.
- [53] T. Sorokina, F. Zeilfelder, *An explicit quasi-interpolation scheme based on C^1 quartic splines on type-1 triangulations*, Comput. Aided Geom. Des., 25 (2008) 1–13.
- [54] H. Speleers, *Multivariate normalized Powell–Sabin B-splines and quasi-interpolants*, Comput. Aided Geom. Des., 30 (2013) 2–19.
- [55] Eero Vainikko, Gennadi Vainikko, *A spline product quasi-interpolation method for weakly singular Fredholm integral equations*, SIAM J. Numer. Anal., 46 (2008) 1799–1820.
- [56] V. Volterra, *Theory of functionals and of integral and integro-differential equations*, Dover Publications, 2005.
- [57] M. Xu , Q. Fang , R.H. Wang , Z.W. Jiang , M.Z. Liu, *A multinomial spline approximation scheme using spline quasi-interpolants*, Appl. Math. Comput., 218 (2012) 5081–5089.

Abstract

Investigating some classes of integral and integro-differential equations via spline quasi-interpolant is the main goal of this study. In order to attain the best approximation order, this work provides a construction on a renovation of the initial type-1 triangulation. We use the continuous blending sum scheme and the tensor product one of spline quasi-interpolants to approximate the kernels of a Fredholm second-kind integral equations system. Additionally, we expand the quasi-interpolant method for periodic data.

Key words: Integral equations, integro-differential equations, spline quasi-interpolants.

Résumé

L'étude de certaines classes d'équations intégrales et intégro-différentielles via une spline quasi-interpolante est l'objectif principal de ce travail. Afin d'atteindre le meilleur ordre d'approximation, ce travail propose une construction sur une rénovation de la triangulation initiale de type 1. Nous utilisons le schéma de somme de mélange continu et celui du produit tensoriel des quasi-interpolants splines pour approcher les noyaux d'un système d'équations intégrales de second type de Fredholm. De plus, nous étendons la méthode quasi-interpolante pour les données périodiques.

Mots clés: Equations intégrales, équations intégro-différentielles, quasi-interpolants splines.

ملخص:

الهدف الرئيسي لهذا العمل هو دراسة بعض أنواع المعادلات التكاملية و التكامل تفاضلية عبر عبر أشباه الاستقطاب المنحنية. من أجل الحصول على أحسن تقريب ، يقترح هذا العمل تشكيل المنحنيات المثلثية من النوع 1. نستعمل مخطط المجموع المستمر وجداء التزويج لأشباه الاستقطاب المنحنية لتقريب حلول بعض أنظمة المعادلات التكاملية من نوع فريدهولم. أكثر من هذا ندرس طريقة شبه الاستقطاب الدورية .

الكلمات المفتاحية: المعادلات التكاملية، المعادلات التكاملية تفاضلية، أشباه الاستقطاب المنحنية.