RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
UNIVERSITÉ BATNA 2
FACULTÉ DE MATHÉMATIQUES-INFORMATIQUE
DÉPARTEMENT DE MATHÉMATIQUES
LABORATOIRE EQUATIONS AUX DERIVEES PARTIELLES ET APPLICATIONS, EDPA
LABORATOIRE DES TECHNIQUES MATHÉMATIQUES, LTM

THÈSE

Présentée en vue d'obtenir du diplôme de doctorat
Option : Équations aux Dérivées Partielles et Applications

Par

TOUIL ASMA

THÈME

EXISTENCE GLOBALE OU EXPLOSION EN TEMPS FINI DES SOLUTIONS D'EQUATION D'EVOLUTION

Soutenue le : 11 Mai 2017

Devant le jury composé de :

Salah-Eddine Rebiai	Pr. Université Batna 2	Président
Amar Youkana	Pr. Université Batna 2	Rapporteur
Abdelaziz Mennouni	MCA Université Batna 2	Examinateur
Mohamed Zerguine	MCA université Batna 2	Examinateur
Benabderrahmane Benyatou	Pr. Université Msila	Examinateur
Salah Drabla	Pr. Université Sétif	Examinateur
Fadila Bentalha	Pr. Université Batna2	Invitée

Existence Globale ou Explosion en Temps Fini des Solutions d'Équation d'Évolution

THESE
Présentée en vue d'obtenir le diplôme de doctorat

Option : Equations aux Dérivées Partielles et Applications
Université de Batna -2-

Par
TOUIL ASMA

11 May 2017

CONTENTS

ACKNOWLEDGMENTS

I owe my gratitude to all those people who have made my journey to this date possible. As I started putting it in words, I felt words can never measure the intensity of my gratitude to all of you.

I would like to express my heartfelt gratitude to my supervisor Prof. Youkana Amar, for his careful guidance and continuing support. His willingness to listen to me and offer advice is very inspiring. I feel honored to have had a chance to work with him and learn from him from my first year in the university untill now. Thank you, Prof. Youkana, for all of your help and support, and for a graduate experience that I will cherish forever.

I would also like to thank my committee members Prof. Salah-Eddine Rebiai, Dr. Abdelaziz Mennouni, Dr. Mohamed Zerguine, Prof. Benabderrahmane Benyatou and Prof.Salah Drabla for taking out time to serve in my committee.

I am sincerely grateful to Prof. Fadila Bentalha for being part of the examination committe.

I would like to thank my teachers from primary school until now, who have helped me in my journey, especially Mr. Alouache Abd Allah, who was a reason for me to choose study Mathematics. Thank you all.

I also would like to thank my friends and colleagues, for all the time that we shared together, for their support and care that helped me to stay focused on my graduate study.

Most importantly, none of this would have been possible without the love and patience of my family. My family, to whom this thesis is dedicated, has been a constant source of love, concern, support and strength, all of these years.

ABSTRACT

The purpose of this thesis is to study the question of global existence of solutions for some non linear evolution equations.

In the first part, we are interested with the study of some reaction-diffusion systems arising from the diffusion of an epidemic phenomena, with homogeneous Neumann boundary conditions and nonlinearities of weakly exponential growth. We establish a result on the global existence and the asymptotic behavior of these solutions via a Lyapunov functional.

In the second part, a threshold condition, given by the basic reproduction number, determines the global stability of the equilibrium points.

In the last, we deal with the stability analysis of equilibrium points of the ODE system corresponding to the system studied in the first part.

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

To control the disease, mathematical epidemic models, was introduced to get informations about the disease in some population of individuals.

In this thesis, we are concerned with some evolution equations that describe the diffusion in time and space of an epidemic in some population of individuals divided into two groups.

In the second chapter, we establish the global existence and the boundedness of solutions of some reaction diffusion systems with homogeneous Neumann boundary conditions and nonlinearities of weakly exponential growth. Furthermore, we give a result on the asymptotic behavior of the solutions. Our approach is based on a Lyapunov functional.

In the third chapter, we investigate the global stability and we give a threshold condition for the disease vanishing or spreading for some epidemic model depending on the basic reproduction rate R.
We wil show that if $R < 1$ the free equilibrium point is globally asymptotically stable, and if $R > 1$ this free equilibrium point is unstable and the endemic equilibrium point is asymptotically stable.

Finally, we investigate the local stability of free and endemic equilibrium points, where the recruitment rates are supposed to be constants or variable paramaters.

1.1 The SIR model / Reaction-Diffusion systems

In the first part, we consider a reaction-difusion system of four equations

$$\begin{cases} \dfrac{\partial S_1}{\partial t} - d_1 \Delta S_1 = \Lambda_1 - \beta_1 \dfrac{S_1 \varphi(U_1)}{T} - \beta_2 \dfrac{S_1 \varphi(U_2)}{T} - \mu S_1 & \text{in } \mathbb{R}^+ \times \Omega \\ \dfrac{\partial U_1}{\partial t} - d_2 \Delta U_1 = \beta_1 \dfrac{S_1 \varphi(U_1)}{T} + \beta_2 \dfrac{S_1 \varphi(U_2)}{T} - \sigma_1 U_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial S_2}{\partial t} - d_3 \Delta S_2 = \Lambda_2 - \beta_3 \dfrac{S_2 \varphi(U_1)}{T} - \beta_4 \dfrac{S_2 \varphi(U_2)}{T} - \mu S_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U_2}{\partial t} - d_4 \Delta U_2 = \beta_3 \dfrac{S_2 \varphi(U_1)}{T} + \beta_4 \dfrac{S_2 \varphi(U_2)}{T} - \sigma_2 U_2 & \text{in } \mathbb{R}^+ \times \Omega. \end{cases} \tag{1.1}$$

with the homogeneous Newmann boundary conditions

$$\frac{\partial S_i}{\partial \nu} = \frac{\partial U_i}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega,\ i = 1, 2, \tag{1.2}$$

and the continuous initial data $S_{1,0}$, $U_{1,0}$, $S_{2,0}$, $U_{2,0}$ such that

$$S_i(0,x) = S_{i,0}(x) \geq 0, U_i(0,x) = U_{i,0}(x) > 0 \qquad \text{in } \Omega,\ i = 1, 2. \tag{1.3}$$

The constants d_1, d_2, d_3, d_4, Λ_1, Λ_2, μ, σ_1, σ_2 are such that

$$d_1 > 0,\ d_2 > 0,\ d_3 > 0,\ d_4 > 0,\ \mu > 0,\ \sigma_1 > 0,\ \sigma_2 > 0,\ \Lambda_1 \geq 0,\ \Lambda_2 \geq 0. \tag{1.4}$$

The nonlinearity φ is assumed to be a nonnegative increasing and continuously differentiable function on $[0, +\infty)$ such that

$$\varphi(0) = 0,\ \varphi(\eta) \leq Ce^{\eta^\gamma},\ 0 < \gamma < 1. \tag{1.5}$$

Here Ω is a bounded domain of class C^1 in $\mathbb{R}^n$, with boundary $\partial\Omega$, $\dfrac{\partial}{\partial \nu}$ denotes the outward normal derivative to $\partial\Omega$.

In the second part, we consider the following reaction-diffusion system

$$\begin{cases} \dfrac{\partial S_1}{\partial t} - D\Delta S_1 = \Lambda_1 - \beta \dfrac{S_1(U_1+U_2)}{T} - \mu S_1 & \text{in } \mathbb{R}^+ \times \Omega \\ \dfrac{\partial U_1}{\partial t} - D\Delta U_1 = \beta \dfrac{S_1(U_1+U_2)}{T} - \sigma U_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial S_2}{\partial t} - D\Delta S_2 = \Lambda_2 - \beta \dfrac{S_2(U_1+U_2)}{T} - \mu S_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U_2}{\partial t} - D\Delta U_2 = \beta \dfrac{S_2(U_1+U_2)}{T} - \sigma U_2 & \text{in } \mathbb{R}^+ \times \Omega. \end{cases} \tag{1.6}$$

supplemented with homogeneous Neumann boundary conditions

$$\frac{\partial S_i}{\partial \nu} = \frac{\partial U_i}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \qquad i = 1, 2, \tag{1.7}$$

and continuous initial data $S_{1,0}$, $U_{1,0}$, $S_{2,0}$, $U_{2,0}$

$$S_i(0,x) = S_{i,0}(x) \geq 0, \quad U_i(0,x) = U_{i,0}(x) > 0 \qquad \text{in } \Omega, \qquad i = 1, 2, \tag{1.8}$$

where Ω is a bounded domain of class C^1 in $\mathbb{R}^n$, with boundary $\partial\Omega$ and $\dfrac{\partial}{\partial \nu}$ denotes the outward normal derivative to $\partial\Omega$.

Those systems may be viewed as a generalisation of the model given by Piqueira, Castano and Monteiro in [35], by taking account of the diffusion terms

The SIR models, are known by the compartmental models, were introduced by Anderson Grey McKendrick[1] and William Ogilvy Kermack[2] [25] in 1927. They are based on the division of some population of individuals on three elements, and the study of their numbers in time.

---

[1] **Anderson Grey McKendrick**(1876-1970) was a Scottish physician and epidemiologist pionnered the use of mathematical methods in epidemiology.

[2] **William Ogilvy Kermack**(1897-1970) was born in Scotland, he studied advanced geometry and mathematics and graduated in Mathematics and Natural Philosophy from Aberdeen University in 1918.

The basic elements for the discription of the SIR model are the susceptible, infective and removed individuals with unit of time, are respectively defined as

- $S(t)$ those individuals who are healthy and can be infected;
- $I(t)$ those individuals who are infected and are able to transmit the disease;
- $R(t)$ those individuals who are immune because have been infected and now have recovered.

In 1981, Webb [39] was the first that included the spatial term to the SIR models, in one dimensional spatial region

$$\begin{cases} \dfrac{\partial}{\partial t}S(x,t)-\dfrac{\partial^2}{\partial x^2}S(x,t)=-aS(x,t)I(x,t) & \text{in } [-L,L]\times\mathbb{R}^+, \\ \dfrac{\partial}{\partial t}I(x,t)-\dfrac{\partial^2}{\partial x^2}I(x,t)=aS(x,t)I(x,t)-\lambda I(x,t) & \text{in } [-L,L]\times\mathbb{R}^+, \\ \dfrac{\partial}{\partial t}R(x,t)-\dfrac{\partial^2}{\partial x^2}R(x,t)=\lambda I(x,t) & \text{in } [-L,L]\times\mathbb{R}^+, \end{cases} \tag{1.9}$$

with the homogeneous boundary conditions

$$\frac{\partial}{\partial x}S(\pm L,t)=\frac{\partial}{\partial x}I(\pm L,t)=\frac{\partial}{\partial x}R(\pm L,t)=0, \quad \text{in } \mathbb{R}^+ \tag{1.10}$$

and the initial data

$$S(x,0)=S_0(x),\ I(x,0)=I_0(x),\ R(x,0)=R_0(x)\text{on } [-L,L], \tag{1.11}$$

where $S(x,t)$, $I(x,t)$ and $R(x,t)$ are the number of susceptible, infective, and recovered individuals with respect to the position x and the time t.

He established the existence of solutions to the problem (1.9)–(1.11) and analyzed their behavior as time goes to infinity, by using some tools of functional analysis and dynamical systems. Specifically, the theory of semigroups of linear and nonlinear operators in Banach spaces and Lyapunov stability techniques for dynamical systems in metric spaces.

Some works have been established on the SIR moddels with diffusion terms for the global existence and the asymptotic behavior of solutions, see Hamaya [16], Mezienne, Hattaf and Yousfi [28]...etc and the references therein.

In [35], Piqueira, Castano and Monteiro considered a population divided into two groups, and the model is presented with a pair of compartments in each group: the susceptible individuals S and the infected once that are not aware of their situation (Unware) U, where the indices 1 and 2 designate which group they belong to.
The model is given by the ODE system bellow

$$\begin{cases} \dfrac{\partial S_1}{\partial t} = \Lambda_1 - \beta_1 \dfrac{S_1 U_1}{T} - \beta_2 \dfrac{S_1 U_2}{T} - \mu S_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U_1}{\partial t} = \beta_1 \dfrac{S_1 U_1}{T} + \beta_2 \dfrac{S_1 U_2}{T} - \sigma_1 U_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial S_2}{\partial t} = \Lambda_2 - \beta_3 \dfrac{S_2 U_1}{T} - \beta_4 \dfrac{S_2 U_2}{T} - \mu S_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U_2}{\partial t} = \beta_3 \dfrac{S_2 U_1}{T} + \beta_4 \dfrac{S_2 U_2}{T} - \sigma_2 U_2 & \text{in } \mathbb{R}^+ \times \Omega. \end{cases} \tag{1.23}$$

With the homogeneous Newmann boundary conditions

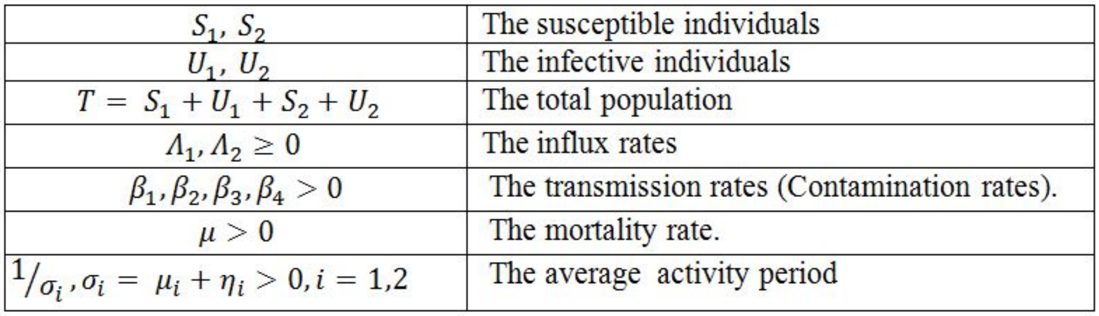
$$\frac{\partial S_i}{\partial \nu} = \frac{\partial U_i}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega,\ i = 1,2, \tag{1.24}$$

and the initial data $S_{1,0}$, $U_{1,0}$, $S_{2,0}$, $U_{2,0}$ verify

$$S_i(0,x) = S_{i,0}(x) \geq 0, U_i(0,x) = U_{i,0}(x) > 0 \qquad \text{in } \Omega,\ i = 1,2. \tag{1.25}$$

Λ_1 and Λ_2 denote the recruitments ratio in the susceptible class (the influx rates) in each group. μ is the mortality rate, the ratio of the number of deaths from the disease to the total number of cases of that disease.
The parameters β_1, β_2, β_3, β_4 describe the contamination rates, the rates at which the disease is spread among the individuals per unit of time, the parameters σ_1, σ_2 are given by $\sigma_i = \mu + \eta_i$, $i = 1,2$, where $1/\eta_i$ is the average activity period of the infected individuals

and $T = S_1 + U_1 + S_2 + U_2$ is the total population (See Figures 1-1, 1-2).



S_1, S_2	The susceptible individuals
U_1, U_2	The infective individuals
$T = S_1 + U_1 + S_2 + U_2$	The total population
$\Lambda_1, \Lambda_2 \geq 0$	The influx rates
$\beta_1, \beta_2, \beta_3, \beta_4 > 0$	The transmission rates (Contamination rates).
$\mu > 0$	The mortality rate.
$^1/_{\sigma_i}, \sigma_i = \mu_i + \eta_i > 0, i = 1,2$	The average activity period

Figure 1.1: Parameters of the model

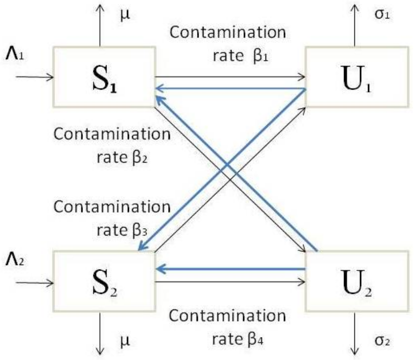


Figure 1.2: Model structure

Many models available in the literature represent dynamics of disease by system of nonlinear ordinary differential equations. Tripathi et .al. [37] have proposed a nonlinear model to study the effect of screening of unaware infectives on the spread of HIV/AIDS in a homogenous population with constant immigration of susceptibles.
Al-sheikh et al. [3] have studied the local and global stability for the nonlinear system of ordinary differential equations. They also studied the effect of screening of unaware infectives on the spread of the disease. (See also [23], [14], [5] and [10],and the reference

therein).

1.2 Boundedness and asymptotic behavior of solutions for a diffusive epidemic model

1.2.1 Global existence and boundedness of solutions

The fundamental mathematical question for these problems is to establish the global existence of solutions of system(1.1)–(1.3). Much works have been done in the literature and positive answers have been under different forms for subgrowth nonlinearity.

For the coupled system 2 × 2, the system (1.1)–(1.3) is somewhat similar to the two-component system below

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_1 \Delta S = F(S,U) & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U}{\partial t} - d_2 \Delta U = G(S,U) & \text{in } \mathbb{R}^+ \times \Omega, \\ a\dfrac{\partial S}{\partial \eta} = b\dfrac{\partial U}{\partial \eta} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ S(0,x) = S_0(x),\ U(0,x) = U_0(x) & \text{on } \Omega. \end{cases} \tag{1.15}$$

$\Omega \subset \mathbb{R}^n$ is a bounded domain of class C^1, $d_1, , d_2 > 0$ positive coefficients, F and G are two continuous differential functions on $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

Some works have been done in literature, in order to study the maximum time of existence of the solutions for the system (1.15) (see Hollis, Martin and Pierre [22], Morgan [31]and Kouachi [27]...etc).

In [22], Hollis, Martin and Pierre (1987) proved global existence and uniform boundedness of solutions for a class of reaction diffusion systems involving two unknowns.

They assumed that the unknown S is uniformly bounded,

1. There exists $\gamma \geq 1$, $L_0 > 0$ such that $|G(\xi,\zeta)| \leq (1+\xi+\zeta)^\gamma$, for all ξ, $\zeta \geq 0$.
2. There exists $\delta_0 \geq 0$, such that $F(\xi,\zeta) + G(\xi,\zeta) \leq \delta_0(1+\xi+\zeta)$, for all ξ, $\zeta \geq 0$.

By using the famous method of duality arguments L_p, they established a result on the existence of solutions for any positive time.

For the particular case where

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_1\Delta S = -S\varphi(U) & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U}{\partial t} - d_2\Delta U = S\varphi(U) & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{1.16}$$

Alikakos [2] in 1979, proved that solutions of (1.16) are global for $|\varphi(\xi)| \leq C\xi^\gamma$ and $1 \geq \gamma < \frac{n+2}{n}$.

Next, Masuda [29] in 1983, showed that the solutions of (1.16) are global for any $\gamma > 1$. For nonlinearities with weakly exponential growth a.e $\varphi(\xi) \leq Ce^{\xi^\gamma}$, $0 < \gamma < 1$, Haraux and Youkana[19] in 1988, gave a positive answer for the global solution for any positive initial data.

When the nonlinearities are as $\varphi(\xi) = e^\xi$, one can cite the works of Barabanova [4], Kanel [24] and Herrero and Lacey et Velazquez [21].

In this section, we prove the global existence of solutions for the reaction-diffusion system (1.1)–(1.3) with nonlinearities of weakly exponential growth. Our appraoch is based on a Lyapunov functional argument which yields the uniform boundedness of solutions. It is given by the following theorem

Theorem 1.2.1. *Let φ satisfies the condition* (2.8), *then the solution* (S_1, U_1, S_2, U_2) *of the system* (2.4)–(2.6) *with nonnegative and bounded initial data* $S_{0,1}, U_{0,1}, S_{0,2}, U_{0,2}$ *is global and uniformly bounded on* $(0, +\infty) \times \Omega$.

1.2.2 Asymptotic behavior of solutions

For a population of one group of individuals, Hamaya [16] considered the following system

$$\begin{cases} \dfrac{\partial S}{\partial t} - D\Delta S = \Lambda - \lambda C(T(t))\dfrac{S(t)I(t)}{T(t)} - \mu S(t) & \text{in } \mathbb{R}^+ \times \Omega \\ \dfrac{\partial I}{\partial t} - D\Delta I = \lambda C(T(t))\dfrac{S(t)I(t)}{T(t)} - \sigma I(t) & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial S}{\partial \eta} = \dfrac{\partial I}{\partial \eta} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ S(0,x) = S_0(x),\ I(0,x) = I_0(x) & \text{on } \Omega. \end{cases} \tag{1.17}$$

where $C(T(t,x)) \in C^1([0,\infty) \times \overline{\Omega}$ is an increasing function of T subject some conditions. He proved that under conditions on the parameters Λ, μ, σ and the function $C(T)$, the solution (S, I) tends to $(\dfrac{\Lambda}{\mu}, 0)$ as t goes to infinity

$$\lim_{t\to\infty} \|S(t,.) - \frac{\Lambda}{\mu}\|_\infty = \lim_{t\to\infty} \|I(t,.)\|_\infty = 0.$$

In [30], Melkemi, Mokrane, and Youkana studied the reaction-diffusion system

$$\begin{cases} \frac{\partial S}{\partial t} - d_1\Delta S = \Lambda - \lambda(t) f(S,I) - \mu S & \text{in } \mathbb{R}^+ \times \Omega \\ \frac{\partial I}{\partial t} - d_2\Delta I = \lambda(t) f(S,I) - \sigma I & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{1.18}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0 \qquad \text{sur } \mathbb{R}^+ \times \partial\Omega, \tag{1.19}$$

and positive initial data

$$S(0,x) = S_0(x), \qquad I(0,x) = I_0(x) \qquad \text{dans } \Omega. \tag{1.20}$$

where d_1, d_2, Λ, μ, σ are constants such that

$$d_1 > 0,\ d_2 > 0,\ \mu > 0,\ \sigma > 0, \Lambda \geq 0. \tag{1.21}$$

with $t \longmapsto \lambda(t)$ is a positive and bounded function in $C(\mathbb{R}^+)$ verifies $0 \leq \lambda(t) \leq \hat{\lambda}$ and the nonlinearity $f(\xi,\eta)$ is a nonnegative differentiable function in $\mathbb{R}^+ \times \mathbb{R}^+$ such that there exist two increasing nonnegative functions φ and ψ in $C^1(\mathbb{R}^+)$ with

$$\xi \geq 0,\ \eta \geq 0 \Rightarrow 0 \leq f(\xi,\eta) \leq \psi(\xi)\varphi(\eta), \tag{1.22}$$

$$\psi(0) = 0,\ \varphi(0) = 0,\ \lim_{\eta \to +\infty} \frac{\ln(1+\varphi(\eta))}{\eta} = 0. \tag{1.23}$$

They proved that (S, I) tends to $(\frac{\Lambda}{\mu}, 0)$ as t goes to infinity

$$\lim_{t \to +\infty} \|I(t,.)\|_\infty = \lim_{t \to +\infty} \|S(t,.) - \frac{\Lambda}{\mu}\|_\infty = 0, \tag{1.24}$$

if one of the two following hypothesis satisfied

$(H-1)$ There exists a real number $p \geq 1$, such that

$$\int_0^{+\infty} (\lambda(s))^p ds < +\infty. \tag{1.25}$$

$(H-2)$ The function $\eta \to \frac{\varphi(\eta)}{\eta}$ is an increasing function on $]0,+\infty[$ and $\lambda(t) = \hat{\lambda} > 0$ is a positive constant independant of de t such that

$$\frac{\hat{\lambda}}{\sigma} \frac{\varphi(N)}{N} \psi(\frac{\Lambda}{\mu}) < 1, \tag{1.26}$$

where $N > 0$ is a positive constant independant of t.

Our main contribution of this thesis cares with the asymptotic behavior of solutions of the system (1.1)–(1.3).

By using a judicious Lyapunov functional argument, we will prove that

Theorem 1.2.2. *Let* (S_1, U_1, S_2, U_2) *be the solution of* (2.4)–(2.6) *in* $(0,+\infty)\times\Omega$. *Assume that the nonlinearity* φ *satisfies*

$$\chi = \sup_{0\le\zeta\le\min(N_1,N_2)} \varphi^{'}(\zeta) \le \min\{\frac{\Lambda_1}{\eta}, \frac{\Lambda_2}{\xi}, \frac{\mu+\sigma_1}{\max(\beta_1,\beta_3)}, \frac{\mu+\sigma_2}{\max(\beta_2,\beta_4)}\}, \tag{1.27}$$

where

$$\eta = (2\beta_1\frac{\Lambda_1}{\sigma_1} + 2\beta_3\frac{\Lambda_2}{\sigma_2} + \beta_1\frac{\Lambda_2}{\sigma_1} + \beta_3\frac{\Lambda_1}{\sigma_2} + \beta_1\theta_1\frac{\Lambda_1}{\mu} + \beta_3\theta_2\frac{\Lambda_2}{\mu}), \tag{1.28}$$

$$\xi = (2\beta_2\frac{\Lambda_1}{\sigma_1} + 2\beta_4\frac{\Lambda_2}{\sigma_2} + \beta_2\frac{\Lambda_2}{\sigma_1} + \beta_4\frac{\Lambda_1}{\sigma_2} + \beta_2\theta_1\frac{\Lambda_1}{\mu} + \beta_4\theta_2\frac{\Lambda_2}{\mu}), \tag{1.29}$$

then

$$\lim_{t\to+\infty} \|S_i(t,.) - \frac{\Lambda_1}{\mu}\|_\infty = \lim_{t\to+\infty} \|U_i(t,.)\|_\infty = 0, \qquad i = 1,2. \tag{1.30}$$

1.3 Threshold condition for global stability of an epidemic model

One of the main problems proposed for epidemic systems is to understand the behavior of the epidemic, its impact and possible future predictions about its spreading.

For such reason, we study the stability of solutions and to know where the solutions goes and under which conditions.

The question of global stability is generally answered by a threshold quantity, called Basic Reproduction Number (denoted by R), which measures the number of new cases an infected individual will generate in a completly susceptible population.

If $R < 1$, the free equilibrium point is globally asymptotically stable, and if $R > 1$ the free equilibrium point becomes instable and the endemic equilibrium point is gllobally asymptotically stable.

In 1999, Hamaya [16] showed the global stability of the equilibrium points of the diffusive epidemic model below

$$\begin{cases} \dfrac{\partial S}{\partial t} - D\Delta S = \Lambda - \lambda C(T(t))\dfrac{S(t)I(t)}{T(t)} - \mu S(t) & \text{in } \mathbb{R}^+ \times \Omega \\ \dfrac{\partial I}{\partial t} - D\Delta I = \lambda C(T(t))\dfrac{S(t)I(t)}{T(t)} - \sigma I(t) & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial S}{\partial \eta} = \dfrac{\partial I}{\partial \eta} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ S(0,x) = S_0(x),\ I(0,x) = I_0(x) & \text{on } \Omega. \end{cases} \tag{1.32}$$

where $C(T)$ is an increasing function of T verifying some conditions.
For this model, the reproductive number R is given by $R = \lambda C(\dfrac{\Lambda}{\mu})/\sigma$.
Recently, Kim, Lin and Zhang [26], gave also a result on the global stability, according to the basic reproducton rate for some reaction diffusion system with free boundary.
By means of Lyapunov functional, Lotfi, Maziane, Hattaf, and Yousfi [28] showed the global stability of both equilibria for the system

$$\begin{cases} \frac{dS}{dI} = \Lambda - \mu S - \dfrac{\beta SI}{1+\alpha_1 S + \alpha_2 I + \alpha_3 SI} \\ \frac{dI}{dt} = \dfrac{\beta SI}{1+\alpha_1 S + \alpha_2 I + \alpha_3 SI} - (\mu + d + r)I \\ \frac{dR}{dt} = rI - \mu R \end{cases} \tag{1.33}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \qquad i = 1,2, \tag{1.34}$$

and initial conditions

$$S(0,x) = \phi_1(x) \geq 0, \quad I(0,x) = \phi_2(x) \geq 0 \qquad \text{in } \Omega,\ i = 1,2 \tag{1.35}$$

We analyse the dynamics of the system (1.6)–(1.8) by giving the basic reproduction number and using a Lyapunov functional method.
The main results of this section are the following

Theorem 1.3.1. *If* $R < 1$ *then for each continious positive intial data* $S_{1,0}, U_{1,0}, S_{2,0}, U_{2,0}$ *the solution of the system verifies*

$$\lim_{t \to +\infty} \| U_i(t,.) \|_\infty = 0, \tag{1.36}$$

$$\lim_{t \to +\infty} \| S_i(t,.) - \frac{\Lambda_i}{\mu} \|_\infty = 0. \tag{1.37}$$

Theorem 1.3.2. *If* $R > 1$ *then the system* (3.7)–(3.9) *has a unique endemic equilibrium point* $(S_1^*, U_1^*, S_2^*, U_2^*)$ *and for each continious positive initial data* $S_{1,0}, U_{1,0}, S_{2,0}, U_{2,0}$ *the solution of the system verifies*

$$\lim_{t \to +\infty} \| U_i(t,.) - U_i^* \|_\infty = \lim_{t \to +\infty} \| S_i(t,.) - S_i^* \|_\infty = 0, \tag{1.38}$$

Thus, the endemic equilibrium point is gobally asumptoticaly stable.

1.4 Local Stability of equilibrium points of an ODE system

In [35], Piqueira, Castano and Monteiro studied the local stability of equilibrium points for the system below

$$\begin{cases} \frac{\partial S_1}{\partial t} = \Lambda_1 - \beta_1 \frac{S_1 U_1}{T} - \beta_2 \frac{S_1 U_2}{T} - \mu S_1, & \text{in } \mathbb{R}^+ \\ \frac{\partial U_1}{\partial t} = \beta_1 \frac{S_1 U_1}{T} + \beta_2 \frac{S_1 U_2}{T} - \sigma_1 U_1, & \text{in } \mathbb{R}^+ \\ \frac{\partial S_2}{\partial t} = \Lambda_2 - \beta_3 \frac{S_2 U_1}{T} - \beta_4 \frac{S_2 U_2}{T} - \mu S_2, & \text{in } \mathbb{R}^+ \\ \frac{\partial U_2}{\partial t} = \beta_3 \frac{S_2 U_1}{T} + \beta_4 \frac{S_2 U_2}{T} - \sigma_2 U_2, & \text{in } \mathbb{R}^+. \end{cases} \tag{1.39}$$

They gave some results on the local stability analysis according to the value of the basic reproduction rate R.

They showed that if $R < 1$, the free equilibrium points are locally asymptotically stable, and if $R > 1$, the free equilibrium points become instable, and the endemic equilibrium points are asymptotically stable.

Here in our work, we generalize the above results of Piqueira, Castano and Monteiro [35] to a class of strong nonlinearities

$$\begin{cases} \frac{\partial S_1}{\partial t} = \Lambda_1 - \beta_1 \dfrac{S_1\psi(U_1)}{T} - \beta_2 \dfrac{S_1\psi(U_2)}{T} - \mu S_1 \\ \frac{\partial U_1}{\partial t} = \beta_1 \dfrac{S_1\psi(U_1)}{T} + \beta_2 \dfrac{S_1\psi(U_2)}{T} - \sigma_1 U_1, \\ \frac{\partial S_2}{\partial t} = \Lambda_2 - \beta_3 \dfrac{S_2\psi(U_1)}{T} - \beta_4 \dfrac{S_2\psi(U_2)}{T} - \mu S_2, \\ \frac{\partial U_2}{\partial t} = \beta_3 \dfrac{S_2\psi(U_1)}{T} + \beta_4 \dfrac{S_2\psi(U_2)}{T} - \sigma_2 U_2, \end{cases} \tag{1.40}$$

with homogeneous Newmann boundary conditions

$$\frac{\partial S_i}{\partial \nu} = \frac{\partial U_i}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \text{ for all } i = 1, 2. \tag{1.41}$$

and positive initial data

$$S_i(0,x) = S_{i,0}(x) > 0, U_i(0,x) = U_{i,0}(x) > 0 \qquad \text{in } \Omega, \text{ for all } i = 1, 2. \tag{1.42}$$

We assume here that ψ is a nonnegative and continuously differentiable function on $[0,+\infty)$ satisfies

$$\psi(0) = 0.$$

Here, we consider that the influx rates Λ_1, Λ_2 are constants or they change their values exponentially with respect to the susceptible individuals values

$$\Lambda_1 = \kappa_1 S_1 e^{-\kappa_2 S_1}, \Lambda_2 = \kappa_3 S_2 e^{-\kappa_4 S_2}$$

The result is given as follow

Theorem 1.4.1.

1. *Assume that* Λ_1, Λ_2 *are constants, then the basic reproduction rate* R_0 *is given by*

$$R_0 = \frac{(\beta_1 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi'(0) + \beta_4 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi'(0))(\beta_2\beta_3 \frac{\Lambda_1\Lambda_2}{(\Lambda_1+\Lambda_2)^2}(\psi'(0))^2)}{(\sigma_1+\sigma_2)(\beta_1 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi'(0) - \sigma_1)(\beta_4 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi'(0) - \sigma_2)}. \tag{1.43}$$

   If $R_0 < 1$, *then the disease free equilibrium point* $(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\frac{\Lambda_1}{\mu}, 0, \frac{\Lambda_2}{\mu}, 0)$ *is locally asymptotically stable and if* $R_0 > 1$, *then the disease free equilibrium point is unstable and the disease endemic equilibrium point* $(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\overline{S}_1, \frac{\Lambda_1 - \mu\overline{S}_1}{\sigma_1}, \overline{S}_2, \frac{\Lambda_2 - \mu\overline{S}_2}{\sigma_2})$ *is locally asymptotically stable.*

2. *Assume that the influx rates are given by* $\Lambda_1 = \kappa_1 S_1 e^{-\kappa_2 S_1}$ *and* $\Lambda_2 = \kappa_3 S_2 e^{-\kappa_4 S_2}$.

   Then R_0 *is given by*

$$R_0 = \frac{(\beta_1 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi'(0) + \beta_4 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi'(0))(\beta_2\beta_3 \frac{\underline{S}_1\underline{S}_2}{(\underline{S}_1+\underline{S}_2)^2}(\psi'(0))^2)}{(\sigma_1+\sigma_2)(\beta_1 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi'(0) - \sigma_1)(\beta_4 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi'(0) - \sigma_2)}. \tag{1.44}$$

   If $R_0 < 1$, *then the disease free equilibrium point* $(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\frac{\ln\frac{\kappa_1}{\mu}}{\kappa_2}, 0, \frac{\ln\frac{\kappa_3}{\mu}}{\kappa_4}, 0)$ *is locally asymptotically stable and if* $R_0 > 1$, *the disease free equilibrium point is unstable and the disease endemic equilibrium point*

   $(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\overline{S}_1, \frac{(\kappa_1 e^{-\kappa_2\overline{S}_1} - \mu)\overline{S}_1}{\sigma_1}, \overline{S}_2, \frac{(\kappa_3 e^{-\kappa_4\overline{S}_2} - \mu)\overline{S}_2}{\sigma_2})$ *is locally asymptotically stable.*

CHAPTER 2

Boundedness and asymptotic behavior of solutions for a diffusive epidemic model

2.1 Introduction

Epidemiology is the science that study the distribution and determinants of diseases health conditions, or events among populations and the application of that study to control health problems. In mathematics, this study conducted on the models that arise from epidemiological phenomena to know some data on the behavior of the disease.

The authors Piqueira, Castano and Monteiro studied the following ODE system given in [35] where they gave some results on the analysis stability

$$\begin{cases} \dfrac{\partial S_1}{\partial t} = \Lambda_1 - \beta_1 \dfrac{S_1 U_1}{T} - \beta_2 \dfrac{S_1 U_2}{T} - \mu S_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U_1}{\partial t} = \beta_1 \dfrac{S_1 U_1}{T} + \beta_2 \dfrac{S_1 U_2}{T} - \sigma_1 U_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial S_2}{\partial t} = \Lambda_2 - \beta_3 \dfrac{S_2 U_1}{T} - \beta_4 \dfrac{S_2 U_2}{T} - \mu S_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U_2}{\partial t} = \beta_3 \dfrac{S_2 U_1}{T} + \beta_4 \dfrac{S_2 U_2}{T} - \sigma_2 U_2 & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{2.1}$$

with homogeneous Newmann boundary conditions

$$\frac{\partial S_i}{\partial \nu} = \frac{\partial U_i}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega,\ i = 1, 2, \tag{2.2}$$

and the initial data $S_{1,0}$, $U_{1,0}$, $S_{2,0}$, $U_{2,0}$ verify

$$S_i(0,x) = S_{i,0}(x) \geq 0, U_i(0,x) = U_{i,0}(x) > 0 \qquad \text{in } \Omega,\ i = 1,2. \tag{2.3}$$

This system arises as a model of a diffusive epidemic in some populations. The population concidered contains susceptible individuals S_1, S_2 and infected ones U_1, U_2 divided into two groups, where the indices 1 and 2 indicate which group they belong to. The constants Λ_1 and Λ_2 represent the influx rates of new susceptible individuals in each group. μ is the mortality rate, the ratio of the number of deaths from the disease to the total number of cases of that disease. The parameters β_1, β_2, β_3, β_4 describe the rate at which the disease is spread among the individuals per unit of time, the parameters σ_1, σ_2 are given by $\sigma_i = \mu + \eta_i$, $i = 1,2$, where $1/\eta_i$ is the average activity period of the infected individuals and $T = S_1 + U_1 + S_2 + U_2$ is the total population (See [9], [12], [35] and [39]).

In our work, we deal with the same model as in [35], but in reaction-diffusion systems and more general nonlinearities. We consider the following reaction-diffusion system of the form

$$\begin{cases} \dfrac{\partial S_1}{\partial t} - d_1 \Delta S_1 = \Lambda_1 - \beta_1 \dfrac{S_1 \varphi(U_1)}{T} - \beta_2 \dfrac{S_1 \varphi(U_2)}{T} - \mu S_1 & \text{in } \mathbb{R}^+ \times \Omega \\ \dfrac{\partial U_1}{\partial t} - d_2 \Delta U_1 = \beta_1 \dfrac{S_1 \varphi(U_1)}{T} + \beta_2 \dfrac{S_1 \varphi(U_2)}{T} - \sigma_1 U_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial S_2}{\partial t} - d_3 \Delta S_2 = \Lambda_2 - \beta_3 \dfrac{S_2 \varphi(U_1)}{T} - \beta_4 \dfrac{S_2 \varphi(U_2)}{T} - \mu S_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U_2}{\partial t} - d_4 \Delta U_2 = \beta_3 \dfrac{S_2 \varphi(U_1)}{T} + \beta_4 \dfrac{S_2 \varphi(U_2)}{T} - \sigma_2 U_2 & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{2.4}$$

with homogeneous Newmann boundary conditions

$$\frac{\partial S_i}{\partial \nu} = \frac{\partial U_i}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega,\ i = 1,2, \tag{2.5}$$

and the continuous initial data $S_{1,0}$, $U_{1,0}$, $S_{2,0}$, $U_{2,0}$ are such that

$$S_i(0,x) = S_{i,0}(x) \geq 0, U_i(0,x) = U_{i,0}(x) > 0 \qquad \text{in } \Omega,\ i = 1,2. \tag{2.6}$$

The constants d_1, d_2, d_3, d_4, Λ_1, Λ_2, μ, σ_1, σ_2 are such that

$$d_1 > 0,\ d_2 > 0,\ d_3 > 0,\ d_4 > 0,\ \mu > 0,\ \sigma_1 > 0,\ \sigma_2 > 0,\ \Lambda_1 \geq 0,\ \Lambda_2 \geq 0. \tag{2.7}$$

The nonlinearity φ is assumed to be a nonnegative increasing and continuously differentiable function on $[0,+\infty)$ such that

$$\varphi(0) = 0,\ \varphi(\eta) \leq Ce^{\eta^\gamma},\ 0 < \gamma < 1. \tag{2.8}$$

Here Ω is a bounded domain of class C^1 in $\mathbb{R}^n$, with boundary $\partial\Omega$, $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative to $\partial\Omega$.

In the case where $d_1 = d_2 = d_3 = d_4$, using maximum principle, one can see that

$$T \leq \max(\frac{\Lambda_1 + \Lambda_2}{\sigma}, \|S_{1,0} + U_{1,0} + S_{2,0} + U_{2,0}\|_\infty), \tag{2.9}$$

where $\sigma = \max(\sigma_1, \sigma_2)$. It is clear that in this case, the global existence and the boundedness of the solutions are verified.

In the other case where the coefficients of the diffusion are different and with nonlinearities of polynomial growth Hollis, Martin and Pierre [22], Alikakos [2], Masuda [29] gave the global existence of the solutions. But with nonlinearities more than polynomial growth the problem becomes more difficult and these methods will be ineffective.

We will show that the solutions of the system (2.4)–(2.6) are global and uniformly bounded on $(0,+\infty) \times \Omega$ and under suitable assumptions on the parameters of this system, we have an extinction of the epidemic. This study generalizes the results in [35].

2.2 Notations and preliminary results

2.2.1 Local existence of solutions

Throughout this study, we denote by

$$\begin{aligned} \|u\|_p^p &= \int_\Omega |u(x)|^p dx, \ 1 \leq p < +\infty \\ \|u\|_\infty &= \operatorname{ess\,sup}_{x\in\Omega} |u(x)|, \\ \|u\|_{C(\overline{\Omega})} &= \max_{x\in\overline{\Omega}} |u(x)|, \end{aligned} \tag{2.10}$$

the usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ respectively.

It is clear that from the general theory of semigroup and the abstract semilinear differential equations, we can deduce the local existence of solutions to the system (2.4)–(2.6) in some interval $]0, T^*[$,where T^* is the eventual blowing-up time in $L^\infty(\Omega)$ (see Henry [20] or Pazy [34]).

2.2.2 Positivity of the solutions

Since the initial data $S_{0,1}, U_{0,1}, S_{0,2}, U_{0,2}$ are nonnegative on Ω, it follows from the maximum principle that the solution (S_1, U_1, S_2, U_2) remains nonnegative on $(0, T^*) \times \Omega$ and for strictement nonnegative initial data $S_{0,1}, U_{0,1}, S_{0,2}, U_{0,2}$, we have for all $i = 1, 2$

$$U_i(t,x) \geq \min_{x\in\Omega} U_{i,o}(x)\, e^{-\sigma_i t} > 0. \qquad \forall (t,x) \in (0,T) \times \Omega \tag{2.11}$$

Again, maximum principle applied to the first and the third equation of (2.4) permits to deduce that the components S_1, S_2 are bounded on $(0, T^*) \times \Omega$, so that for all $(t,x) \in (0,T^*) \times \Omega$

$$0 \leq S_i(t,x) \leq K_i = max(\|S_{0,i}\|_\infty, \frac{\Lambda_i}{\mu}), \qquad i = 1, 2. \tag{2.12}$$

2.3 Boundedness of the solutions

It is well known that to prove global existence of the solution to the system (2.4)–(2.6), it suffices to derive a uniform estimate of $\|\beta_1 \frac{S_1\varphi(U_1)}{T}+\beta_2\frac{S_1\varphi(U_2)}{T}-\sigma_1 U_1\|_p$ and $\|\beta_3\frac{S_2\varphi(U_1)}{T}+\beta_4\frac{S_2\varphi(U_2)}{T}-\sigma_2 U_2\|_p$ on $]0,T^*[$, for some $p>\frac{n}{2}$.

Before stating the results, let us define for any $t\in(0,T^*)$ the functional

$$J(t)=\sum_{i=1}^{2}\int_\Omega (1+\omega_i(S_i+S_i^2))e^{\epsilon_i U_i}dx,\ i=1,2 \tag{2.13}$$

where ω_1, ω_2, ϵ_1, and ϵ_2 are positive constants such that

$$o<\omega_1\leq min(\frac{\sigma_1}{2\Lambda_1(1+2K_1)},\frac{8d_1d_2}{(1+2K_1)^2(d_1+d_2)^2}), \tag{2.14}$$

$$o<\omega_2\leq min(\frac{\sigma_2}{2\Lambda_2(1+2K_2)},\frac{8d_3d_4}{(1+2K_2)^2(d_3+d_4)^2}), \tag{2.15}$$

$$0<\epsilon_i\leq\frac{\omega_i}{(1+\omega_i(K_i+K_i^2))}, \tag{2.16}$$

for all $i=1,2$.

Our main results can be stated as follow

Theorem 2.3.1. *Let* (S_1,U_1,S_2,U_2) *be a solution of* (2.4)–(2.6) *on* $(0,T^*)$, *then there exist two positive constants* a_1, a_2 *such that*

$$\frac{d}{dt}J(t)\leq -a_1J(t)+a_2,\qquad \textit{for all } t\ in(0,T^*). \tag{2.17}$$

Corollary 2.3.2. *Let* φ *satisfies the condition* (2.8), *then the solution* (S_1,U_1,S_2,U_2) *of the system* (2.4)–(2.6) *with nonnegative and bounded initial data* $S_{0,1},U_{0,1},S_{0,2},U_{0,2}$ *is global and uniformly bounded on* $(0,+\infty)\times\Omega$.

Proof. (Proof of theorem 2.3.1)

Let (S_1,U_1,S_2,U_2) be a solution of (2.4)–(2.6) on $(0,T^*)$. Differentiating J with respect to t

and a simple use of Green's formula yields

$$\frac{d}{dt}J(t) = G + H,$$

such that

$$\begin{aligned} G &= G_1 + G_2, && (2.18)\\ H &= H_1 + H_2, && (2.19)\end{aligned}$$

with

$$\begin{aligned} G_1 &= -2\omega_1 d_1 \int_\Omega |\nabla S_1|^2 \, e^{\epsilon_1 U_1} \, dx - \epsilon_1 \, (d_1 + d_2) \, \omega_1 \int_\Omega (1 + 2S_1) \, \nabla U_1 \, \nabla S_1 \, e^{\epsilon_1 U_1} \, dx \\ &- d_2 \, \epsilon_1^2 \int_\Omega (1 + \omega_1 (S_1 + S_1^2)) \, |\nabla U_1|^2 \, e^{\epsilon_1 U_1} \, dx, \\ G_2 &= -2\omega_2 d_3 \int_\Omega |\nabla S_2|^2 \, e^{\epsilon_2 U_2} \, dx - \epsilon_2 \, (d_3 + d_4) \, \omega_2 \int_\Omega (1 + 2S_2) \, \nabla U_2 \, \nabla S_2 \, e^{\epsilon_2 U_2} \, dx \\ &- d_4 \, \epsilon_2^2 \int_\Omega (1 + \omega_2 (S_2 + S_2^2)) \, |\nabla U_2|^2 \, e^{\epsilon_2 U_2} \, dx, \end{aligned}$$

and

$$H_i = \int_\Omega \rho_i \, (1 + \omega_i (S_i + S_i^2)) e^{\epsilon_i U_i} \, dx, \qquad i = 1, 2 \tag{2.20}$$

where

$$\begin{aligned} \rho_i &= (\epsilon_i - \frac{\omega_i (1 + 2S_i)}{(1 + \omega_i (S_i + S_i^2))}) \, l_i + (\Lambda_i \, \frac{\omega_i (1 + 2S_i)}{(1 + \omega_i (S_i + S_i^2))} - \mu S_i \frac{\omega_i (1 + 2S_i)}{(1 + \omega_i (S_i + S_i^i))} - \sigma_i) \\ &+ \sigma_i (1 - \epsilon_i \, U_i) e^{\epsilon_i \, U_i} \, e^{-\epsilon_i \, U_i}, && (2.21) \end{aligned}$$

with

$$l_i = \begin{cases} (\beta_1 \, \dfrac{S_i \varphi(U_1)}{T} + \beta_2 \dfrac{S_i \varphi(U_2)}{T}) \geq 0 & \text{if} \quad i = 1, \\ (\beta_3 \, \dfrac{S_i \varphi(U_1)}{T} + \beta_4 \dfrac{S_i \varphi(U_2)}{T}) \geq 0 & \text{if} \quad i = 2. \end{cases} \tag{2.22}$$

We start with the term G_1,

$$\begin{aligned} G_1 &\leq -2\omega_1 d_1 \int_\Omega |\nabla S_1|^2 \, e^{\epsilon_1 U_1} \, dx + |\epsilon_1 \, (d_1 + d_2) \, \omega_1 \int_\Omega (1 + 2S_1) \, \nabla U_1 \, \nabla S_1 \, e^{\epsilon_1 U_1} \, dx| \\ &- d_2 \, \epsilon_1^2 \int_\Omega (1 + \omega_1 (S_1 + S_1^2)) \, |\nabla U_1|^2 \, e^{\epsilon_1 U_1} \, dx, \end{aligned}$$

Applying Young inequality, we get

$$\begin{aligned}|\epsilon_1\,(d_1+d_2)\,\omega_1\int_\Omega(1+2S_1)\,\nabla U_1\,\nabla S_1\,e^{\epsilon_1 U_1}\,dx| &\leq d_2\varepsilon_1^2\int_\Omega(1+\omega_1(S_1+S_1^2))|\nabla U_1|^2\,e^{\epsilon_1 U_1}\,dx\\ &+ \frac{1}{4}\frac{(d_1+d_2)^2}{d_2}\omega_1^2\int_\Omega\frac{(1+2S_1)^2}{((1+\omega_1(S_1+S_1^2))}|\nabla S_1|^2\,e^{\epsilon_1 U_1}\,dx.\end{aligned}$$

Thus

$$\begin{aligned}G_1 &\leq -2\omega_1 d_1\int_\Omega|\nabla S_1|^2\,e^{\epsilon_1 U_1}\,dx+\frac{1}{4}\frac{(d_1+d_2)^2}{d_2}\omega_1^2\int_\Omega\frac{(1+2S_1)^2}{((1+\omega_1(S_1+S_1^2))}|\nabla S_1|^2\,e^{\epsilon_1 U_1}\,dx,\\ &\leq -2\omega_1 d_1\int_\Omega|\nabla S_1|^2\,e^{\epsilon_1 U_1}\,dx+\frac{1}{4}\frac{(d_1+d_2)^2}{d_2}\omega_1^2(1+2K_1)^2\int_\Omega|\nabla S_1|^2\,e^{\epsilon_1 U_1}\,dx\\ &= (-2\omega_1 d_1+\frac{1}{4}\frac{(d_1+d_2)^2}{d_2}\omega_1^2(1+2K_1)^2)\int_\Omega|\nabla S_1|^2\,e^{\epsilon_1 U_1}\,dx,\end{aligned}$$

where K_1 is given by (2.12).

So that, by the hypothesis (3.7), we can conclude that

$$G_1\leq 0\qquad\text{for all } t\in(0,T^*). \tag{2.23}$$

We use the same raisonning for G_2.

Consequently, for all $i=1,2$

$$G_i\leq 0\qquad\text{for all } t\in(0,T^*).$$

On the other hand, concerning the terms $H_i, i=1,2$, we can deduce from (2.12), (3.7), (3.8), and (3.9) that for all $i=1,2$

$$\epsilon_i-\frac{\omega_i(1+2S_i)}{(1+\omega_i(S_i+S_i^2))}\leq 0, \tag{2.24}$$

$$\Lambda_i\,\frac{\omega_i(1+2S_i)}{(1+\omega_i(S_i+S_i^2))}-\mu S_i\frac{\omega_i(1+2S_i)}{(1+\omega_i(S_i+S_i^2))}-\sigma_i\leq-\frac{\sigma_i}{2}. \tag{2.25}$$

Application of the mean value theorem for all $x\in]0,1[$ gives

$$e^x\leq\frac{1}{1-x}. \tag{2.26}$$

In the other cases where $x = 0$ or $x \geq 1$, the inequaity (2.26) is verified.

Then from (2.26), for all $\xi \geq 0$

$$(1 - \epsilon_i\, \xi)e^{\epsilon_i\, \xi} \leq 1, \qquad i = 1,2. \tag{2.27}$$

Hence from (2.24), (2.25) and (2.27) there exist two positive constants $b_i,\ i = 1,2$

$$b_i = \sigma_i(1 + \omega_i(K_i + K_i^2))|\Omega|,$$

such that

$$H_i \leq -\frac{\sigma_i}{2}\int_\Omega (1 + \omega_i(S_i + S_i^2))e^{\epsilon_i U_i}\,dx + b_i, \qquad i = 1,2 \tag{2.28}$$

from which we infer that there are two constants $a_1,\ a_2$ such that

$$\frac{d}{dt}J(t) = G + H_1 + H_2 \leq -a_1 J(t) + a_2 \qquad t \in (0, T^*), \tag{2.29}$$

with

$$a_1 = \max(\frac{\sigma_1}{2}, \frac{\sigma_2}{2}),\ a_2 = \sum_{i=1}^{2} b_i. \tag{2.30}$$

Whence, (2.17) is satisfied.

Thus, the proof of the theorem is completed. □

Proof. (Proof of corollary 2.3.2) Let φ satisfies the condition (2.8) and let $(S_1,\ U_1,\ S_2,\ U_2)$ be the solution of the system (2.4)–(2.6) in $(0, T^*)$.

From (2.8), it is easy to prove that there exists a constant $C_i(n),\ i = 1,2$ such that

$$\|\varphi(U_i)\|_n^n \leq C_i(n) + J(t) \quad for\ all\ t\ in\ (0, T^*). \tag{2.31}$$

Hence, thanks to the theorem 2.3.1 and as it has been mentioned above, the result established in [18] or [20] permits to deduce that there exist two postive constants $N_i,\ i = 1,2$ such that for all $i = 1,2$

$$\|U_i(t,x)\|_\infty \leq N_i. \tag{2.32}$$

Consequently, the solution of the system (2.4)–(2.6) is global and uniformly bounded on $(0, +\infty) \times \Omega$, and the proof of the corollary is completed. □

2.4 Asymptotic Behavior of solutions

In this section, we discuss the asymptotic behavior of the solutions for the system (2.4)–(2.6).

Before we state our main result of this section, let us define for any $t \in (0,+\infty)$

$$\begin{aligned} F(t) &= \int_\Omega U_1 S_1 dx + \int_\Omega U_2 S_2 dx + \int_\Omega S_1 U_2 dx + \int_\Omega S_2 U_1 dx \\ &+ \frac{\theta_1}{2}\int_\Omega (S_1 - \frac{\Lambda_1}{\mu})^2 dx + \frac{\theta_2}{2}\int_\Omega (S_2 - \frac{\Lambda_2}{\mu})^2 dx + \frac{1}{2}\int_\Omega U_1^2 dx + \frac{1}{2}\int_\Omega U_2^2 dx \\ &+ 2\frac{\Lambda_1}{\sigma_1}\int_\Omega U_1 dx + 2\frac{\Lambda_2}{\sigma_2}\int_\Omega U_2 dx + \frac{\Lambda_1}{\sigma_2}\int_\Omega U_2 dx + \frac{\Lambda_2}{\sigma_1}\int_\Omega U_1 dx, \end{aligned} \tag{2.33}$$

where θ_1 and θ_2 verify

$$\theta_1 \geq \max(\frac{9}{4}\frac{(d_1+d_2)^2}{d_1 d_2}, \frac{9}{4}\frac{(d_1+d_4)^2}{d_1 d_4}, 1), \tag{2.34}$$

$$\theta_2 \geq \max(\frac{9}{4}\frac{(d_3+d_4)^2}{d_3 d_4}, \frac{9}{4}\frac{(d_2+d_3)^2}{d_2 d_3}, 1). \tag{2.35}$$

Theorem 2.4.1. *Let* (S_1, U_1, S_2, U_2) *be the solution of* (2.4)–(2.6) *in* $(0,+\infty)\times\Omega$. *Assume that the nonlinearity* φ *satisfies*

$$\chi = \sup_{0\leq\zeta\leq\min(N_1,N_2)} \varphi'(\zeta) \leq \min\{\frac{\Lambda_1}{\eta}, \frac{\Lambda_2}{\xi}, \frac{\mu+\sigma_1}{\max(\beta_1,\beta_3)}, \frac{\mu+\sigma_2}{\max(\beta_2,\beta_4)}\}, \tag{2.36}$$

where

$$\begin{aligned} \eta &= (2\beta_1\frac{\Lambda_1}{\sigma_1} + 2\beta_3\frac{\Lambda_2}{\sigma_2} + \beta_1\frac{\Lambda_2}{\sigma_1} + \beta_3\frac{\Lambda_1}{\sigma_2} + \beta_1\theta_1\frac{\Lambda_1}{\mu} + \beta_3\theta_2\frac{\Lambda_2}{\mu}), & (2.37) \\ \xi &= (2\beta_2\frac{\Lambda_1}{\sigma_1} + 2\beta_4\frac{\Lambda_2}{\sigma_2} + \beta_2\frac{\Lambda_2}{\sigma_1} + \beta_4\frac{\Lambda_1}{\sigma_2} + \beta_2\theta_1\frac{\Lambda_1}{\mu} + \beta_4\theta_2\frac{\Lambda_2}{\mu}), & (2.38) \end{aligned}$$

then

$$\lim_{t\to+\infty} \|S_i(t,.) - \frac{\Lambda_1}{\mu}\|_\infty = \lim_{t\to+\infty} \|U_i(t,.)\|_\infty = 0. \qquad i = 1,2 \tag{2.39}$$

Proof. Differentiating F with respect to t and using Green's formula, we can show that

$$F^{'}(t) = I_1 + I_2, \tag{2.40}$$

where

$$\begin{aligned}
I_1 &= -d_2\int_\Omega \nabla U_1 \nabla S_1 dx - d_1\int_\Omega \nabla S_1 \nabla U_1 dx - d_4\int_\Omega \nabla U_2 \nabla S_2 dx - d_3\int_\Omega \nabla S_2 \nabla U_2 dx \\
&- d_1\int_\Omega \nabla S_1 \nabla U_2 dx - d_4\int_\Omega \nabla U_2 \nabla S_1 dx - d_3\int_\Omega \nabla S_2 \nabla U_1 dx - d_2\int_\Omega \nabla U_1 \nabla S_2 dx \\
&- d_2\int_\Omega |\nabla U_1|^2 dx - d_4\int_\Omega |\nabla U_2|^2 dx - d_1\,\theta_1\int_\Omega |\nabla S_1|^2 dx - d_3\,\theta_2\int_\Omega |\nabla S_2|^2 dx,
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_\Omega (\beta_1 \frac{S_1}{T}\varphi(U_1) + \beta_2 \frac{S_1}{T}\varphi(U_2)) S_1 dx - \theta_1\int_\Omega (\beta_1 \frac{S_1^2}{T}\varphi(U_1) + \frac{S_1^2}{T}\varphi(U_2)) dx \\
&+ \int_\Omega (\beta_3 \frac{S_2}{T}\varphi(U_1) + \beta_4 \frac{S_2}{T}\varphi(U_2)) S_2 dx - \theta_2\int_\Omega (\beta_3 \frac{S_2^2}{T}\varphi(U_1) + \beta_4\frac{S_2^2}{T}\varphi(U_2)) dx \\
&- \int_\Omega (\beta_1 \frac{S_1}{T}\varphi(U_1) + \beta_2 \frac{S_1}{T}\varphi(U_2)) U_1 dx - \int_\Omega (\beta_3 \frac{S_2}{T}\varphi(U_1) + \beta_4 \frac{S_2}{T}\varphi(U_2)) U_2 dx \\
&+ \Lambda_1\int_\Omega U_1 dx + \Lambda_1\int_\Omega U_2 dx + \Lambda_2\int_\Omega U_2 dx + \Lambda_2\int_\Omega U_1 dx \\
&- 2\Lambda_1\int_\Omega U_1 dx - 2\Lambda_2\int_\Omega U_2 dx - \Lambda_2\int_\Omega U_1 dx - \Lambda_1\int_\Omega U_2 dx \\
&+ \int_\Omega (\beta_3 S_2 \frac{S_1}{T}\varphi(U_1) + \beta_4 S_2 \frac{S_1}{T}\varphi(U_2)) dx + \int_\Omega (\beta_1 S_1 \frac{S_2}{T}\varphi(U_1) + \beta_2 S_1 \frac{S_2}{T}\varphi(U_2)) dx \\
&- \mu\int_\Omega S_1 U_1 dx - \mu\int_\Omega S_2 U_2 dx - \mu\int_\Omega S_1 U_2 dx - \mu\int_\Omega S_2 U_1 dx \\
&- \sigma_1\int_\Omega U_1 S_2 dx - \sigma_1\int_\Omega U_2 S_1 dx - \sigma_2\int_\Omega U_1 S_2 dx - \sigma_2\int_\Omega U_2 S_1 dx \\
&- \int_\Omega (\beta_1 \frac{S_1}{T}\varphi(U_1) + \beta_2 \frac{S_1}{T}\varphi(U_2)) U_1 dx - \int_\Omega (\beta_3 \frac{S_2}{T}\varphi(U_1) + \beta_4 \frac{S_2}{T}\varphi(U_2)) U_2 dx \\
&+ \int_\Omega (\beta_1 \frac{S_1}{T}\varphi(U_1) + \beta_2 \frac{S_1}{T}\varphi(U_2)) U_1 dx + \int_\Omega (\beta_3 \frac{S_2}{T}\varphi(U_1) + \beta_4 \frac{S_2}{T}\varphi(U_2)) U_2 dx \\
&+ \theta_1 \frac{\Lambda_1}{\mu}\int_\Omega (\beta_1 \frac{S_1}{T}\varphi(U_1) + \beta_2 \frac{S_1}{T}\varphi(U_2)) dx + \theta_2 \frac{\Lambda_2}{\mu}\int_\Omega (\beta_3 \frac{S_2}{T}\varphi(U_1) + \beta_4 \frac{S_2}{T}\varphi(U_2)) dx \\
&+ 2\frac{\Lambda_1}{\sigma_1}\int_\Omega (\beta_1 \frac{S_1}{T}\varphi(U_1) + \beta_2 \frac{S_1}{T}\varphi(U_2)) dx + 2\frac{\Lambda_2}{\sigma_2}\int_\Omega (\beta_3 \frac{S_2}{T}\varphi(U_1) + \beta_4 \frac{S_2}{T}\varphi(U_2)) dx \\
&+ \frac{\Lambda_2}{\sigma_1}\int_\Omega (\beta_1 \frac{S_1}{T}\varphi(U_1) + \beta_2 \frac{S_1}{T}\varphi(U_2)) dx + \frac{\Lambda_1}{\sigma_2}\int_\Omega (\beta_3 \frac{S_2}{T}\varphi(U_1) + \beta_4 \frac{S_2}{T}\varphi(U_2)) dx \\
&- \mu\theta_1\int_\Omega (S_1 - \frac{\Lambda_1}{\mu})^2 dx - \mu\theta_2\int_\Omega (S_2 - \frac{\Lambda_2}{\mu})^2 dx - \sigma_1\int_\Omega U_1^2 dx - \sigma_2\int_\Omega U_2^2 dx.
\end{aligned}$$

We start with the term I_1, we can rewrite it as follow

$$\begin{aligned} I_1 &= -\frac{1}{3}d_2\int_\Omega |\nabla U_1|^2 dx-(d_1+d_2)\int_\Omega \nabla S_1\nabla U_1 dx-\frac{1}{3}d_1\theta_1\int_\Omega |\nabla S_1|^2 dx \\ &- \frac{1}{3}d_4\int_\Omega |\nabla U_2|^2 dx-(d_3+d_4)\int_\Omega \nabla S_2\nabla U_2 dx-\frac{1}{3}d_3\theta_2\int_\Omega |\nabla S_2|^2 dx \\ &- \frac{1}{3}d_2\int_\Omega |\nabla U_1|^2 dx-(d_2+d_3)\int_\Omega \nabla U_1\nabla S_2 dx-\frac{1}{3}d_3\theta_2\int_\Omega |\nabla S_2|^2 dx \\ &- \frac{1}{3}d_4\int_\Omega |\nabla U_2|^2 dx-(d_1+d_4)\int_\Omega \nabla U_2\nabla S_1 dx-\frac{1}{3}d_1\theta_1\int_\Omega |\nabla S_1|^2 dx \\ &- \frac{1}{3}d_2\int_\Omega |\nabla U_1|^2 dx-\frac{1}{3}d_1\theta_1\int_\Omega |\nabla S_1|^2 dx-\frac{1}{3}d_4\int_\Omega |\nabla U_2|^2 dx-\frac{1}{3}d_3\theta_2\int_\Omega |\nabla S_2|^2 dx. \end{aligned}$$

We can observe that I_1 involves four quadratic forms $Q_1,\ Q_2,\ Q_3,\ Q_4$

$$\begin{aligned} Q_1 &= \frac{1}{3}d_2|\nabla U_1|^2+(d_1+d_2)\nabla S_1\nabla U_1+\frac{1}{3}d_1\theta_1|\nabla S_1|^2, \\ Q_2 &= \frac{1}{3}d_4|\nabla U_2|^2+(d_3+d_4)\nabla S_2\nabla U_2+\frac{1}{3}d_3\theta_2|\nabla S_2|^2, \\ Q_3 &= \frac{1}{3}d_2|\nabla U_1|^2+(d_2+d_3)\nabla U_1\nabla S_2+\frac{1}{3}d_3\theta_2|\nabla S_2|^2, \\ Q_4 &= \frac{1}{3}d_4|\nabla U_2|^2+(d_1+d_4)\nabla U_2\nabla S_1+\frac{1}{3}d_1\theta_1|\nabla S_1|^2, \end{aligned}$$

where their discriminants are

$$\begin{aligned} \Delta_1 &= (d_1+d_2)^2-\frac{4}{9}d_1d_2\theta_1,\ \Delta_2=(d_3+d_4)^2-\frac{4}{9}d_3d_4\theta_2, \\ \Delta_3 &= (d_2+d_3)^2-\frac{4}{9}d_2d_3\theta_2,\ \Delta_4=(d_1+d_4)^2-\frac{4}{9}d_1d_4\theta_1. \end{aligned}$$

So that, by the hypothesis (2.34) and (2.35) the discriminants are negative and by that the quadratic forms are positive.

Consequently

$$I_1 \le -\frac{1}{3}d_2\int_\Omega |\nabla U_1|^2 dx-\frac{1}{3}d_1\theta_1\int_\Omega |\nabla S_1|^2 dx-\frac{1}{3}d_4\int_\Omega |\nabla U_2|^2 dx-\frac{1}{3}d_3\theta_2\int_\Omega |\nabla S_2|^2 dx. \quad (2.41)$$

Now, we will treat the term I_2, we have for all $i=1,2$

$$\frac{S_i}{T} \le 1. \quad (2.42)$$

As $\varphi(0)=0$, the mean value theorem and (2.42) imply

$$\begin{aligned}
I_2 &\leq (1-\theta_1)\int_\Omega (\beta_1 \frac{S_1\varphi(U_1)}{T}+\beta_2\frac{S_1\varphi(U_2)}{T})S_1dx+(1-\theta_2)\int_\Omega(\beta_3\frac{S_2\varphi(U_1)}{T}+\beta_4\frac{S_2\varphi(U_2)}{T})S_2dx\\
&+ (\beta_1\chi-(\mu+\sigma_1))\int_\Omega S_1U_1dx+(\beta_2\chi-(\mu+\sigma_2))\int_\Omega S_1U_2dx\\
&+ (\beta_3\chi-(\mu+\sigma_1))\int_\Omega S_2U_1dx+(\beta_4\chi-(\mu+\sigma_2))\int_\Omega S_2U_2dx\\
&+ [(\beta_1\frac{\Lambda_1}{\sigma_1}a+\beta_3\frac{\Lambda_2}{\sigma_2}b+\beta_1\frac{\Lambda_2}{\sigma_1}+\beta_3\frac{\Lambda_1}{\sigma_2}+\beta_1\theta_1\frac{\Lambda_1}{\mu}+\beta_3\theta_2\frac{\Lambda_2}{\mu})\chi-\Lambda_1]\int_\Omega U_1dx\\
&+ [(\beta_2\frac{\Lambda_1}{\sigma_1}a+\beta_4\frac{\Lambda_2}{\sigma_2}b+\beta_2\frac{\Lambda_2}{\sigma_1}+\beta_4\frac{\Lambda_1}{\sigma_2}+\beta_2\theta_1\frac{\Lambda_1}{\mu}+\beta_4\theta_2\frac{\Lambda_2}{\mu})\chi-\Lambda_2]\int_\Omega U_2dx\\
&- \mu\theta_1\int_\Omega(S_1-\frac{\Lambda_1}{\mu})^2dx-\mu\theta_2\int_\Omega(S_2-\frac{\Lambda_2}{\mu})^2dx-\sigma_1\int_\Omega U_1^2dx-\sigma_2\int_\Omega U_2^2dx.
\end{aligned}$$

If θ_1, θ_2 and χ verify the estimates (2.34), (2.35), and (2.36), then

$$I_2\leq -\mu\theta_1\int_\Omega(S_1-\frac{\Lambda_1}{\mu})^2dx-\mu\theta_2\int_\Omega(S_2-\frac{\Lambda_2}{\mu})^2dx-\sigma_1\int_\Omega U_1^2dx-\sigma_2\int_\Omega U_2^2dx, \tag{2.43}$$

then, by (2.41) and (2.43)

$$\begin{aligned}
F^{'}(t) &= I_1+I_2\\
&\leq -\frac{1}{3}d_2\int_\Omega|\nabla U_1|^2dx-\frac{1}{3}d_1\theta_1\int_\Omega|\nabla S_1|^2dx-\frac{1}{3}d_4\int_\Omega|\nabla U_2|^2dx-\frac{1}{3}d_3\theta_2\int_\Omega|\nabla S_2|^2dx\\
&- \mu\theta_1\int_\Omega(S_1-\frac{\Lambda_1}{\mu})^2dx-\mu\theta_2\int_\Omega(S_2-\frac{\Lambda_2}{\mu})^2dx-\sigma_1\int_\Omega u_1^2dx-\sigma_2\int_\Omega u_2^2dx,\\
&\leq 0. \qquad (2.44)
\end{aligned}$$

By integrating over $(0,t)$

$$\begin{aligned}
&F(t)+\mu\theta_1\int_0^t\int_\Omega(S_1-\frac{\Lambda_1}{\mu})^2dx+\mu\theta_2\int_0^t\int_\Omega(S_2-\frac{\Lambda_2}{\mu})^2dx+\sigma_1\int_0^t\int_\Omega u_1^2dx+\sigma_2\int_0^t\int_\Omega u_2^2dx\\
+&\frac{1}{3}d_2\int_0^t\int_\Omega|\nabla U_1|^2dx+\frac{1}{3}d_1\theta_1\int_0^t\int_\Omega|\nabla S_1|^2dx+\frac{1}{3}d_4\int_0^t\int_\Omega|\nabla U_2|^2dx+\frac{1}{3}d_3\theta_2\int_0^t\int_\Omega|\nabla S_2|^2dx\leq F(0).
\end{aligned} \tag{2.45}$$

From (2.45), we can deduce that, for all $(t,x)\in(0,+\infty)\times\Omega$ and $i=1,2$, we have

$$\int_\Omega(S_i(t,.)-\frac{\Lambda_i}{\mu})^2dx<+\infty,\ \int_0^{+\infty}\int_\Omega(S_i(t,.)-\frac{\Lambda_i}{\mu})^2dxds<+\infty,\ \int_0^{+\infty}\int_\Omega|\nabla(S_i(t,.)-\frac{\Lambda_i}{\mu})|^2dxds<+\infty,$$

and

$$\int_0^{+\infty}\int_\Omega U_i^2(t,.)dxds < +\infty, \int_0^{+\infty}\int_\Omega |\nabla U_i(t,.)|^2 dxds < +\infty. \tag{2.46}$$

So that, by Barbalate's lemma, we conclude that

$$\lim_{t\to+\infty} \|S_i(t,.) - \frac{\Lambda_i}{\mu}\|_2 = 0, \qquad i = 1,2 \tag{2.47}$$

$$\lim_{t\to+\infty} \|U_i(t,.)\|_2 = 0. \qquad i = 1,2 \tag{2.48}$$

Combining (2.47), (3.38) and the fact that the orbits $\{S_i(t,x), t > 0, i = 1,2\}$ and $\{U_i(t,x), t > 0, i = 1,2\}$ are relatively compact on $C(\overline{\Omega})$ [18], we conclude that

$$\lim_{t\to+\infty} \|S_i(t,.) - \frac{\Lambda_i}{\mu}\|_\infty = 0 \qquad \text{for all } i = 1,2, \tag{2.49}$$

$$\lim_{t\to+\infty} \|U_i(t,.)\|_\infty = 0 \qquad \text{for all } i = 1,2, \tag{2.50}$$

and the theorem is completely proved. □

CHAPTER 3

Threshold condition for global stability of an epidemic model

3.1 Introduction

Over the last years, the infectious disease models had a huge interest in both biological and mathematical researches. More investigations on the use of mathematical models to understand the dynamic of infectious diseases (such us HIV/AIDS, tuberculosis (TB), malaria, etc) were carried out.

In a population, the interaction between the individuals causes the widespread of the infectious disease.

Mathematically, this interaction best discribed by the so-called SIR model, wich was initially studied by Kermack and McKendrik (1927) [25], where the population is divised into three componants :

S (Susceptible individuals) : individuals that are able to catch the disease, once they have it, they move to the infected compartement.

I (Infected individuals): can spread the disease to susceptible individuals.

R (Recovered individuals): in this compartement the individuals are assumed to be immune for life.

These models were studied by several reasearchers, and in many different forms. They start by the simple SIR model:

$$\begin{cases} \dfrac{\partial S}{\partial t} = -\beta S I & \text{in } \mathbb{R}^+, \\ \dfrac{\partial I}{\partial t} = \beta S I - \gamma I & \text{in } \mathbb{R}^+, \\ \dfrac{\partial R}{\partial t} = \gamma I & \text{in } \mathbb{R}^+, \end{cases} \tag{3.1}$$

Here β and γ are two positive constants, where β indicates the contact rate, and γ represents the recovery rate.

As a result of the contact between infected and susceptible individuals, new infections occur with the rate βSI, they move from the susceptible class to the infected one. In other process, the infected individuals can enter the removed class (they will acquire an immunity from the disease) at the rate γI.

Other more general model, is a model where the susceptible class is generated by birth at a rate Λ (new recreatement in the susceptible class) and all the classes decrease with a natural death at the rates μS, μI and μR. We describe it with the following model

$$\begin{cases} \dfrac{\partial S}{\partial t} = \Lambda - \beta S I - \mu S & \text{in } \mathbb{R}^+, \\ \dfrac{\partial I}{\partial t} = \beta S I - \gamma I - \mu I & \text{in } \mathbb{R}^+, \\ \dfrac{\partial R}{\partial t} = \gamma I - \mu R & \text{in } \mathbb{R}^+. \end{cases} \tag{3.2}$$

In this direction, many researches has been done, where the mathematicians have given results on the existence and the stability of the solutions, of the ordinary systems (see [6], [35] and [38]).

Including the spatial diffusion, Webb [39] assumed that spatial mobility is governed by random diffusion coefficients respectively D_S, D_I, D_R for the suscebtible, infected and recovered classes. The spatial factors are used to describe the mobility of the population

$$\begin{cases} \dfrac{\partial S}{\partial t} - D_S \Delta S = -\beta S I & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial I}{\partial t} - D_I \Delta I = \beta S I - r I & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial R}{\partial t} - D_R \Delta R = r I & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{3.3}$$

(Ω is a bounded domain of class C^1 in $\mathbb{R}^n$) with homogeneous Neumann boundary conditions and the initial data $S_0(x) \geq 0, I_0(x) \geq 0,\ R_0(x) = 0$ for all x in Ω.

Recently, Kim, Lin and Zhang in [26] gave a result of global existence, uniqueness of solutions and sufficient conditions for the vanishment or spread of the disease of an SIR model with free boundary. Lotfi, Maziane, Hattaf and Yousfi [28] showed the global existence, positivity and boundedness of solutions for the reaction diffusion system below

$$\begin{cases} \dfrac{\partial S}{\partial t} = D_S \Delta S_1 + \Lambda - \mu S - \dfrac{\beta S I}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 S I} & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial I}{\partial t} = D_I \Delta I_1 + \dfrac{\beta S I}{1 + \alpha_1 S + \alpha_2 I + \alpha_3 S I} - (\mu + d + r) I & \text{in } \mathbb{R}^+ \times \Omega, \end{cases} \tag{3.4}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \nu} = \frac{\partial I}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \qquad i = 1, 2, \tag{3.5}$$

and initial conditions

$$S(0,x) = \phi_1(x) \geq 0, \quad I(0,x) = \phi_2(x) \geq 0 \qquad \text{in } \Omega, \qquad i = 1, 2. \tag{3.6}$$

We consider the following reaction-diffusion system

$$\begin{cases} \dfrac{\partial S_1}{\partial t} - D \Delta S_1 = \Lambda_1 - \beta \dfrac{S_1 (U_1 + U_2)}{T} - \mu S_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U_1}{\partial t} - D \Delta U_1 = \beta \dfrac{S_1 (U_1 + U_2)}{T} - \sigma U_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial S_2}{\partial t} - D \Delta S_2 = \Lambda_2 - \beta \dfrac{S_2 (U_1 + U_2)}{T} - \mu S_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial U_2}{\partial t} - D \Delta U_2 = \beta \dfrac{S_2 (U_1 + U_2)}{T} - \sigma U_2 & \text{in } \mathbb{R}^+ \times \Omega. \end{cases} \tag{3.7}$$

supplemented with the homogeneous Neumann boundary conditions

$$\frac{\partial S_i}{\partial \nu} = \frac{\partial U_i}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \qquad i = 1,2, \tag{3.8}$$

and the continuous initial data $S_{1,0}$, $U_{1,0}$, $S_{2,0}$, $U_{2,0}$

$$S_i(0,x) = S_{i,0}(x) \geq 0, \quad U_i(0,x) = U_{i,0}(x) > 0 \qquad \text{in } \Omega, \qquad i = 1,2. \tag{3.9}$$

Here Ω is a bounded domain of class C^1 in $\mathbb{R}^n$, with boundary $\partial\Omega$ and $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative to $\partial\Omega$, Δ is the Laplacien, it presents the spatial diffusion of the epidemic in the population and the homogeneous Neumann boundary conditions (2.5) implies that the system in question is self-contained and there is no emigration across the boundary.

This system is coming as a model of a diffusive epidemic phenomena in some population of individuals. The population considered contains susceptible individuals S_1, S_2 and infected ones U_1, U_2 divided into two groups, where the indices 1 and 2 indicate which group they belong to. D is the coefficient of the spatial diffusion. The constants Λ_1 and Λ_2 represent the influx rates of new susceptible individuals in each group. μ is the mortality rate, the ratio of the number of deaths from the disease to the total number of cases per unit of time of that disease. The parameter β describes the rate at which the disease is spread among the individuals per unit of time, the parameter σ is given by $\sigma = \mu + \eta$, where $1/\eta$ is the average activity period of the infected individuals and T is the total population (See [9], [12], [35] and [39]).

So, the constants D, Λ_1, Λ_2, μ, σ are such that

$$D > 0,\ \mu > 0,\ \Lambda_1 \geq 0,\ \Lambda_2 \geq 0,\ \sigma \geq 0. \tag{3.10}$$

In this paper, we will show that the solutions of the system (3.7)–(3.9) are global and uniformly bounded on $(0,+\infty) \times \Omega$. Besides of that, we will give a threshold condition defined by the reproduction number R, allow us to obtain the extinction or the persistence of the epidemic in the population.

The basic reproduction number, denoted R, is the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual.

If $R < 1$, then on average an infected individual produces less than one new infected individual over the course of its infectious period, and the infection cannot grow. Conversely, if $R > 1$, then each infected individual produces, on average, more than one new infection, and the disease can invade the population.

3.2 Notations and preliminary results

3.2.1 Local existence of solutions

Throughout this study, we denote by

$$\begin{aligned} \|u\|_p^p &= \int_\Omega |u(x)|^p dx,\ 1 \le p < +\infty \\ \|u\|_\infty &= \operatorname{ess\,sup}_{x\in\Omega} |u(x)|, \\ \|u\|_{C(\overline{\Omega})} &= \max_{x\in\overline{\Omega}} |u(x)|, \end{aligned} \tag{3.11}$$

the usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ respectively.

The study of local existence and uniqueness of solutions to the problem (3.7)–(3.9) in $C(\overline{\Omega})$ is classical. As a consequence of the theory of analytical semigroups the solutions are classical on $(0, T^*)$, where T^* denotes the eventual blowing-up time in $L^\infty(\Omega)$. (see Henry [20] or Pazy [34]).

3.2.2 Positivity of the solutions

It follows from maximum principle that if the initial data $S_{1,0}, U_{1,0}, S_{2,0}, U_{2,0}$ are nonnegative, then the solutions S_1, U_1, S_2 and U_2 are nonnegative on $(0, T^*) \times \Omega$, and we introduce the following lemma

Lemma 3.2.1. *If* (S_1, U_1, S_2, U_2) *is a solution of the system* (3.7)–(3.9)*, then for all* $(t,x) \in (0, T^*) \times \Omega$*, we have*

$$\begin{aligned} 0 \le S_i(t,x) &\le K_i = max(\|S_{0,i}\|_\infty, \frac{\Lambda_i}{\mu}),\ \forall (t,x) \in (0,T) \times \Omega,\ i = 1,2. && (3.12) \\ U_i(t,x) &\ge \min_{x\in\Omega} U_{i,o}(x)\, e^{-\sigma\, t} > 0. \qquad \forall (t,x) \in (0,T) \times \Omega, i = 1,2. && (3.13) \end{aligned}$$

Proof. Immediate from the maximum principle. □

3.3 Boundedness of the solutions

Lemma 3.3.1. *For nonnegative and bounded initial data* $S_{0,1}, U_{0,1}, S_{0,2}, U_{0,2}$*, the solution* (S_1, U_1, S_2, U_2) *of the system* (3.7)–(3.9) *is global and uniformly bounded on* $(0, +\infty) \times \Omega$*.*

Proof. We add all the equations of the system (3.7)–(3.9), we get

$$\frac{\partial T}{\partial t} - D\Delta T = \Lambda_1 + \Lambda_2 - \mu T - \eta(U_1 + U_2). \tag{3.14}$$

Maximum principle applied to the above equation may give

$$T \le \max(\frac{\Lambda_1 + \Lambda_2}{\sigma}, \|S_{1,0} + U_{1,0} + S_{2,0} + U_{2,0}\|_\infty), \tag{3.15}$$

which achieves the global existence and the boundedness of the solutions S_1, U_1, S_2, U_2 of the system (3.7)–(3.9) on $(0, +\infty) \times \Omega$. □

3.4 Global stability

Without lose of generality, we assume in this section that

$$T(t,x) \le \frac{\Lambda_1 + \Lambda_2}{\mu}, \qquad (t,x) \in (0, +\infty) \times \Omega \tag{3.16}$$

Before we state the main results of this section, let

$$R = \frac{\beta}{\sigma}. \tag{3.17}$$

R is the basic reproduction number. The main results of this section are the following

Theorem 3.4.1. *If* $R < 1$*, then for each continuous positive intial data* $S_{1,0}, U_{1,0}, S_{2,0}, U_{2,0}$ *the solution of the system verifies*

$$\lim_{t\to+\infty} \|U_i(t,.)\|_\infty = 0, \tag{3.18}$$

$$\lim_{t\to+\infty} \|S_i(t,.) - \frac{\Lambda_i}{\mu}\|_\infty = 0. \tag{3.19}$$

Theorem 3.4.2. *If* $R > 1$*, then the system* (3.7)–(3.9) *has a unique endemic equilibrium point* $(S_1^*, U_1^*, S_2^*, U_2^*)$ *and for each continuous positive initial data* $S_{1,0}, U_{1,0}, S_{2,0}, U_{2,0}$ *the solution of the system verifies*

$$\lim_{t\to+\infty} \|U_i(t,.) - U_i^*\|_\infty = \lim_{t\to+\infty} \|S_i(t,.) - S_i^*\|_\infty = 0, \tag{3.20}$$

Thus, the endemic equilibrium point is globally asymptotically stable.

Proof. (Proof of theorem 3.4.1)

Adding the second and the fourth equation of the system (3.7), we get

$$\begin{aligned}\frac{\partial}{\partial t}(U_1+U_2) - D\Delta(U_1+U_2) &= \beta\frac{(S_1+S_2)}{T}(U_1+U_2) - \sigma(U_1+U_2),\\ &\le \sigma(\frac{\beta}{\sigma}-1)(U_1+U_2).\end{aligned}$$

We set $U = U_1 + U_2$, then

$$\frac{\partial}{\partial t}U - D\Delta U \le \sigma(R-1)U, \tag{3.21}$$

where R is the reproduction number defined by (3.17).

Now, we define the following function for all $t \in [0, T]$

$$L(t) = \frac{1}{2}\int_\Omega U^2 dx. \tag{3.22}$$

Differentiating L with respect to t

$$\begin{aligned}L'(t) &= \int_\Omega U\frac{\partial U}{\partial t}dx,\\ &\le \int_\Omega U(D\Delta U + \sigma(R-1)U)dx\end{aligned}$$

Green's formula yields

$$L'(t) + D\int_{\Omega} |\nabla U|^2 dx + \sigma(1-R)\int_{\Omega} U^2 dx \le 0.$$

Integrating over (o, t)

$$L(t) + D\int_0^t \int_{\Omega} |\nabla U|^2 dx + \sigma(1-R)\int_0^t \int_{\Omega} U^2 dx \le L(0).$$

By Barbalate's lemma and the condition that $R < 1$, we get

$$\lim_{t\to+\infty} \|U(t,.)\|_2 = 0.$$

Thus

$$\lim_{t\to+\infty} \|U_i(t,.)\|_2 = 0.$$

Beside of that, the orbits $\{U_i(t,.), t > 0,\ i = 1,2\}$ are relatively compact, so that (3.18) is verified.

To prove (3.19), it suffices to show that T tends to $\dfrac{\Lambda_1+\Lambda_2}{\mu}$ as the time goes to $+\infty$. The limit (3.18) permits to deduce that for any $\varepsilon > 0$, there exists a large time t_0, such that, for all $t > t_0$

$$(U_1 + U_2) < \varepsilon, \tag{3.23}$$

By the maximum principle, we prove that

$$T(t,x) \ge \hat{T}(t), t > t_0, x \in \overline{\Omega}, \tag{3.24}$$

avec $\hat{T}(t)$ is the solution of the following ordinary equation

$$\begin{cases} \dfrac{\partial \hat{T}(t)}{\partial t} = -\mu\hat{T}(t) + \Lambda_1 + \Lambda_2 - \eta\,\varepsilon, \\ \hat{T}(t_0) = \hat{T}_0(t), \end{cases} \tag{3.25}$$

with $0 < \hat{T}_0 \le \inf_{x\in\overline{\Omega}} T(t_0, x) \le \dfrac{\Lambda_1+\Lambda_2}{\mu}$.
Applying variation of constants formula, one can write

$$\hat{T} = \hat{C}e^{-\mu t} + \frac{\Lambda_1+\Lambda_2}{\mu} - \eta\,\frac{\varepsilon}{\mu},$$

with

$$\hat{C} = e^{-\mu t_0}(\hat{T}_0 - \frac{\Lambda_1 + \Lambda_2}{\mu} + \eta\,\frac{\varepsilon}{\mu}).$$

By letting $t \to +\infty$ and then $\varepsilon \to 0$ in the inequalities (3.16) and (3.24), we obtain

$$\lim_{t\to+\infty} T = \frac{\Lambda_1 + \Lambda_2}{\mu}.$$

This complete the proof. □

Proof. (Proof of theorem (3.4.2))

$(S_1^*,\ U_1^*,\ S_2^*,\ U_2^*)$ will be an equilibrium point of the system (3.7), if it verifies the system below

$$\begin{cases} \Lambda_1 - \beta\dfrac{S_1^*(U_1^* + U_2^*)}{T^*} - \mu S_1^* = 0, \\ \beta\dfrac{S_1^*(U_1^* + U_2^*)}{T^*} - \sigma U_1^* = 0, \\ \Lambda_2 - \beta\dfrac{S_2^*(U_1^* + U_2^*)}{T^*} - \mu S_2^* = 0, \\ \beta\dfrac{S_2^*(U_1^* + U_2^*)}{T^*} - \sigma U_2^* = 0. \end{cases} \tag{3.26}$$

Thus, the addition of the four equations gives

$$(S_1^* + S_2^*) = \frac{\Lambda_1 + \Lambda_2}{\mu} - \frac{\sigma}{\mu}(U_1^* + U_2^*), \tag{3.27}$$

and

$$T^* = \frac{\Lambda_1 + \Lambda_2}{\mu} + \frac{\mu - \sigma}{\mu}(U_1^* + U_2^*), \tag{3.28}$$

In the other hand, the addition of the second and the fourth equation gives

$$\left(\frac{(S_1^* + S_2^*)}{T^*} - \frac{\sigma}{\beta}\right)(U_1^* + U_2^*) = 0,$$

then

$$(S_1^* + S_2^*) = \frac{\sigma}{\beta}T^*. \tag{3.29}$$

We substitute the equations (3.27) and (3.28) in (3.29), we obtain

$$\frac{\Lambda_1 + \Lambda_2}{\mu} - \frac{\sigma}{\mu}(U_1^* + U_2^*) = \frac{\sigma}{\beta}\left(\frac{\Lambda_1 + \Lambda_2}{\mu} + \frac{\mu - \sigma}{\mu}(U_1^* + U_2^*)\right). \tag{3.30}$$

So that, to prove the existence of a unique endemic equilibrium point to the system (2.4), it suffices to prove that there exists a unique positive constant $U^* = (U_1^* + U_2^*)$ that verifies the equation (3.30).

For that, we pose the following function

$$f(U) = \frac{\left(\dfrac{\Lambda_1+\Lambda_2}{\mu} - \dfrac{\sigma}{\mu}U\right)}{\left(\dfrac{\Lambda_1+\Lambda_2}{\mu} - \dfrac{\sigma-\mu}{\mu}U\right)} - \frac{\sigma}{\beta}. \tag{3.31}$$

By the condition (3.17) on the basic reproduction rate, the function f is a decreasing function on the interval $\left(0, \dfrac{\Lambda_1+\Lambda_2}{\sigma}\right)$ and

$$f(0) = 1 - \frac{\sigma}{\beta} > 0,\ g(\frac{\Lambda_1+\Lambda_2}{\sigma}) = -\frac{\sigma}{\beta} < 0. \tag{3.32}$$

Then, by the mean value theorem, there existe a unique $U^* > 0$ such that

$$f(U^*) = 0. \tag{3.33}$$

By contradiction reasoning, we can easily show that U_1^* is unique. So, the fact that the sum $(U_1^* + U_2^*)$ is unique, we deduce that for $R > 1$, $(S_1^*, U_1^*, S_2^*, U_2^*)$ is the unique equilibrium point for the system (2.4).

Now, we will prove the limits (3.20).

$(S_1^*, U_1^*, S_2^*, U_2^*)$ is an endemic equilibruim point of the system (2.4), thus

$$\Lambda_1 + \Lambda_2 - \mu T^* - \eta(U_1^* + U_2^*) = 0. \tag{3.34}$$

By the equation (3.14) and (3.34)

$$\frac{\partial T}{\partial t} - D\Delta T = -\mu(T - T^*) - \eta(U - U^*), \tag{3.35}$$

with $U = (U_1 + U_2)$, $U^* = (U_1^* + U_2^*)$.

The addition of the second and the fourth equation of the system (3.7) gives

$$\frac{\partial}{dt}(U_1+U_2) = D\Delta(U_1+U_2) + \beta(\frac{(S_1+S_2)}{T} - \frac{\sigma}{\beta})(U_1+U_2), \tag{3.36}$$

Because $(S_1^*, U_1^*, S_2^*, U_2^*)$ is an endemic equilibruim point of the system (2.4), the equation (3.29) permits to write

$$\frac{\sigma}{\beta} = \frac{(S_1^* + S_2^*)}{T^*}. \tag{3.37}$$

Thus

$$\frac{\partial U}{dt} = D\Delta U + \beta(\frac{S}{T} - \frac{S^*}{T^*})U, \tag{3.38}$$

with $S = (S_1 + S_2)$, $S^* = (S_1^* + S_2^*)$.

Let $G(T) = \frac{(T - T^*)}{TT^*}U^*$. The function G is a positive function if $T \geq T^*$ and it is a negative function if $T \leq T^*$.

Then we can rewrite (3.38) as follows

$$\frac{\partial U}{dt} = D\Delta U + \beta G(T)U - \beta\frac{(U - U^*)}{T}U. \tag{3.39}$$

We define the function H by

$$H(t) = \int_\Omega \left(\frac{\beta}{\eta}\int_{T^*}^{T} G(s)ds + U - U^* - U^* \ln\frac{U}{U^*}\right)dx. \tag{3.40}$$

Differentiating H with respect to t, using the equation (3.35) and (3.39), we get

$$H'(t) = \frac{\beta}{\eta}D\int_\Omega \Delta TG(t)dx - \frac{\beta\mu}{\eta}\int_\Omega G(t)(T - T^*)dx + D\int_\Omega \Delta U\frac{U - U^*}{U}dx - \beta\int_\Omega \frac{(U - U^*)^2}{T}dx.$$

Then

$$H'(t) = -\frac{\beta\mu}{\eta}\int_\Omega G(t)(T - T^*)dx - \beta\int_\Omega \frac{(U - U^*)^2}{T}dx + H_1 + H_2, \tag{3.41}$$

avec

$$\begin{aligned} H_1 &= \frac{\beta}{\eta}D\int_\Omega \Delta TG(t)dx, \\ H_2 &= D\int_\Omega \Delta U\frac{U - U^*}{U}dx. \end{aligned}$$

Using Green's formula, we obtain

$$\begin{array}{rcl} H_1 & = & \frac{\beta}{\eta} D \int_\Omega \Delta T G(t) dx, \\ & = & \frac{\beta\, U^*}{\eta\, T^*} D \int_\Omega \Delta T \frac{(T-T^*)}{T} dx, \\ & = & \frac{\beta\, U^*}{\eta\, T^*} D \left(\int_{\partial\Omega} \frac{\partial T}{\partial \nu} \, \frac{(T-T^*)}{T} dx - \int_\Omega \nabla T \, \nabla \left(\frac{(T-T^*)}{T} \right) dx \right), \\ & = & -\frac{\beta\, U^*}{\eta\, T^*} D \int_\Omega \nabla T \, \frac{\nabla T}{T^2} dx, \end{array}$$

Thus

$$H_1 = -\frac{\beta\, U^*}{\eta\, T^*} D \int_\Omega \frac{|\nabla T|^2}{T^2} dx \le 0,$$

By the same reasoning

$$H_2 = D \int_\Omega \Delta U \frac{U-U^*}{U} dx, = -D \int_\Omega \frac{|\nabla U|^2}{U^2} dx \le 0,$$

We conclude that

$$H'(t) = -\frac{\beta\, \mu}{\eta} \int_\Omega G(t)(T-T^*) dx - \beta \int_\Omega \frac{(U-U^*)^2}{T} dx + H_1 + H_2 \le 0, \tag{3.42}$$

Then, the limits (3.20) are verified and the theorem is completely proved. □

CHAPTER 4

Local stability of equilibrium points of an ODE system

This chapter is devoted to the stability of the disease equilibrium point which determines the epidemic behavior in a local neighborhood of the equilibrium point for the following ODE system below coressponding to the system (2.4)–(2.6)

$$\begin{cases} \frac{\partial S_1}{\partial t} = \Lambda_1 - \beta_1 \dfrac{S_1\psi(U_1)}{T} - \beta_2 \dfrac{S_1\psi(U_2)}{T} - \mu S_1 \\ \frac{\partial U_1}{\partial t} = \beta_1 \dfrac{S_1\psi(U_1)}{T} + \beta_2 \dfrac{S_1\psi(U_2)}{T} - \sigma_1 U_1, \\ \frac{\partial S_2}{\partial t} = \Lambda_2 - \beta_3 \dfrac{S_2\psi(U_1)}{T} - \beta_4 \dfrac{S_2\psi(U_2)}{T} - \mu S_2, \\ \frac{\partial U_2}{\partial t} = \beta_3 \dfrac{S_2\psi(U_1)}{T} + \beta_4 \dfrac{S_2\psi(U_2)}{T} - \sigma_2 U_2. \end{cases} \tag{4.1}$$

with the homogeneous Newmann boundary conditions

$$\frac{\partial S_i}{\partial \nu} = \frac{\partial U_i}{\partial \nu} = 0 \qquad \text{on } \mathbb{R}^+ \times \partial\Omega, \text{ for all } i = 1, 2. \tag{4.2}$$

and the positive initial data

$$S_i(0,x) = S_{i,0}(x) > 0, U_i(0,x) = U_{i,0}(x) > 0 \qquad \text{in } \Omega, \text{ for all } i = 1, 2. \tag{4.3}$$

We assume here that ψ is a nonnegative and continuously differentiable function on $[0, +\infty)$ satisfying

$$\psi(0) = 0.$$

Here, we consider that the influx rates Λ_1, Λ_2 are constants or they change their values exponentially with respect to the susceptible individuals values

$$\Lambda_1 = \kappa_1 S_1 e^{-\kappa_2 S_1}, \Lambda_2 = \kappa_3 S_2 e^{-\kappa_4 S_2} \tag{4.4}$$

4.1 Equilibrium points

Before stating our main result of this chapter, we start our analysis by calculating the equilibrium points of the system (4.1)–(4.3).

Resolving the system

$$\begin{cases} \Lambda_1 - \beta_1 \dfrac{S_1 \psi(U_1)}{T} - \beta_2 \dfrac{S_1 \psi(U_2)}{T} - \mu S_1 = 0 \\ \beta_1 \dfrac{S_1 \psi(U_1)}{T} + \beta_2 \dfrac{S_1 \psi(U_2)}{T} - \sigma_1 U_1 = 0, \\ \frac{\partial S_2}{\partial t} = \Lambda_2 - \beta_3 \dfrac{S_2 \psi(U_1)}{T} - \beta_4 \dfrac{S_2 \psi(U_2)}{T} - \mu S_2 = 0, \\ \frac{\partial U_2}{\partial t} = \beta_3 \dfrac{S_2 \psi(U_1)}{T} + \beta_4 \dfrac{S_2 \psi(U_2)}{T} - \sigma_2 U_2 = 0. \end{cases} \tag{4.5}$$

We get the disease free equilibruim points corresponding to the different influx rates Λ_1, Λ_2 wich referred to the case where the infected populations are zero.

$$(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\frac{\Lambda_1}{\mu}, 0, \frac{\Lambda_2}{\mu}, 0) \qquad \text{and} \qquad (\frac{\ln \frac{\kappa_1}{\mu}}{\kappa_2}, 0, \frac{\ln \frac{\kappa_3}{\mu}}{\kappa_4}, 0),$$

with Λ_1 and Λ_2 are constants or $\Lambda_1 = \kappa_1 S_1 e^{-\kappa_2 S_1}$ and $\Lambda_2 = \kappa_3 S_2 e^{-\kappa_4 S_2}$.

It is also possible to obtain the endemic equilibrium points corresponding to the situation in which the disease is settled in the population. In this case, we consider that the number of susceptible individuals achieves the equilibrium for the values $\overline{S}_1$ and $\overline{S}_2$.

Thus

$$(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\overline{S}_1, \frac{\Lambda_1 - \mu \overline{S}_1}{\sigma_1}, \overline{S}_2, \frac{\Lambda_2 - \mu \overline{S}_2}{\sigma_2})$$

if the influx rates are constant, and

$$(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\overline{S}_1, \frac{(\kappa_1 e^{-\kappa_2 \overline{S}_1} - \mu) \overline{S}_1}{\sigma_1}, \overline{S}_2, \frac{(\kappa_3 e^{-\kappa_4 \overline{S}_2} - \mu) \overline{S}_2}{\sigma_2})$$

if the influx rates are $\Lambda_1 = \kappa_1 S_1 e^{-\kappa_2 S_1}$ and $\Lambda_2 = \kappa_3 S_2 e^{-\kappa_4 S_2}$.

4.2 Local Stability of equilibrium points

Our main result is as follows

Theorem 4.2.1.

1. *Assume that* Λ_1, Λ_2 *are constants, then the basic reproduction rate* R_0 *is given by*

$$R_0 = \frac{(\beta_1 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi^{'}(0) + \beta_4 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi^{'}(0))(\beta_2\beta_3 \frac{\Lambda_1\Lambda_2}{(\Lambda_1+\Lambda_2)^2}(\psi^{'}(0))^2)}{(\sigma_1+\sigma_2)(\beta_1 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi'(0)-\sigma_1)(\beta_4 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi'(0)-\sigma_2)}. \tag{4.6}$$

If $R_0 < 1$*, then the disease free equilibrium point* $(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\frac{\Lambda_1}{\mu}, 0, \frac{\Lambda_2}{\mu}, 0)$ *is locally asymptotically stable and if* $R_0 > 1$*, then the disease free equilibrium point is unstable and the disease endemic equilibrium point* $(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\overline{S}_1, \frac{\Lambda_1 - \mu\overline{S}_1}{\sigma_1}, \overline{S}_2, \frac{\Lambda_2 - \mu\overline{S}_2}{\sigma_2})$ *is locally asymptotically stable.*

2. *Assume that the influx rates are given by* $\Lambda_1 = \kappa_1 S_1 e^{-\kappa_2 S_1}$ *and* $\Lambda_2 = \kappa_3 S_2 e^{-\kappa_4 S_2}$.
Then R_0 *is given by*

$$R_0 = \frac{(\beta_1 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi^{'}(0) + \beta_4 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi^{'}(0))(\beta_2\beta_3 \frac{\underline{S}_1\underline{S}_2}{(\underline{S}_1+\underline{S}_2)^2}(\psi^{'}(0))^2)}{(\sigma_1+\sigma_2)(\beta_1 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi'(0)-\sigma_1)(\beta_4 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi'(0)-\sigma_2)}. \tag{4.7}$$

If $R_0 < 1$*, then the disease free equilibrium point* $(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\frac{\ln\frac{\kappa_1}{\mu}}{\kappa_2}, 0, \frac{\ln\frac{\kappa_3}{\mu}}{\kappa_4}, 0)$ *is locally asymptotically stable and if* $R_0 > 1$*, the disease free equilibrium point is unstable and the disease endemic equilibrium point*

$(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\overline{S}_1, \frac{(\kappa_1 e^{-\kappa_2 \overline{S}_1} - \mu)\overline{S}_1}{\sigma_1}, \overline{S}_2, \frac{(\kappa_3 e^{-\kappa_4 \overline{S}_2} - \mu)\overline{S}_2}{\sigma_2})$ is locally asymptotically stable.

Proof. 1. The analysis can be done with calculating the eigenvalues for the Jacobian matrix of the system (4.1) evaluating at our disease free equilibrium points. Since

$$(\mu+\lambda)^2[\lambda^2 - (\alpha+\rho)\lambda + \alpha\rho - \gamma\delta] \tag{4.8}$$

where
$\alpha = \beta_1 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi'(0) - \sigma_1$, $\rho = \beta_4 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi'(0) - \sigma_2$, $\gamma = \beta_2 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi'(0)$, $\delta = \beta_3 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi'(0)$. The double multiplicity eigenvalues $\lambda_{1,2} = -\mu$ are always real and negative. Then the study is refered to know the sign of the roots of the other term $(\lambda^2 - (\alpha+\rho)\lambda + \alpha\rho - \gamma\delta)$, for that we use the Routh-Hurwitz criterion [33],testing if there are roots with positive real part in a polynomial equation without solving it, then it suffices that

$$\begin{array}{rrcl} & -(\alpha+\rho) & > & 0, \\ \text{and} & \alpha\rho - \gamma\delta & > & 0. \end{array}$$

Consequently, the stability criteria are expressed by the inequalities:

$$\beta_1 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi'(0) + \beta_4 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi'(0) < \sigma_1 + \sigma_2, \tag{4.9}$$

$$\frac{\beta_2\beta_3}{(\Lambda_1+\Lambda_2)^2}\Lambda_1\Lambda_2(\psi'(0))^2 < (\beta_1 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi'(0) - \sigma_1)(\beta_4 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi'(0) - \sigma_2), \tag{4.10}$$

In epidemiology literatures, we usually analyse the model by examining the number of individuals that become infected from introducing one infected into a totally susceptible population during an infectious period [7], [13], [35]. We find that if the number of new infections is greater than one, then the disease will persist. Alternatively, if the number of new infections is less than one, then the disease will die out. We refer to this number as the basic reproduction number (R_0) and it determines the stability of the system. From (4.9) and (4.10), we obtain the expression of R_0 given by

$$R_0 = \frac{(\beta_1 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi^{'}(0) + \beta_4 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi^{'}(0))(\beta_2\beta_3 \frac{\Lambda_1\Lambda_2}{(\Lambda_1+\Lambda_2)^2}(\psi^{'}(0))^2)}{(\sigma_1+\sigma_2)(\beta_1 \frac{\Lambda_1}{\Lambda_1+\Lambda_2}\psi'(0) - \sigma_1)(\beta_4 \frac{\Lambda_2}{\Lambda_1+\Lambda_2}\psi'(0) - \sigma_2)} \tag{4.11}$$

2. If the influx rates are $\Lambda_1 = \kappa_1 S_1 e^{-\kappa_2 S_1}$ and $\Lambda_2 = \kappa_3 S_2 e^{-\kappa_4 S_2}$, with the same reasoning, the polynomial equation corresponding to the eigenvalues of the Jacobian calculated in the equilibrium point $(\underline{S}_1, \underline{U}_1, \underline{S}_2, \underline{U}_2) = (\frac{\ln\frac{\kappa_1}{\mu}}{\kappa_2}, 0, \frac{\ln\frac{\kappa_3}{\mu}}{\kappa_4}, 0)$ is given by:

$$(\lambda + \mu \ln\frac{\kappa_1}{\mu})(\lambda + \mu \ln\frac{\kappa_3}{\mu})[\lambda^2 - (\alpha+\rho)\lambda + \alpha\rho - \gamma\delta] \tag{4.12}$$

where α, ρ, γ and δ in this case are given by

$$\alpha = \beta_1 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi^{'}(0) - \sigma_1,\ \ \rho = \beta_4 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi^{'}(0) - \sigma_2,\ \ \gamma = \beta_2 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi'(0),\ \ \delta = \beta_3 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi^{'}(0).$$

The immediately eigenvalues $\lambda_1 = -\mu\ln(\frac{\kappa_1}{\mu})$, $\lambda_2 = -\mu\ln(\frac{\kappa_3}{\mu})$ obtained from (4.12) can be non negative if $\kappa_1 > \mu$ and $\kappa_3 > \mu$, and this is possible according to our model's paramaters.

By using the Routh-Hurwitz criterion the stability conditions are reduced to

$$\beta_1 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi^{'}(0) + \beta_4 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi^{'}(0) < \sigma_1 + \sigma_2, \tag{4.13}$$

$$\beta_2\beta_3 \frac{\underline{S}_1\underline{S}_2}{(\underline{S}_1+\underline{S}_2^2)}(\psi^{'}(0))^2 < (\beta_1 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi^{'}(0) - \sigma_1)(\beta_4 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi^{'}(0) - \sigma_2). \tag{4.14}$$

Consequently, the expression of R_0 in this case is given by

$$R_0 = \frac{(\beta_1 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi^{'}(0) + \beta_4 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi^{'}(0))(\beta_2\beta_3 \frac{\underline{S}_1\underline{S}_2}{(\underline{S}_1+\underline{S}_2)^2}(\psi^{'}(0))^2)}{(\sigma_1+\sigma_2)(\beta_1 \frac{\underline{S}_1}{\underline{S}_1+\underline{S}_2}\psi'(0) - \sigma_1)(\beta_4 \frac{\underline{S}_2}{\underline{S}_1+\underline{S}_2}\psi'(0) - \sigma_2)}. \tag{4.15}$$

Finally, in the two cases, if R_0 is less then 1, the disease free equilibrium point is asymptotically stable and the disease disapears, if R_0 is greater than 1, the disease goes into the endemic equilibrium situation and the theorem is completly proved.

$\square$

CONCLUSION

In this thesis we are interested by a general Susceptible-Infected-Recovered (SIR)model which describes the diffusion of an epidemic in a population.

Our first contribution is devoted to the proof of the global existence of solutions for a reaction-diffusion system, and the establishment of sufficient conditions that lead to the eradication of the disease from population.

By using the theory of Lyapunov function, we study in the second contribution, the global asymptotic stability for a particular reaction diffusion system of the system in the first part, with a threshold condition.

In the third contribution, we discuss the stability analysis of the epidemic model. We present both disease-free equilibria and the endemic equilibria of the proposed model. The local dynamics of a general SIR is determined by the basic reproduction number R_0 which depends on the parameter values. For $R_0 < 1$ the disease-free equilibrium is locally asymptotically stable while for $R_0 > 1$ the endemic equilibrium exists.

BIBLIOGRAPHY

[1] S. Abdelmalek, M. Kirane and A. Youkana, *A Lyapunov Functional for a triangular reaction-diffusion system with nonlinearities of exponential growth*, Mathematical Methods in the Applied Sciences,vol. 36, no. 1, pp. 80?85, 2012.

[2] N. D. Alikakos, L^p *bounds of solutions of reaction-diffusion equations*, Communications in Partial Differential Equations, vol. 84,no.1,pp. 150-161, 1981.

[3] S. Al-Sheikh, F. Musali and M. Alsolami, *Stability Analysis of an HIV/AIDS Epidemic Model with Screening*, International Mathematical Forum, Vol. 6, no. 66, 3251 - 3273, 2011.

[4] A. Barabanova, *On the global existence of solutions of reaction-diffusion equation with exponential nonlinearity*, Proc. Amer. Math. Soc. 122 (1994), 827-831.

[5] S. Basak, B. K. Datta and P. K. Ghose, Mathematical Analysis of an HIV/AIDS Epidemic Model, American Journal of Mathematics and Statistics, 5(5): 253-258, 2015.

[6] F. Bozkurt and F. Peker, *Mathematical modeling of HIV epidemic and stability analysis, Advances in Difference Equations*, A Springer Open Journal, doi: 10.1186/1687-1847-(2014)-95.

[7] F. Brauer, C. Castillo-Chaves, *Mathematical Models in Population Biology and Epidemiology*, Springer-Verlag, Berlin, 2000.

[8] H. Brezis, *Analyse fonctionnelle Théorie et Application*, DUNOD, Paris, 1999.

[9] S. Busenberg and K. Cooke, *Vertically Transmitted Diseases, vol. 23 of Biomathematics,* Springer, Berlin, Germany, 1993.

[10] L. Cai, X. Li, M. Ghosh and B. Guo, Stability analysis of an HIV/AIDS epidemic model with treatment , Journal of Computational and Applied Mathematics 229 (2009) 313-323, 2009.

[11] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations,* Wiley Series in Mathematical and Computational Biology, John Wiley et Sons, Chichester, UK, 2003.

[12] C. Castillo-Chavez, K. Cooke, W. Huang and S. A. Levin, *The role of long periods of infectiousness in the dynamics of acquired immunodeficiency syndrome (AIDS),*in: C. Castillo-Chavez, S.A. Levin,C. Shoemaker (Eds.), Mathematical Approaches to Problems in Resource Management and Epidemiology,Lecture Notes in Biomathematics, vol. 81, Springer, New York Inc., 1989.

[13] V. D. Driessche and J. Watmough, *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*, Mathematical Biosciences 180:29-48, 2002.

[14] S. A. Egbetade and F. M. Babalola, *A Stability Analysis for a Two-Sex Mathematical Model of Hiv*, The International Journal Of Engineering And Science (IJES), Volume3, Issue 2, Pages 20-23, 2014.

[15] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of population Dynamics, vol. 74 of Mathematics and Its Applications, Kluwer Academic publishers, Dorecht, The Netherlands, 1992.

[16] Y. Hamaya, *On the asymptotic behavior of a diffusive epidemic model (AIDS)*, Nonlinear Analysis 36 (1999).

[17] A. Haraux and T. Cazenave, *Introduction aux problèmes d'évolution semi-linéaires*. Ellipses (1990).

[18] A. Haraux and M. Kirane, *Estimations C^1 pour des problèmes paraboliques semi-linéaires*, Ann. Fac. Sci. Toulouse 5, 265-280, 1983.

[19] A. Haraux and A. Youkana, *On a result of K.Masuda concerning reaction-diffusion equations*, Tohoku Mathematical Journal.Second Series, vol. 40,no. 1,pp. 159-163,1988.

[20] D. Henry, *Geometric Theory of Semi-linear Parabolic Equations*. Lecture Notes in Mathematics 840, Springer-Verlag, New-York, 1984.

[21] M. A. Herrero, A. Lacey and J. J. L. Velazquez, *Global existence for reaction-diffusion systems modelling ignition*, Arch. Rational Mech. Anal. 142 (1998), 219–251.

[22] S. L. Hollis, R. H. Martin and M. Pierre, *Global existence and boundedness in reaction-diffusion systems*, SIAM Journal on Mathematical Analysis,vol. 18,no. 3,pp.744-761, 1987.

[23] H. F. Huo and R. Chen, *Stability of an HIV/AIDS treatment model with different stages*, Hindawi Publishing Corporation, Discrete Dynamics in Nature and Society, Volume 2015, 9 pages, 2014.

[24] J. I. Kanel, *On global initial-boundary-value problems for reaction-diffusion systems with balence conditions*, Nonlinear Analisis, vol.37,no. 8,pp.971-995,1999.

[25] W. O. Kermack and A. G. McKendrick, *Contributions to the mathematical theory of epidemics (Part 1)*, Proc.R. Soc. Lond. Ser. A, 115, 700-720, 1927.

[26] K. I. Kim, Z. Lin and Q. Zhang, *An SIR epidemic model with free boundary*, Nonlinear Analysis RealWorld Applications; 14(5). DOI: 10.1016/j.nonrwa.2013.02.003, 02/2013.

[27] S. Kouachi, *Existence of global solutions to reaction-diffusion systems via a Lyapunov functional*, Electronic Journal of Differential Equations, No. 68, pp. 1-10, Vol. 2001(2001).

[28] E. M. Lotfi, M. Maziane, K. Hattaf and N. Yousfi, *Partial differential equations of an epidemic model with spatial diffusion.* International Journal of Partial Differential Equations 2014; 2014:Article ID 186437, 6.

[29] K. Masuda, *On the global existence and asymptotic behavior of solutions of reaction-diffusion equations*, Hokkaydo Mathematical Journal, vol. 12, no. 3,pp. 360-370,1983.

[30] L. Melkemi, A. Z. Mokrane and A. Youkana, *Boundedness and Large-Time Behavior Results for a Diffusive Epidemic Model* Journal of Applied Mathematics, vol 2007,Article ID 17930, 15 pages.

[31] J. Morgan, M. Parrott and G. Webb, *Drug resistance in diffusive epidemic population models*, World Scientific, Singapore, 1998, pp. 613-628.

[32] J. D. Murray, *Mathematical Biology I. An Introduction*, Interdisciplinary Applied Mathematics, Volume 17.

[33] K. Ogata,*Modern Control Engineering*, Prentice-Hall, New Jersy, 1997.

[34] A. Pazy, *Semigroups of linear Operators and Applications to Partial Differntial Equations*, vol.44 of Applied Mathematical Sciences,Springer,New York,NY,USA, 1983.

[35] J. R. C. Piqueira, M. C. Castano and L. H. A. Monteiro, *Modeling The Spreading of HIV in Homosexual Populations with Heterogeneous Preventive Attitude* , Journal of Biological Systems,12, 439-456, 2004.

[36] A. Touil and A. Youkana, *Boundedness and asymptotic behavior of solutions for a diffusive epidemic model*, Mathematical Methods in the Applied Sciences, 9 pages, 2016.

[37] A. Tripathi, R. Naresh, and D. Sharma, *Modeling the effect of screening of unaware infectives on the spread of HIV infection*, Appl. Math. 184(2007), 1053-1068.

[38] R. Ullah, G. Zaman and S. Islam, *Stability Analysis Of A General SIR Epidemic Model*, VFAST Transactions on Mathematics, Volume 1, Number 1, June 2013, 16-20.

[39] G. F. Webb, *A reaction-diffusion model for a deterministic diffusive epidemic*, Journal of Mathematical Analysis and Applications, vol. 84, no. 1, pp. 150-161, 1981.