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THÈME

Stabilisation des systèmes interconnectés avec retard.

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Dedication

To my mother

To my father

To my brothers and my sisters

To all my family and friends.

To the soul of my teacher " Moumen Youcef "

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Abstract

In this thesis, we have established polynomial or exponential stability results for delayed or undelayed interconnected systems governed by partial differential equations with boundary coupling (wave/wave, Schrödinger/Schrödinger, wave/Schrödinger). Multipliers technique, Lyapunov functionnals and frequency domain approach are used for the proofs.

Keywords: Wave equation, Schrödinger equation, stabilization, boundary feedback, internal feedback, transmission systems, time delay.

Contents

Introduction

. The main aim of this thesis is to estabilish stability results for delayed or undelayed interconnected systems governed by partial differential equations with boundary coupling.
Interconnected systems described by partial differential equations with distributed or boundary connections arise in many physical process such as composite laminates in smart materials and structures ([77],[78]), fuid-structure interactions ([86],[87], [25], [6]) and structural acoustic systems [31].
Kim [36] used energy methods combined with multipliers technique and compactness argument to show the exponential decay of the energy of a thermoelastic bar and plate. Lebeau and Zuazua [42] analyzed the stability and controllability of two and three dimensional system of linear thermoelasticity in a bounded and smooth domain with Dirichlet boundary conditions. Ammari and Nicaise [5] used multipliers technique to estabilish under some geometric condition an exponential stability result for a transmission wave-Kirchoff plate equation where boundary feedback controls are applied to both wave and plate equations. Stability and controllability problems for a heat-wave system which arise in fluid- structure interactions are investigated in [86], [87], [25] and [6]. Guo et al [29] considered a two-dimensional coupled wave-plate system with boundary coupling subject to a distributed dissipative feedback acting either through the plate or through the wave equation. Using frequency domain method combiined with multipliers technique, they showed under some assumptions on the geometry of the spatial domain that the system energy decays exponentially when the damping acts through the wave equation and polynomially when the damping acts on the plate equation.
A Riesz basis approach has been adopted in:

- [77] and [78] to establish analyticity, controllability and stability results for a three layer sandwich beam,
- [85] and [79] to study stability of an interconnected system of Euler-Bernoulli beam or plate and heat equation with boundary coupling.
- [76] to show that the C_0-semigroup generated by the oprtator of the system governed the transmission Schrödinger-heat equation is exponentially stable and of Gevrey class $\delta > 2$,
- [82] to study a stabilization problem for a coupled system of Euler-Bernoulli beam through boundary coupling with an internally damped wave equation.
- [64] to prove the uniform exponential stability of the multilayer Rao-Nakra sandwich beam with boundary damping applied at one end point.

Time-delay appears in many practical systems such as biological and engineering systems ([1], [7], [70]), and it may be a source of instability ([30]). In fact it is well-known that for systems described by partial differential equations such as the wave equation an arbitrarily small delay in the feedback may destabilize the system, see for instance [20], [21], and [57]. Stability analysis of wave type equations with delay has been extensively studied over the past decades, we refer to [4], [8] and the references therein. In the following, we give a brief review of some of the most relevant publications.
Nicaise and Pignotti [57] considered in a bounded domain of $\mathbb{R}^n, n \geq 2$, the wave equation with a discrete time delay term in the boundary or internal dampings. In both cases, they showed that if the coefficient

of the delayed damping term is smaller than the one of the undelayed damping term, then the solution decays exponentially in an appropriate functional space. These results are obtained by proving some observability estimates. In the opposite case, they constructed sequences of delays that destabilize these systems. Similar results were obtained in[60] for the Schrödinger equation with discrete-time delay in the boundary or internal feedbacks. Xu et [83] established similar results for the one-dimensional wave equation via the Riesz basis approach. Ning and Yan [61] and Ning et al [62] used Riemannian geometric energy method approach to extend the results of [57] to the case of the wave equation with variable coefficients in the principal elliptic part. Benseghir [9] proved exponential stability of the solution of the one-dimensional transmission wave equation with internal damping and discrete delay by introducing suitable Lyapunov functionals.

The thesis is organized as follows. In the first chapter, we recall basic properties of C_0-semigroups of linear operators and their applications to abstract Cauchy problems in Hilbert spaces. We also define some stability concepts for abstract Cauchy problems we are interested in and state some of their characterizations.

Chapter two and three are devoted to the transmission Schrdinger equation with a time delay in the Neumann boundary feedback. In chapter two, we consider the case where the delay is time-ivariant. We prove under some assumptions that the solutions decay exponentially in an appropriate energy space. To establish this result, we introduce a suitable energy function and use multipliers technique method and compactness-uniqueness argument.

Chapter three analyzes the case of time-varying delay. By defining appropriate energy and Lyapunov functionnals, we show subject to some conditions that the solution is exponentially stable.

In chapter four, we consider a system of transmission of the wave equation with Neumann feedback control that con- tains a distributed delay term and that acts on the exterior boundary. We prove under some assumptions that the solutions decay exponentially by adopting the approach used in chapter two.

In chapters five and six, we study stabilization problems for the Schrödinger equation coupled by the interface with a wave equation and with boundary or distributed damping. In chapter five we consider the case where the dissipation is acting on the wave equation through the Neumann boundary condition. We formulate the coupled system as an abstract evolution equation in an appropriate Hilbert space and use linear semigroup theory to show the well-posedness of the system. Then under some assumptions on the geometry of the spatial domain, we prove exponential stability of the solution. The proof of this result is based on a frequency domain approach which consists in verifying that the imaginary axis is included in the resolvent set of the system and analyzing the behavior of the resolvent operator of the system on the imaginary axis. The analysis of the resolvent is carried out by combining contradiction argument with the multipliers technique. This result extends Theorem 3.2 in [80] to multimensional spatial domains.

In chapter six, we treat the case where the damping is distributed and it acts either on one of the equations. We prove by using frequency domain method that the system is exponentially stable if the damping acts on the wave equation but only polynomially stable if the damping acts on the Schrödinger equation.

Notations

$\mathbb{N}$	the set of positive integers.
$\mathbb{R}$	the set of real numbers.
$\mathbb{C}$	the set of complex numbers.
$\Re$	real part.
$\Im$	imaginary part.
$\overline{u}$	the conjugate of a complex number u.
$\vert . \vert$	the absolute value for a real number or the modulus of a complex number.
$X^{'}$	dual space of a Hilbert space X.
$\langle ., . \rangle$	inner product.
$\Vert . \Vert$	the norm.
$\mathcal{L}(X)$	space of bounded linear operators from a Hilbert space X into itself X.
Ω	open bounded domain of $\mathbb{R}^n$.
$C^\infty(\Omega)$	the space of infinitely differentiable functions on Ω.
$L^p(\Omega)$	class of Lebesgue measurable complex (or real) -valued functions with $\int_\Omega \vert u(x) \vert^p dx < \infty, 1 \leq p < \infty$.
$L^\infty(\Omega)$	class of bounded measurable functions from Ω to $\mathbb{C}$ or $\mathbb{R}$ with $\vert u(x) \vert \leq Const$ a.e. in Ω.
$W^{k,p}(\Omega)$	Sobolev space of order k.
$C([0,\infty);X)$	class of continuous functions from [0,∞) *to* X .
∇	the gradient operator.
Δ	the Laplace operator.
div	the divergence operator.

$C^1([0,\infty);X)$	class of continuously differentiable functions from $[0,\infty)\, to\, X$
$\partial_t . = \frac{\partial .}{\partial t}$	the first partial derivative
$\partial_t^2 . = \frac{\partial^2 .}{\partial t^2}$	the second partial derivative
$\dfrac{\partial .}{\partial \nu}$	is the normal derivative

Chapter 1

Preliminaries

In this chapter, we recall for later use some well known results from the theory of semigroups of linear operators on Hilbert space and their applications to abstract Cauchy problems in Hilbert spaces. We also define the stability concepts for abstract Cauchy problems in Hilbert spaces we are interested in and provide some of their characterizations.

1.1 Semigroups of continuous linear operators

Let $\mathcal{H}$ be a Hilbert space.

Definition 1.1.1. *A one-parameter family* $S(t) =$ *for* $0 < t < \infty$ *of* $\mathcal{L}(\mathcal{H})$ *is a* C_0 *-(or strongly continuous) semigroup on* $\mathcal{H}$ *if*
1) $S(t+s) = S(t)S(s)$ *for every* $t, s > 0$.
2) $S(0) = I$, *(*I *is the identity operator in* $\mathcal{H}$*).*
3) $\underset{t \longrightarrow 0}{lim} \|S(t)x - x\| = 0$ *for all* $x \in \mathcal{H}$.

Definition 1.1.2. *Let* $S(t)$ *be a* C_0*-semigroup defined on* $\mathcal{H}$*. The infinitesimal generator* A *of* $S(t)$ *is the linear operator defined by*

$$Ax = \underset{x \longrightarrow h}{lim} \frac{S(h)x - x}{h}, \quad for \quad x \in D(A),$$

with $D(A) = \{x \in \mathcal{H}, \underset{x \longrightarrow h}{lim} \dfrac{S(h)x - x}{h}, \quad exists\ in \mathcal{H}\}$.

Theorem 1.1.3. *(Engel and Nagel [24]) Let* $S(t)$ *be a* C_0*-semigroup on* $\mathcal{H}$*. There exist constants* $\omega \in \mathbb{R}$ *and* $M \geq 1$ *such that the following holds:*

$$\|S(t)\| \leq M\, e^{\omega t}.$$

If $\omega = 0$ *and* $M = 1$, *then* $S(t)$ *is called a* C_0*-semigroup of contraction.*

Theorem 1.1.4. *(Lumer-Phillips)([65]) A linear operator* $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ *generates a strongly continuous semigroup of contractions* $(S(t))_{t \geq 0}$ *on* $\mathcal{H}$ *if and only if* A *is maximal dissipative, i.e.,it satisfies*

- $\Re\langle Ax, x\rangle_{\mathcal{H}} \leq 0, \ ,\forall x \in D(\mathcal{A})$,
- $\lambda I - A$ *is onto for some (hence all)* $\lambda > 0$.

1.2 Abstract Cauchy problems

Let $\mathcal{H}$ be a Hilbert space and let $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ be a linear operator. Consider the homogeneous Cauchy problem

$$\begin{cases} \frac{du}{dt}(t) = A\,u(t), & t \geq 0, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

where $u_0 \in \mathcal{H}$.

Definition 1.2.1. *A function $u : [0, T] \longrightarrow \mathcal{H}$ is a strong solution of (1.1) on $[0, T]$ if u is continuously differentiable on $[0, T]$ and for all $t \in [0, T]$ $u(t) \in D(A)$ and satisfies (1.1).*

Theorem 1.2.2. *(Curtain and Zwart [19]) If A is the infinitesimal generator of a C_0 semigroup $S(t)$ on $\mathcal{H}$, then for all $u_0 \in D(A)$, the abstract Cauchy problem (1.1) has a unique strong solution given by*

$$u(t) = S(t)u_0, \qquad t \geq 0. \tag{1.2}$$

Definition 1.2.3. *A function $u \in C([0, T]; \mathcal{H})$ is a weak solution of (1.1) on $[0, T]$ if for every $y \in D(A^*)$ where A^* is the adjoint of A, the function $(u(t), y)$ is absolutely continuous on $[0, T]$ and*

$$\frac{d}{dt}\langle u(t), y\rangle = (u(t), A^* y)\, a.e.\ on\, [0, T].$$

Theorem 1.2.4. *(Curtain and Zwart [19]) If A is the infinitesimal generator of a C_0 semigroup $S(t)$ on $\mathcal{H}$, then for every $u_0 \in \mathcal{H}$, the problem (1.1) has unique weak solution given by (1.2).*

1.3 Stability concepts

Consider in a Hilbert space $\mathcal{H}$, the differential equation

$$\frac{du}{dt}(t) = Au(t), \quad t \geq 0, \tag{1.3}$$

where $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is the infinitesimal generator of a C_0 semigroup $S(t)$ on $\mathcal{H}$.
Many concepts of stability have been defined for systems described by (1.3), and we are interested in the following:

Definition 1.3.1. *The system (1.3)is said to be*

- *uniformly exponentially stable, if there exist constants $\delta > 0, M > 0$ such that*

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-\delta t}, \quad \forall t \geq 0,$$

- *polynomially stable if there exist constants $\alpha > 0, M > 0$ such that*

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq \frac{M}{t^\alpha}, \quad \forall t \geq 0.$$

We have the following results

Proposition 1.3.2. *(Engel and Nagel [24]) For a linear C_0- semigroup $(S(t))_{t\geq 0}$, the following assertions are equivalent.*

- *$(S(t))_{t\geq 0}$ is uniformly exponentially stable.*
- *There exists $T > 0$ such that $\|S(T)\| < 1$.*

Definition 1.3.3. *Let A be a closed linear operator on a(complex) normed linear space $\mathcal{H}$. We say that λ is in the resolvent set $\rho(A)$ of A, if $(\lambda I - A)^{-1}$ exists and is a bounded linear operator on a dense domain of $\mathcal{H}$.*

Theorem 1.3.4 and Theorem 1.3.5 caracterize the exponential or polynomial stability of C_0-semigroups in terms of the hehavior of the resolvents of the generators.

Theorem 1.3.4. *([66], [52]) Let $S(t)$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$ with infinitesimal generator A. Then there exists $M \geq 1$ and $\omega > 0$ such that*

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-\omega t} \textit{ for all } t > 0$$

if and only if

$$\{i\beta : \beta \in \mathbb{R}\} \subset \rho(A) \tag{1.4}$$

and

$$\underset{|\beta|\to+\infty}{lim} \sup \left\|(i\beta I - A)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < +\infty \tag{1.5}$$

Theorem 1.3.5. *([10]) Let $S(t)$ be as in Theorem 1.3.4. Then there exist $C > 0$ and $\theta > 0$ such that,*

$$\|S(t)u_0\|_{\mathcal{H}} \leq \frac{C}{t^{\frac{1}{\theta}}} \|u_0\|_{D(A)} \textit{ for all } t > 0.$$

if and only if (1.4) is satisfied and

$$\underset{|\beta|\to+\infty}{lim} \sup \frac{1}{\beta^{\theta}} \left\|(i\beta I - A)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < +\infty \tag{1.6}$$

The following result due to Munoz Rivera and Racke [56] provides sufficient conditions for a C_0-semigroup to be nonexponentially stable.

Theorem 1.3.6. *Let $\mathcal{H}_1$ be a Hilbert space and $\mathcal{H}_2$ a closed subspace of $\mathcal{H}_1$. Let $S_j = (S_j(t))_{t\geq 0}$ be a C_0-semigroup on $\mathcal{H}_j, j = 1, 2$.*
Let $r_{ess}(S_2(t))$ denote the essential spectral radius of $(S_2(t))$ defined by

$$r_{ees}(S_2(t)) = \sup\{|\lambda| : \lambda \in \mathbb{C}; S_2(t) - \lambda I_{\mathcal{H}_2} \textit{ is not a Fredholm operator}\} \tag{1.7}$$

Assume that there exist $t_0 > 0$ such that for $t \geq t_0$, we have

$$\begin{aligned} &(i) r_{ees}(S_2(t)) \geq 1, \\ &(ii) S_1(t) - S_2(t) : \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \textit{ is compact} \end{aligned}$$

Then S_1 is not exponentially stable.

Chapter 2

Stability of the transmission Schrödinger equation with a delay term in the boundary feedback

2.1 Introduction and statement of the main result

The aim of this chapter is to present a stability result for the transmission Schrödinger equation with a discrete time delay term in the boundary feedback.
Let Ω be an open bounded domain of $\mathbb{R}^n$ with a boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. Let Γ_0 with $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \overline{\Gamma}_0 \cap \overline{\Gamma}_2 = \emptyset$ be a regular hypersurface of class C^2 which separates Ω into two domains Ω_1 and Ω_2 such that $\Gamma_1 \subset \partial\Omega_1$ and $\Gamma_2 \subset \partial\Omega_2$. Furthermore, we assume that there exists a real vector field $h \in (C^2(\overline{\Omega}))^n$ such that:
$(H.1)$ The Jacobian matrix J of h satisfies

$$\Re \int_\Omega J(x)\zeta(x).\overline{\zeta}(x)d\Omega \geq \alpha \int_\Omega |\zeta(x)|^2 \, d\Omega,$$

for some constant $\alpha > 0$ and for all $\zeta \in L^2(\Omega; \mathbb{C}^n)$;
$(H.2)$ $h(x).\nu(x) \leq 0$ on Γ_1.
$(H.3)$ $(a_1 - a_2)h(x).\nu(x) \geq 0$ on Γ_0.
where ν is the unit normal on Γ or Γ_0 pointing towards the exterior of Ω or Ω_1.

Let $a_1, a_2 > 0$ be given. Consider the system of transmission of the Schrödinger equation with a delay term in the boundary conditions:

$$\begin{aligned}
&\partial_t y_k(x,t) - i a_k \Delta y_k(x,t) = 0 && \text{in } \Omega_k \times (0,+\infty), k = 1,2, && (2.1)\\
&y_k(x,0) = y_{0k}(x) && \text{in } \Omega_k, k = 1,2, && (2.2)\\
&y_1(x,t) = 0 && \text{on } \Gamma_1 \times (0,+\infty), && (2.3)\\
&\frac{\partial y_2(x,t)}{\partial \nu} = -\alpha \partial_t y_2(x,t) - \beta \partial_t y_2(x,t-\tau) && \text{on } \Gamma_2 \times (0,+\infty), && (2.4)\\
&y_1(x,t) = y_2(x,t), && \text{on } \Gamma_0 \times (0,+\infty), && (2.5)\\
&a_1 \frac{\partial y_1(x,t)}{\partial \nu} = a_2 \frac{\partial y_2(x,t)}{\partial \nu} && \text{on } \Gamma_0 \times (0,+\infty), && (2.6)\\
&\partial_t y_2(x,t-\tau) = f(x,t-\tau) && \text{on } \Gamma_2 \times (0,\tau). && (2.7)
\end{aligned}$$

where:

- α and β are positive constants,
- τ is the time-delay,
- y^0, f are the initial data which belong to suitable spaces.

There has been extensive work on stabilization problems for the Schrödinger equation. Machtyngier and Zuazua [53] adopted the multipliers method to prove an exponential decay result in the energy space $H^1(\Omega)$ by means of a Neumann feedback control involving the velocity of the solution. Lasiecka et al [40] used $L^2(\Omega)$- Carleman estimates for the general linear Schrödinger equation to provide a stabilization result in the energy space $L^2(\Omega)$ with a Neumann dissipation involving the solution. For other stability results related to Schrödinger types equations see [3], [37], [41], [63], [14], [15] and the references therein.

Stabilization problems for the Schrödinger equation with time delay have also been studied and many nice results have been obtained, see for example [27], [59], [84], [17], [18], [16] and [28] among others. We state in particular the reference [59], in which the authors considered the multi-dimensional Schrödinger equation with a discrete time delay term in the boundary or internal feedbacks. In both cases, they showed that if the coefficient of the delayed feedback term is smaller than the one of the undelayed damping term, then the solution decays exponentially in an appropriate functional space. These results are obtained by proving some observability estimates. In the opposite case, they constructed a sequence of delays that destabilize these systems. Regarding the transmission Schrödinger equation, the only references we are aware of are [13] and [2]. In [13], the authors proved exponential decay of the energy of the solutions under linear boundary dissipation in the Neumann boundary condition by adopting a frequency domain approach which is based upon a resolvent criterion. Reference [2] gives a uniform stabilization result with a dissipative feedback acting in the Dirichlet boundary condition by establishing exact controllability of the corresponding open-loop system.

In this chapter, we use multipliers technique method and compactness-uniqueness arguments to prove that solutions of $(2.1)-(2.7)$ decay exponentially in an appropriate energy space. To this aim, assume as in [59] that in this chapter, we use multipliers technique method and compactness-uniqueness arguments to prove the solutions of $(2.1)-(2.7)$ decay exponentially in an appropriate energy space. To this aim, assume as in [60] that

$$\alpha > \beta \tag{2.8}$$

and define the energy of a solution

$$y(x,t) = \begin{cases} y_1(x,t), & (x,t) \in \Omega_1 \times (0,+\infty) \\ y_2(x,t), & (x,t) \in \Omega_2 \times (0,+\infty) \end{cases}$$

of $(2.1)-(2.7)$ by

$$E(t) = \frac{a_1}{2}\int_{\Omega_1} |\nabla y_1(x,t)|^2 \, dx + \frac{a_2}{2}\int_{\Omega_2} |\nabla y_2(x,t)|^2 \, dx + \frac{\xi}{2}\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x,t-\rho s)|^2 \, d\rho \, d\Gamma \tag{2.9}$$

where

$$a_2\tau\beta < \xi < a_2\tau(2\alpha-\beta) \tag{2.10}$$

The main result of this chapter can be stated as follows.

Theorem 2.1.1. *In addition to* $(H.1)$, $(H.2)$, $(H.3)$ *and* (2.8). *Then there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq Me^{-\omega t}E(0)$$

Theorem 2.1.1 is proved in Section 3. In Section 2, we investigate the well-posedness of system $(2.1)-(2.7)$ using semigroup theory.

2.2 Well-posedness

Inspired from [57], we introduce the auxiliary variable

$$z(x,\rho,t) = \partial_t y(x, t-\tau\rho). \tag{2.11}$$

With this new unknown, problem (2.1)-(2.7) is equivalent to

$$\begin{aligned}
&\partial_t y_k(x,t) - ia_k\Delta y_k(x,t) = 0 && \text{in } \Omega_k \times (0,+\infty), k=1,2, && (2.12)\\
&y_k(x,0) = y_{0k}(x) && \text{in } \Omega_k, k=1,2, && (2.13)\\
&y_1(x,t) = 0 && \text{on } \Gamma_1 \times (0,+\infty), && (2.14)\\
&\partial_t z(x,\rho,t) + \frac{1}{\tau}\partial_\rho z(x,\rho,t) = 0 && \text{on } \Gamma_2 \times (0,1)\times(0,+\infty) && (2.15)\\
&\frac{\partial y_2(x,t)}{\partial\nu} = -i\alpha a_2\Delta y_2(x,t) - \beta z(x,1,t) && \text{on } \Gamma_2 \times (0,+\infty), && (2.16)\\
&y_1(x,t) = y_2(x,t) && \text{on } \Gamma_0 \times (0,+\infty), && (2.17)\\
&a_1\frac{\partial y_1(x,t)}{\partial\nu} = a_2\frac{\partial y_2(x,t)}{\partial\nu} && \text{on } \Gamma_0 \times (0,+\infty), && (2.18)\\
&z(x,0,t) = \partial_t y_2(x,t) && \text{on } \Gamma_2 \times (0,+\infty), && (2.19)\\
&z(x,\rho,0) = f_0(x,\rho,s) && \text{on } \Gamma_2 \times (0,1). && (2.20)
\end{aligned}$$

Let

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\} \tag{2.21}$$

and

$$\mathcal{V} = \{(u_1,u_2) \in H^1_{\Gamma_1}(\Omega_1) \times H^1(\Omega_2); u_1 = u_2 \text{ on } \Gamma_0\}$$

The space for well-posedness of (2.12)-(2.20) is taken to be the space

$$\mathcal{H} = \mathcal{V} \times L^2(\Gamma_2; L^2(0,1))$$

$\mathcal{H}$ is a Hilbert space with the following norm inducing inner product

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \\ z \end{pmatrix}; \begin{pmatrix} \widetilde{u}_1 \\ \widetilde{u}_2 \\ \widetilde{z} \end{pmatrix} \right\rangle = a_1\int_{\Omega_2} \nabla u_1(x).\nabla\widetilde{u}_1(x)dx + a_2\int_{\Omega_2}\nabla u_2(x).\nabla\widetilde{u}_2(x)dx + \xi\int_{\Gamma_2}\int_0^1 z(x,\rho)\widetilde{z}(x,\rho)d\rho d\Gamma$$

In $\mathcal{H}$, define a linear operator A by

$$A(u_1,u_2,z)^T = (ia_1\Delta u_1, ia_2\Delta u_2, -\tau^{-1}\partial_\rho z)^T, \tag{2.22}$$

$$\begin{aligned}
D(A) \quad = \quad & \{(u_1,u_2,z)^T \in \mathcal{V} \times L^2(\Gamma_2; H^1(0,1)); \Delta u_1 \in H^1_{\Gamma_1}(\Omega_1), \Delta u_2 \in H^1(\Omega_2),\\
& z(.,0) = ia_2\Delta u_2, \text{ satisfying (2.24), (2.25) and (2.27) below}\} \qquad (2.23)
\end{aligned}$$

$$\begin{aligned}
a_1\frac{\partial u_1}{\partial\nu} &= a_2\frac{\partial u_2}{\partial\nu} \text{ on } \Gamma_0, && (2.24)\\
a_1\Delta u_1 &= a_2\Delta u_2 \text{ on } \Gamma_0, && (2.25)\\
z(.,0) &= ia_2\Delta u_2 \text{ on } \Gamma_2, && (2.26)\\
\frac{\partial u_2}{\partial\nu} &= -\alpha z(.,0) - \beta z(.,1), \text{ on } \Gamma_2. && (2.27)
\end{aligned}$$

Note that for $(u_1,u_2,z) \in D(A)$, we have the following boundary regularity:

- $\Delta u_k|_{\partial\Omega_k} \in H^{1/2}(\partial\Omega_k), k = 1, 2,$ (trace theorem),
- $\frac{\partial u_1}{\partial\nu}|_{\partial\Omega_1} \in H^{-1/2}(\partial\Omega_1)$, $\frac{\partial u_2}{\partial\nu}|_{\partial\Gamma_0} \in H^{-1/2}(\Gamma_0)$ (see e.g., [35] p. 71, Theorem. 3.8.1]),
- $\frac{\partial u_2}{\partial\nu}|_{\Gamma_2} \in L^2(\Gamma_2)$ since $z \in L^2(\Gamma_2)$.

Using the operator A, we rewrite (2.12) − (2.20) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}Y(t) = AY(t) \\ Y(0) = Y_0 \end{cases} \tag{2.28}$$

where

$$Y(t) = (y, z)^T \text{ and } Y^0 = (y^0(x), f(x, \rho))^T$$

Proposition 2.2.1. *The operator A defined by (2.22) and (2.23) generates a strongly continuous semigroup on $\mathcal{H}$. Thus, for every $Y_0 \in \mathcal{H}$, problem (2.28) has a unique solution Y whose regularity depends on the initial datum Y_0 as follows:*

$$\begin{array}{rcl} Y(.) & \in & C([0, +\infty); \mathcal{H}) \text{ if } Y_0 \in \mathcal{H}, \\ Y(.) & \in & C([0, +\infty); D(A)) \cap C^1([0, +\infty); \mathcal{H}) \text{ if } Y_0 \in D(A). \end{array}$$

Proof. Let $Y = \begin{pmatrix} u_1 \\ u_2 \\ z \end{pmatrix} \in D(A)$. Then

$$\begin{aligned} \Re\langle Y, AY\rangle \;=\; & -\Re\{a_1^2 i\int_{\Omega_1}\nabla u_1(x).\nabla(\Delta\overline{u}_1(x))dx\} - \Re\{a_2^2 i\int_{\Omega_2}\nabla u_2(x).\nabla(\Delta\overline{u}_2(x))dx\} - \\ & \Re\frac{\xi}{\tau}\int_{\Gamma_2}\int_0^1 z(x,\rho)\partial_\rho\overline{z}(x,\rho,s)d\rho ds d\Gamma \end{aligned} \tag{2.29}$$

Applying Green's theorem to the first two integrals on the right-hand side of (2.29) and using the fact that the normal vector on Γ_0 is oriented towards the interior of Ω_2, we obtain

$$\begin{aligned} & -\Re\{a_1^2 i\int_{\Omega_1}\nabla u_1(x).\nabla(\Delta\overline{u}_1(x))dx\} - \Re\{a_2^2 i\int_{\Omega_2}\nabla u_2(x).\nabla(\Delta\overline{u}_2(x))dx\} = - \\ & \Re\{a_1^2 i\int_{\Gamma_1}\frac{\partial u_1(x)}{\partial\nu}\Delta\overline{u}_1(x)d\Gamma + a_1^2 i\int_{\Gamma_0}\frac{\partial u_1(x)}{\partial\nu}\Delta\overline{u}_1(x)d\Gamma - a_1^2 i\int_{\Omega_1}|\Delta u_1(x)|^2\,dx\,\} - \\ & \Re\{a_2^2 i\int_{\Gamma_2}\frac{\partial u_2(x)}{\partial\nu}\Delta\overline{u}_2(x)d\Gamma - a_2^2 i\int_{\Gamma_0}\frac{\partial u_2(x)}{\partial\nu}\Delta\overline{u}_2(x)d\Gamma - a_2^2 i\int_{\Omega_2}|\Delta u_2(x)|^2\,dx\} \end{aligned} \tag{2.30}$$

Note that the integrals over Γ_1 (resp.Γ_0) on the right-hand side of (2.30) are to be interpreted in the sense of duality pairing between $H^{1/2}(\Gamma_1)$ and $H^{-1/2}(\Gamma_1)$ (resp. $H^{1/2}(\Gamma_0)$ and $H^{-1/2}(\Gamma_0)$)
(2.30) together with (2.23) − (2.27) yields

$$\begin{aligned} & -\Re\{a_1^2 i\int_{\Omega_1}\nabla u_1(x).\nabla(\Delta\overline{u}_1(x))dx\} - \Re\{a_2^2 i\int_{\Omega_2}\nabla u_2(x).\nabla(\Delta\overline{u}_2(x))dx\} = - \\ & a_2\alpha\int_{\Gamma_2}|z(x,0)|^2\,d\Gamma - a_2\beta\Re\int_{\Gamma_2}z(x,1)\overline{z}(x,0)d\Gamma \end{aligned} \tag{2.31}$$

Integrating by parts in ρ the third integral on the right-hand side of (2.29), we get

$$\Re\int_{\Gamma_2}\int_0^1 z(x,\rho)\partial_\rho\overline{z}(x,\rho)d\rho d\Gamma = \frac{1}{2}\int_{\Gamma_2}\{|z(x,1)|^2 - |z(x,0)|^2\}d\Gamma \tag{2.32}$$

Inserting (2.31) and (2.32) into (2.29) results in

$$\Re\langle Y, AY\rangle = -a_2\alpha\int_{\Gamma_2}|z(x,0)|^2\,d\Gamma - a_2\beta\Re\int_{\Gamma_2} z(x,1)\overline{z}(x,0)d\Gamma -$$
$$\frac{\xi}{2\tau}\int_{\Gamma_2}\{|z(x,1)|^2 - |z(x,0)|^2\}d\Gamma$$

from which follows after using the Cauchy-Schwarz inequality

$$\Re\langle Y, AY\rangle \le -(a_2\alpha - \frac{\xi}{2\tau} - \frac{a_2\beta}{2})\int_{\Gamma_2}|z(x,0)|^2\,d\Gamma - (\frac{\xi}{2\tau} - \frac{a_2\beta}{2})\int_{\Gamma_2}|z(x,1)|^2\,d\Gamma \tag{2.33}$$

(2.33) together with (2.10) implies that

$$\Re\langle AY, Y\rangle \le 0$$

Thus A is dissipative.
Now we show that $\lambda I - A$ is onto for some $\lambda > 0$. Given $(f_1, f_2, g)^T \in \mathcal{H}$, we seek $Y = (u_1, u_2, z)^T \in D(A)$ such that

$$(\lambda I - A)Y = (f, g)^T \tag{2.34}$$

or equivalently

$$\begin{aligned}
&\lambda u_k(x) - ia_k\Delta u_k(x) = f_k(x) && \text{in } \Omega_k, k = 1,2, && (2.35)\\
&\lambda z(x,\rho) + \tau^{-1}\partial_\rho z(x,\rho) = g(x) && \text{on } \Gamma_2\times(0,1), && (2.36)\\
&u_1(x) = 0 && \text{on } \Gamma_1, && (2.37)\\
&\frac{\partial u_2(x)}{\partial\nu} = -\alpha z(x,1) - \beta z(x,0) && \text{on } \Gamma_2, && (2.38)\\
&u_1(x) = u_2(x) && \text{on } \Gamma_0, && (2.39)\\
&a_1\frac{\partial u_1(x)}{\partial\nu} = a_2\frac{\partial u_2(x)}{\partial\nu} && \text{on } \Gamma_0, && (2.40)\\
&a_1\Delta u_1(x) = a_2\Delta u_2(x) && \text{on } \Gamma_0. && (2.41)
\end{aligned}$$

Suppose that we have found (u_1, u_2) with the appropriate regularity, then we can determine z. Indeed, from (2.36) and (2.26) we have

$$\begin{cases} \partial_\rho z(x,\rho) = -\lambda\tau z(x,\rho) + \tau g(x,\rho), \\ z(x,0) = ia_2\Delta u_2(x), \end{cases}$$

The unique solution of the above initial value problem is given by

$$z(x,\rho) = ia_2 e^{-\lambda\tau\rho}\Delta u_2(x) + \tau e^{-\lambda\tau\rho}\int_0^\rho e^{\lambda\tau s}g(x,s)ds$$

and in particular

$$z(x,1) = ia_2 e^{-\lambda\tau}\Delta u_2(x) + \tau e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}g(x,s)ds, \quad x\in\Gamma_2$$

and

$$\frac{\partial u_2(x)}{\partial\nu} = -ia_2(\alpha + \beta e^{-\lambda\tau})\Delta u_2(x) - \beta\tau e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}g(x,s)ds \quad \text{for } x\in\Gamma_2, \tag{2.42}$$

From (2.35), we have

$$\begin{aligned}
&\lambda u_1(x) - ia_1\Delta u_1(x) = f_1(x), && x\in\Omega_1, && (2.43)\\
&\lambda u_2(x) - ia_2\Delta u_2(x) = f_2(x), && x\in\Omega_2, && (2.44)
\end{aligned}$$

Let $(\varphi_1, \varphi_2) \in \mathcal{V}$. Then, multiplying (2.43) (resp. (2.44) by φ_1 (resp. by φ_2) and integrating formally in Ω_1(resp. in Ω_2), we obtain after using (2.23) and (2.42)

$$\begin{aligned}&\lambda a_1 \int_{\Omega_1} \nabla u_1(x).\nabla\overline{\varphi}_1(x)dx - ia_1^2 \int_{\Gamma_0} \Delta u_1(x).\frac{\partial\overline{\varphi}_1(x)}{\partial\nu}d\Gamma + ia_1^2\int_{\Omega_1}\Delta u_1(x)\Delta\overline{\varphi}_1(x)d\Gamma = \\ &a_1\int_{\Omega_1}\nabla f_1(x).\nabla\overline{\varphi}_1(x)dx \end{aligned} \tag{2.45}$$

$$\begin{aligned}&\lambda a_2 \int_{\Omega_2} \nabla u_2(x).\nabla\overline{\varphi}_2(x)dx + \frac{a_2 e^{\lambda\tau}}{\alpha e^{\lambda\tau}+\beta}\int_{\Gamma_2}\frac{\partial u_2(x)}{\partial\nu}\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma + ia_2^2\int_{\Gamma_0}\Delta u_2(x).\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma + \\ &ia_2^2\int_{\Omega_2}\Delta u_2(x)\Delta\overline{\varphi}_2(x)d\Gamma = a_2\int_{\Omega_2}\nabla f_2(x).\nabla\overline{\varphi}_2(x)dx - \frac{a_2\beta\tau}{e^{\lambda\tau}\alpha+\beta}\int_{\Gamma_2}\int_0^1 e^{\lambda\tau s}g(x,s)\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}dsd\Gamma\end{aligned} \tag{2.46}$$

Summing up (2.45) and (2.46) yields

$$\Lambda((u_1,u_2),(\varphi_1,\varphi_2)) = \mathcal{F}(\varphi_1,\varphi_2) \tag{2.47}$$

where

$$\begin{aligned}&\Lambda((u_1,u_2),(\varphi_1,\varphi_2)) = \lambda a_1\int_{\Omega_1}\nabla u_1(x).\nabla\overline{\varphi}_1(x)dx + \lambda a_2\int_{\Omega_2}\nabla u_2(x).\nabla\overline{\varphi}_2(x)dx + \\ &ia_1^2\int_{\Omega_1}\Delta u_1(x)\Delta\overline{\varphi}_1(x)d\Gamma + ia_2^2\int_{\Omega_2}\Delta u_2(x)\Delta\overline{\varphi}_2(x)d\Gamma + \frac{a_2 e^{\lambda\tau}}{\alpha e^{\lambda\tau}+\beta}\int_{\Gamma_2}\frac{\partial u_2(x)}{\partial\nu}\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma - \\ &ia_1^2\int_{\Gamma_0}\Delta u_1(x).\frac{\partial\overline{\varphi}_1(x)}{\partial\nu}d\Gamma + ia_2^2\int_{\Gamma_0}\Delta u_2(x).\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma\end{aligned} \tag{2.48}$$

and $\mathcal{F}:\mathcal{V}\to\mathbb{C}$ is the linear form defined by

$$\mathcal{F}(\varphi_1,\varphi_2) = a_1\int_{\Omega_1}\nabla f_1(x).\nabla\overline{\varphi}_1(x)dx + a_2\int_{\Omega_2}\nabla f_2(x).\nabla\overline{\varphi}_2(x)dx - \frac{a_2\beta\tau}{e^{\lambda\tau}\alpha+\beta}\int_{\Gamma_2}\int_0^1 e^{\lambda\tau s}g(x,s)\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}dsd\Gamma$$

We note that the bilinear form Λ is not continuous on $\mathcal{V}$ neither in $\mathcal{F}$. To overcome this difficulty, we adapt an idea of [14]. We introduce the space

$$\mathcal{Z} = \{(\varphi_1,\varphi_2)\in\mathcal{V} : \Delta\varphi_k \in L^2(\Omega_k), k=1,2, a_1\frac{\partial\varphi_1}{\partial\nu} = a_2\frac{\partial\varphi_2}{\partial\nu} \text{ on } \Gamma_0, \frac{\partial\varphi_2}{\partial\nu}\in L^2(\Gamma_2)\}$$

on which we define the inner product

$$\begin{gathered}\langle(\varphi_1,\varphi_2),(\psi_1,\psi_2)\rangle = a_1\int_{\Omega_1}\nabla\varphi_1(x).\nabla\overline{\psi}_1(x)dx + a_2\int_{\Omega_2}\nabla\varphi_2(x).\nabla\overline{\psi}_2(x)dx + \\ ia_1^2\int_{\Omega_1}\Delta\varphi_1(x)\Delta\overline{\psi}_1(x)dx + ia_2^2\int_{\Omega_2}\Delta\varphi_2(x)\Delta\overline{\psi}_2(x)dx + \frac{a_2e^{\lambda\tau}}{\alpha e^{\lambda\tau}+\beta}\int_{\Gamma_2}\frac{\partial\varphi_2(x)}{\partial\nu}\frac{\partial\overline{\psi}_2(x)}{\partial\nu}d\Gamma\end{gathered}$$

Then $\mathcal{Z}$ is a Hilbert space.
Applying Cauchy-Schwarz inequality to each inner product on the right-hand side of (2.48), we obtain

$$\begin{aligned}&|\Lambda((u_1,u_2),(\varphi_1,\varphi_2))| \leq \lambda a_1\left\|\nabla u_1\right\|_{L^2(\Omega_1)}\left\|\nabla\varphi_1\right\|_{L^2(\Omega_1)} + \lambda a_2\left\|\nabla u_2\right\|_{L^2(\Omega_2)}\left\|\nabla\varphi_2\right\|_{L^2(\Omega_2)} + \\ &a_1^2\left\|\Delta u_1\right\|_{L^2(\Omega_1)}\left\|\Delta\varphi_1\right\|_{L^2(\Omega_1)} + a_2^2\left\|\Delta u_2\right\|_{L^2(\Omega_2)}\left\|\Delta\varphi_2\right\|_{L^2(\Omega_2)} + \frac{a_2\beta\tau}{e^{\lambda\tau}\alpha+\beta}\left\|\frac{\partial u_2}{\partial\nu}\right\|_{L^2(\Gamma_2)}\left\|\frac{\partial\overline{\varphi}_2}{\partial\nu}\right\|_{L^2(\Gamma_2)}\end{aligned} \tag{2.49}$$

(2.49) implies $\Lambda(.,.)$ is continuous on $\mathcal{Z}$.

For the coercivity of Λ, observe that

$$\Lambda((u_1,u_2),(u_1,u_2)) = \lambda a_1 \left\|\nabla u_1\right\|^2_{L^2(\Omega_1)} + \lambda a_2 \left\|\nabla u_2\right\|^2_{L^2(\Omega_2)} + i a_1^2 \left\|\Delta u_1\right\|^2_{L^2(\Omega_1)} + i a_2^2 \left\|\Delta u_2\right\|^2_{L^2(\Omega_2)} + \frac{a_2\beta\tau}{e^{\lambda\tau}\alpha+\beta}\left\|\frac{\partial u_2}{\partial\nu}\right\|^2_{L^2(\Gamma_2)}$$

Hence

$$\begin{aligned}\Lambda((u_1,u_2),(u_1,u_2)) \quad \geq \quad & \frac{1}{2}\{\lambda a_1 \left\|\nabla u_1\right\|^2_{L^2(\Omega_1)} + \lambda a_2 \left\|\nabla u_2\right\|^2_{L^2(\Omega_2)} + \frac{a_2\beta\tau}{e^{\lambda\tau}\alpha+\beta}\left\|\frac{\partial u_2}{\partial\nu}\right\|^2_{L^2(\Gamma_2)}\} + \\ & \frac{1}{2}\{a_1^2 \left\|\Delta u_1\right\|^2_{L^2(\Omega_1)} + a_2^2 \left\|\Delta u_2\right\|^2_{L^2(\Omega_2)}\}\end{aligned}$$

$\mathcal{F}$ is also continue on $\mathcal{Z}$. Therefore, we conclude from the Lax-Millgram Theorem (see [71], p. 344) that for all $\mathcal{F}\in\mathcal{Z}'$, where $\mathcal{Z}'$ is the dual of $\mathcal{Z}$, there exists a unique solution $(u_1,u_2)\in\mathcal{Z}$ to (2.47) for all $(\varphi_1,\varphi_2)\in\mathcal{Z}$. Since $\mathcal{V}'\subset\mathcal{Z}'$, then for all $\mathcal{F}\in\mathcal{V}'$, there exists a unique solution $(u_1,u_2)\in\mathcal{Z}$ to (2.47) for all $(\varphi_1,\varphi_2)\in\mathcal{Z}$.

Moreover, by restricting the variational forms (2.45) (resp. (2.46)) to functions for which $\frac{\partial\varphi_1}{\partial\nu}=0$ (resp. $\frac{\partial\varphi_2}{\partial\nu}=0$), we obtain

$$\lambda u_1(x) - i a_1 \Delta u_1(x) = f_1(x), \qquad x\in\Omega_1, \tag{2.50}$$

$$\lambda u_2(x) - i a_2 \Delta u_2(x) = f_2(x), \qquad x\in\Omega_2, \tag{2.51}$$

from which we deduce that $(\Delta u_1,\Delta u_2)\in\mathcal{V}$ since $(u_1,u_2)\in\mathcal{V}$ and $(f_1,f_2)\in\mathcal{V}$.

We return to the variational form (2.47) after using some integrations by parts:

$$\begin{aligned}&\lambda a_1\int_{\Omega_1}\nabla u_1(x).\nabla\overline{\varphi}_1(x)dx - i a_1^2\int_{\Omega_1}\nabla\Delta u_1(x).\nabla\overline{\varphi}_1(x)dx + \lambda a_2\int_{\Omega_1}\nabla u_2(x).\nabla\overline{\varphi}_2(x)dx - \\ & i a_2^2\int_{\Omega_2}\nabla\Delta u_2(x).\nabla\overline{\varphi}_2(x)dx + i a_2^2\int_{\Gamma_2}\Delta u_2(x)\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma + \frac{a_2 e^{\lambda\tau}}{\alpha e^{\lambda\tau}+\beta}\int_{\Gamma_2}\frac{\partial u_2(x)}{\partial\nu}\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma = \\ & a_1\int_{\Omega_1}\nabla f_1(x).\nabla\overline{\varphi}_1(x)dx + a_2\int_{\Omega_2}\nabla f_2(x).\nabla\overline{\varphi}_2(x)dx - \frac{a_2\beta\tau}{e^{\lambda\tau}\alpha+\beta}\int_{\Gamma_2}\int_0^1 e^{\lambda\tau s}g(x,s)\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}dsd\Gamma\end{aligned} \tag{2.52}$$

(2.52) together with (2.50) and (2.51), yields

$$\frac{a_2 e^{\lambda\tau}}{\alpha e^{\lambda\tau}+\beta}\int_{\Gamma_2}\frac{\partial u_2(x)}{\partial\nu}\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma = -\int_{\Gamma_2}\{i a_2^2\Delta u_2(x) - \frac{a_2\beta\tau}{e^{\lambda\tau}\alpha+\beta}\int_0^1 e^{\lambda\tau s}g(x,s)ds\}\frac{\partial\overline{\varphi}_2(x)}{\partial\nu}d\Gamma \tag{2.53}$$

(2.53) implies that

$$\frac{\partial u_2(x)}{\partial\nu} = -i a_2(\alpha+\beta e^{-\lambda\tau})\Delta u_2(x) - \beta\tau e^{-\lambda\tau}\int_0^1 e^{\lambda\tau s}g(x,s)ds \qquad \text{for } x\in\Gamma_2,$$

as desired and consequently $(u_1,u_2)\in D(A)$. Thus, by the Lumer-Phillips Theorem (see for instance [65], Theorem 1.4.3), A generates a strongly continuous semigroup of contractions on $\mathcal{H}$. □

2.3 Proof of the main result

We prove Theorem 2.1.1 for smooth initial data. The general case follows by a standard density argument. First, we show that the energy function defined by (2.9) is decreasing.

Proposition 2.3.1. *The energy corresponding to any regular solution of problem (2.1)-(2.7) is decreasing and there exists a positive constant* K *such that*

$$\frac{d}{dt}E(t) \leq -K\int_{\Gamma_2}\left\{|\partial_t y_2(x,t)|^2 + \int_{\tau_1}^{\tau_2} |\partial_t y_2(x,t-s)|^2\, ds\right\} d\Gamma \tag{2.54}$$

where

$$K = \min\{a_2\alpha - \frac{a_2\beta}{2} - \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{a_2\beta}{2}\}$$

Proof. Differentiating $E(t)$ with respect to time, we obtain

$$\frac{d}{dt}E(t) = a_1\Re\int_{\Omega_1} \nabla\overline{y}_1(x,t).\nabla\partial_t y_1(x,t)dx + a_2\Re\int_{\Omega_2} \nabla\overline{y}_2(x,t).\nabla\partial_t y_2(x,t)dx + $$
$$\xi\Re\int_{\Gamma_2}\int_0^1 \partial_t y_2(x,t-\tau\rho)\partial_t^2 y_2(x,t-\tau\rho)d\rho d\Gamma \tag{2.55}$$

Applying Green's Theorem to the first two integrals on the right-hand side of (2.55), we obtain

$$\frac{d}{dt}E(t) = a_1\Re\int_{\Gamma_1} \frac{\partial\overline{y}_1(x,t)}{\partial\nu}\partial_t y_1(x,t)d\Gamma + a_1\Re\int_{\Gamma_0} \frac{\partial\overline{y}_1(x,t)}{\partial\nu}\partial_t y_1(x,t)d\Gamma +$$
$$a_2\Re\int_{\Gamma_2} \frac{\partial\overline{y}_2(x,t)}{\partial\nu}\partial_t y_2(x,t)d\Gamma - a_2\Re\int_{\Gamma_0} \frac{\partial\overline{y}_2(x,t)}{\partial\nu}\partial_t y_2(x,t)d\Gamma +$$
$$\xi\Re\int_{\Gamma_2}\int_0^1 \partial_t y_2(x,t-\tau\rho)\partial_t^2 y_2(x,t-\tau\rho)d\rho d\Gamma \tag{2.56}$$

Recalling the boundary conditions (2.3)-(2.4) and the transmission conditions (2.5)-(2.6), we get

$$\frac{d}{dt}E(t) = -a_2\alpha\int_{\Gamma_2} |\partial_t y_2(x,t)|^2\, d\Gamma - a_2\beta\Re\int_{\Gamma_2} \partial_t\overline{y}_2(x,t-\tau)\partial_t y_2(x,t)d\Gamma +$$
$$\xi\Re\int_{\Gamma_2}\int_0^1 \partial_t y_2(x,t-\tau\rho)\partial_t^2 y_2(x,t-\tau\rho)d\rho d\Gamma \tag{2.57}$$

Now we have

$$\begin{aligned} \partial_\rho y(x,t-\tau\rho) &= -\tau\partial_t y(x,t-\tau\rho), \\ \partial_\rho^2 y(x,t-\tau\rho) &= \tau^2\partial_t^2 y(x,t-\tau\rho) \end{aligned}$$

Therefore

$$\begin{aligned} \Re\int_{\Gamma_2}\int_0^1 \partial_t\overline{y}_2(x,t-\tau\rho)\partial_t^2 y_2(x,t-\tau\rho)d\rho d\Gamma &= -\frac{1}{\tau^3}\Re\int_{\Gamma_2}\int_0^1 \partial_\rho\overline{y}_2(x,t-\tau\rho)\partial_\rho^2 y_2(x,t-\tau\rho)d\rho d\Gamma \\ &= -\frac{1}{2\tau^3}\Re\int_{\Gamma_2}\int_0^1 \frac{d}{d\rho}\,|\partial_\rho y_2(x,t-\tau\rho)|^2\, d\rho d\Gamma \\ &= \frac{1}{2\tau^3}\Re\int_{\Gamma_2} \{|\partial_\rho y_2(x,t)|^2 - |\partial_\rho y_2(x,t-\tau)|^2\}d\Gamma \\ &= \frac{1}{2\tau}\Re\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 - |\partial_t y_2(x,t-\tau)|^2\}d\Gamma. \end{aligned} \tag{2.58}$$

Inserting (2.58) into (2.57) yields

$$\frac{d}{dt}E(t) = -a_2\alpha\int_{\Gamma_2} |\partial_t y_2(x,t)|^2\, d\Gamma - a_2\beta\Re\int_{\Gamma_2} \partial_t\overline{y}_2(x,t-\tau)\partial_t y_2(x,t)d\Gamma +$$
$$\frac{\xi}{2\tau}\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 - |\partial_t y_2(x,t-s)|^2\}d\Gamma \tag{2.59}$$

from which we obtain after using Cauchy-Schwarz's inequality

$$\frac{d}{dt}E(t) \leq -K \int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau)|^2\} d\Gamma$$

where

$$K = \min\{a_2\alpha - \frac{a_2\beta}{2} - \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{a_2\beta}{2}\}$$

□

Step 2.
Set

$$E(t) = \mathcal{E}(t) + E_d(t) \tag{2.60}$$

where

$$\mathcal{E}(t) = \frac{a_1}{2} \int_{\Omega_1} |\nabla y_1(x,t)|^2 \, dx + \frac{a_2}{2} \int_{\Omega_2} |\nabla y_2(x,t)|^2 \, dx \tag{2.61}$$

and

$$E_d(t) = \frac{\xi}{2} \int_{\Gamma_2} \int_0^1 |\partial_t y_2(x, t-\tau\rho)|^2 \, d\rho \, d\Gamma$$

$E_d(t)$ can be rewritten via a change of variable as

$$E_d(t) = \frac{\xi}{2\tau} \int_{\Gamma_2} \int_t^{t+\tau} |\partial_t y_2(x,t-s)|^2 \, ds d\Gamma \tag{2.62}$$

From (2.62), we obtain

$$E_d(t) \leq C \int_0^T \int_{\Gamma_2} |\partial_t y_2(x,t-s)|^2 \, d\Gamma ds \tag{2.63}$$

for $0 \leq t + \tau \leq T$. Here and throughout the rest of the chapter C is a positive constant different at different occurrences.

Step 3.
We multiply both sides of (2.1) by $h(x).\nabla\overline{y}_k(x,t)$ and integrate by parts over $\Omega_k \times (0,T), .k = 1, 2$. We obtain (see the Appendix A)

$$\begin{aligned}
&2a_k\Re\int_0^T\int_{\Omega_k} J(x)\nabla y_k(x,t).\nabla\overline{y}_k(x,t))dxdt = 2a_k\Re\int_0^T\int_{\partial\Omega_k}\frac{\partial y_k(x,t)}{\partial\nu}h(x).\nabla\overline{y}_k(x,t)d\Gamma dt- \\
&a_k\int_0^T\int_{\partial\Omega_k}|\nabla y_k(x,t)|^2 h(x).\nu(x)d\Gamma dt + \Im\int_0^T\int_{\partial\Omega_k} y_k(x,t)\partial_t\overline{y_k}(x,t)h(x).\nu(x)d\Gamma dt+ \\
&a_k\Re\int_0^T\int_{\partial\Omega_k} y_k(x,t)\frac{\partial\overline{y}_k(x,t)}{\partial\nu}divh(x)d\Gamma dt - a_k\Re\int_0^T\int_{\Omega_k} y_k(x,t)\nabla\overline{y}_k(x,t).\nabla(divh(x))dxdt- \\
&\Im\int_{\Omega_k} y_k(x,t)h(x).\nabla\overline{y}_k(x,t)dx\Big|_0^T
\end{aligned} \tag{2.64}$$

Applying identity (2.64) with $y_k = y_1$ and with $y_k = y_2$, we get respectively

$$\begin{aligned}
&2a_1\Re\int_0^T\int_{\Omega_1}\nabla y_1(x,t)..J(x)\nabla\overline{y}_1(x,t)dxdt = a_1\Re\int_0^T\int_{\Gamma_1}\left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 h(x).\nu(x)d\Gamma dt+ \\
&2a_1\Re\int_0^T\int_{\Gamma_0}\frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla\overline{y}_1(x,t)d\Gamma dt - a_1\int_0^T\int_{\Gamma_0}|\nabla y_1(x,t)|^2 h(x).\nu(x)d\Gamma dt+ \\
&\Im\int_0^T\int_{\Gamma_0} y_1(x,t)\partial_t\overline{y}_1(x,t)h(x).\nu(x)d\Gamma dt + a_1\Re\int_0^T\int_{\Gamma_0}\frac{\partial\overline{y}_1(x,t)}{\partial\nu}y_1(x,t)divh(x)d\Gamma dt- \\
&a_1\Re\int_0^T\int_{\Omega_1} y_1(x,t)\nabla\overline{y}_1(x,t).\nabla(divh(x))d\Gamma dt - \Im\int_{\Omega_1} y_1(x,t)h(x).\nabla\overline{y}_1(x,t)dx\Big|_0^T
\end{aligned} \tag{2.65}$$

$$\begin{aligned}
&2a_2\Re\int_0^T\int_{\Omega_2}\nabla y_2(x,t).J(x)\nabla\overline{y}_2(x,t)dxdt = 2a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla\overline{y}_2(x,t)d\Gamma dt-\\
&2a_2\Re\int_0^T\int_{\Gamma_0}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla\overline{y}_2(x,t)d\Gamma dt - a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt+\\
&a_2\int_0^T\int_{\Gamma_0}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt + \Im\int_0^T\int_{\Gamma_2}y_2(x,t)\partial_t\overline{y}_2(x,t)h(x).\nu(x)d\Gamma dt-\\
&\Im\int_0^T\int_{\Gamma_0}y_2(x,t)\partial_t\overline{y}_2(x,t)h(x).\nu(x)d\Gamma dt + a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial\overline{y}_2(x,t)}{\partial\nu}y_2(x,t)divh(x)d\Gamma dt-\\
&a_2\Re\int_0^T\int_{\Gamma_0}\frac{\partial\overline{y}_2(x,t)}{\partial\nu}y_2(x,t)divh(x)d\Gamma dt - a_2\Re\int_0^T\int_{\Omega_2}y_2(x,t)\nabla\overline{y}_2(x,t).\nabla(divh(x))dxdt-\\
&\Im\int_{\Omega_2}y_2(x,t)h(x).\nabla\overline{y}_2(x,t)dx\Big|_0^T
\end{aligned}\tag{2.66}$$

after using the boundary condition (2.3).

Summing up (2.65) and (2.66), we obtain

$$\begin{aligned}
&2a_1\Re\int_0^T\int_{\Omega_1}\nabla y_1(x,t).J(x)\nabla\overline{y}_1(x,t)dxdt + 2a_2\Re\int_0^T\int_{\Omega_2}\nabla y_2(x,t).J(x)\nabla\overline{y}_2(x,t)dxdt =\\
&2a_1\Re\int_0^T\int_{\Gamma_1}\left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 h(x).\nu(x)d\Gamma dt + 2a_1\Re\int_0^T\int_{\Gamma_0}\frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla\overline{y}_1(x,t)d\Gamma dt-\\
&a_1\int_0^T\int_{\Gamma_0}\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x)d\Gamma dt - 2a_2\Re\int_0^T\int_{\Gamma_0}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla\overline{y}_2(x,t)d\Gamma dt+\\
&a_2\int_0^T\int_{\Gamma_0}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt + 2a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla\overline{y}_2(x,t)d\Gamma dt-\\
&-a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt + \Im\int_0^T\int_{\Gamma_2}y_2(x,t)\partial_t\overline{y}_2(x,t)h(x).\nu(x)d\Gamma dt+\\
&a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial\overline{y}_2(x,t)}{\partial\nu}y_2(x,t)divh(x)d\Gamma dt - \Im\int_{\Omega_1}y_1(x,t)h(x).\nabla\overline{y}_1(x,t)dx\Big|_0^T-\\
&\Im\int_{\Omega_2}y_2(x,t)h(x).\nabla\overline{y}_2(x,t)dx\Big|_0^T - a_1\Re\int_0^T\int_{\Omega_1}y_1(x,t)\nabla\overline{y}_1(x,t).\nabla(divh(x))d\Gamma dt-\\
&a_2\Re\int_0^T\int_{\Omega_2}y_2(x,t)\nabla\overline{y}_2(x,t).\nabla(divh(x))d\Gamma dt
\end{aligned}\tag{2.67}$$

We conclude from the boundary condition (2.5) that

$$\nabla(y_2(x,t)-y_1(x,t)) = \frac{\partial(y_2(x,t)-y_1(x,t))}{\partial\nu}\nu(x)\quad\text{on }\Gamma_0\times(0,T),$$

then

$$\left|\nabla y_2(x,t)\right|^2 = \left|\nabla y_1(x,t)\right|^2 + \left|\frac{\partial y_2(x,t)}{\partial\nu}\right|^2 - \left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2\qquad\text{on }\Gamma_0\times(0,T),$$

so on $\Gamma_0 \times (0,T)$,

$$\begin{aligned}&2a_1\Re(\frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla\overline{y}_1(x,t)) - 2a_2\Re(\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla\overline{y}_2(x,t)) - a_1\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x)+\\ &a_2\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x) = (a_2-a_1)\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x) - \frac{(a_2-a_1)^2}{a_2}\left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 h(x).\nu(x)\end{aligned} \tag{2.68}$$

after using the boundary condition (2.6).

Insertion of (2.68) into (2.67) results in

$$\begin{aligned}&2a_1\Re\int_0^T\int_{\Omega_1}\nabla y_1(x,t).J(x)\nabla\overline{y}_1(x,t)dxdt + 2a_2\Re\int_0^T\int_{\Omega_2}\nabla y_2(x,t).J(x)\nabla\overline{y}_2(x,t)dxdt =\\ &2a_1\Re\int_0^T\int_{\Gamma_1}\left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 h(x).\nu(x)d\Gamma dt + (a_2-a_1)\int_0^T\int_{\Gamma_0}\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x)d\Gamma dt -\\ &\frac{(a_2-a_1)^2}{a_2}\int_0^T\int_{\Gamma_0}\left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 h(x).\nu(x)d\Gamma dt + 2a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla\overline{y}_2(x,t)d\Gamma dt -\\ &a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt + \Im\int_0^T\int_{\Gamma_2}y_2(x,t)\partial_t\overline{y}_2(x,t)h(x).\nu(x)d\Gamma dt +\\ &a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial\overline{y}_2(x,t)}{\partial\nu}y_2(x,t)divh(x)d\Gamma dt - \Im\int_{\Omega_1}y_1(x,t)h(x).\nabla\overline{y}_1(x,t)dx\Big|_0^T -\\ &\Im\int_{\Omega_2}y_2(x,t)h(x).\nabla\overline{y}_2(x,t)dx\Big|_0^T - a_1\Re\int_0^T\int_{\Omega_1}y_1(x,t)\nabla\overline{y}_1(x,t).\nabla(divh(x))d\Gamma dt -\\ &a_2\Re\int_0^T\int_{\Omega_2}y_2(x,t)\nabla\overline{y}_2(x,t).\nabla(divh(x))d\Gamma dt\end{aligned} \tag{2.69}$$

Using the fact that

$$\left\|\nabla y_2\right\|^2_{L^2(\Gamma_0)} = \left\|\frac{\partial y_2}{\partial\nu}\right\|^2_{L^2(\Gamma_0)} + \left\|\nabla_\sigma y_2\right\|^2_{L^2(\Gamma_0)} \tag{2.70}$$

where $\nabla_\sigma y_2$ is the tangential gradient of y_2, (2.69) becomes

$$\begin{aligned}&2a_1\Re\int_0^T\int_{\Omega_1}\nabla y_1(x,t).J(x)\nabla\overline{y}_1(x,t)dxdt + 2a_2\Re\int_0^T\int_{\Omega_2}\nabla y_2(x,t).J(x)\nabla\overline{y}_2(x,t)dxdt =\\ &2a_1\Re\int_0^T\int_{\Gamma_1}\left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 h(x).\nu(x)d\Gamma dt - (a_1-a_2)\int_0^T\int_{\Gamma_0}\{\left|\nabla_\sigma y_1(x,t)\right|^2 + \frac{a_1}{a_2}\left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2\}h(x).\nu(x)d\Gamma dt+\\ &2a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla\overline{y}_2(x,t)d\Gamma dt - a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt+\\ &\Im\int_0^T\int_{\Gamma_2}y_2(x,t)\partial_t\overline{y}_2(x,t)h(x).\nu(x)d\Gamma dt + a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial\overline{y}_2(x,t)}{\partial\nu}y_2(x,t)divh(x)d\Gamma dt-\\ &a_1\Re\int_0^T\int_{\Omega_1}y_1(x,t)\nabla\overline{y}_1(x,t).\nabla(divh(x))d\Gamma dt - a_2\Re\int_0^T\int_{\Omega_2}y_2(x,t)\nabla\overline{y}_2(x,t).\nabla(divh(x))d\Gamma dt-\\ &\Im\int_{\Omega_1}y_1(x,t)h(x).\nabla\overline{y}_1(x,t)dx\Big|_0^T - \Im\int_{\Omega_2}y_2(x,t)h(x).\nabla\overline{y}_2(x,t)dx\Big|_0^T\end{aligned} \tag{2.71}$$

It follows from Assumptions $(H.2)$ and $(H.3)$ that

$$\begin{aligned}
&2a_1\Re\int_0^T\int_{\Omega_1}\nabla y_1(x,t).J(x)\nabla\overline{y}_1(x,t)dxdt+2a_2\Re\int_0^T\int_{\Omega_2}\nabla y_2(x,t).J(x)\nabla\overline{y}_2(x,t)dxdt\leq\\
&2a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla\overline{y}_2(x,t)d\Gamma dt-a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2h(x).\nu(x)d\Gamma dt+\\
&\Im\int_0^T\int_{\Gamma_2}y_2(x,t)\partial_t\overline{y}_2(x,t)h(x).\nu(x)d\Gamma dt+a_2\Re\int_0^T\int_{\Gamma_2}\frac{\partial\overline{y}_2(x,t)}{\partial\nu}y_2(x,t)divh(x)d\Gamma dt-\\
&a_1\Re\int_0^T\int_{\Omega_1}y_1(x,t)\nabla\overline{y}_1(x,t).\nabla(divh(x))d\Gamma dt-a_2\Re\int_0^T\int_{\Omega_2}y_2(x,t)\nabla\overline{y}_2(x,t).\nabla(divh(x))d\Gamma dt-\\
&\Im\int_{\Omega_1}y_1(x,t)h(x).\nabla\overline{y}_1(x,t)dx\bigg|_0^T-\Im\int_{\Omega_2}y_2(x,t)h(x).\nabla\overline{y}_2(x,t)dx\bigg|_0^T
\end{aligned}\tag{2.72}$$

Step 4.

We now estimate both sides of (2.72). From $(H.1)$, we have for the terms on the left-hand side of (2.72)

$$\begin{aligned}
&2a_1\Re\int_0^T\int_{\Omega_1}\nabla y_1(x,t).J(x)\nabla\overline{y}_1(x,t)dxdt+2a_2\Re\int_0^T\int_{\Omega_2}\nabla y_2(x,t).J(x)\nabla\overline{y}_2(x,t)dxdt\geq\\
&2\min\{a_1,a_2\}\Re\{\int_0^T\int_{\Omega_1}\nabla y_1(x,t).J(x)\nabla\overline{y}_1(x,t)dxdt+\int_0^T\int_{\Omega_2}\nabla y_2(x,t).J(x)\nabla\overline{y}_2(x,t)dxdt\}\geq\\
&2\min\{a_1,a_2\}\Re\{\int_0^T\int_{\Omega}\nabla y(x,t).J(x)\nabla\overline{y}(x,t)dxdt\geq\\
&2\min\{a_1,a_2\}\alpha\int_0^T\int_{\Omega}\left|\nabla y(x,t)\right|^2dx\geq 2\alpha\min\{a_1,a_2\}\{\frac{1}{a_1}a_1\int_0^T\int_{\Omega_1}\left|\nabla y_1(x,t)\right|^2dx+\frac{1}{a_2}a_2\int_0^T\int_{\Omega_2}\left|\nabla y_2(x,t)\right|^2dx\}\geq\\
&2\alpha\min\{a_1,a_2\}\max\{a_1,a_2\}\{a_1\int_0^T\int_{\Omega_1}\left|\nabla y_1(x,t)\right|^2dx+a_2\int_0^T\int_{\Omega_2}\left|\nabla y_2(x,t)\right|^2dx\}\geq\\
&2\alpha a_1a_2\{a_1\int_0^T\int_{\Omega_1}\left|\nabla y_1(x,t)\right|^2dx+a_2\int_0^T\int_{\Omega_2}\left|\nabla y_2(x,t)\right|^2dx\}
\end{aligned}$$

Hence

$$2a_1\Re\int_0^T\int_{\Omega_1}\nabla y_1(x,t).J(x)\nabla\overline{y}_1(x,t)dxdt+2a_2\Re\int_0^T\int_{\Omega_2}\nabla y_2(x,t).J(x)\nabla\overline{y}_2(x,t)dxdt\geq 4\alpha a_1a_2\int_0^T\mathcal{E}(t)dt\tag{2.73}$$

Now, we estimate each integral term on the right-hand side of the inequality (2.72) separately.

First term. We have by the Cauchy-Schwarz,Young and Poincaré inequalities

$$\left|\Im\int_{\Omega_1}y_1(x,t)h(x).\nabla\overline{y}_1(x,t)dx\bigg|_0^T-\Im\int_{\Omega_2}y_2(x,t)h(x).\nabla\overline{y}_2(x,t)dx\bigg|_0^T\right|\leq C(\mathcal{E}(T)+\mathcal{E}(0))\tag{2.74}$$

recalling (2.61).

Second term. Using Cauchy-Schwarz and Young inequalities yields

$$\begin{aligned}
&\left|a_1\Re\int_0^T\int_{\Omega_1}y_1(x,t)\nabla\overline{y}_1(x,t).\nabla(divh(x))d\Gamma dt+a_2\Re\int_0^T\int_{\Omega_2}y_2(x,t)\nabla\overline{y}_2(x,t).\nabla(divh(x))d\Gamma dt\right|\leq\\
&\frac{\eta}{2}\int_0^T\mathcal{E}(t)dt+\frac{C}{\eta}\int_0^T\int_{\Omega}\left|y(x,t)\right|^2dxdt
\end{aligned}\tag{2.75}$$

where η is a positive constant that will be fixed later.

Third term. Recalling the boundary conditions (2.4) and using again Cauchy-Schwarz and Young inequalities gives

$$2a_2\left|\Re\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla\overline{y}_2(x,t)d\Gamma dt\right| = 2a_2\int_0^T\int_{\Gamma_2}\{-\alpha\partial_t y_2(x,t)-\beta\partial_t y_2(x,t-\tau)\}h(x).\nabla y_2(x,t)d\Gamma dt \leq$$

$$C\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+|\partial_t y_2(x,t-\tau)|^2\}d\Gamma dt + C\int_0^T\int_{\Gamma_2}|\nabla y_2(x,t)|^2\,d\Gamma dt \tag{2.76}$$

Fourth term. Proceeding as for (2.76), we obtain

$$a_2\left|\Re\int_0^T\int_{\Gamma_2}\frac{\partial\overline{y}_2(x,t)}{\partial\nu}y_2(x,t)divh(x)d\Gamma dt\right| \leq$$

$$C\int_0^T\int_{\Gamma_2}|y_2(x,t)|^2\,d\Gamma dt + \int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+|\partial_t y_2(x,t-\tau)|^2\}d\Gamma dt \tag{2.77}$$

Fifth term.

$$\left|\Im\int_0^T\int_{\Gamma_2}y_2(x,t)\partial_t\overline{y}_2(x,t)h(x).\nu(x)d\Gamma dt\right| \leq C\int_0^T\int_{\Gamma_2}\{|y_2(x,t)|^2+|\partial_t y_2(x,t)|^2\}d\Gamma dt \tag{2.78}$$

Sixth term.

$$a_2\int_0^T\int_{\Gamma_2}|\nabla y_2(x,t)|^2\,h(x).\nu(x)d\Gamma dt \leq C\int_0^T\int_{\Gamma_2}|\nabla y_2(x,t)|^2\,d\Gamma dt \tag{2.79}$$

Inserting (2.73) – (2.79) into (2.72), we obtain

$$(\quad\alpha-\eta(\frac{1}{2}+C))\int_0^T\int_\Omega a(x)\,|\nabla y(x,t)|^2\,dxdt \leq C\{\mathcal{E}(T)+\mathcal{E}(0)\}+\frac{C}{\eta}\int_0^T\int_\Omega|y(x,t)|^2\,dxdt +$$

$$C\int_0^T\int_{\Gamma_2}|y_2(x,t)|^2\,d\Gamma dt + C\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+|\partial_t y_2(x,t-\tau)|^2\}d\Gamma dt + C\int_0^T\int_{\Gamma_2}|\nabla y_2(x,t)|^2\,d\Gamma dt$$

Choosing η sufficiently small to make $\alpha-\eta(\frac{1}{2}+C)>0$, yields

$$\int_0^T\mathcal{E}(T)dt \leq C\{\mathcal{E}(T)+\mathcal{E}(0)\}+\frac{C}{\eta}\int_0^T\int_\Omega|y(x,t)|^2\,dxdt + C\int_0^T\int_{\Gamma_2}|y_2(x,t)|^2\,d\Gamma dt +$$

$$C\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+|\partial_t y_2(x,t-\tau)|^2\}d\Gamma dt + C\int_0^T\int_{\Gamma_2}|\nabla y_2(x,t)|^2\,d\Gamma dt \tag{2.80}$$

Recalling (2.70) and (2.4), we readily obtain from (2.80)

$$\int_0^T\mathcal{E}(t)dt \leq C\{\mathcal{E}(T)+\mathcal{E}(0)\}+\frac{C}{\eta}\int_0^T\int_\Omega|y(x,t)|^2\,dxdt + C\int_0^T\int_{\Gamma_2}|y_2(x,t)|^2\,d\Gamma dt +$$

$$C\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+|\partial_t y_2(x,t-\tau)|^2\}d\Gamma dt + C\int_0^T\int_{\Gamma_2}|\nabla_\sigma y_2(x,t)|^2\,d\Gamma dt \tag{2.81}$$

Step 4.
For fixed $\epsilon > 0$ small we apply estimate (2.81) over the interval $(\epsilon, T-\epsilon)$ rather than $(0,T)$. We obtain

$$\int_{\epsilon}^{T-\epsilon} \mathcal{E}(t)dt \leq C\{\mathcal{E}(T-\epsilon) + \mathcal{E}(\epsilon)\} + C\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2} |y_2(x,t)|^2 \, d\Gamma dt +$$
$$C\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau)|^2\} d\Gamma dt + C\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2} |\nabla_\sigma y_2(x,t)|^2 \, d\Gamma dt +$$
$$C\int_{\epsilon}^{T-\epsilon}\int_{\Omega} |y(x,t)|^2 \, dxdt \tag{2.82}$$

From Lemma 7.2, inequality 7.5 in [72], we have

$$\int_{\epsilon}^{T-\epsilon}\int_{\Gamma_2} |\nabla_\sigma y_2(x,t)|^2 \, d\Gamma dt \leq C(\epsilon,\delta,T)\{\int_0^T\int_{\Gamma_2}\{\left|\frac{\partial y_2(x,t)}{\partial \nu}\right|^2 + |\partial_t y_2(x,t)|^2\} d\Gamma dt +$$
$$\|y_2\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))}\} \tag{2.83}$$

where δ is an arbitrarily small positive constants and $C(\epsilon,\delta,T)$ denotes a positive constant that depends on ϵ, δ and T.

Inserting (2.83) into (2.82) and recalling the boundary condition (2.4), yields

$$\int_{\epsilon}^{T-\epsilon} \mathcal{E}(t)dt \leq C\{\mathcal{E}(T-\epsilon) + \mathcal{E}(\epsilon)\} + C\int_0^T\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau)|^2\} d\Gamma dt +$$
$$C\int_0^T\int_{\Gamma_2} |y_2(x,t)|^2 \, d\Gamma dt + C\int_0^T\int_{\Omega} |y(x,t)|^2 \, dxdt + C(\epsilon,\delta,T)\, \|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))} \tag{2.84}$$

From (2.59), we have for $0 \leq S < T$,

$$E(S) = E(T) + a_2\alpha\int_S^T\int_{\Gamma_2} |\partial_t y_2(x,t)|^2 \, d\Gamma dt + a_2\beta\Re\int_S^T\int_{\Gamma_2} \partial_t\overline{y}_2(x,t-\tau)\partial_t y_2(x,t) ds d\Gamma dt -$$
$$\frac{\xi}{2\tau}\int_S^T\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 - (|\partial_t y_2(x,t-\tau)|^2\} d\Gamma dt \tag{2.85}$$

(2.85) implies

$$E(S) \leq E(T) + C\int_0^T\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau)|^2\} d\Gamma dt \tag{2.86}$$

Using (2.60) and (2.86), we deduce from (2.84)

$$\int_{\epsilon}^{T-\epsilon} \mathcal{E}(t)dt \leq C(E(T) + \int_0^T\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau)|^2\} d\Gamma dt +$$
$$\int_0^T\int_{\Gamma_2} |y_2(x,t)|^2 \, d\Gamma dt + \int_0^T\int_{\Omega} |y(x,t)|^2 \, dxdt) + C(\epsilon,\delta,T)\, \|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))} \tag{2.87}$$

On the other hand for a fixed ϵ

$$\int_0^{\epsilon} \mathcal{E}(t)dt + \int_{T-\epsilon}^T \mathcal{E}(t) \leq 2\epsilon E(0)$$

and by (2.86),

$$\int_0^{\epsilon} \mathcal{E}(t)dt + \int_{T-\epsilon}^T \mathcal{E}(t) \leq C\{E(T) + \int_0^T\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau)|^2\} d\Gamma dt\}$$

Hence

$$\int_0^T \mathcal{E}(t)dt \leq C\{E(T)+\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+\int_{\tau_1}^{\tau_2}|\partial_t y_2(x,t-\tau)|^2\}d\Gamma dt+$$
$$\int_0^T\int_{\Gamma_2}|y_2(x,t)|^2\,d\Gamma dt+\int_0^T\int_\Omega |y(x,t)|^2\,dxdt+C(\epsilon,\delta,T)\,\|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))}\} \tag{2.88}$$

Notice that for any $\theta>0$, (see [45], p. 112, Theorem 16.3)

$$\|y_2\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))}\leq \theta\int_0^T\int_{\Omega_2}|\nabla y_2(x,t)|^2\,dxdt+C(\theta)\int_0^T\int_{\Omega_2}|y_2(x,t)|^2\,dxdt$$

and consequently

$$\|y_2\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))}\leq \frac{2\theta}{\min\{a_1,a_2\}}\int_0^T E(t)dt+C(\theta)\int_0^T\int_\Omega |y(x,t)|^2\,dxdt \tag{2.89}$$

Collecting (2.9), (2.74), (2.60), (2.61), (2.63), (2.88) and (2.89), we obtain for an appropriate choice of θ and for T large enough

$$E(T)\leq C\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+|\partial_t y_2(x,t-\tau)|^2\}d\Gamma dt+$$
$$C(\int_0^T\int_{\Gamma_2}|y_2(x,t)|^2\,d\Gamma dt+\int_0^T\int_\Omega |y(x,t)|^2\,dxdt) \tag{2.90}$$

Step 6.

We prove by a compactness-uniqueness argument that there exists a constant $C>0$ such that

$$\|y\|^2_{L^2(0,T;L^2(\Omega))}+\|y\|^2_{L^2(0,T;L^2(\Gamma_{2-}))}\leq C\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+|\partial_t y_2(x,t-\tau)|^2\}d\Gamma dt \tag{2.91}$$

Assume that there exists a sequence y^n of solutions of problem (2.1) – (2.7) with

$$\begin{aligned} y_k^n(x,0) &= y_{k0}^n(x), \quad x\in\Omega_k,\\ y_k^n(x,t-\tau) &= f^n(x,t-\tau), \quad x\in\Gamma_2, t\in(0,\tau), k=1,2. \end{aligned}$$

such that

$$\|y^n\|^2_{L^2(0,T;L^2(\Omega))}+\|y_2^n\|^2_{L^2(0,T;L^2(\Gamma_2))}=1,$$
$$\int_0^T\int_{\Gamma_2}\{|\partial_t y_2^n(x,t)|^2+|\partial_t y_2^n(x,t-\tau)|^2\}d\Gamma dt\to 0 \text{ as } n\to+\infty \tag{2.92}$$

Since each solution satisfies (2.90), we deduce from (2.86) and (2.92) that the sequence $Y_0^n=(y_0^n,f^n)$ is bounded in $\mathcal{H}$. Hence there is a subsequence still denoted by Y_0^n which converges weakly to some $Y_0=(y_0,f)$. Let y be the solution of problem (2.1) – (2.7) corresponding to such initial conditions. We have from Proposition 2.2.1

$$y\in C(0,T;H^1_{\Gamma_1}(\Omega))$$

Then

$$\begin{aligned} y_n &\to y \text{ in } L^\infty(0,T;H^1_{\Gamma_1}(\Omega)) \qquad \text{weak-star},\\ y_n &\to y \text{ strongly in } L^2(0,T;L^2(\Gamma_2))\cap L^2(0,T,L^2(\Omega)) \end{aligned}$$

This fact along with the compactness $H^1_{\Gamma_1}(\Omega) \to H^{1-\varepsilon}(\Omega)$ for $\varepsilon > 0$, implies that there exists a subsequence still denoted by y^n such that $y^n \to y$ strongly in $L^\infty(0,T;H^{1-\varepsilon}(\Omega))$. Then we have from (2.92)

$$\|y\|^2_{L^2(0,T;L^2(\Omega))} + \|y_2\|^2_{L^2(0,T;L^2(\Gamma_2))} = 1 \tag{2.93}$$

and

$$\int_0^T \int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau)|^2\} d\Gamma dt = 0$$

Thus y satisfies

$$\partial_t y_2(x,t) = 0 \qquad \text{on } \Gamma_2 \times (0,T)$$

and

$$\frac{\partial y_2(x,t)}{\partial \nu} = 0 \qquad \text{on } \Gamma_2 \times (0,T)$$

Let

$$u_k(x,t) = y_k(x,t). \tag{2.94}$$

Then

$$\begin{cases} \partial_t u_k(x,t) - a_k i \Delta u_k(x,t) = 0, & (x,t) \in \Omega_k \times (0,T), k = 1,2, \\ u_1(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,T), \\ \dfrac{\partial u_2(x,t)}{\partial \nu} = u_2(x,t) = 0, & (x,t) \in \Gamma_2 \times (0,T), \\ u_1(x,t) = u_2(x,t), & (x,t) \in \Gamma_0 \times (0,T), \\ a_1 \dfrac{\partial u_1(x,t)}{\partial \nu} = a_2 \dfrac{\partial u_2(x,t)}{\partial \nu} & (x,t) \in \Gamma_0 \times (0,T). \end{cases}$$

From Holmgren's uniqueness theorem (see [44] Thm. 8.2, p. 92, and Thm. 7.1, p.391) applied to problem

$$\begin{cases} \partial_t u_2(x,t) - a_2 i \Delta u_2(x,t) = 0, & (x,t) \in \Omega_2 \times (0,T), \\ \dfrac{\partial u_2(x,t)}{\partial \nu} = u_2(x,t) = 0, & (x,t) \in \Gamma_2 \times (0,T), \end{cases}$$

we obtain for T large enough

$$u_2(x,t) = 0, \quad (x,t) \in \Omega_2 \times (0,T) \tag{2.95}$$

and hence

$$u_1(x,t) = \frac{\partial u_1(x,t)}{\partial \nu} = 0, \quad (x,t) \in \Gamma_0 \times (0,T)$$

We apply again Holmgren's uniqueness theorem this time to problem

$$\begin{cases} \partial_t u_1(x,t) - a_1 i \Delta u_1(x,t) = 0, & (x,t) \in \Omega_1 \times (0,T), \\ u_1(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,T), \\ u_1(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,T), \\ \dfrac{\partial u_1(x,t)}{\partial \nu} = 0, & (x,t) \in \Gamma_0 \times (0,T) \end{cases}$$

we obtain

$$u_1(x,t) = 0, \; (x,t) \in \Omega_1 \times (0,T) \tag{2.96}$$

(2.94) together with (2.95) and (2.96) implies that

$$y_k(x,t) = y_k(x)$$

Thus $y_k, k = 1,2$ verify

$$\begin{cases} -a_k \Delta y_k(x) = 0, & x \in \Omega_k, k = 1,2, \\ y_1(x) = 0, & x \in \Gamma_1, \\ \frac{\partial y_2(x)}{\partial \nu} = 0, & x \in \Gamma_2, \\ y_1(x) = y_2(x), & x \in \Gamma_0, \\ a_1 \frac{\partial y_1(x)}{\partial \nu} = a_2 \frac{\partial y_2(x)}{\partial \nu}, & x \in \Gamma_0, \end{cases}$$

and so $y_k(x) = 0$ for $x \in \Omega_k, k = 1, 2$ and consequently $y(x) = 0$ for $x \in \Omega$. This contradicts (2.93).
Step 8.
The estimate (2.54) together with (2.90) and (2.91) yields

$$E(T) \leq \frac{C}{k + C} E(0) \tag{2.97}$$

The desired conclusion follows now from (2.97) and Proposition 1.3.2 since $0 < \frac{C}{k+C} < 1$.

2.4 Appendix A

. **Proof of the identity** (2.64)
We multiply both sides of (2.1) by $h(x).\nabla y_k(x, t)$ and integrate over $\Omega_k \times (0, T); k = 1, 2,$

$$\int_0^T \int_{\Omega_k} \partial_t y_k(x,t) h(x).\nabla \overline{y}_k(x,t) dxdt - ia_k \int_0^T \int_{\Omega_k} \Delta y_k(x,t)) h(x).\nabla \overline{y}_k(x,t) dxdt = 0 \tag{2.98}$$

Integrating by parts in t, we obtain for the first term at the right of (2.98)

$$\int_0^T \int_{\Omega} \partial_t y_k(x,t) h(x).\nabla \overline{y}_k(x,t) dxdt = \int_{\Omega} y_k(x,t) h(x).\nabla \overline{y}_k(x,t) dx \Big|_0^T - \int_0^T \int_{\Omega} y_k(x,t) h(x).\nabla \partial_t \overline{y}_k(x,t) dxdt \tag{2.99}$$

Applying Green's Theorem to the second integral on the right -hand side of (2.99), yields

$$\begin{aligned}
&\int_0^T \int_{\Omega_k} y_k(x,t) h(x).\nabla \partial_t \overline{y}_k(x,t) dxdt = \int_0^T \int_{\partial\Omega} y_k(x,t) \partial_t \overline{y_k}(x,t) h(x).\nu(x) d\Gamma dt - \\
&\int_0^T \int_{\Omega} y_k(x,t) \partial_t \overline{y}_k(x,t) divh(x) dxdt - \int_0^T \int_{\Omega} \partial_t \overline{y}_k(x,t) h(x).\nabla y_k(x,t) dxdt = \\
&\int_0^T \int_{\partial\Omega} y_k(x,t) \partial_t \overline{y_k}(x,t) h(x).\nu(x) d\Gamma dt - \int_0^T \int_{\Omega} y_k(x,t) \partial_t \overline{y}_k(x,t) divh(x) dxdt + \\
&ia_k \int_0^T \int_{\Omega_k} \Delta \overline{y}_k(x,t)) h(x).\nabla y_k(x,t) dxdt
\end{aligned} \tag{2.100}$$

Inserting (2.100) into (2.98), yields

$$\begin{aligned}
&2a_k \Re \int_0^T \int_{\Omega_k} \Delta y_k(x,t)) h(x).\nabla \overline{y}_k(x,t) dxdt = \Im \int_{\Omega_k} y_k(x,t) h(x).\nabla \overline{y}_k(x,t) dx \Big|_0^T - \\
&\Im \int_0^T \int_{\partial\Omega_k} y_k(x,t) \partial_t \overline{y_k}(x,t) h(x).\nu(x) d\Gamma dt - a_k \Re \int_0^T \int_{\Omega_k} y_k(x,t) \Delta \overline{y}_k(x,t) divh(x) dxdt
\end{aligned} \tag{2.101}$$

For the term at the left of (2.101), we have after using Green's Theorem

$$\begin{aligned}
&2a_k \Re \int_0^T \int_{\Omega_k} \Delta y_k(x,t) h(x).\nabla \overline{y}_k(x,t) dxdt = 2a_k \Re \int_0^T \int_{\partial\Omega_k} \frac{\partial y_k(x,t)}{\partial \nu} h(x).\nabla \overline{y}_k(x,t) d\Gamma dt - \\
&2a_k \Re \int_0^T \int_{\Omega_k} \nabla y_k(x,t).\nabla(h(x).\nabla \overline{y}_k(x,t)) dxdt
\end{aligned} \tag{2.102}$$

Applying the identity

$$\Re\{\nabla w(x).\nabla(h(x).\nabla \overline{w}(x))\} = \Re\{J(x)\nabla w(x).\nabla \overline{w}(x)\} + \frac{1}{2} h(x).\nabla(|\nabla w(x)|^2)$$

to the last integral on the right hand side of (2.102), we find

$$2a_k\Re\int_0^T\int_{\Omega_k}\Delta y_k(x,t)h(x).\nabla\overline{y}_k(x,t)dxdt = 2a_k\Re\int_0^T\int_{\partial\Omega_k}\frac{\partial y_k(x,t)}{\partial\nu}h(x).\nabla\overline{y}_k(x,t)d\Gamma dt-$$
$$2a_k\Re\int_0^T\int_{\Omega_k}J(x)\nabla y_k(x,t).\nabla\overline{y}_k(x,t))dxdt - a_k\int_0^T\int_{\Omega_k}h(x).\nabla(|\nabla y_k(x,t)|^2)$$

Another use of Green's theorem yields

$$2a_k\Re\int_0^T\int_{\Omega_k}\Delta y_k(x,t)h(x).\nabla\overline{y}_k(x,t)dxdt = 2a_k\Re\int_0^T\int_{\partial\Omega_k}\frac{\partial y_k(x,t)}{\partial\nu}h(x).\nabla\overline{y}_k(x,t)d\Gamma dt-$$
$$2a_k\Re\int_0^T\int_{\Omega_k}J(x)\nabla y_k(x,t).\nabla\overline{y}_k(x,t))dxdt - a_k\int_0^T\int_{\partial\Omega_k}|\nabla y_k(x,t)|^2\,h(x).\nu(x)d\Gamma dt+$$
$$a_k\int_0^T\int_{\Omega_1}|\nabla y_k(x,t)|^2\,divh(x)dxdt \tag{2.103}$$

For the last integral at the right (2.101), using Green'sTheorem once more results in

$$a_k\Re\int_0^T\int_{\Omega_k}y_k(x,t)\Delta\overline{y}_k(x,t)divh(x)dxdt = a_k\Re\int_0^T\int_{\partial\Omega_k}y_k(x,t)\frac{\partial\overline{y}_k(x,t)}{\partial\nu}divh(x)d\Gamma dt-$$
$$a_k\Re\int_0^T\int_{\Omega_k}|\nabla y_k(x,t)|^2\,divh(x)dxdt - a_k\Re\int_0^T\int_{\Omega_k}y_k(x,t)\nabla\overline{y}_k(x,t).\nabla(divh(x))dxdt \tag{2.104}$$

Inserting (2.103) and (2.104) into (2.101), we obtain identity (2.64).

Chapter 3

Stabilization of the transmission Schrödinger equation with boundary time-varying delay

3.1 Introduction and statement of the exponential stability result

In [59], Nicaise et al considered the multi-dimensional wave equation with a time-varying delay term in the boundary condition. Under suitable assumptions they established the exponential stability of the solution. This result is obtained by introducing suitable energies and suitable Lyapunov functionnals. In this chapter, we adpot the approach of [59] to study the stability problem for the transmission Schrödinger equation with a time-varying delay term in the boundary feedback.
Let $\Omega, \Omega_1, \Omega_2, \Gamma, \Gamma_0, \Gamma_1$ and Γ_2 be as in Chapter 2, and assume that there exists $x_0 \in \mathbb{R}^n$ such that for $m(x) = x - x_0$, we have:

$$m(x).\nu(x) \leq 0 \qquad \text{on } \Gamma_1 \text{ and on } \Gamma_0 \tag{3.1}$$

$$m(x).\nu(x) \geq \delta > 0 \qquad \text{on } \Gamma_2 \tag{3.2}$$

Let $a_1, a_2 > 0$ be given. Consider the system of transmission of the Schrödinger equation with a time-varying delay term in the boundary conditions:

$$\partial_t y_k(x,t) - i a_k \Delta y_k(x,t) = 0 \qquad \text{in } \Omega_k \times (0,+\infty), k = 1,2, \tag{3.3}$$

$$y_k(x,0) = y_{0k}(x) \qquad \text{in } \Omega_k, k = 1,2, \tag{3.4}$$

$$y_1(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{3.5}$$

$$\frac{\partial y_2(x,t)}{\partial \nu} = -\alpha \partial_t y_2(x,t) - \beta \partial_t y_2(x, t-\tau(t)) \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{3.6}$$

$$y_1(x,t) = y_2(x,t), \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{3.7}$$

$$a_1 \frac{\partial y_1(x,t)}{\partial \nu} = a_2 \frac{\partial y_2(x,t)}{\partial \nu} \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{3.8}$$

$$\partial_t y_2(x, t-\tau) = f(x, t-\tau) \qquad \text{on } \Gamma_2 \times (0,\tau). \tag{3.9}$$

where:

- α and β are positive constants,
- y^0, f are the initial data which belong to suitable spaces,

- $\tau(.)$ is the time-varying which is as in [59] subject to the following assumptions:
  There exist positive constants $\widehat{\tau}$ and $\widetilde{\tau}$ such that

$$\widehat{\tau} \leq \tau(t) \leq \widetilde{\tau} \quad \text{for all } t > 0, \tag{3.10}$$

$$\tau'(t) \leq d < 1 \qquad \text{for all } t > 0, \tag{3.11}$$

$$\tau(.) \in W^{2,\infty}([0,T]). \tag{3.12}$$

To state our stability result, we assume as in [59] that

$$\alpha\sqrt{1-d} > \beta \tag{3.13}$$

and define the energy of a solution

$$y(x,t) = \begin{cases} y_1(x,t), & (x,t) \in \Omega_1 \times (0,+\infty) \\ y_2(x,t), & (x,t) \in \Omega_2 \times (0,+\infty) \end{cases}$$

of (3.3) – (3.9) by

$$E(t) = \frac{a_1}{2}\int_{\Omega_1} |\nabla y_1(x,t)|^2 \, dx + \frac{a_2}{2}\int_{\Omega_2} |\nabla y_2(x,t)|^2 \, dx + \frac{\xi}{2}\tau(t)\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x,t-\rho s)|^2 \, d\rho \, d\Gamma \tag{3.14}$$

where and

$$\frac{a_2\beta}{\sqrt{1-d}} < \xi < 2\alpha a_2 - \frac{\beta}{\sqrt{1-d}} \tag{3.15}$$

The main result of this chapter can be stated as follows.

Theorem 3.1.1. *In addition to* (3.1), (3.2), *(3.10), (3.11), (3.12) and* (3.13), *assume that*

$$a_1 > a_2$$

Then there exist constants $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq Me^{-\omega t}E(0)$$

for any regular solution of (3.3) – (3.9)

Theorem 3.1.1 is proved in Section 3. In Section 2, we study existence, uniqueness and regularity of solutions for system (3.3) – (3.9) using semigroup theory.

3.2 Well-posedness

Inspired from [57], we introduce the auxiliary variable

$$z(x,\rho,t) = \partial_t y(x, t-\tau(t)\rho). \tag{3.16}$$

With this new unknown, problem (3.3)-(3.9) is equivalent to

$$
\begin{aligned}
&\partial_t y_k(x,t) - i a_k \Delta y_k(x,t) = 0 && \text{in } \Omega_k \times (0,+\infty), k=1,2, && (3.17)\\
&y_k(x,0) = y_{0k}(x) && \text{in } \Omega_k, k=1,2, && (3.18)\\
&y_1(x,t) = 0 && \text{on } \Gamma_1 \times (0,+\infty), && (3.19)\\
&\tau(t)\partial_t z(x,\rho,t) + (1-\tau'(t)\rho)\partial_\rho z(x,\rho,t) = 0 && \text{in } \Gamma_2 \times (0,1)\times(0,+\infty) && (3.20)\\
&\frac{\partial y_2(x,t)}{\partial \nu} = -i\alpha a_2 \Delta y_2(x,t) - \beta z(x,1,t) && \text{on } \Gamma_2 \times (0,+\infty), && (3.21)\\
&y_1(x,t) = y_2(x,t) && \text{on } \Gamma_0 \times (0,+\infty), && (3.22)\\
&a_1 \frac{\partial y_1(x,t)}{\partial \nu} = a_2 \frac{\partial y_2(x,t)}{\partial \nu} && \text{on } \Gamma_0 \times (0,+\infty), && (3.23)\\
&z(x,0,t) = \partial_t y_2(x,t) && \text{on } \Gamma_2 \times (0,+\infty), && (3.24)\\
&z(x,\rho,0) = f_0(x,-\rho\tau(0)) && \text{in } \Gamma_2 \times (0,1). && (3.25)
\end{aligned}
$$

Let

$$\mathcal{V} = \{(u_1,u_2) \in H^1_{\Gamma_1}(\Omega_1) \times H^1(\Omega_2); u_1 = u_2 \text{ on } \Gamma_0\}$$

The space for well-posedness of (3.17)-(3.25) is taken to be the space

$$\mathcal{H} = \mathcal{V} \times L^2(\Gamma_2; L^2(0,1))$$

$\mathcal{H}$ is a Hilbert space with the following inner product

$$\left\langle \begin{pmatrix} u_1 \\ u_2 \\ z \end{pmatrix}; \begin{pmatrix} \widetilde{u}_1 \\ \widetilde{u}_2 \\ \widetilde{z} \end{pmatrix} \right\rangle = a_1 \int_{\Omega_2} \nabla u_1(x).\nabla \widetilde{u}_1(x) dx + a_2 \int_{\Omega_2} \nabla u_2(x).\nabla \widetilde{u}_2(x) dx + \xi \int_{\Gamma_2}\int_0^1 z(x,\rho)\widetilde{z}(x,\rho) d\rho d\Gamma$$

In $\mathcal{H}$, define a linear operator $A(t)$ by

$$A(t)(u_1,u_2,z)^T = (ia_1\Delta u_1, ia_2\Delta u_2, \frac{\tau'(t)\rho - 1}{\tau(t)}\partial_\rho z)^T, \tag{3.26}$$

$$
\begin{aligned}
D(A(t)) \quad = \quad & \{(u_1,u_2,z)^T \in \mathcal{V} \times L^2(\Gamma_2; H^1(0,1)); \Delta u_1 \in H^1_{\Gamma_1}(\Omega_1), \Delta u_2 \in H^1(\Omega_2),\\
& \text{satisfying (3.28)-(3.31) below}\} \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
&a_1 \frac{\partial u_1}{\partial \nu} = a_2 \frac{\partial u_2}{\partial \nu} \text{ on } \Gamma_0, && (3.28)\\
&a_1 \Delta u_1 = a_2 \Delta u_2 \text{ on } \Gamma_0, && (3.29)\\
&z(.,0) = i a_2 \Delta u_2 \text{ on } \Gamma_2, && (3.30)\\
&\frac{\partial u_2}{\partial \nu} = -\alpha z(.,0)) - \beta z(.,1), \text{ on } \Gamma_2. && (3.31)
\end{aligned}
$$

Notice that for $(u_1,u_2,z) \in D(A(t))$, we have the following boundary regularity:

- $\Delta u_k|_{\partial\Omega_k} \in H^{1/2}(\partial\Omega_k), k=1,2,$ (trace theorem),
- $\frac{\partial u_1}{\partial \nu}|_{\partial\Omega_1} \in H^{-1/2}(\partial\Omega_1)$, $\frac{\partial u_2}{\partial \nu}|_{\Gamma_0} \in H^{-1/2}(\Gamma_0)$ (see e.g., [35] p. 71, Theorem. 3.8.1]),
- $\frac{\partial u_2}{\partial \nu}|_{\Gamma_2} \in L^2(\Gamma_2)$ since $z \in L^2(\Gamma_2)$.

Using the operator $A(t)$, we rewrite (3.17) – (3.25) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}Y(t) = A(t)Y(t) \\ Y(0) = Y_0 \end{cases} \tag{3.32}$$

where

$$Y(t) = (y, z)^T \text{ and } Y^0 = (y^0(x), f_0(., -.\tau(0)))^T$$

Notice that problem (3.32) is equivalent to

$$\begin{cases} \frac{d}{dt}\widetilde{Y}(t) = \widetilde{A}(t)\widetilde{Y}(t) \\ \widetilde{Y}(0) = Y_0 \end{cases} \tag{3.33}$$

where

$$\widetilde{A}(t) = A(t) - \kappa(t)I, \kappa(t) = \frac{(\tau'(t)^2 + 1)^{\frac{1}{2}}}{2\tau(t)},$$

in the sense that if $\widetilde{Y}(t)$ is a solution of (3.33) then $Y(t) = e^{\theta(t)}\widetilde{Y}(t)$ where $\theta(t) = \int_0^t \kappa(s)ds$ is a solution of (3.32).

To establish existence and uniqueness of solutions for problem (3.33), we employ the result stated next.([34]).

Theorem 3.2.1. *Let $\mathcal{K}(t) : D(\mathcal{K}(t)) \subset \mathcal{H} \to \mathcal{H}$ be a time-varying linear operator such that :*

i. $D(\mathcal{K}(t))$ is independent of t,

ii. $D(\mathcal{K}(0))$ is a dense subset of $\mathcal{H}$,

iii. For all $t \in [0, T]$ $\mathcal{K}(t)$ is the infinitesimal generator of a C_0-semigroup on $\mathcal{H}$,

iv. There exist constants C and m independent of t such that for all $t \in [0, T]$, the semigroup $\{S_t(s)\}_{s\geq 0}$ generated by $\mathcal{K}(t)$ satisfies

$$\|S_t(s)u\|_{\mathcal{H}} \leq Ce^{ms} \|u\|_{\mathcal{H}},$$

for all $u \in \mathcal{H}$ and $s \geq 0$,

v. $\frac{d}{dt}\mathcal{K}(t) \in L^\infty_*([0, T], B(D(\mathcal{K}(0)), \mathcal{H})$.

Then problem

$$\begin{cases} \frac{d}{dt}U(t) = \mathcal{K}(t)U(t) \\ U(0) = U_0 \end{cases}$$

has a unique solution

$$U \in C([0, T], D(\mathcal{K}(t))) \cap C^1([0, T], \mathcal{H}),$$

for any initial datum in $D(\mathcal{K}(0))$.

Below, we prove that the conditions required by Theorem 3.2.1 are met by the operator $\widetilde{A}(t)$. Since $D(\widetilde{A}(t)) = D(A(t))$, then it follows from (3.27) that

$$D(\widetilde{A}(t)) = D(\widetilde{A}(0)), \tag{3.34}$$

that is the domain of $A(t)$ is independent of t.

Lemma 3.2.2. *$D(\widetilde{A}(0))$ is dense in $\mathcal{H}$.*

Proof. It is sufficient to show that $D(A(0))$ is a dense subset of $\mathcal{H}$. We proceed as in ([59]). Let $(f_1, f_2, g) \in \mathcal{H}$ be orthogonal to all elements of $D(A(0))$, *i.e.*

$$a_1 \int_{\Omega_1} \nabla y_1(x).\nabla\overline{f_1}(x)dx + a_2 \int_{\Omega_1} \nabla y_2(x).\nabla\overline{f_2}(x)dx + \int_{\Gamma_2}\int_0^1 z(x,\rho)\overline{g(x,\rho)}d\rho d\Gamma = 0, \tag{3.35}$$

for all $(y_1, y_2, g) \in D(A(0))$.

For $y_1 = 0, y_2 = 0$ and $z \in \mathcal{D}(\Gamma_2 \times (0,1)), (y_1, y_2, g) \in D(A(0))$, and

$$\int_{\Gamma_2}\int_0^1 z(x,\rho)\overline{g(x,\rho)}d\rho d\Gamma = 0$$

Since $\mathcal{D}(\Gamma_2 \times (0,1))$ is dense in $L^2(\Gamma_2, L^2(0,1))$, we conclude that $g = 0$. In the same manner, we obtain $y_2 = 0$ if we take $f_1 = 0$ and $z = 0$. Therefore the identity (3.35) is reduced to

$$\int_{\Omega_1} \nabla\overline{y_1}(x).\nabla\overline{f_1}(x)dx = 0$$

By taking in (3.35), $y_2 = 0$ and $z = 0$, we get

$$\int_{\Omega_1} \nabla y_1(x).\nabla\overline{f_1}(x)dx = 0 \qquad \text{for all } (y_1, 0, 0)^T \in D(A(0)). \tag{3.36}$$

But $(y_1, 0, 0)^T \in D(A(0))$ if and only if $y_1 \in \mathcal{D} = \{f \in \mathcal{V}; \Delta u_1 \in \mathcal{V}, \frac{\partial f}{\partial \nu} = 0 \text{ on } \Gamma_0\}$. Since $\mathcal{D}_1 = \{f \subset \mathcal{V} \cap H^2(\Omega_1) : \frac{\partial f}{\partial \nu} = 0 \text{ on } \Gamma_0\} \subset \mathcal{D}$ and $\mathcal{D}_1$ is dense in $\mathcal{V}$. Then $\mathcal{D}$ is dense in $\mathcal{V}$. Combining this fact with (3.36), we conclude that $f_1 = 0$. ☐

Lemma 3.2.3. *Define on the Hilbert space $\mathcal{H}$ the following time-dependent inner product*

$$\left\langle \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \\ g \end{pmatrix} \right\rangle_t = a_1 \int_{\Omega_1} \nabla y_1(x).\nabla\overline{f_1}(x)dx + a_2 \int_{\Omega_1} \nabla y_2(x).\nabla\overline{f_2}(x)dx + \xi\tau(t)\int_{\Gamma_2}\int_0^1 z(x,\rho)\overline{g(x,\rho)}d\rho d\Gamma$$

Then $\widetilde{A}(t)$ is dissipative for fixed t.

Proof. Let $Y = (y_1, y_2, z)^T \in D(A(t))$. Then

$$\begin{aligned} \Re\langle Y, A(t)Y\rangle_t &= -\Re\{a_1^2 i \int_{\Omega_1} \nabla y_1(x).\nabla(\Delta\overline{y_1(x)})dx\} - \Re\{a_2^2 i \int_{\Omega_2} \nabla y_2(x).\nabla(\Delta\overline{y_2(x)})dx\} - \\ &\quad \Re\xi\tau(t)\int_{\Gamma_2}\int_0^1 z(x,\rho)\overline{\frac{\tau'(t)\rho - 1}{\tau(t)}\partial_\rho z(x,\rho)}d\rho d\Gamma \end{aligned} \tag{3.37}$$

Applying Green's theorem to the first two integrals on the right-hand side of (3.37) and using the fact that the normal vector on Γ_0 is oriented towards the interior of Ω_1, we obtain

$$\begin{aligned} &- \Re\{a_1^2 i \int_{\Omega_1} \nabla y_1(x).\nabla(\Delta\overline{y_1(x)})dx\} - \Re\{a_2^2 i \int_{\Omega_2} \nabla u_2(x).\nabla(\Delta\overline{y_2(x)})dx\} = - \\ &\Re\{a_1^2 i \int_{\Gamma_1} \frac{\partial y_1(x)}{\partial \nu}\Delta\overline{y_1(x)}d\Gamma + a_1^2 i \int_{\Gamma_0} \frac{\partial y_1(x)}{\partial \nu}\Delta\overline{y_1(x)}d\Gamma - a_1^2 i \int_{\Omega_1} |\Delta y_1(x)|^2 \, dx\ \} - \\ &\Re\{a_2^2 i \int_{\Gamma_2} \frac{\partial y_2(x)}{\partial \nu}\Delta\overline{y_2(x)}d\Gamma - a_2^2 i \int_{\Gamma_0} \frac{\partial y_2(x)}{\partial \nu}\Delta\overline{y_2(x)}d\Gamma - a_2^2 i \int_{\Omega_2} |\Delta u_2(x)|^2 \, dx\} \end{aligned} \tag{3.38}$$

Note that the integrals over Γ_1 (resp.Γ_0) on the right-hand side of (3.38) are to be interpreted in the sense of duality pairing between $H^{1/2}(\Gamma_1)$ and $H^{-1/2}(\Gamma_1)$ (resp. $H^{1/2}(\Gamma_0)$ and $H^{-1/2}(\Gamma_0)$)
(3.38) together with (3.27) – (3.31) yields

$$\begin{aligned}&-\Re\{a_1^2 i\int_{\Omega_1}\nabla y_1(x).\nabla(\Delta\overline{y_1(x)})dx\}-\Re\{a_2^2 i\int_{\Omega_2}\nabla u_2(x).\nabla(\Delta\overline{y_2(x)})dx\}=-\\&a_2\alpha\int_{\Gamma_2}|z(x,0)|^2\,d\Gamma-a_2\beta\Re\int_{\Gamma_2}z(x,1)\overline{z}(x,0)d\Gamma\end{aligned}\tag{3.39}$$

Integrating by parts in ρ the third integral on the right-hand side of (3.37), we get

$$2\Re\xi\int_{\Gamma_2}\int_0^1 z(x,\rho)\frac{\tau'(t)\rho-1}{\tau(t)}\overline{\partial_\rho z(x,\rho)}d\rho d\Gamma=\int_{\Gamma_2}\{|z(x,1)|^2\,(1-\tau'(t))-|z(x,0)|^2\}d\Gamma-\tau'(t)\int_{\Gamma_2}\int_0^1|z(x,\rho)|^2\,d\rho d\Gamma\tag{3.40}$$

Inserting (3.39) and (3.40) into (3.37) results in

$$\begin{aligned}&\Re\left\langle Y,A(t)Y\right\rangle_t=-a_2\alpha\int_{\Gamma_2}|z(x,0)|^2\,d\Gamma-a_2\beta\Re\int_{\Gamma_2}z(x,1)\overline{z}(x,0)d\Gamma-\\&\frac{\xi}{2}\int_{\Gamma_2}\{|z(x,1)|^2\,(1-\tau'(t))-|z(x,0)|^2\}d\Gamma-\frac{\xi\tau'(t)}{2}\int_{\Gamma_2}\int_0^1|z(x,\rho)|^2\,d\rho d\Gamma\end{aligned}$$

from which follows after using the Cauchy-Schwarz inequality

$$\Re\left\langle Y,A(t)Y\right\rangle_t\leq-(a_2\alpha-\frac{\xi}{2}-\frac{a_2\beta}{2\sqrt{1-d}})\int_{\Gamma_2}|z(x,0)|^2\,d\Gamma-(\frac{\xi(1-d)}{2}-\frac{a_2\beta\sqrt{1-d}}{2})\int_{\Gamma_2}|z(x,1)|^2\,d\Gamma-\kappa(t)\left\|Y\right\|_t^2\tag{3.41}$$

where(3.41) together with (3.15) implies that

$$\Re\left\langle\widetilde{A}(t)Y,Y\right\rangle_t\leq 0\tag{3.42}$$

Thus the operator $\widetilde{A}(t)$ is dissipative. □

Lemma 3.2.4. *The operator $\widetilde{A}(t)$ is maximal for each fixed t.*

Proof. Since $\kappa(t)>0$, then the maximality of $\widetilde{A}(t)$ follows from that of $A(t)$. To this end, let $(f_1,f_2,g)^T\in\mathcal{H}$, and consider for some $\lambda>0$ the equation

$$(\lambda I-A(t))Y=(f_1,f_2,g)^T$$

where $Y=(y_1,y_2,z)$ or equivalently

$$\lambda y_k(x)-ia_k\Delta y_k(x)=f_k(x)\qquad\text{in }\Omega_k,k=1,2,\tag{3.43}$$

$$\lambda z(x,\rho)+\frac{1-\tau'(t)\rho}{\tau(t)}\partial_\rho z(x,\rho)=g(x,\rho)\qquad\text{on }\Gamma_2\times(0,1),\tag{3.44}$$

$$y_1(x)=0\qquad\text{on }\Gamma_1,\tag{3.45}$$

$$\frac{\partial y_2(x)}{\partial\nu}=-\alpha z(x,0)-\beta z(x,1)\qquad\text{on }\Gamma_2,\tag{3.46}$$

$$y_1(x)=y_2(x)\qquad\text{on }\Gamma_0,\tag{3.47}$$

$$a_1\frac{\partial y_1(x)}{\partial\nu}=a_2\frac{\partial y_2(x)}{\partial\nu}\qquad\text{on }\Gamma_0,\tag{3.48}$$

$$a_1\Delta u_1(x)=a_2\Delta u_2(x)\qquad\text{on }\Gamma_0.\tag{3.49}$$

Suppose that we have found (y_1, y_2) with the appropriate regularity, then we can determine z. Indeed, from (3.44) and (3.30) we have

$$\begin{cases} \partial_\rho z(x,\rho) = \frac{\lambda\tau(t)}{1-\tau'(t)\rho} z(x,\rho) + \frac{\tau(t)}{1-\tau'(t)\rho} g(x,\rho), & x \in \Gamma_2, \rho \in (0,1), \\ z(x,0) = ia_2 \Delta u_2(x), & x \in \Gamma_2. \end{cases}$$

The unique solution of the above initial value problem is given by

$$\begin{aligned} & z(x,\rho) = e^{-\lambda\rho\tau(t)} z(x,0) + \tau(t) e^{-\lambda\rho\tau(t)} \int_0^\rho e^{\lambda s\tau(t)} g(x,s)ds, && \text{if } \tau'(t) = 0, \\ & z(x,\rho) = z(x,0)\exp(\frac{\lambda\tau(t)\ln(1-\tau'(t)\rho)}{\tau'(t)}) + \\ & \exp(\frac{\lambda\tau(t)\ln(1-\tau'(t)\rho)}{\tau'(t)}) \int_0^\rho \frac{g(x,s)\tau(t)}{1-\tau'(t)s} \exp(\frac{-\lambda\tau(t)\ln(1-\tau'(t)s)}{\tau'(t)})ds, && \text{otherwise.} \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{\partial y_2(x)}{\partial \nu} = -ia_2(\alpha + \beta e^{-\lambda\tau(t)})\Delta y_2(x) - \beta\tau e^{-\lambda\tau(t)} \int_0^1 e^{\lambda\tau(t)s} g(x,s)ds, \quad x \in \Gamma_2, \text{ if } \tau'(t) = 0, \\ & \frac{\partial y_2(x)}{\partial \nu} = z(x,0)\exp(\frac{\lambda\tau(t)\ln(1-\tau'(t))}{\tau'(t)}) + \\ & \exp(\frac{\lambda\tau(t)\ln(1-\tau'(t))}{\tau'(t)}) \int_0^1 \frac{g(x,s)\tau(t)}{1-\tau'(t)s} \exp(\frac{-\lambda\tau(t)\ln(1-\tau'(t)s)}{\tau'(t)})ds, \quad x \in \Gamma_2, \qquad \text{otherwise.} \end{aligned}$$

From (3.43), we have

$$\lambda y_1(x) - ia_1\Delta y_1(x) = f_1(x), \qquad x \in \Omega_1, \tag{3.50}$$

$$\lambda y_2(x) - ia_2\Delta y_2(x) = f_2(x), \qquad x \in \Omega_2, \tag{3.51}$$

and we proceed as in the previous chapter to prove that (3.50)-(3.51) has a unique solution $(y_1, y_2) \in \{(u_1, u_2) \in \mathcal{V} \times L^2(\Gamma_2; H^1(0,1)); \Delta u_1 \in H^1_{\Gamma_1}(\Omega_1), \Delta u_2 \in H^1(\Omega_2)\}$ and satisfying (3.28)-(3.31), and consequently (3.43)-(3.49) has a unique solution $(y_1, y_2) \in D(A(t))$ and thus $\lambda I - A(t)$ is onto for some $\lambda > 0$ and for all $t > 0$. This shows that $A(t)$ is maximal for each fixed t. □

From Lemma 3.2.3 and Lemma 3.2.4, we obtain the following result.

Proposition 3.2.5. *For each fixed* $t \in [0,T]$, *the operator* $\widetilde{A}(t)$ *generates a* C_0*-semigroup* $\widetilde{S}_t(s)$ *on* $\mathcal{H}$.

Lemma 3.2.6. *There exist constants* C *and* m *independent of* t *such that for all* $t \in [0,T]$, *the semigroup* $\{S_t(s)\}_{s\geq 0}$ *generated by* $\mathcal{L}(t)$ *satisfies*

$$\left\| S_t(s)u \right\|_{\mathcal{H}} \leq Ce^{ms} \left\| u \right\|_{\mathcal{H}}, \tag{3.52}$$

for all $u \in \mathcal{H}$ *and* $s \geq 0$.

Proof. Let $\varphi = (y_1, y_2, z) \in D(\mathcal{K}(\prime)$, then

$$\begin{aligned} \left\|\varphi\right\|_s^2 &= a_1 \int_{\Omega_1} \left|\nabla y_1(x)\right|^2 dx + a_2 \int_{\Omega_2} \left|\nabla y_2(x)\right|^2 dx + \xi\tau(s) \int_{\Gamma_2}\int_0^1 \left|z(x,\rho)\right|^2 d\rho d\Gamma, \\ \left\|\varphi\right\|_r^2 &= a_1 \int_{\Omega_1} \left|\nabla y_1(x)\right|^2 dx + a_2 \int_{\Omega_2} \left|\nabla y_2(x)\right|^2 dx + \xi\tau(r) \int_{\Gamma_2}\int_0^1 \left|z(x,\rho)\right|^2 d\rho d\Gamma, \end{aligned}$$

and

$$\frac{\|\varphi\|_s^2}{\|\varphi\|_r^2} \le 1 + \frac{\tau(s) - \tau(r)}{\tau(r)}$$

From the mean value theorem, we have

$$\tau(s) - \tau(r) = \tau'(a)(s - r), \quad \text{where } a \in (r, s)$$

and thus

$$\frac{\|\varphi\|_s^2}{\|\varphi\|_r^2} \le 1 + \frac{|\tau'(a)|}{\tau(r)} |s - r|$$

By (3.11), τ' is bounded and therefore

$$\frac{\|\varphi\|_s^2}{\|\varphi\|_r^2} \le 1 + \frac{|\tau'(a)|}{\tau(r)} |s - r| \le 1 + \frac{d}{\widehat{\tau}} |s - r|$$

which gives

$$\|\varphi\|_s^2 \le e^{\frac{d}{\widehat{\tau}}|s-r|} \|\varphi\|_r^2 . \tag{3.53}$$

and the desired inequality (3.52) follows from (3.53) with $C = 1$ and $m = \frac{d}{\widehat{\tau}}$. □

Lemma 3.2.7. *For the operator $\widetilde{A}(t)$ we have*

$$\frac{d}{dt}\widetilde{A}(t) \in L_*^\infty([0,T], B(D(A(0)), \mathcal{H}).$$

Proof. We have

$$\frac{d}{dt}\widetilde{A}(t) = \frac{d}{dt}A(t) - \kappa'(t)I$$

where

$$\kappa'(t) = \frac{\tau''(t)\tau'(t)}{2\tau(t)\sqrt{\tau'(t)^2 + 1}} - \frac{\tau'(t)\sqrt{\tau'(t)^2 + 1}}{2\tau(t)^2}$$

and

$$\frac{d}{dt}A(t) = (0, 0, \frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1))}{\tau(t)^2})^T$$

By (3.11) and (3.12), $k'(t)$ and $\frac{\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1))}{\tau(t)^2}$ are bounded on $[0,T]$. Thus

$$\frac{d}{dt}\widetilde{A}(t) \in L_*^\infty([0,T], B(D(A(0)), \mathcal{H}).$$

as desired. □

The main result of this section can now be stated.

Theorem 3.2.8. *For any initial datum $Y_0 \in D(A(0))$, problem (3.32) has a unique solution*

$$Y \in C([0, +\infty), D(A(0))) \cap C^1([0, +\infty), \mathcal{H}). \tag{3.54}$$

Proof. It follows from (3.34), Lemma 3.2.2, Proposition 3.2.5, Lemma 3.2.6, Lemma 3.2.7 that $\widetilde{A}(t)$ satisfies all the hypothesis of Theorem 3.2.1. Therefore for any initial datum $Y_0 \in D(\widetilde{A}(0))$ problem (3.33) has a unique solution

$$\widetilde{Y} \in C([0, +\infty), D(\widetilde{A}(0))) \cap C^1([0, +\infty), \mathcal{H}). \tag{3.55}$$

and the desired conclusion follows from the equality $Y(t) = e^{\theta(t)}\widetilde{Y}(t)$ □

3.3 Proof of the exponential stability result

We proceed in several steps.
Step 1.

First, we show that the energy function defined by (3.14) is decreasing.

Proposition 3.3.1. *The energy corresponding to any regular solution of problem (3.3)-(3.9) is decreasing and there exists a positive constant K such that*

$$\frac{d}{dt}E(t) \leq -K \int_{\Gamma_2} \left\{ |\partial_t y_2(x,t)|^2 + |\partial_t y_2(x, t-\tau(t))|^2 \right\} d\Gamma$$

where

$$\begin{array}{rcll} K & = & \min\{a_2\alpha - \dfrac{a_2\beta}{2\sqrt{1-d}} - \dfrac{\xi}{2}, \dfrac{\xi(1-\tau'(t))-}{2} - \dfrac{a_2\beta\sqrt{1-d}}{2}\} & \text{if } \tau(t) \neq 0, \\ K & = & a_2(\alpha+\beta) & \text{if } \tau(t) = 0. \end{array}$$

Proof. Differentiating $E(t)$, we obtain

$$\begin{aligned} &\frac{d}{dt}E(t) = a_1 \Re \int_{\Omega_1} \nabla \overline{y}_1(x,t).\nabla \partial_t y_1(x,t)dx + a_2 \Re \int_{\Omega_2} \nabla \overline{y}_2(x,t).\nabla \partial_t y_2(x,t)dx + \\ &\xi\tau(t)\Re \int_{\Gamma_2}\int_0^1 \partial_t \overline{y}_2(x,t-\tau(t)\rho)\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau(t)\rho)d\rho d\Gamma + \frac{\xi}{2}\tau'(t)\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x,t-\tau(t)\rho)|^2 d\rho d\Gamma \end{aligned} \tag{3.56}$$

Applying Green's Theorem to the first two integrals on the right-hand side of (3.56), we obtain

$$\begin{aligned} &\frac{d}{dt}E(t) = a_1\Re\int_{\Gamma_1} \frac{\partial \overline{y}_1(x,t)}{\partial\nu}\partial_t y_1(x,t)d\Gamma + a_1\Re\int_{\Gamma_0}\frac{\partial \overline{y}_1(x,t)}{\partial\nu}\partial_t y_1(x,t)d\Gamma + \\ &a_2\Re\int_{\Gamma_2}\frac{\partial \overline{y}_2(x,t)}{\partial\nu}\partial_t y_2(x,t)d\Gamma - a_2\Re\int_{\Gamma_0}\frac{\partial \overline{y}_2(x,t)}{\partial\nu}\partial_t y_2(x,t)d\Gamma + \\ &\xi\tau(t)\Re\int_{\Gamma_2}\int_0^1 \partial_t\overline{y}_2(x,t-\tau(t)\rho)\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma + \\ &\frac{\xi}{2}\tau'(t)\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x,t-\tau(t)\rho)|^2 d\rho d\Gamma \end{aligned}$$

Recalling the boundary conditions (3.5)-(3.6) and the transmission conditions (3.7)-(3.8), we get

$$\begin{aligned} &\frac{d}{dt}E(t) = -a_2\alpha\int_{\Gamma_2}|\partial_t y_2(x,t)|^2 d\Gamma - a_2\beta\Re\int_{\Gamma_2}\partial_t\overline{y}_2(x,t-\tau)\partial_t y_2(x,t)d\Gamma + \\ &\xi\tau(t)\Re\int_{\Gamma_2}\int_0^1 \partial_t\overline{y}_2(x,t-\tau(t)\rho)\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma + \\ &\frac{\xi}{2}\tau'(t)\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x,t-\tau(t)\rho)|^2 d\rho d\Gamma \end{aligned} \tag{3.57}$$

Now we have

$$\partial_\rho y(x,t-\tau(t)\rho) = -\tau(t)\partial_t y(x,t-\tau(t)\rho), \tag{3.58}$$

$$\partial_\rho^2 y(x,t-\tau(t)\rho) = \tau(t)^2\partial_t^2 y(x,t-\tau(t)\rho) \tag{3.59}$$

Therefore

$$
\begin{aligned}
&\Re \int_0^1 \partial_t \overline{y}_2(x,t-\tau(t)\rho)\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma = \\
&-\frac{1}{\tau(t)^3}\Re \int_0^1 \partial_\rho \overline{y}_2(x,t-\tau(t)\rho)\partial_\rho^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma \\
&= -\frac{1}{2\tau(t)^3}\Re \int_0^1 (1-\tau'(t)\rho)\frac{d}{d\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma \\
&= -\frac{1}{2\tau(t)^3}\Re[(1-\tau'(t)\rho)\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2]_0^1 - \frac{\tau'(t)}{2\tau(t)^3}\int_0^1 \left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2 d\rho \\
&= \frac{1}{2\tau(t)^3}[\left|\partial_\rho y_2(x,t)\right|^2 - (1-\tau'(t))\left|\partial_\rho y_2(x,t-\tau(t))\right|^2] - \frac{\tau'(t)}{2\tau(t)^3}\int_0^1 \left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2 d\rho \\
&= \frac{1}{2\tau(t)}[\left|\partial_t y_2(x,t)\right|^2 - (1-\tau'(t))\left|\partial_t y_2(x,t-\tau(t))\right|^2] - \frac{\tau'(t)}{2\tau(t)}\int_0^1 \left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho \qquad (3.60)
\end{aligned}
$$

Inserting (3.60) into (3.57) yields

$$
\begin{aligned}
&\frac{d}{dt}E(t) = -a_2\alpha\int_{\Gamma_2}\left|\partial_t y_2(x,t)\right|^2 d\Gamma - a_2\beta\Re\int_{\Gamma_2}\partial_t\overline{y}_2(x,t-\tau(t))\partial_t y_2(x,t)d\Gamma + \\
&\frac{\xi}{2}\int_{\Gamma_2}[\left|\partial_t y_2(x,t)\right|^2 - (1-\tau'(t))\left|\partial_t y_2(x,t-\tau(t))\right|^2]d\Gamma
\end{aligned}
$$

from which we obtain after using Cauchy-Schwarz's inequality

$$
\begin{aligned}
\frac{d}{dt}E(t) \quad \le \quad & -a_2\alpha\int_{\Gamma_2}\left|\partial_t y_2(x,t)\right|^2 d\Gamma + \frac{a_2\beta}{2\sqrt{1-d}}\int_{\Gamma_2}\left|\partial_t y_2(x,t)\right|^2 d\Gamma + \frac{a_2\beta\sqrt{1-d}}{2}\int_{\Gamma_2}\left|\partial_t y_2(x,t-\tau(t))\right|^2 d\Gamma - \\
& \frac{\xi(1-\tau'(t))}{2}\int_{\Gamma_2}\left|\partial_t y_2(x,t-\tau(t))\right|^2 d\Gamma + \frac{\xi}{2}\int_{\Gamma_2}\left|\partial_t y_2(x,t)\right|^2 d\Gamma
\end{aligned}
$$

and consequently

$$
\frac{d}{dt}E(t) \le -K\int_{\Gamma_2}\{\left|\partial_t y_2(x,t)\right|^2 + \left|\partial_t y_2(x,t-\tau(t))\right|^2\}d\Gamma
$$

with

$$
K = \min\{a_2\alpha - \frac{a_2\beta}{2\sqrt{1-d}} - \frac{\xi}{2}, \frac{\xi(1-\tau'(t))-}{2} - \frac{a_2\beta\sqrt{1-d}}{2}\}
$$

Assumption (3.15) implies that the constant K is positive, which concludes the proof of Proposition 3.3.1. □

Step 2.
Set

$$
\mathbb{E}(t) = E(t) + \gamma\{\Im\int_{\Omega_1} y_1(x)m(x).\nabla y_1(x)dx + \Im\int_{\Omega_2} y_2(x)m(x).\nabla y_2(x)dx + \mathcal{E}(t)\}
$$

where γ is a positive constant that will be chosen later and $\mathcal{E}(t)$ is given by

$$
\mathcal{E}(t) = \xi\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\, d\Gamma \qquad (3.61)
$$

Lemma 3.3.2. *For γ small enough, the functional $\mathbb{E}$ is equivalent to the energy E, that is there exist two positive constants μ_1 and μ_2 such that*

$$
\mu_1\mathbb{E}(t) \le E(t) \le \mu_2\mathbb{E}(t)
$$

Proof. See the Appendix B. ☐

Lemma 3.3.3. *For any regular solution of problem (3.3)-(3.9), there exist positive constants* C_0 *and* C_1 *such that*

$$\Psi(t) \leq -C_0E_s(t) + C_1\{\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x, t-\tau(t))|^2\}d\Gamma\} \tag{3.62}$$

where

$$\Psi(t) = \sum_{k=1}^{2}\Im\int_{\Omega_k} y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx$$

and

$$E_s(t) = \sum_{k=1}^{2}a_k\int_{\Omega_k}|\nabla y_k(t,x)|^2\,dx$$

Proof. We have

$$\frac{d}{dt}\Psi(t) = \sum_{k=1}^{2}\Im\int_{\Omega_k}\{\partial_t y_k(t,x)m(x).\nabla\overline{y_k(t,x)} + y_k(t,x)m(x).\nabla\overline{\partial_t y_k(t,x)}\}dx$$

By using Green's theorem, we get after using the boundary condition (3.5)

$$\Im\int_{\Omega_1} y_1(t,x)m(x).\nabla\overline{\partial_t y_1(t,x)}dx = \Im\int_{\partial\Omega_1} y_1(t,x)\overline{\partial_t y_1(t,x)}m(x).\nu(x)d\Gamma - \Im\int_{\Omega_1} y_1(t,x)\overline{\partial_t y_1(t,x)}div\, m(x)dx -$$
$$\Im\int_{\Omega_1}\overline{\partial_t y_1(t,x)}m(x).\nabla y_1(t,x)dx$$
$$= -\Im\int_{\Gamma_0} y_1(t,x)\overline{\partial_t y_1(t,x)}m(x).\nu(x)d\Gamma - n\Im\int_{\Omega_1} y_1(t,x)\overline{\partial_t y_1(t,x)}dx + \Im\int_{\Omega_1}\partial_t y_1(t,x)m(x).\nabla\overline{y_1(t,x)}dx \tag{3.63}$$

and

$$\Im\int_{\Omega_2} {}_2y_2(t,x)m(x).\nabla\overline{\partial_t y_2(t,x)}dx = \Im\int_{\partial\Omega_2} y_2(t,x)\overline{\partial_t y_2(t,x)}m(x).\nu(x)d\Gamma - \Im\int_{\Omega_2} y_2(t,x)\overline{\partial_t y_2(t,x)}div\, m(x)dx -$$
$$\Im\int_{\Omega_2}\overline{\partial_t y_2(t,x)}m(x).\nabla y_2(t,x)dx$$
$$= \Im\int_{\Gamma_0} y_2(t,x)\overline{\partial_t y_2(t,x)}m(x).\nu(x)d\Gamma + \Im\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(t,x)}m(x).\nu(x)d\Gamma - n\Im\int_{\Omega_2} y_2(t,x)\overline{\partial_t y_2(t,x)}dx +$$
$$\Im\int_{\Omega_2}\partial_t y_2(t,x)m(x).\nabla\overline{y_2(t,x)}dx \tag{3.64}$$

Summing up (3.63) and (3.64) and recalling the boundary conditions (3.7) and (3.8), we obtain

$$\frac{d}{dt}\sum_{k=1}^{2}\Im\int_{\Omega_k} y_k(t,x)m(x).\nabla\overline{\partial_t y_k(t,x)}dx = \Im\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(t,x)}m(x).\nu(x)d\Gamma +$$
$$\sum_{k=1}^{2}\Im\int_{\Omega_k}\{-ny_k(t,x)\overline{\partial_t y_k(t,x)} + \partial_t y_k(t,x)m(x).\nabla\overline{y_k(t,x)}\}dx$$

On the other hand, using equation (3.3), we get

$$\Im\int_{\Omega_1} y_1(t,x)\overline{\partial_t y_1(t,x)}dx = -a_1\Re\int_{\Omega_1} y_1(t,x)\Delta\overline{y_1(t,x)}dx$$
$$= a_1\Re\int_{\Gamma_0}\overline{y_1(t,x)}\frac{\partial y_1(t,x)}{\partial\nu}d\Gamma + a_1\int_{\Omega_1}|\nabla y_1(t,x)|^2\,dx \tag{3.65}$$

and

$$\Im \int_{\Omega_2} y_2(t,x)\overline{\partial_t y_2(t,x)}dx = -a_2\Re \int_{\Omega_2} y_2(t,x)\Delta\overline{y_2(t,x)}dx$$
$$= -a_2\Re\int_{\Gamma_0} y_2(t,x)\frac{\partial\overline{y_2(t,x)}}{\partial\nu}d\Gamma - a_2\Re\int_{\Gamma_2} y_2(t,x)\frac{\partial\overline{y_2(t,x)}}{\partial\nu}d\Gamma + a_2\int_{\Omega_2}\left|\nabla y_2(t,x)\right|^2 dx$$
$$= -a_2\Re\int_{\Gamma_0} y_2(t,x)\frac{\partial\overline{y_2(t,x)}}{\partial\nu}d\Gamma + a_2\alpha\Re\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(x,t)}d\Gamma + a_2\beta\Re\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(x,t-\tau(t))}d\Gamma +$$
$$a_2\int_{\Omega_2}\left|\nabla y_2(t,x)\right|^2 dx \tag{3.66}$$

Combining (3.65) and (3.66) and using the transmission conditions (3.7) and (3.8) gives

$$\sum_{k=1}^{2}\Im\int_{\Omega_k}\partial_t y_k(t,x)\overline{y_k(t,x)}dx = a_2\alpha\Re\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(x,t)}d\Gamma + a_2\beta\Re\int_{\Gamma_2} y_2(t,x)\overline{\partial_t y_2(x,t-\tau(t))}d\Gamma +$$
$$\sum_{k=1}^{2} a_k\int_{\Omega_k}\left|\nabla y_k(t,x)\right|^2 dx$$

We also have from (3.3)

$$\sum_{k=1}^{2}\Im\int_{\Omega_k}\partial_t y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx = \sum_{k=1}^{2} a_k\Re\int_{\Omega_k}\Delta y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx$$

Thus

$$\frac{d}{dt}\Psi(t) = \Im\int_{\Gamma_2}\partial_t y_2(t,x)\overline{y_2(t,x)}m(x).\nu(x)d\Gamma - na_2\alpha\Re\int_{\Gamma_2}\overline{y_2(t,x)}\partial_t y_2(x,t)d\Gamma -$$
$$na_2\beta\Re\int_{\Gamma_2}\overline{y_2(t,x)}\partial_t y_2(x,t-\tau(t))d\Gamma - \sum_{k=1}^{2} na_k\int_{\Omega_k}\left|\nabla y_k(t,x)\right|^2 dx +$$
$$2\sum_{k=1}^{2} a_k\Re\int_{\Omega_k}\Delta y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx$$

Now we have

$$2\Re\int_{\Omega_k}\Delta y_k(t,x)m(x).\nabla\overline{y_k(t,x)}dx = 2\Re\int_{\partial\Omega_k}\frac{\partial y_k(x,t)}{\partial\nu}m(x).\nabla\overline{y_k(t,x)}d\Gamma + (n-2)\int_{\Omega_k}\left|\nabla y_k(t,x)\right|^2 dx -$$
$$\int_{\partial\Omega_k}\left|\nabla y_k(t,x)\right|^2 d\Gamma$$

Specializing this identity to $k=1$ and $k=2$, we find

$$2\Re\int_{\Omega_1}\Delta y_1(t,x)m(x).\nabla\overline{y_1(t,x)}dx = \int_{\Gamma_1}\left|\nabla y_1(t,x)\right|^2 m(x).\nu(x)d\Gamma - 2\Re\int_{\Gamma_0}\frac{\partial y_1(x,t)}{\partial\nu}m(x).\nabla\overline{y_1(t,x)}d\Gamma +$$
$$\int_{\Gamma_0}\left|\nabla y_1(t,x)\right|^2 m(x).\nu(x)d\Gamma + (n-2)\int_{\Omega_1}\left|\nabla y_1(t,x)\right|^2 dx$$

$$2\Re\int_{\Omega_2}\Delta y_2(t,x)m(x).\nabla\overline{y_2(t,x)}dx = 2\Re\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}m(x).\nabla\overline{y_2(t,x)}d\Gamma - \int_{\Gamma_2}\left|\nabla y_2(t,x)\right|^2 d\Gamma +$$
$$2\Re\int_{\Gamma_0}\frac{\partial y_2(x,t)}{\partial\nu}m(x).\nabla\overline{y_2(t,x)}d\Gamma - \int_{\Gamma_0}\left|\nabla y_2(t,x)\right|^2 d\Gamma + (n-2)\int_{\Omega_2}\left|\nabla y_2(t,x)\right|^2 dx$$

and consequently

$$\begin{aligned}
&2\sum_{k=1}^{2} a_k \Re \int_{\Omega_k} \Delta y_k(t,x) m(x).\nabla\overline{y_k(t,x)} dx = a_1 \int_{\Gamma_1} \left|\nabla y_1(t,x)\right|^2 m(x).\nu(x) d\Gamma - \\
&2a_1 \Re \int_{\Gamma_0} \frac{\partial y_1(x,t)}{\partial\nu} m(x).\nabla\overline{y_1(t,x)} d\Gamma + a_1 \int_{\Gamma_0} \left|\nabla y_1(t,x)\right|^2 m(x).\nu(x) d\Gamma + 2a_2 \Re \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial\nu} m(x).\nabla\overline{y_2(t,x)} d\Gamma - \\
&a_2 \int_{\Gamma_2} \left|\nabla y_2(t,x)\right|^2 d\Gamma + 2a_2 \Re \int_{\Gamma_0} \frac{\partial y_2(x,t)}{\partial\nu} m(x).\nabla\overline{y_2(t,x)} d\Gamma - a_2 \int_{\Gamma_0} \left|\nabla y_2(t,x)\right|^2 d\Gamma + \\
&(n-2)\sum_{k=1}^{2} a_k \int_{\Omega_k} \left|\nabla y_k(t,x)\right|^2 dx
\end{aligned} \tag{3.67}$$

We conclude from the boundary condition (3.7) that

$$\nabla(y_2(x,t) - y_1(x,t)) = \frac{\partial(y_2(x,t) - y_1(x,t))}{\partial\nu}\nu(x) \quad \text{on } \Gamma_0 \times (0,T),$$

then

$$\left|\nabla y_2(x,t)\right|^2 = \left|\nabla y_1(x,t)\right|^2 + \left|\frac{\partial y_2(x,t)}{\partial\nu}\right|^2 - \left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 \qquad \text{on } \Gamma_0 \times (0,T),$$

so on $\Gamma_0 \times (0,T)$,

$$\begin{aligned}
&2a_1 \Re\left(\frac{\partial y_1(x,t)}{\partial\nu} m(x).\nabla\overline{y_1(x,t)}\right) - 2a_2 Re\left(\frac{\partial y_2(x,t)}{\partial\nu} m(x).\nabla\overline{y_2(x,t)}\right) - a_1 \left|\nabla y_1(x,t)\right|^2 m(x).\nu(x) + \\
&a_2 \left|\nabla y_2(x,t)\right|^2 mx).\nu(x) = (a_2 - a_1)\left|\nabla y_1(x,t)\right|^2 m(x).\nu(x) - \frac{(a_2-a_1)^2}{a_2}\left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 m(x).\nu(x)
\end{aligned} \tag{3.68}$$

after using the boundary condition (3.8).

Inserting (3.68) into (3.67) yields

$$\begin{aligned}
&2\sum_{k=1}^{2} a_k \Re \int_{\Omega_k} \Delta y_k(t,x) m(x).\nabla\overline{y_k(t,x)} dx = a_1 \int_{\Gamma_1} \left|\nabla y_1(t,x)\right|^2 m(x).\nu(x) d\Gamma + \\
&2a_2 \Re \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial\nu} m(x).\nabla\overline{y_2(t,x)} d\Gamma - a_2 \int_{\Gamma_2} \left|\nabla y_2(t,x)\right|^2 m(x).\nu(x) d\Gamma + (n-2)\sum_{k=1}^{2} a_k \int_{\Omega_k} \left|\nabla y_k(t,x)\right|^2 dx + \\
&(a_1 - a_2)\int_{\Gamma_0} \left|\nabla y_1(x,t)\right|^2 m(x).\nu(x) d\Gamma + \frac{(a_2-a_1)^2}{a_2}\int_{\Gamma_0} \left|\frac{\partial y_1(x,t)}{\partial\nu}\right|^2 m(x).\nu(x) d\Gamma
\end{aligned} \tag{3.69}$$

From (3.62) and (3.69), and invoking assumption (3.1), we deduce that

$$\begin{aligned}
&\frac{d}{dt}\Psi(t) \leq \Im \int_{\Gamma_2} \partial_t y_2(t,x)\overline{y_2(t,x)} m(x).\nu(x) d\Gamma + 2a_2 \Re \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial\nu} m(x).\nabla\overline{y_2(t,x)} d\Gamma - a_2\delta \int_{\Gamma_2} \left|\nabla y_2(t,x)\right|^2 d\Gamma - \\
&n a_2 \alpha \Re \int_{\Gamma_2} \overline{y_2(t,x)}\partial_t y_2(x,t) d\Gamma - n a_2 \beta \Re \int_{\Gamma_2} \overline{y_2(t,x)}\partial_t y_2(x, t-\tau(t)) d\Gamma - 2\sum_{k=1}^{2} a_k \int_{\Omega_k} \left|\nabla y_k(t,x)\right|^2 dx
\end{aligned} \tag{3.70}$$

For the first term on the right-hand side of (3.70), we use Young's inequality, trace theorem and Poincaré's inequality to get the following estimate

$$\begin{aligned}&\left|\Im\int_{\Gamma_2}\partial_t y_2(t,x)\overline{y_2(t,x)}m(x).\nu(x)d\Gamma\right| \leq \frac{\mathcal{M}^2}{2\epsilon}\int_{\Gamma_2}\left|\partial_t y_2(t,x)\right|^2 d\Gamma+\frac{\epsilon}{2}\int_{\Gamma_2}\left|y_2(t,x)\right|^2 d\Gamma\\ &\leq \frac{\mathcal{M}^2}{2\epsilon}\int_{\Gamma_2}\left|\partial_t y_2(t,x)\right|^2 d\Gamma+\frac{\epsilon}{2}\int_{\Omega_2}\left|\nabla y_2(t,x)\right|^2 dx+\frac{\epsilon}{2}\int_{\Gamma_2}\left|y_2(t,x)\right|^2 d\Gamma\\ &\leq \frac{\mathcal{M}^2}{2\epsilon}\int_{\Gamma_2}\left|\partial_t y_2(t,x)\right|^2 d\Gamma+\frac{\epsilon}{2a_2}a_2\int_{\Omega_2}\left|\nabla y_2(t,x)\right|^2 dx+\frac{\epsilon}{2a_1}C_p a_1\int_{\Omega_1}\left|\nabla y_1(t,x)\right|^2 dx \end{aligned} \tag{3.71}$$

where $\mathcal{M} = \sup|m(x)|$, Cp is the Poincaré's constant and ϵ is an arbitrary positive constant.

For the second term, we have

$$2a_2\Re\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}m(x).\nabla\overline{y_2(t,x)}d\Gamma \leq \frac{a_2\mathcal{M}^2}{\delta}\int_{\Gamma_2}\left|\frac{\partial y_2(x,t)}{\partial\nu}\right|^2 d\Gamma+a_2\delta\int_{\Gamma_2}\left|\nabla y_2(t,x)\right|^2 d\Gamma \tag{3.72}$$

Inserting (3.71) and (3.72) into (3.70) leads to for ϵ small enough

$$\begin{aligned}&\frac{d}{dt}\Psi(t) \leq -C_0\sum_{k=1}^{2}a_k\int_{\Omega_k}\left|\nabla y_k(t,x)\right|^2 dx+\frac{\mathcal{M}^2}{2\epsilon}\int_{\Gamma_2}\left|\partial_t y_2(t,x)\right|^2 d\Gamma+\frac{a_2\mathcal{M}^2}{\delta}\int_{\Gamma_2}\left|\frac{\partial y_2(x,t)}{\partial\nu}\right|^2 d\Gamma-\\ &na_2\alpha\Re\int_{\Gamma_2}\overline{y_2(t,x)}\partial_t y_2(x,t)d\Gamma-na_2\beta\Re\int_{\Gamma_2}\overline{y_2(t,x)}\partial_t y_2(x,t-\tau(t))d\Gamma\end{aligned}$$

Recalling the boundary condition (3.6) and using the Cauchy-Scwarz and Young inequalities give us the desired estimate (3.62).

Lemma 3.3.4. *For any regular solution of problem (3.3)-(3.9),*

$$\frac{d}{dt}\mathcal{E}(t) \leq -2\mathcal{E}(t)+\xi\int_{\Gamma_2}\left|\partial_t y_2(x,t)\right|^2 d\Gamma$$

Proof. Differentiating both sides of (3.61) yields

$$\xi\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\, d\Gamma$$

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t) = {}&\xi\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\, d\Gamma-2\xi\tau(t)\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\rho\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\, d\Gamma+\\ &2\xi\tau(t)\Re\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_t\overline{y_2(x,t-\tau(t)\rho)}\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma.\end{aligned} \tag{3.73}$$

.

We have from (3.58) and (3.59)

$$\Re\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_t\overline{y_2(x,t-\tau(t)\rho)}\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma$$

$$=-(\tau(t))^{-3}\Re\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_\rho\overline{y_2(x,t-\tau(t)\rho)}\partial_\rho^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho d\Gamma$$

$$=-\frac{1}{2}(\tau(t))^{-3}\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_\rho\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)d\rho d\Gamma$$

$$=-\frac{1}{2}(\tau(t))^{-3}\int_{\Gamma_2}[e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)]_0^1 d\Gamma+$$

$$\frac{1}{2}(\tau(t))^{-3}\int_{\Gamma_2}\int_0^1\{-\tau'(t)e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2-2\tau(t)e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)\}d\rho d\Gamma$$

$$=-\frac{1}{2}(\tau(t))^{-3}\int_{\Gamma_2}\{e^{-2\tau(t)}\left|\partial_\rho y_2(x,t-\tau(t))\right|^2(1-\tau'(t))-\left|\partial_\rho y_2(x,t)\right|^2\}d\Gamma-$$

$$\frac{\tau'(t)}{2}(\tau(t))^{-3}\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma-$$

$$(\tau(t))^{-2}\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_\rho y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)\}d\rho d\Gamma$$

and then

$$\Re\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\partial_t\overline{y_2(x,t-\tau(t)\rho)}\partial_t^2 y_2(x,t-\tau(t)\rho)(1-\tau'(t)\rho)d\rho=$$

$$-\frac{1}{2}(\tau(t))^{-1}\int_{\Gamma_2}\{e^{-2\tau(t)}\left|\partial_t y_2(x,t-\tau(t))\right|^2(1-\tau'(t))-\left|\partial_t y_2(x,t)\right|^2\}d\Gamma-$$

$$\frac{\tau'(t)}{2}(\tau(t))^{-1}\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma-\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)\}d\rho d\Gamma \tag{3.74}$$

Inserting (3.74) into (3.73) leads to

$$\frac{d}{dt}\mathcal{E}(t)=\xi\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\, d\Gamma-2\xi\tau(t)\tau'(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\rho\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho\, d\Gamma-$$

$$\xi\int_{\Gamma_2}\{e^{-2\tau(t)}\left|\partial_t y_2(x,t-\tau(t))\right|^2(1-\tau'(t))-\left|\partial_t y_2(x,t)\right|^2\}d\Gamma-\tau'(t)\xi\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma-$$

$$2\xi\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2(1-\tau'(t)\rho)d\rho d\Gamma$$

and so

$$\frac{d}{dt}\mathcal{E}(t)=-2\xi\tau(t)\int_{\Gamma_2}\int_0^1 e^{-2\tau(t)\rho}\left|\partial_t y_2(x,t-\tau(t)\rho)\right|^2 d\rho d\Gamma-\xi\int_{\Gamma_2}e^{-2\tau(t)}\left|\partial_t y_2(x,t-\tau(t))\right|^2(1-\tau'(t))d\Gamma+$$

$$\xi\int_{\Gamma_2}\left|\partial_t y_2(x,t)\right|^2 d\Gamma$$

which gives the desired estimate. □

Completion of the proof of Theorem 3.1.1.

From Proposition 3.3.1, Lemma 3.3.3 and 3.3.4, we have

$$\frac{d}{dt}\mathbb{E}(t) \leq -K\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau(t))|^2\}d\Gamma +$$

$$\gamma\left\{-C_0E_s(t) + (C_1+\xi)\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2 + |\partial_t y_2(x,t-\tau(t))|^2\}d\Gamma - 2\mathcal{E}(t)\right\} \tag{3.75}$$

Then for $\gamma(C_1+\xi) < K$, we get from (3.75)

$$\frac{d}{dt}\mathbb{E}(t) \leq -\gamma C_0 E_s(t) - 2\gamma\mathcal{E}(t)$$

On the other hand, from the assumption (3.10), on $\tau(t)$, we deduce that

$$\mathcal{E}(t) \geq \xi\tau(t)e^{-2\tilde{\tau}}\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x,t-\tau(t)\rho)|^2\, d\rho d\Gamma$$

Therefore

$$\frac{d}{dt}\mathbb{E}(t) \leq -\gamma C_0 E_s(t) - \xi\tau(t)e^{-2\tilde{\tau}}\int_{\Gamma_2}\int_0^1 |\partial_t y_2(x,t-\tau(t)\rho)|^2\, d\rho d\Gamma \leq -\min\{\gamma C_0, \frac{e^{-2\tilde{\tau}}}{2}\}E(t) \leq -C\mathbb{E}(t)$$

where

$$C = \alpha_1 \min\{\gamma C_0, \frac{e^{-2\tilde{\tau}}}{2}\}$$

This implies

$$\mathbb{E}(t) \leq e^{-Ct}\mathbb{E}(0)$$

and consequently

$$E(t) \leq \frac{\alpha_2}{\alpha_1}e^{-Ct}E(0)$$

3.4 Appendix B

. **Proof of Lemma 3.3.2**

Using Cauchy-Schwarz's, Young's and Poincaré's inequalities, we obtain

$$\left|\Im\int_{\Omega_1} y_1(x,t)m(x).\nabla\overline{y_1(x,t)}dx\right| \leq Const_m\{\frac{a_1}{2}\int_{\Omega_1}|\nabla y_1(x,t)|^2\, dx\} \tag{3.76}$$

$$\left|\Im\int_{\Omega_1} y_2(x,t)m(x).\nabla\overline{y_2(x,t)}dx\right| \leq Const_m\int_{\Omega_2}\{|y_2(x,t)|^2 + |\nabla y_2(x,t)|^2\}dx \tag{3.77}$$

But

$$\begin{aligned}\int_{\Omega_2}\{|y_2(x,t)|^2 + |\nabla y_2(x,t)|^2\}dx &\leq Const\{\int_{\Omega_2}|\nabla y_2(x,t)|^2\, dx + \int_{\Gamma_0}|y_2(x,t)|^2\, d\Gamma\}\\ &\leq Const\{\int_{\Omega_2}|\nabla y_2(x,t)|^2\, dx + \int_{\Gamma_0}|y_1(x,t)|^2\, d\Gamma\}\\ &\leq Const\{\int_{\Omega_2}|\nabla y_2(x,t)|^2\, dx + \int_{\Omega_1}|\nabla y_1(x,t)|^2\, dx\}\\ &\leq Const\{\frac{a_2}{2}\int_{\Omega_2}|\nabla y_2(x,t)|^2\, dx + \frac{a_1}{2}\int_{\Omega_1}|\nabla y_1(x,t)|^2\, dx\} \end{aligned} \tag{3.78}$$

Combining (3.76), (3.77) and (3.78) gives us

$$\begin{aligned}&\left|\Im\int_{\Omega_1} y_1(x,t)m(x).\nabla\overline{y_1(x,t)}dx+\Im\int_{\Omega_2} y_2(x,t)m(x).\nabla\overline{y_2(x,t)}dx\right|\leq\\ &Const_m\{\frac{a_2}{2}\int_{\Omega_2}\left|\nabla y_2(x,t)\right|^2dx+\frac{a_1}{2}\int_{\Omega_1}\left|\nabla y_1(x,t)\right|^2dx\}\end{aligned}\tag{3.79}$$

On the other hand, it follows from (3.10), that

$$\mathcal{E}(t)\leq\xi\tau(t)\int_{\Gamma_2}\int_0^1|\partial_t y_2(x,t-\tau(t)\rho)d\rho d\Gamma|$$

which together with (3.79) implies

$$\mu_1\mathbb{E}(t)\leq E(t)$$

Now from (3.10), we have

$$\mathcal{E}(t)\geq\xi\tau(t)e^{-2\widetilde{\tau}}\int_{\Gamma_2}\int_0^1|\partial_t y_2(x,t-\tau(t)\rho)d\rho d\Gamma|$$

and

$$\begin{aligned}&\left|\Im\int_{\Omega_1} y_1(x,t)m(x).\nabla\overline{y_1(x,t)}dx+\Im\int_{\Omega_1} y_2(x,t)m(x).\nabla\overline{y_2(x,t)}dx\right|\geq\\ &-Const_m\{\frac{a_2}{2}\int_{\Omega_2}\left|\nabla y_2(x,t)\right|^2dx+\frac{a_1}{2}\int_{\Omega_1}\left|\nabla y_1(x,t)\right|^2dx\}\end{aligned}$$

Therefore for γ small enough,

$$E(t)\leq\mu_2\mathbb{E}(t)$$

Chapter 4

Exponential stability of the transmission wave equation with a distributed delay term in the boundary damping

4.1 Introduction and statement of the stability result

In [58], Nicaise and Pignotti considered the wave equation with distributed delayed boundary or internal feedbacks. Adopting the approach they developed in [57], they proved exponential stability of the solution for both systems. Furthermore, for the internal delayed feedback case, they showed some instability results. Liu [49] used Riemannian geometric energy method approach to extend the results of [58] to the case of the wave equation with variable coefficients in the principal elliptic part. Ghecham et al [26] investigated the exponential stability property of compactly coupled wave equations with a distributed delay term in the boundary or internal feedbacks by applying inequalities obtained from Carleman estimates for a system of coupled non-conservative hyperbolic equations due to Lasiecka and Triggiani [39]. Benseghir [9] and Liu [48] proved exponential stability of the solution of the one-dimensional transmission wave equation with internal damping and discrete or distributed delay by introducing suitable Lyapunov functionals. Rebiai and Sidiali [68] considered a multidimensional transmission wave equation with a Neumann feedback control that contains a discrete delay term and that acts on the exterior boundary. They showed, under some assumptions, that the solution decays exponentially in some appropriate energy space. To obtain this result, they used multipliers technique and compactness uniqueness argument. Motivated by the references [48] and [68], we study in this chapter stability problems for the system of transmission of the multidimensional wave equation with a distributed delay term in the linear boundary feedback.

Let $\Omega, \Omega_1, \Omega_2, \Gamma, \Gamma_0, \Gamma_1$, Γ_2 and ν be as in Chapter 2 and assume the following:
(H)There exists a real vector field $h \in (C^2(\overline{\Omega}))^n$ such that:
$(H.1)$ The Jacobian matrix J of h satisfies

$$\int_\Omega J(x)\zeta(x).\zeta(x)d\Omega \geq \alpha \int_\Omega |\zeta(x)|^2 \, d\Omega,$$

for some constant $\alpha > 0$ and for all $\zeta \in L^2(\Omega; \mathbb{R}^n)$;
$(H.2)$ $h(x).\nu(x) \leq 0$ on Γ_1,

Let $a_1, a_2 > 0$ be given. Consider the system of transmission of the wave equation with a delay term in the boundary conditions:

$$\partial_t^2 y_i(x,t) - a_i \Delta y_i(x,t) = 0 \qquad \text{in } \Omega_i \times (0,+\infty), i = 1,2, \tag{4.1}$$

$$y_i(x,0) = y_i^0(x), \partial_t y_i(x,0) = y^1(x) \qquad \text{in } \Omega, i = 1,2, \tag{4.2}$$

$$y_1(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{4.3}$$

$$\frac{\partial y_2(x,t)}{\partial \nu} + b(x) y_2(x,t) = -\mu_0 \partial_t y_2(x,t) - \int_{\tau_1}^{\tau_2} \mu(s) \partial_t y_2(x,t-s) ds \quad \text{on } \Gamma_2 \times (0,+\infty), \tag{4.4}$$

$$y_1(x,t) = y_2(x,t), \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{4.5}$$

$$a_1 \frac{\partial y_1(x,t)}{\partial \nu} = a_2 \frac{\partial y_2(x,t)}{\partial \nu} \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{4.6}$$

$$\partial_t y_2(x,t-\tau) = f_0(x,t-\tau) \qquad \text{on } \Gamma_2 \times (0,\tau). \tag{4.7}$$

where:

- $b(.)$ is an $L^\infty(\Gamma_2)$-function that satisfies $b(x) \geq b_0 > 0$ *a.e.* on Γ_2,
- μ_0 is a positive constant,
- τ_1 and τ_2 are two real numbers with $0 \leq \tau_1 < \tau_2$,
- $\mu : [\tau_1, \tau_2] \to \mathbb{R}$ is an L^∞function nonnegative almost everywhere,
- y_i^0, y_i^1, f_0 are the initial data which belong to suitable spaces.

From the physical point of view, the transmission problem $(4.1)-(4.7)$ describes the wave propagation from one medium into another different medium, for instance from air into glass (see [22]) and [47]). In the absence of delay, that is $\mu = 0$, Liu and Williams ([46]) have shown that the solution of $(4.1)-(4.6)$ decays exponentially to zero in the energy space $H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)$ provided that $\{\Omega, \Gamma_0, \Gamma_1, \Gamma_2\}$ satisfies $(H.1), (H.2)$ and $(H.3)$ $(a_1 - a_2)h(x).\nu(x) \geq 0$ on Γ_0, $(H.4)$ $h(x).\nu(x) \geq \gamma > 0$ on Γ_2.
In the presence of discrete time delay that is the boundary condition (6.6) is replaced by

$$\frac{\partial y_2(x,t)}{\partial \nu} + b(x) y(x,t) = -\mu_0 \partial_t y_2(x,t) - \mu_1 \partial_t y_2(x,t-\tau) \qquad \text{on } \Gamma_2 \times (0,+\infty),$$

the solution of (4.1)-(4.7) decays exponentially in an appropriate energy space provided that $\mu_0 > \mu_1$ ([68]).

The purpose of this chapter is to investigate the uniform exponential stability of problem $(4.1)-(4.7)$. To this aim, assume as in [58] that

$$\mu_0 > \int_{\tau_1}^{\tau_2} \mu(s) ds, \tag{4.8}$$

and define the energy of a solution

$$y(x,t) = \begin{cases} y_1(x,t), & (x,t) \in \Omega_1 \times (0,+\infty), \\ y_2(x,t), & (x,t) \in \Omega_2 \times (0,+\infty), \end{cases}$$

of $(4.1)-(4.7)$ by

$$E(t) = \frac{1}{2} \int_\Omega \left[|\partial_t y(x,t)|^2 + a(x) |\nabla (y(x,t)|^2 \right] dx + \frac{a_2}{2} \int_{\Gamma_2} b(x) |y_2(x,t)|^2 d\Gamma + \frac{a_2}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} s(\mu(s) + c_0) \int_0^1 |\partial_t y_2(x,t-\rho s)|^2 d\rho \, ds d\Gamma, \tag{4.9}$$

where

$$a(x) = \begin{cases} a_1, & x \in \Omega_1, \\ a_2, & x \in \Omega_2, \end{cases}$$

and c_0 is a positive constant chosen so that

$$\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)ds - \frac{c_0}{2}(\tau_2 - \tau_1) > 0. \tag{4.10}$$

We show that if in addition to (4.8), $\{\Omega, \Gamma_0, \Gamma_1, \Gamma_2\}$ satisfies $(H.1), (H.2)$ and $(H.3)$, then there is an exponential decay rate for $E(t)$. The proof of this result combines multipliers technique and compactness-uniqueness argument.

The main result of this chapter can be stated as follows.

Theorem 4.1.1. *Assume* $(H.1)$, $(H.2), (H.3)$ *and* (4.8). *Then there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq Me^{-\omega t}E(0).$$

Theorem 4.1.1 is proved in Section 3. In Section 2, we study the existence, uniqueness and regularity of solutions to problem (6.1) – (6.8) using semigroup theory.

This chapter was published in [54].

4.2 Well-posedness

Inspired from [58], we introduce the auxiliary variable $z(x, \rho, t, s) = \partial_t y(x, t-\rho s)$. With this new unknown, problem (4.1) – (4.7) is equivalent to

$$\partial_t^2 y_i(x,t) - a_i \Delta y_i(x,t) = 0 \qquad \text{in } \Omega_i \times (0, +\infty), i = 1, 2, \tag{4.11}$$

$$y_i(x,0) = y_i^0(x), \partial_t y_i(x,0) = y^1(x) \qquad \text{in } \Omega, i = 1, 2, \tag{4.12}$$

$$y_1(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0, +\infty), \tag{4.13}$$

$$\partial_t z(x, \rho, t, s) + \frac{1}{s}\partial_\rho z(x, \rho, t, s) = 0 \qquad \text{on } \Gamma_2 \times (0,1) \times (0, +\infty), \tag{4.14}$$

$$\frac{\partial y_2(x,t)}{\partial \nu} + b(x)y_2(x,t) = -\mu_0 \partial_t y_2(x,t) - \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)ds \text{ on } \Gamma_2 \times (0, +\infty), \tag{4.15}$$

$$y_1(x,t) = y_2(x,t) \qquad \text{on } \Gamma_0 \times (0, +\infty), \tag{4.16}$$

$$a_1 \frac{\partial y_1(x,t)}{\partial \nu} = a_2 \frac{\partial y_2(x,t)}{\partial \nu} \qquad \text{on } \Gamma_0 \times (0, +\infty), \tag{4.17}$$

$$z(x,0,t,s) = \partial_t y_2(x,t) \qquad \text{on } \Gamma_2 \times (0, +\infty) \times (\tau_1, \tau_2), \tag{4.18}$$

$$z(x,\rho,0,s) = f_0(x,\rho,s) \qquad \text{on } \Gamma_2 \times (0,1) \times (\tau_1, \tau_2). \tag{4.19}$$

Now, we endow the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2; L^2((0,1) \times (\tau_1, \tau_2)))$$

with the inner product

$$\begin{aligned}\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}; \begin{pmatrix} \widetilde{u} \\ \widetilde{v} \\ \widetilde{z} \end{pmatrix} \right\rangle &= \int_\Omega (a(x)\nabla u(x)\nabla \widetilde{u}(x) + v(x)\widetilde{v}(x))\, dx + a_2 \int_{\Gamma_2} b(x)u(x)\widetilde{u}(x)\, d\Gamma + \\ & a_2 \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} s\mu(s) \int_0^1 z(x,\rho,s)\widetilde{z}(x,\rho,s) d\rho ds d\Gamma,\end{aligned}$$

and define a linear operator in $\mathcal{H}$ by

$$\begin{aligned} D(A) = \{&(u,v,z)^T \in H^2(\Omega_1, \Omega_2, \Gamma_1) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times (\tau_1, \tau_2); H^1(0,1)); \\ &\frac{\partial u(x)}{\partial \nu} + b(x)u(x) = -\mu_0 v(x) - \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,s)ds, v(x) = z(x,0,s) \text{ on } \Gamma_2\} \end{aligned}, \tag{4.20}$$

$$A\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} v \\ a(x)\Delta u \\ -s^{-1}\partial_\rho z \end{pmatrix}. \tag{4.21}$$

The spaces used for the definition of $\mathcal{H}$ and $D(A)$ are

$$\begin{aligned} &H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}, \\ &H^2(\Omega_1, \Omega_2, \Gamma_1) = \{u \in H^1_{\Gamma_1}(\Omega) : u_i \in H^2(\Omega_i), i = 1, 2, \text{ and } a_1\frac{\partial u_1}{\partial \nu} = a_2\frac{\partial u_2}{\partial \nu} \text{ on } \Gamma_0\}. \end{aligned} \tag{4.22}$$

Then we can rewrite (4.11) – (4.19) as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}Y(t) = AY(t), \\ Y(0) = Y_0, \end{cases} \tag{4.23}$$

where

$$Y(t) = (y, \partial_t y, z)^T \text{ and } Y_0 = (y^0(x), y^1(x), , f^0(x, \rho, s))^T.$$

Proposition 4.2.1. *The operator* A *defined by* (4.20) *and* (4.21) *generates a strongly continuous semigroup on* $\mathcal{H}$*. Thus, for every* $Y_0 \in \mathcal{H}$*, problem* (4.23) *has a unique solution* Y *whose regularity depends on the initial datum* Y_0 *as follows:*

$$\begin{aligned} Y(.) &\in C([0, +\infty); \mathcal{H}) \text{ if } Y_0 \in \mathcal{H}, \\ Y(.) &\in C([0, +\infty); D(A)) \cap C^1([0, +\infty); \mathcal{H}) \text{ if } Y_0 \in D(A). \end{aligned}$$

Proof. Let $Y = \begin{pmatrix} u \\ v \\ z \end{pmatrix} \in D(A)$. Then

$$\begin{aligned} \langle AY, Y\rangle &= \int_\Omega a(x)\nabla u(x).\nabla v(x)dx + a_2\int_{\Gamma_2} b(x)u(x).u(x)\,d\Gamma + \int_\Omega v(x)(a(x)\Delta u(x))dx - \\ &\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s\mu(s)\int_0^1 z(x, \rho, s)s^{-1}\partial_\rho z(x, \rho, s)d\rho ds d\Gamma. \end{aligned} \tag{4.24}$$

Applying Green's theorem to the third integral on the right-hand side of (4.24) and recalling (4.20), we obtain

$$\begin{aligned} &\int_\Omega (a(x)\Delta u(x))v(x)dx = a_1\int_{\Gamma_1} v_1(x)\frac{\partial u_1(x)}{\partial \nu}d\Gamma - a_1\int_{\Omega_1} \nabla u_1(x).\nabla v_1(x)dx + \\ &a_2\int_{\Gamma_2} v_2(x)\frac{\partial u_2(x)}{\partial \nu}d\Gamma - a_2\int_{\Omega_2} \nabla u_2(x).\nabla v_2(x)dx \\ = \;&a_2\int_{\Gamma_2} v(x)\{-b(x)u(x) - \mu_0 v(x) - \int_{\tau_1}^{\tau_2} \mu(s)z(x, 1, s)ds\}d\Gamma - \\ &\int_\Omega a(x)\nabla u(x).\nabla v(x)dx. \end{aligned} \tag{4.25}$$

Integrating by parts in ρ the fourth integral on the right-hand side of (4.24), we get

$$\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} \mu(s)\int_0^1 \partial_\rho z(x, \rho, s)z(x, \rho, s)d\rho ds d\Gamma = \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} \mu(s)\{z^2(x, 1, s) - z^2(x, 0, s)\}ds d\Gamma. \tag{4.26}$$

Inserting (4.25) and (4.26) into (4.24) results in

$$\begin{aligned} &\langle AY, Y\rangle = a_2\int_{\Gamma_2} v_2(x)\{-\mu_0 v_2(x) - \int_{\tau_1}^{\tau_2} \mu(s)z(x, 1, s)ds\}d\Gamma - \\ &\frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} \mu(s)\{z^2(x, 1, s) - z^2(x, 0, s)\}ds d\Gamma. \end{aligned}$$

Using Cauchy-Schwarz inequality, we obtain

$$\left|\int_{\Gamma_2} v_2(x)\int_{\tau_1}^{\tau_2}\mu(s)z(x,1,s)d\Gamma\right| \leq \frac{1}{2}\int_{\Gamma_2} v_2^2(x)(\int_{\tau_1}^{\tau_2}\mu(s)ds)d\Gamma+$$
$$\frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\mu(s)z^2(x,1,s)dsd\Gamma, \tag{4.27}$$

and consequently

$$\langle AY,Y\rangle \leq a_2(-\mu_0+\int_{\tau_1}^{\tau_2}\mu(s)ds)\int_{\Gamma_2} v_2^2(x)d\Gamma. \tag{4.28}$$

(4.28) together with (4.8) implies that

$$\langle AY,Y\rangle \leq 0.$$

Thus A is dissipative.
Now we show that for a fixed $\lambda > 0$ and $(g,h,k)^T \in \mathcal{H}$, there exists $Y=(u,v,w)^T \in D(A)$ such that

$$(\lambda I - A)Y = (g,h,k)^T, \tag{4.29}$$

or equivalently

$$\lambda u - v = g, \tag{4.30}$$
$$\lambda v - a(x)\Delta u = h, \tag{4.31}$$
$$\lambda w + \frac{1}{\tau}\partial_\rho w = k. \tag{4.32}$$

Suppose that we have found u with the appropriate regularity, then we can determine w. Indeed, from (4.20) and (4.32) we have

$$\begin{cases} \partial_\rho w(x,\rho,s) = -\lambda s w(x,\rho) + \tau k(x,\rho,s), \\ \quad w(x,0,s) = v(x). \end{cases}$$

The unique solution of the above initial value problem is given by

$$w(x,\rho,s) = e^{-\lambda s\rho}v(x) + se^{-\lambda s\rho}\int_0^\rho e^{\lambda s\sigma}k(x,\sigma,s)ds,$$

and in particular

$$w(x,1,s) = e^{-\lambda s}v(x) + w_0(x,s), \quad x \in \Gamma_2,$$

where

$$w_0(x,s) = se^{-\lambda s}\int_0^1 e^{\lambda\sigma s}k(x,\sigma,s)ds.$$

From (4.30) and (4.31), we have

$$\lambda^2 u(x) - div(a(x)\nabla u(x)) = h(x) + \lambda g(x), \tag{4.33}$$

or equivalently

$$\lambda^2 u_1(x) - a_1\Delta u_1(x) = h_1(x) + \lambda g_1(x), \qquad x \in \Omega_1, \tag{4.34}$$
$$\lambda^2 u_2(x) - a_2\Delta u_2(x) = h_2(x) + \lambda g_2(x), \qquad x \in \Omega_2. \tag{4.35}$$

Let $(\varphi_1,\varphi_2) \in \mathbb{V}$ where

$$\mathbb{V} = \{(\varphi_1,\varphi_2) \in H^1_{\Gamma_1}(\Omega_1) \times H^1(\Omega_2) : \varphi_1 = \varphi_2 \text{ on } \Gamma_0\}.$$

Then, multiplying (4.34) by φ_1 (resp. (4.35) by φ_2) and integrating in Ω_1(resp. in Ω_2), we obtain

$$a_1\int_{\Omega_1}(\lambda^2 u_1\varphi_1+\nabla u_1.\nabla\varphi_1)dx+a_1\int_{\Gamma_0}\frac{\partial u_1}{\partial\nu}\varphi_1 d\Gamma=\int_{\Omega}(h_1+\lambda g_1)\varphi_1 dx, \tag{4.36}$$

$$\begin{aligned}&a_2\int_{\Omega_2}(\lambda^2 u_2\varphi_2+\nabla u_2.\nabla\varphi_2)dx-a_2\int_{\Gamma_0}\frac{\partial u_2}{\partial\nu}\varphi_2 d\Gamma+\\&\int_{\Gamma_2}(b(x)+\mu_0\lambda-\lambda\int_{\tau_1}^{\tau_2}\mu(s)e^{-\lambda s}ds)u_2\varphi_2 d\Gamma\\&=\int_{\Omega}(h_2+\lambda g_2)\varphi_2 dx+a_2\int_{\Gamma_2}(-\mu_0 g_2+\int_{\tau_1}^{\tau_2}\mu(s)(w_0(x,s)-e^{-\lambda s}g_2)ds)\varphi_2 dx.\end{aligned} \tag{4.37}$$

Summing up (4.36) and (4.37), and recalling (4.20) and (4.22), yields

$$\Lambda((u_1,u_2),(\varphi_1,\varphi_2))=\mathcal{F}(\varphi_1,\varphi_2), \tag{4.38}$$

where

$$\begin{aligned}\Lambda((u_1,u_2),(\varphi_1,\varphi_2))&=a_1\int_{\Omega_1}(\lambda^2 u_1\varphi_1+\nabla u_1.\nabla\varphi_1)dx+a_2\int_{\Omega_2}(\lambda^2 u_2\varphi_2+\nabla u_2.\nabla\varphi_2)dx+\\&\int_{\Gamma_2}(b(x)+\mu_0\lambda-\lambda\int_{\tau_1}^{\tau_2}\mu(s)e^{-\lambda s}ds)u_2\varphi_2 d\Gamma,\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(\varphi_1,\varphi_2)&=\int_{\Omega}(h_1+\lambda g_1)\varphi_1 dx+\int_{\Omega}(h_2+\lambda g_2)\varphi_2 dx+\\&a_2\int_{\Gamma_2}(-\mu_0 g_2+\int_{\tau_1}^{\tau_2}\mu(s)(w_0(x,s)-e^{-\lambda s}g_2)ds)\varphi_2 dx.\end{aligned}$$

Since Λ is a continuous bilinear coercive form on $H^1_{\Gamma_1}(\Omega_1)\times H^1(\Omega_2)$ and $\mathcal{F}$ is a continuous linear form on $H^1_{\Gamma_1}(\Omega_1)\times H^1(\Omega_2)$, the Lax-Milgram Theorem guarantees the existence and the uniqueness of the solution $(u_1,u_2)\in H^1_{\Gamma_1}(\Omega_1)\times H^1(\Omega_2)$ of (4.38). It follows from (4.36) and (4.37) after using some integration by parts, that (u_1,u_2) satisfies

$$\begin{aligned}\lambda^2 u_1-a_1\Delta u_1&=h_1+\lambda g_1,\\ \lambda^2 u_2-a_2\Delta u_2&=h_2+\lambda g_2,\end{aligned}$$

as well as the boundary conditions once we have set $v_1=\lambda u_1-g_1$ and $v_2=\lambda u_2-g_2$. This means that (4.29) holds and consequently $\lambda I-A$ is surjective. Thus, by the Lumer-Phillips Theorem (see for instance [65], Theorem 1.4.3), generates a strongly continuous semigroup of contractions on $\mathcal{H}$. □

4.3 Proof of the stability result

We prove Theorem 4.1.1 for smooth initial data. The general case follows by a standard density argument. First, we show that the energy function defined by (4.9) is decreasing.

Proposition 4.3.1. *The energy corresponding to any regular solution of problem (4.1)-(4.7) is decreasing and there exists a positive constant* K *such that*

$$\frac{d}{dt}E(t)\le -K\int_{\Gamma_2}\left\{|\partial_t y_2(,,t)|^2+\int_{\tau_1}^{\tau_2}|\partial_t y_2(x,t-s)|^2\,ds\right\}d\Gamma, \tag{4.39}$$

where

$$K=\min\{a_2(\mu_0-\int_{\tau_1}^{\tau_2}\mu(s)ds-\frac{c_0}{2}(\tau_2-\tau_1)),\frac{a_2c_0}{2}\}.$$

Proof. Differentiating $E(t)$ with respect to time, we obtain

$$\frac{d}{dt}E(t) = \int_{\Omega_1}\{\partial_t y_1(x,t)\partial_t^2 y_1(x,t) + a_1\nabla y_1(x,t).\nabla\partial_t y_1(x,t)\}dx +$$
$$\int_{\Omega_2}\{\partial_t y_2(x,t)\partial_t^2 y_2(x,t) + a_2\nabla y_2(x,t).\nabla\partial_t y_2(x,t)\}dx +$$
$$a_2\int_{\Gamma_2} b(x)y_2(x,t)\partial_t y_2(x,t)d\Gamma +$$
$$a_2\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\mu(s)+c_0)(\int_0^1 \partial_t y_2(x,t-\rho s)\partial_t^2 y_2(x,t-\rho s)d\rho)dsd\Gamma.$$

Applying Green's Theorem and recalling the boundary conditions (4.3)-(4.4) and the transmission conditions (4.5)-(4.6), we get

$$\begin{aligned}\frac{d}{dt}E(t) &= -a_2\mu_0\int_{\Gamma_2}|\partial_t y_2(x,t)|^2 d\Gamma - a_2\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\mu(s)\partial_t y_2(x,t-s)\partial_t y_2(x,t)dsd\Gamma + \\ &\quad a_2\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\mu(s)+c_0)\left(\int_0^1 \partial_t y_2(x,t-\rho s)\partial_t^2 y_2(x,t-\rho s)d\rho\right)dsd\Gamma.\end{aligned} \tag{4.40}$$

Now we have

$$\begin{aligned}\partial_\rho y(x,t-\rho s) &= -s\partial_t y(x,t-\rho s),\\ \partial_\rho^2 y(x,t-\rho s) &= s^2\partial_t^2 y(x,t-\rho s).\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} s(\mu(s)+c_0)\left(\int_0^1 \partial_t y_2(x,t-\rho s)\partial_t^2 y_2(x,t-\rho s)d\rho\right)dsd\Gamma = \\ &-\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\frac{1}{s^2}(\mu(s)+c_0)\left(\int_0^1 \partial_\rho y_2(x,t-\rho s)\partial_\rho^2 y_2(x,t-\rho s)d\rho\right)dsd\Gamma = \\ &\frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\frac{1}{s^2}(\mu(s)+c_0)\left((\partial_\rho y_2(x,t-s))^2 - (\partial_\rho y_2(x,t))^2\right)dsd\Gamma = \\ &\frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\mu(s)+c_0)\left((\partial_t y_2(x,t))^2 - (\partial_t y_2(x,t-s))^2\right)dsd\Gamma.\end{aligned} \tag{4.41}$$

Inserting (4.41) into (4.40), we obtain

$$\begin{aligned}\frac{d}{dt}E(t) = -a_2\mu_0&\int_{\Gamma_2}|\partial_t y_2(x,t)|^2 d\Gamma - a_2\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\mu(s)\partial_t y_2(x,t-s)\partial_t y_2(x,t)dsd\Gamma + \\ &\frac{a_2}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}(\mu(s)+c_0)\left((\partial_t y_2(x,t))^2 - (\partial_t y_2(x,t-s))^2\right)dsd\Gamma.\end{aligned} \tag{4.42}$$

For the second integral on the right-hand side of (4.42), we have from the Cauchy-Schwartz's inequality

$$\begin{aligned}&\left|\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\mu(s)\partial_t y_2(x,t-s)\partial_t y_2(x,t)dsd\Gamma\right| \le \frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\mu(s)\left|\partial_t y_2(x,t-s)\right|^2 dsd\Gamma + \\ &\frac{1}{2}\int_{\Gamma_2}\int_{\tau_1}^{\tau_2}\mu(s)\left|\partial_t y_2(x,t)\right|^2 dsd\Gamma.\end{aligned} \tag{4.43}$$

Combining (4.43) and (4.42) yields

$$\frac{d}{dt}E(t) \leq -a_2(\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)ds - \frac{c_0}{2}(\tau_2 - \tau_1)) \int_{\Gamma_2} |\partial_t y_2(x,t)|^2 \, d\Gamma - \frac{a_2 c_0}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} |\partial_t y_2(x,t-s)|^2 \, ds d\Gamma,$$

which implies

$$\frac{d}{dt}E(t) \leq -K \int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + \int_{\tau_1}^{\tau_2} |\partial_t y_2(x,t-s)|^2 \, ds\} d\Gamma,$$

where

$$K = \min\{a_2(\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)ds - \frac{c_0}{2}(\tau_2 - \tau_1)), \frac{a_2 c_0}{2}\}.$$

□

Step 2.
Set

$$E(t) = \mathcal{E}(t) + E_d(t) + \frac{a_2}{2} \int_{\Gamma_2} b(x) \, |y(x,t)|^2 \, d\Gamma, \tag{4.44}$$

where

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} \{a(x) \, |\nabla y(x,t)|^2 + |\partial_t y(x,t)|^2\} dx, \tag{4.45}$$

and

$$E_d(t) = \frac{a_2}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} s(\mu(s) + c_0) \int_0^1 |\partial_t y_2(x,t-\rho s)|^2 \, d\rho \, ds d\Gamma.$$

$E_d(t)$ can be rewritten via a change of variable as

$$E_d(t) = \frac{a_2}{2} \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} (\mu(s) + c_0) \int_t^{t+s} |\partial_t y_2(x,\kappa - s)|^2 \, d\kappa \, ds d\Gamma. \tag{4.46}$$

From (4.46), we obtain

$$E_d(t) \leq C \int_0^T \int_{\Gamma_2} \int_{\tau_1}^{\tau_2} |\partial_t y_2(x,t-s)|^2 \, ds d\Gamma dt, \tag{4.47}$$

for $0 \leq t + \tau_2 \leq T$. Here and throughout the rest of the chapter C is a positive constant different at different occurrences.

Step 3.
We multiply both sides of (4.1) by $2h(x).\nabla y_i(x,t) + (div h(x) - \alpha)y_i(x,t)$ and integrate by parts over

$\Omega_i \times (0,T), .i = 1,2$. We obtain (see the Appendix C)

$$
\begin{aligned}
&2\int_0^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)dxdt+\\
&\alpha\int_0^T\int_\Omega\{|\partial_t y_2(x,t)|^2 - a(x)\left|\nabla y(x,t)\right|^2\}dxdt =\\
&-\left[\int_\Omega\{2\partial_t y(x,t)h(x).\nabla y(x,t) + (divh(x)-\alpha)\partial_t y(x,t)y(x,t)\}dx\right]_0^T -\\
&\int_0^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(divh(x)-\alpha)dxdt+\\
&a_1\int_0^T\int_{\Gamma_1}(\frac{\partial y_1(x,t)}{\partial\nu})^2 h(x).\nu(x)d\Gamma dt - (a_1-a_2)\int_0^T\int_{\Gamma_0}\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x)d\Gamma dt-\\
&\frac{(a_1-a_2)^2}{a_2}\int_0^T\int_{\Gamma_0}(\frac{\partial y_1(x,t)}{\partial\nu})^2 h(x).\nu(x)d\Gamma dt + \int_0^T\int_{\Gamma_2}\partial_t y_2(x,t)h(x).\nu(x)d\Gamma dt+\\
&2a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)d\Gamma dt - a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt+\\
&a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(divh(x)-\alpha)y_2(x,t)d\Gamma dt,
\end{aligned}
\tag{4.48}
$$

after using the boundary conditions (4.3),(4.5) and (4.6).
It follows from Assumptions $(H.2)$ and $(H.3)$ that

$$
\begin{aligned}
&2\int_0^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)dxdt+\\
&\alpha\int_0^T\int_\Omega\{|\partial_t y(x,t)|^2 - a(x)\left|\nabla y(x,t)\right|^2\}dxdt\leq\\
&\left[\int_\Omega\{2\partial_t y(x,t)h(x).\nabla y(x,t) + (divh(x)-\alpha)\partial_t y(x,t)y(x,t)\}dx\right]_0^T -\\
&\int_0^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(divh(x)-\alpha)dxdt+\\
&2a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)d\Gamma dt+\\
&a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(divh(x)-\alpha)y_2(x,t)d\Gamma dt+\\
&\int_0^T\int_{\Gamma_2}\partial_t y_2(x,t)h(x).\nu(x)d\Gamma dt-\\
&a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt.
\end{aligned}
\tag{4.49}
$$

Step 4. We now estimate both sides of (4.49). From $(H.1)$, we have for the terms on the left-hand side of (4.49)

$$
\begin{aligned}
\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)dx &= \int_\Omega J(x)(\sqrt{a(x)}\nabla y(x,t)).(\sqrt{a(x)}\nabla y(x,t))dx\\
&\geq \alpha\int_\Omega a(x)\left|\nabla y(x,t)\right|^2 dx.
\end{aligned}
$$

Hence

$$2\int_0^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t)dxdt+$$
$$\alpha\int_0^T\int_\Omega\{|\partial_t y(x,t)|^2-a(x)\left|\nabla y(x,t)\right|^2\}dxdt\geq\alpha\mathcal{E}(T). \tag{4.50}$$

Now, we estimate each integral term on the right-hand side of the inequality (4.49) separately.

First term. We have by the Cauchy-Schwarz, Young and Poincaré inequalities

$$\left|\left[\int_\Omega\{2\partial_t y(x,t)h(x).\nabla y(x,t)+(divh(x)-\alpha)\partial_t y(x,t)y(x,t)\}dx\right]_0^T\right|$$
$$\leq C(\mathcal{E}(T)+\mathcal{E}(0)), \tag{4.51}$$

recalling (4.45).

Second term. Using Cauchy-Schwarz and Young inequalities yields

$$\left|\int_0^T\int_\Omega a(x)y(x,t)\nabla y(x,t).\nabla(divh(x)-\alpha)dxdt\right|\leq\frac{\eta}{2}\int_0^T\int_\Omega a(x)\left|\nabla y(x,t)\right|^2dxdt+$$
$$\frac{C}{\eta}\int_0^T\int_\Omega(y(x,t))^2dxdt, \tag{4.52}$$

where η is a positive constant that will be fixed later.

Third term. Recalling the boundary conditions (4.4) and using again Cauchy-Schwarz and Young inequalities gives

$$2a_2\left|\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)d\Gamma dt\right|=$$
$$2a_2\left|\int_0^T\int_{\Gamma_2}\{-b(x)y_2(x,t)-\mu_0\partial_t y_2(x,t)-\int_{\tau_1}^{\tau_2}\mu(s)\partial_t y_2(x,t-s)ds\}h(x).\nabla y_2(x,t)d\Gamma dt\right|$$
$$\leq 2a_2\left|\int_0^T\int_{\Gamma_2}b(x)y_2(x,t)h(x).\nabla y_2(x,t)d\Gamma dt\right|+$$
$$2a_2\left|\int_0^T\int_{\Gamma_2}\{\mu_0\partial_t y_2(x,t)+\int_{\tau_1}^{\tau_2}\mu(s)\partial_t y_2(x,t-s)ds\}h(x).\nabla y_2(x,t)d\Gamma dt\right|$$
$$\leq C\int_0^T\int_{\Gamma_2}\left|y_2(x,t)\right|^2d\Gamma dt+\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+\int_{\tau_1}^{\tau_2}\left|\partial_t y_2(x,t-s)\right|^2ds\}d\Gamma dt+$$
$$C\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2d\Gamma dt. \tag{4.53}$$

Fourth term. Proceeding as for (4.53), we obtain

$$\begin{aligned}
&a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(divh(x)-\alpha)y_2(x,t)d\Gamma dt =\\
&a_2\int_0^T\int_{\Gamma_2}\{-b(x)y_2(x,t){-}\mu_0\partial_t y_2(x,t){-}\\
&\int_{\tau_1}^{\tau_2}\mu(s)\partial_t y_2(x,t-s)ds\}(divh(x)-\alpha)y_2(x,t)d\Gamma dt =\\
&a_2\int_0^T\int_{\Gamma_2}b(x)\left|y_2(x,t)\right|^2(divh-\alpha)dtd\Gamma+\\
&a_2\int_0^T\int_{\Gamma_2}\{-\mu_0\partial_t y_2(x,t)-\int_{\tau_1}^{\tau_2}\mu(s)\partial_t y_2(x,t-s)ds\}(divh(x)-\alpha)y_2(x,t)d\Gamma dt\le\\
&C\int_0^T\int_{\Gamma_2}\left|y_2(x,t)\right|^2dtd\Gamma+\int_0^T\int_{\Gamma_2}\{\left|\partial_t y_2(x,t)\right|^2+\int_{\tau_1}^{\tau_2}\left|\partial_t y_2(x,t-s)\right|^2ds\}d\Gamma dt. \qquad (4.54)
\end{aligned}$$

Fifth term.

$$\int_0^T\int_{\Gamma_2}(\partial_t y_2(x,t))^2h(x).\nu(x)d\Gamma dt\le C\int_0^T\int_{\Gamma_2}\left|\partial_t y_2(x,t)\right|^2d\Gamma dt. \qquad (4.55)$$

Sixth term.

$$a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2h(x).\nu(x)d\Gamma dt\le C\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2d\Gamma dt. \qquad (4.56)$$

Inserting (4.50) – (4.56) into (4.49), we obtain

$$\begin{aligned}
&(\alpha-\eta(\frac{1}{2}+C))\int_0^T\int_\Omega a(x)\left|\nabla y(x,t)\right|^2dxdt+\\
&\alpha\int_0^T\int_{\Gamma_2}\left|\partial_t y(x,t)\right|^2d\Gamma dt\le C\{\mathcal{E}T)+\mathcal{E}(0)\}+\\
&\frac{C}{\eta}\int_0^T\int_\Omega(y(x,t))^2dxdt+C\int_0^T\int_{\Gamma_2}\left|y_2(x,t)\right|^2d\Gamma dt+\\
&C\int_0^T\int_{\Gamma_2}\{\left|\partial_t y_2(x,t)\right|^2+\int_{\tau_1}^{\tau_2}\left|\partial_t y_2(x,t-s)\right|^2ds\}d\Gamma dt+\\
&C\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2d\Gamma dt.
\end{aligned}$$

Choosing η sufficiently small to make $\alpha-\eta(\frac{1}{2}+C)>0$, yields

$$\begin{aligned}
&\int_0^T\mathcal{E}(T)dt\le C\{\mathcal{E}(T))+\mathcal{E}(0)\}+\frac{C}{\eta}\int_0^T\int_\Omega(y(x,t))^2dxdt+\\
&C\int_0^T\int_{\Gamma_2}\left|y_2(x,t)\right|^2d\Gamma dt+\\
&C\int_0^T\int_{\Gamma_2}\{\left|\partial_t y_2(x,t)\right|^2+\int_{\tau_1}^{\tau_2}\left|\partial_t y_2(x,t-s)\right|^2ds\}d\Gamma dt+\\
&C\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2d\Gamma dt. \qquad (4.57)
\end{aligned}$$

Using the fact that

$$\left\|\nabla y_2\right\|^2_{L^2(\Gamma_2)}=\left\|\frac{\partial y_2}{\partial\nu}\right\|^2_{L^2(\Gamma_2)}+\left\|\nabla_\sigma y_2\right\|^2_{L^2(\Gamma_2)},$$

where $\nabla_\sigma y_2$ is the tangential gradient of y_2, (4.57) becomes

$$\begin{aligned}
&\int_0^T \mathcal{E}(t)dt \le C\{\mathcal{E}(T)+\mathcal{E}(0)\} + \frac{C}{\eta}\int_0^T\int_\Omega (y(x,t))^2 dxdt+\\
&C\int_0^T\int_{\Gamma_2} |y_2(x,t)|^2\, d\Gamma dt+\\
&C\int_0^T\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + \int_{\tau_1}^{\tau_2} |\partial_t y_2(x,t-s)|^2\, ds\} d\Gamma dt+\\
&C\int_0^T\int_{\Gamma_2} |\nabla_\sigma y_2(x,t)|^2\, d\Gamma dt.
\end{aligned} \tag{4.58}$$

Step 5.
For fixed $\epsilon > 0$ small we apply estimate (4.58) over the interval $(\epsilon, T-\epsilon)$ rather than $(0,T)$. We obtain

$$\begin{aligned}
&\int_\epsilon^{T-\epsilon} \mathcal{E}(t)dt \le C\{\mathcal{E}(T-\epsilon)+\mathcal{E}(\epsilon)\} + C\int_\epsilon^{T-\epsilon}\int_{\Gamma_2} |y_2(x,t)|^2\, d\Gamma dt+\\
&C\int_\epsilon^{T-\epsilon}\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + \int_{\tau_1}^{\tau_2} |\partial_t y_2(x,t-s)|^2\, ds\} d\Gamma dt+\\
&C\int_\epsilon^{T-\epsilon}\int_{\Gamma_2} |\nabla_\sigma y_2(x,t)|^2\, d\Gamma dt + C\int_\epsilon^{T-\epsilon}\int_\Omega y^2(x,t)dxdt.
\end{aligned} \tag{4.59}$$

From Lemma 7.2, inequality 7.5 in [38], we have

$$\begin{aligned}
&\int_\epsilon^{T-\epsilon}\int_{\Gamma_2} |\nabla_\sigma y_2(x,t)|^2\, d\Gamma dt \le\\
&C(\epsilon,\delta,T)\{\int_0^T\int_{\Gamma_2} \{(\frac{\partial y_2(x,t)}{\partial \nu})^2 + (\partial_t y_2(x,t))^2\}\} d\Gamma dt + \|y_2\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))}\},
\end{aligned} \tag{4.60}$$

where δ is an arbitrarily small positive constants and $C(\epsilon,\delta,T)$ denotes a positive constant that depends on ϵ, δ and T.
Inserting (4.60) into (4.59) and recalling the boundary condition (4.4), yields

$$\begin{aligned}
&\int_\epsilon^{T-\epsilon} \mathcal{E}(t)dt \le C\{\mathcal{E}(T-\epsilon)+\mathcal{E}(\epsilon)\}+\\
&C\int_0^T\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + \int_{\tau_1}^{\tau_2} |\partial_t y_2(x,t-s)|^2\, ds\} d\Gamma dt+\\
&C\int_0^T\int_{\Gamma_2} |y_2(x,t)|^2\, d\Gamma dt + C\int_0^T\int_\Omega y^2(x,t)dxdt+\\
&C(\epsilon,\delta,T)\, \|y\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))}\,.
\end{aligned} \tag{4.61}$$

From (4.42), we have

$$\begin{aligned}
&E(0) = E(T) + a_2\mu_0\int_0^T\int_{\Gamma_2} |\partial_t y_2(x,t)|^2\, d\Gamma dt + a_2\int_0^T\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} \mu(s)\partial_t y_2(x,t-s)\partial_t y_2(x,t) ds d\Gamma dt-\\
&\frac{a_2}{2}\int_0^T\int_{\Gamma_2}\int_{\tau_1}^{\tau_2} (\mu(s)+c_0)\left((\partial_t y_2(x,t))^2 - (\partial_t y_2(x,t-s))^2\right) ds d\Gamma dt.
\end{aligned} \tag{4.62}$$

(4.62) implies

$$E(0) \le E(T) + C\int_0^T\int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + \int_{\tau_1}^{\tau_2} |\partial_t y_2(x,t-s)|^2\, ds\} d\Gamma dt. \tag{4.63}$$

Combining (4.44), (4.61), (4.39) and (4.63), yields

$$\begin{aligned}&\int_\epsilon^{T-\epsilon}\mathcal{E}(t)dt \leq C(E(T)+\\&\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+\int_{\tau_1}^{\tau_2}|\partial_t y_2(x,t-s)|^2\,ds\}d\Gamma dt+\\&\int_0^T\int_{\Gamma_2}|y_2(x,t)|^2\,d\Gamma dt+\int_0^T\int_\Omega y^2(x,t)dxdt+\\&C(\epsilon,\delta,T)\left\|y\right\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))}.\end{aligned}\tag{4.64}$$

On the other hand for a fixed ϵ

$$\int_0^\epsilon\mathcal{E}(t)dt+\int_{T-\epsilon}^T\mathcal{E}(t)\leq 2\epsilon E(0),$$

and by (4.63),

$$\int_0^\epsilon\mathcal{E}(t)dt+\int_{T-\epsilon}^T\mathcal{E}(t)\leq C\{E(T)+\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+\int_{\tau_1}^{\tau_2}|\partial_t y_2(x,t-s)|^2\,ds\}d\Gamma dt\}.$$

Hence

$$\begin{aligned}&\int_0^T\mathcal{E}(t)dtdt \leq C(E(T)+\\&\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+\int_{\tau_1}^{\tau_2}|\partial_t y_2(x,t-s)|^2\,ds\}d\Gamma dt+\\&\int_0^T\int_{\Gamma_2}|y_2(x,t)|^2\,d\Gamma dt+\int_0^T\int_\Omega|y(x,t)|^2\,dxdt+\\&C(\epsilon,\delta,T)\left\|y\right\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))},\end{aligned}\tag{4.65}$$

Notice that for any $\theta>0$ (see [45], p. 112, Theorem 16.3)

$$\left\|y_2\right\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))}\leq\theta\int_0^T\int_{\Omega_2}|\nabla y_2(x,t)|^2\,dxdt+C(\theta)\int_0^T\int_{\Omega_2}|y_2(x,t)|^2\,dxdt,$$

and consequently

$$\begin{aligned}&\left\|y_2\right\|^2_{L^2(0,T;H^{1/2+\delta}(\Omega_2))}\leq\frac{\theta}{a_2}\int_0^T\int_\Omega a(x)\,|\nabla y(x,t)|^2\,dxdt+\\&C(\theta)\int_0^T\int_\Omega|y(x,t)|^2\,dxdt.\end{aligned}\tag{4.66}$$

Collecting (4.9), (4.44), (4.45), (4.47), (4.65) and (4.66), we obtain for an appropriate choice of θ and for T large enough

$$\begin{aligned}E(T)\leq C\{|\partial_t y_2(x,t)|^2+\int_{\tau_1}^{\tau_2}|\partial_t y_2(x,t-s)|^2\,ds\}d\Gamma dt+\\C(\int_0^T\int_{\Gamma_2}|y_2(x,t)|^2\,d\Gamma dt+\int_0^T\int_\Omega|y(x,t)|^2\,dxdt).\end{aligned}\tag{4.67}$$

Step 6.
We prove by a compactness-uniqueness argument that there exists a constant $C>0$ such that

$$\begin{aligned}&\left\|y\right\|^2_{L^2(0,T;L^2(\Omega))}+\left\|y\right\|^2_{L^2(0,T;L^2(\Gamma_2))}\leq\\&C\int_0^T\int_{\Gamma_2}\{|\partial_t y_2(x,t)|^2+\int_{\tau_1}^{\tau_2}|\partial_t y_2(x,t-s)|^2\,ds\}d\Gamma dt.\end{aligned}\tag{4.68}$$

Assume that there exists a sequence y_n of solutions of problem (4.1) – (4.7) with

$$\begin{aligned} y_{in}(x,0) &= y^0_{in}(x), \partial_t y_{in}(x,0) = y^1_{in}(x), \quad x \in \Omega_i, \\ y_{in}(x,t-\tau) &= f^0_{in}(x,t-\tau), \quad x \in \Omega_i, t \in (0,\tau), i = 1,2, \end{aligned}$$

such that

$$\|y_{in}\|^2_{L^2(0,T;L^2(\Omega))} + \|y_{in}\|^2_{L^2(0,T;L^2(\Gamma_2))} = 1, \quad i = 1,2,$$

$$\int_0^T \int_{\Gamma_2} \{|\partial_t y_{2n}(x,t)|^2 + \int_{\tau_1}^{\tau_2} |\partial_t y_{2n}(x,t-s)|^2 \, ds\} d\Gamma dt \to 0 \text{ as } n \to +\infty. \tag{4.69}$$

Since each solution satisfies (4.67), we deduce from (4.63) and (4.69) that the sequence $Y^0_n = (y^0_n, y^1_n, f^0_n)$ is bounded in H. Hence there is a subsequence still denoted by Y^0_n which converges weakly to some $Y^0 = (y^0, y^1, f^0)$. Let y be the solution of problem (4.1) – (4.7) corresponding to such initial conditions. We have from Proposition 4.2.1

$$y \in C(0,T; H^1_{\Gamma_1}(\Omega)) \cap C^1(0,T; L^2(\Omega)).$$

Then

$$\begin{aligned} y_n &\to y \text{ in } L^\infty(0,T; H^1_{\Gamma_1}(\Omega)) \qquad \text{weak-star}, \\ y_n &\to y \text{ strongly in } L^2(0,T; L^2(\Gamma_2)) \cap L^2(0,T, L^2(\Omega)). \end{aligned}$$

This fact along with the compactness $H^1_{\Gamma_1}(\Omega) \to H^{1-\varepsilon}(\Omega)$ for $\varepsilon > 0$, implies that there exists a subsequence still denoted by y_n such that $y_n \to y$ strongly in $L^\infty(0,T; H^{1-\varepsilon}(\Omega))$. Then we have from (4.69)

$$\|y\|^2_{L^2(0,T;L^2(\Omega))} + \|y\|^2_{L^2(0,T;L^2(\Gamma_2))} = 1, \tag{4.70}$$

and

$$\int_0^T \int_{\Gamma_2} \{|\partial_t y_2(x,t)|^2 + \int_{\tau_1}^{\tau_2} |\partial_t y_2(x,t-s)|^2 \, ds\} d\Gamma dt = 0.$$

Thus y satisfies

$$\partial_t y_2(x,t) = 0 \qquad \text{on } \Gamma_2 \times (0,T),$$

and

$$\frac{\partial y_2(x,t)}{\partial \nu} + b(x) y_2(x,t) = 0 \qquad \text{on } \Gamma_2 \times (0,T).$$

Let

$$u_i(x,t) = y_i(x,t). \tag{4.71}$$

Then

$$\begin{cases} \partial_t^2 u_i(x,t) - a_i \Delta u_i(x,t) = 0, & (x,t) \in \Omega_i \times (0,T), i = 1,2, \\ u_1(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,T), \\ \dfrac{\partial u_2(x,t)}{\partial \nu} = u_2(x,t) = 0, & (x,t) \in \Gamma_2 \times (0,T), \\ u_1(x,t) = u_2(x,t), & (x,t) \in \Gamma_0 \times (0,T), \\ a_1 \dfrac{\partial u_1(x,t)}{\partial \nu} = a_2 \dfrac{\partial u_2(x,t)}{\partial \nu} & (x,t) \in \Gamma_0 \times (0,T). \end{cases}$$

From Holmgren's uniqueness theorem (see [44] Thm. 8.2, p. 92, and Thm. 7.1, p.391) applied to problem

$$\begin{cases} \partial_t^2 u_2(x,t) - a_2 \Delta u_2(x,t) = 0, & (x,t) \in \Omega_2 \times (0,T), \\ \dfrac{\partial u_2(x,t)}{\partial \nu} = u_2(x,t) = 0, & (x,t) \in \Gamma_2 \times (0,T), \end{cases}$$

we obtain for T large enough

$$u_2(x,t) = 0, \quad (x,t) \in \Omega_2 \times (0,T), \tag{4.72}$$

and hence

$$u_1(x,t) = \frac{\partial u_1(x,t)}{\partial \nu} = 0, \quad (x,t) \in \Gamma_0 \times (0,T).$$

We apply again Holmgren's uniqueness theorem this time to problem

$$\begin{cases} \partial_t^2 u_1(x,t) - a_1 \Delta u_1(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u_1(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,T), \\ u_1(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,T), \\ \dfrac{\partial u_1(x,t)}{\partial \nu} = 0, & (x,t) \in \Gamma_0 \times (0,T), \end{cases}$$

we obtain

$$u_1(x,t) = 0, \ (x,t) \in \Omega_1 \times (0,T). \tag{4.73}$$

(4.71) together with (4.72) and (4.73) implies that

$$y_i(x,t) = y_i(x).$$

Thus $y_i, (i = 1,2)$, verifies

$$\begin{cases} -a_i \Delta y_i(x) = 0, & x \in \Omega_i, i = 1,2, \\ y_1(x) = 0, & x \in \Gamma_1, \\ \frac{\partial y_2(x)}{\partial \nu} = 0, & x \in \Gamma_2, \\ y_1(x) = y_2(x), & x \in \Gamma_0, \\ a_1 \frac{\partial y_1(x)}{\partial \nu} = a_2 \frac{\partial y_2(x)}{\partial \nu}, & x \in \Gamma_0, \end{cases}$$

and so $y_i(x) = 0$ for $x \in \Omega_i, i = 1,2$, and consequently $y(x) = 0$ for $x \in \Omega$. This contradicts (4.70).

Step 8.

The estimate (4.39) together with (4.67) and (4.68) yields

$$E(T) \leq \frac{C}{k+C} E(0). \tag{4.74}$$

The desired conclusion follows now from (4.74) since $0 < \frac{C}{k+C} < 1$.

4.4 Appendix C

Proof of the identity (4.48).

We multiply both sides of (4.1) by $2h(x).\nabla y_i(x,t) + (divh(x) - \alpha) y_i(x,t)$ and integrate over $\Omega_i \times (0,T), i = 1,2$; we have

$$2\int_0^T \int_{\Omega_i} \partial_t^2 y_i(x,t) h(x).\nabla y_i(x,t) dxdt + \int_0^T \int_{\Omega_i} \partial_t^2 y_i(x,t)(divh(x) - \alpha) y_i(x,t) dxdt -$$

$$2\int_0^T \int_{\Omega_i} a_i \Delta y_i(x,t)) h(x).\nabla y_i(x,t) dxdt - \int_0^T \int_{\Omega_i} a_i \Delta y_i(x,t))(divh(x) - \alpha) y_i(x,t) dxdt = 0. \tag{4.75}$$

We sum up (4.75) for i, we obtain

$$\begin{aligned}
&2\int_0^T\int_\Omega \partial_t^2 y(x,t)h(x).\nabla y(x,t)dxdt+\int_0^T\int_\Omega \partial_t^2 y(x,t)(divh(x)-\alpha)y(x,t)dxdt-\\
&2\int_0^T\int_{\Omega_1} a_1\Delta y_1(x,t))h(x).\nabla y_1(x,t)dxdt-\\
&\int_0^T\int_{\Omega_1} a_1\Delta y_1(x,t))(divh(x)-\alpha)y_1(x,t)dxdt-\\
&2\int_0^T\int_{\Omega_2} a_2\Delta y_2(x,t))h(x).\nabla y_2(x,t)dxdt-\\
&\int_0^T\int_{\Omega_2} a_2\Delta y_2(x,t))(divh(x)-\alpha)y_2(x,t)dxdt=0. \qquad (4.76)
\end{aligned}$$

Below, we compute the terms on the left-hand side of (4.76).

- Term $2\int_0^T\int_\Omega \partial_t^2 y(x,t)h(x).\nabla y(x,t)dxdt$
  Integration by parts in t yields

$$\begin{aligned}
&2\int_0^T\int_\Omega \partial_t^2 y(x,t)h(x).\nabla y(x,t)dxdt=2\left[\int_\Omega \partial_t y(x,t)h(x).\nabla y(x,t)dx\right]_0^T-\\
&2\int_0^T\int_\Omega \partial_t y(x,t)h(x).\nabla\partial_t y(x,t)dxdt=\\
&2\left[\int_\Omega \partial_t y(x,t)h(x).\nabla y(x,t)dx\right]_0^T-\int_0^T\int_\Omega h(x).\nabla((\partial_t y(x,t))^2)dxdt. \qquad (4.77)
\end{aligned}$$

  Applying Green's theorem to the second integral on the right-hand side of (4.77), we obtain

$$\begin{aligned}
&2\int_0^T\int_\Omega \partial_t^2 y(x,t)h(x).\nabla y(x,t)dxdt=2\left[\int_\Omega \partial_t y(x,t)h(x).\nabla y(x,t)dx\right]_0^T-\\
&\int_0^T\int_\Gamma(\partial_t y(x,t))^2 h(x).\nu(x)d\Gamma dt+\int_0^T\int_\Omega(\partial_t y(x,t))^2 divh(x)dxdt. \qquad (4.78)
\end{aligned}$$

- Term $\int_0^T\int_\Omega \partial_t^2 y(x,t)(divh(x)-\alpha)y(x,t)dxdt$
  Using again integration by parts in t, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \partial_t^2 y(x,t)(divh(x)-\alpha)y(x,t)dxdt=\\
&\left[\int_\Omega \partial_t y(x,t)(divh(x)-\alpha)y(x,t)dx\right]_0^T-\int_0^T\int_\Omega(\partial_t y(x,t))^2 div(h(x)-\alpha)dxdt. \qquad (4.79)
\end{aligned}$$

- Term
  $2\int_0^T\int_{\Omega_1} a_1\Delta y_1(x,t))h(x).\nabla y_1(x,t)dxdt+2\int_0^T\int_{\Omega_2} a_2\Delta y_2(x,t))h(x).\nabla y_2(x,t)dxdt$
  From Green's theorem, we have,

$$\begin{aligned}
&2a_1\int_0^T\int_{\Omega_1}\Delta y_1(x,t)h(x).\nabla y_1(x,t)dxdt=2a_1\int_0^T\int_{\Gamma_1}\frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)d\Gamma dt+\\
&2a_1\int_0^T\int_{\Gamma_0}\frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)d\Gamma dt-\\
&2a_1\int_0^T\int_{\Omega_1}\nabla y_1(x,t).\nabla(h(x).\nabla y_1(x,t))dxdt. \qquad (4.80)
\end{aligned}$$

Applying the identity

$$\nabla w(x).\nabla(h(x).\nabla w(x)) = J(x)\nabla w(x).\nabla w(x) + \frac{1}{2}h(x).\nabla(|\nabla w(x)|^2)$$

to the last integral on the right hand side of (4.80), we find

$$2a_1 \int_0^T \int_{\Omega_1} \Delta y_1(x,t)h(x).\nabla y_1(x,t)dxdt = 2a_1 \int_0^T \int_{\Gamma_1} \frac{\partial y_1(x,t)}{\partial \nu} h(x).\nabla y_1(x,t)d\Gamma dt +$$
$$2a_1 \int_0^T \int_{\Gamma_0} \frac{\partial y_1(x,t)}{\partial \nu} h(x).\nabla y_1(x,t)d\Gamma dt - 2a_1 \int_0^T \int_{\Omega_1} J(x)\nabla y_1(x,t).\nabla y_1(x,t))dxdt -$$
$$a_1 \int_0^T \int_{\Omega_1} h(x).\nabla(|\nabla y_1(x,t)|^2).$$

Another use of Green's theorem yields

$$2a_1 \int_0^T \int_{\Omega_1} \Delta y_1(x,t)h(x).\nabla y_1(x,t)dxdt = 2a_1 \int_0^T \int_{\Gamma_1} \frac{\partial y_1(x,t)}{\partial \nu} h(x).\nabla y_1(x,t)d\Gamma dt +$$
$$2a_1 \int_0^T \int_{\Gamma_0} \frac{\partial y_1(x,t)}{\partial \nu} h(x).\nabla y_1(x,t)d\Gamma dt - 2a_1 \int_0^T \int_{\Omega_1} J(x)\nabla y_1(x,t).\nabla y_1(x,t))dxdt -$$
$$a_1 \int_0^T \int_{\Gamma_1} |\nabla y_1(x,t)|^2 h(x).\nu(x)d\Gamma dt - a_1 \int_0^T \int_{\Gamma_0} |\nabla y_1(x,t)|^2 h(x).\nu(x)d\Gamma dt +$$
$$a_1 \int_0^T \int_{\Omega_1} |\nabla y_1(x,t)|^2 divh(x)dxdt. \tag{4.81}$$

For the intgral term $2\int_0^T \int_{\Omega_2} a_2\Delta y_2(x,t))h(x).\nabla y_2(x,t)dxdt$, we proceed as above to find

$$2a_2 \int_0^T \int_{\Omega_2} \Delta y_2(x,t)h(x).\nabla y_2(x,t)dxdt = 2a_2 \int_0^T \int_{\Gamma_2} \frac{\partial y_2(x,t)}{\partial \nu} h(x).\nabla y_2(x,t)d\Gamma dt -$$
$$2a_2 \int_0^T \int_{\Gamma_0} \frac{\partial y_2(x,t)}{\partial \nu} h(x).\nabla y_2(x,t)d\Gamma dt - 2a_2 \int_0^T \int_{\Omega_2} J(x)\nabla y_2(x,t).\nabla y_2(x,t)dxdt -$$
$$a_2 \int_0^T \int_{\Gamma_2} |\nabla y_2(x,t)|^2 h(x).\nu(x)d\Gamma dt + a_2 \int_0^T \int_{\Gamma_0} |\nabla y_2(x,t)|^2 h(x).\nu(x)d\Gamma dt +$$
$$a_2 \int_0^T \int_{\Omega_2} |\nabla y_2(x,t)|^2 divh(x)dxdt. \tag{4.82}$$

Summing up (4.81) and (4.82) yields

$$\begin{aligned}
&2\int_0^T\int_\Omega div(a(x)\nabla y(x,t))h(x).\nabla y(x,t)dxdt = \\
&2a_1\int_0^T\int_{\Gamma_1}\frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)d\Gamma dt + 2a_1\int_0^T\int_{\Gamma_0}\frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t)d\Gamma dt- \\
&2a_1\int_0^T\int_{\Omega_1}J(x)\nabla y_1(x,t).\nabla y_1(x,t))dxdt- \\
&a_1\int_0^T\int_{\Gamma_1}\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x)d\Gamma dt - a_1\int_0^T\int_{\Gamma_0}\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x)d\Gamma dt+ \\
&a_1\int_0^T\int_{\Omega_1}\left|\nabla y_1(x,t)\right|^2 divh(x)dxdt + 2a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)d\Gamma dt- \\
&2a_2\int_0^T\int_{\Gamma_0}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)d\Gamma dt- \\
&2a_2\int_0^T\int_{\Omega_2}J(x)\nabla y_2(x,t).\nabla y_2(x,t))dxdt- \\
&a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt + a_2\int_0^T\int_{\Gamma_0}\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x)d\Gamma dt+ \\
&a_2\int_0^T\int_{\Omega_2}\left|\nabla y_2(x,t)\right|^2 divh(x)dxdt.
\end{aligned}\tag{4.83}$$

We conclude from the boundary conditions (4.3) and (4.5) that

$$\nabla y_1(x,t) = \frac{\partial y_1(x,t)}{\partial\nu}\nu(x), \qquad \text{on } \Gamma_1\times(0,T), \tag{4.84}$$

and

$$\nabla(y_2(x,t)-y_1(x,t)) = \frac{\partial(y_2(x,t)-y_1(x,t))}{\partial\nu}\nu(x), \quad \text{on } \Gamma_0\times(0,T).$$

Then

$$\begin{aligned}
&\left|\nabla y_2(x,t)\right|^2 = \left|\nabla y_1(x,t)\right|^2 + 2(\frac{\partial y_2(x,t)}{\partial\nu} - \frac{\partial y_1(x,t)}{\partial\nu})\frac{\partial y_1(x,t)}{\partial\nu}+ \\
&(\frac{\partial y_2(x,t)}{\partial\nu} - \frac{\partial y_1(x,t)}{\partial\nu})^2 = \left|\nabla y_1(x,t)\right|^2 + (\frac{\partial y_2(x,t)}{\partial\nu})^2- \\
&(\frac{\partial y_1(x,t)}{\partial\nu})^2, \quad \text{on } \Gamma_0\times(0,T),
\end{aligned}$$

so on $\Gamma_0\times(0,T)$,

$$\begin{aligned}
&2a_1\frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t) - 2a_2\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t) - a_1\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x)+ \\
&a_2\left|\nabla y_2(x,t)\right|^2 h(x).\nu(x) = 2a_1\frac{\partial y_1(x,t)}{\partial\nu}h(x).\nabla y_1(x,t) - 2a_2\frac{\partial y_2(x,t)}{\partial\nu}(\nabla y_1(x,t)+ \\
&(\frac{\partial y_2(x,t)}{\partial\nu} - \frac{\partial y_1(x,t)}{\partial\nu})\nu(x)).h(x) - a_1\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x) + a_2(\left|\nabla y_1(x,t)\right|^2 + \\
&(\frac{\partial y_2(x,t)}{\partial\nu})^2 - (\frac{\partial y_1(x,t)}{\partial\nu}))h(x).\nu(x) = -2a_1(\frac{a_1}{a_2}-1)(\frac{\partial y_1(x,t)}{\partial\nu})^2 h(x).\nu(x)+ \\
&(a_2-a_1)\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x) + (\frac{a_1^2}{a_2}-a_2)(\frac{\partial y_1(x,t)}{\partial\nu})^2 h(x).\nu(x) = \\
&(a_2-a_1)\left|\nabla y_1(x,t)\right|^2 h(x).\nu(x) - \frac{(a_2-a_1)^2}{a_2}(\frac{\partial y_1(x,t)}{\partial\nu})^2 h(x).\nu(x).
\end{aligned}\tag{4.85}$$

Insertion of (4.84) and (4.85) into (4.83) results in

$$2\int_0^T\int_\Omega div(a(x)\nabla y(x,t))h(x).\nabla y(x,t)dxdt = a_1\int_0^T\int_{\Gamma_1}(\frac{\partial y_1(x,t)}{\partial\nu})^2h(x).\nu(x)d\Gamma dt-$$
$$(a_1-a_2)\int_0^T\int_{\Gamma_0}\left|\nabla y_1(x,t)\right|^2h(x).\nu(x)d\Gamma dt-\frac{(a_2-a_1)^2}{a_2}\int_0^T\int_{\Gamma_0}(\frac{\partial y_1(x,t)}{\partial\nu})^2h(x).\nu(x)d\Gamma dt+$$
$$2a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}h(x).\nabla y_2(x,t)d\Gamma dt-a_2\int_0^T\int_{\Gamma_2}\left|\nabla y_2(x,t)\right|^2h(x).\nu(x)d\Gamma dt-$$
$$2\int_0^T\int_\Omega a(x)J(x)\nabla y(x,t).\nabla y(x,t))dxdt+\int_0^T\int_\Omega a(x)\left|\nabla y(x,t)\right|^2divh(x)dxdt. \tag{4.86}$$

- Term
  $a_1\int_0^T\int_{\Omega_1}\Delta y_1(x,t)(divh(x)-\alpha)y_1(x,t)dxdt+a_2\int_0^T\int_{\Omega_2}\Delta y_2(x,t)(divh(x)-\alpha)y_2(x,t)dxdt$
  It follows from Green's theorem that

$$a_1\int_0^T\int_{\Omega_1}\Delta y_1(x,t)(divh(x)-\alpha)y_1(x,t)dxdt+$$
$$a_2\int_0^T\int_{\Omega_2}\Delta y_2(x,t)(divh(x)-\alpha)y_2(x,t)dxdt=$$
$$a_1\int_0^T\int_{\Gamma_1}\frac{\partial y_1(x,t)}{\partial\nu}(divh(x)-\alpha)y_1(x,t)d\Gamma dt+$$
$$a_1\int_0^T\int_{\Gamma_0}\frac{\partial y_1(x,t)}{\partial\nu}(divh(x)-\alpha)y_1(x,t)d\Gamma dt-$$
$$a_1\int_0^T\int_{\Omega_1}\left|\nabla y_1(x,t)\right|^2(divh(x)-\alpha)dxdt-$$
$$a_1\int_0^T\int_{\Omega_1}y_1(x,t)\nabla y_1(x,t).\nabla(divh(x)-\alpha)dxdt+$$
$$a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(divh(x)-\alpha)y_2(x,t)d\Gamma dt-$$
$$a_2\int_0^T\int_{\Gamma_0}\frac{\partial y_2(x,t)}{\partial\nu}(divh(x)-\alpha)y_2(x,t)d\Gamma dt-$$
$$a_2\int_0^T\int_{\Omega_2}\left|\nabla y_2(x,t)\right|^2(divh(x)-\alpha)dxdt-$$
$$a_2\int_0^T\int_{\Omega_2}y_2(x,t)\nabla y_2(x,t).\nabla(divh(x)-\alpha)dxdt.$$

Thus from $(4.3),(4.5)$ and (4.6), we conclude that

$$\begin{aligned}
&a_1\int_0^T\int_{\Omega_1}\Delta y_1(x,t)(divh(x)-\alpha)y_1(x,t)dxdt+\\
&a_2\int_0^T\int_{\Omega_2}\Delta y_2(x,t)(divh(x)-\alpha)y_2(x,t)dxdt=\\
&a_2\int_0^T\int_{\Gamma_2}\frac{\partial y_2(x,t)}{\partial\nu}(divh(x)-\alpha)y_2(x,t)d\Gamma dt-\\
&\int_0^T\int_{\Omega}a(x)\left|\nabla y(x,t)\right|^2(divh(x)-\alpha)dxdt-\\
&\int_0^T\int_{\Omega}a(x)y(x,t)\nabla y(x,t).\nabla(divh(x)-\alpha)dxdt. \qquad (4.87)
\end{aligned}$$

The desired identity follows now from $(4.76),(4.78),(4.79),(4.86)$ and (4.87).

Chapter 5

Stabilization of an interconnected system of Schrödinger and wave equations with boundary coupling

5.1 Introduction and statement of the main result

The aim of this chapter is to study the stability of an interconnected system of Schrödinger and wave equations with boundary coupling subject to a dissipative boundary feedback acting on the wave equation through the Neumann boundary condition in multidimensional situations. In the case of one space dimension, Wang and Wang [80] used a frequency domain approach to prove that the solution decays exponentially in an appropriate Hilbert space. They also showed that if the dissipative boundary feedback is designed only at the Schrödinger equation then the system is polynomially stable but not exponentially stable.

Let $\Omega, \Omega_1, \Omega_2, \Gamma, \Gamma_0, \Gamma_1$ and Γ_2 be as in Chapter 2

Consider in Ω_1 the Schrödinger equation coupled with the wave equation in Ω_2,

$$
\begin{aligned}
&\partial_t z(x,t) - i\Delta z(x,t) = 0 && \text{in } \Omega_1 \times (0,+\infty), && (5.1)\\
&\partial_t^2 y(x,t) - \Delta y(x,t) = 0 && \text{in } \Omega_2 \times (0,+\infty), && (5.2)\\
&z(x,0) = z_0(x) && \text{in } \Omega_1, && (5.3)\\
&y(x,0) = y_0(x), \partial_t y(x,0) = y_1(x) && \text{in } \Omega_2, && (5.4)\\
&z(x,t) = 0 && \text{in } \Gamma_1 \times (0,+\infty), && (5.5)\\
&\frac{\partial y(x,t)}{\partial \nu} = -\mu \partial_t y(x,t) && \text{in } \Gamma_2 \times (0,+\infty), && (5.6)\\
&z(x,t) = y(x,t), && \text{in } \Gamma_0 \times (0,+\infty), && (5.7)\\
&\frac{\partial z(x,t)}{\partial \nu} = \frac{\partial y(x,t)}{\partial \nu} && \text{in } \Gamma_0 \times (0,+\infty), && (5.8)
\end{aligned}
$$

where

- μ is a positive constant,
- z_0, y_0, y_1 are the initial data which belong to suitable spaces.

(5.7) and (5.8) are the transmission or interface conditions while (5.6) is a standard dissipation law for the wave equation.

The energy of a solution (y, z) of (5.1)-(5.8) at time t is defined by

$$E(t) = \frac{1}{2}\int_{\Omega_1} |\nabla z(x,t)|^2 \, dx + \frac{1}{2}\int_{\Omega_2} \{|\partial_t y(x,t)|^2 + |\nabla y(x,t)|^2\} dx.$$

Formally, we have

$$\frac{d}{dt}E(t) = -\mu \int_{\Gamma_2} |\partial_t y(x,t)|^2 \, d\Gamma,$$

which shows that system (5.1)-(5.8) is dissipative.

Set

$$\begin{aligned} &H^1_{\Gamma_1}(\Omega_1) = \{f \in H^1(\Omega_1) : f = 0 \text{ on } \Gamma_1\}, \\ &\mathcal{V} = \{(f_1, f_2)^T \in H^1_{\Gamma_1}(\Omega_1) \times H^1(\Omega_2) : f_1 = f_2 \text{ on } \Gamma_0\}, \end{aligned}$$

and consider system (5.1)-(5.8) in the Hilbert space

$$\mathcal{H} = \mathcal{V} \times L^2(\Omega_2)$$

with the inner product induced norm

$$\|(f_1, f_2, f_3)\|^2_{\mathcal{H}} = \int_{\Omega_1} |\nabla f_1(x)|^2 \, dx + \int_{\Omega_2} \{|\nabla f_2(x)|^2 + |f_3(x)|^2\} dx. \tag{5.9}$$

Remark 5.1.1. *In $\mathcal{H}$, the norm (5.9) is equivalent to the norm of* $H^1_{\Gamma_1}(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2)$.

The operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ generating the dynamics described by (5.1)-(5.8) is given by

$$A(z, y, u)^T = (i\Delta z, u, \Delta y)^T, \tag{5.10}$$

$$\begin{aligned} D(A) = \{&(z, y, u)^T \in \mathcal{H}; z \in H^2(\Omega_1), y \in H^2(\Omega_2), i\Delta z \in H^1_{\Gamma_1}(\Omega_1), u \in H^1(\Omega_2) \\ &\text{satisfying (5.12)-(5.14) below}\}. \end{aligned} \tag{5.11}$$

$$\frac{\partial z}{\partial \nu} = \frac{\partial y}{\partial \nu} \text{ on } \Gamma_0, \tag{5.12}$$

$$u = i\Delta z \text{ on } \Gamma_0, \tag{5.13}$$

$$\frac{\partial y}{\partial \nu} = -\mu u \text{ on } \Gamma_2. \tag{5.14}$$

Note that for $(z, y, u)^T \in D(A)$, we have by trace theory the following boundary regularity:

- $z|_{\Gamma_j} \in H^{3/2}(\Gamma_j), j = 0, 1,$
- $\Delta z|_{\Gamma_j} \in H^{1/2}((\Gamma_j), j = 0, 1,$
- $\frac{\partial z}{\partial \nu}|_{\Gamma_j} \in H^{1/2}(\Gamma_j), j = 0, 1,,$
- $y|_{\Gamma_j} \in H^{3/2}(\Gamma_j), j = 0, 2,$
- $\frac{\partial y}{\partial \nu}|_{\Gamma_j} \in H^{1/2}(\Gamma_j), j = 0, 2,$
- $u|_{\Gamma_j} \in H^{1/2}(\Gamma_j), j = 0, 2.$

Then system (5.1)-(5.8) can be re-written as an abstract Cauchy problem in $\mathcal{H}$ as follows

$$\left\{ \begin{array}{l} \frac{d}{dt}Y(t) = AY(t), \\ Y(0) = Y_0, \end{array} \right. \tag{5.15}$$

where

$$Y(t) = (z(.,t), y(.,t), \partial_t y(.,t))^T \text{ and } Y_0 = (z_0, y_0, y_1)^T.$$

The main result of this chapter can be stated as follows.

Theorem 5.1.2. *Assume that there exists* $x^0 \in \mathbb{R}^n$*, such that for* $m(x) = x - x^0$,

$$m(x) \cdot \nu(x) \geq \gamma > 0 \text{ on } \Gamma_2, \tag{5.16}$$

$$m(x) \cdot \nu(x) < 0 \quad \text{on } \Gamma_0 \text{ and on } \Gamma_1. \tag{5.17}$$

Then there exist constants $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq Me^{-\omega t}E(0) \qquad \text{for all } t \geq 0$$

for all solutions of (5.1)-(5.8) with $(z_0, y_0, y_1) \in \mathcal{H}$.

The proof of Theorem 5.1.2 combines frequency domain method and multipliers technique and is given in Section 3. In Section 2, we formulate the coupled system $(5.1) - (5.8)$ as an abstract evolution equation in an appropriate Hilbert space and use linear semigroup theory to establish its well-posedness. This chapter is the subject of the paper [55].

5.2 Well-posedness

Proposition 5.2.1. *The operator* A *defined by* $(5.10) - (5.11)$ *is maximal dissipative. Hence it generates a contraction* C_0-*semigroup on* $\mathcal{H}$*. Consequently, for every* $Y_0 \in \mathcal{H}$*, problem* (5.15) *has a unique solution* Y *whose regularity depends on the initial datum* Y_0 *as follows:*

$$Y(.) \in C([0, +\infty); \mathcal{H}) \text{ if } Y_0 \in \mathcal{H},$$
$$Y(.) \in C([0, +\infty); D(A)) \cap C^1([0, +\infty); \mathcal{H}) \text{ if } Y_0 \in D(A).$$

Proof. Let $Y = (z, y, u)^T \in D(A)$. Then

$$\langle AY, Y\rangle = \int_{\Omega_1} \nabla(i\Delta z(x)) \cdot \nabla\overline{z(x)}dx + \int_{\Omega_2} \nabla u(x) \cdot \nabla\overline{y(x)}\, dx + \int_{\Omega_2} \Delta y(x)\overline{u(x)}dx. \tag{5.18}$$

Applying Green's theorem to the first and the third integral on the right-hand side of (5.18) and using the fact that the normal vector on Γ_0 is oriented towards the interior of Ω_2, we obtain

$$\begin{aligned}\langle AY, Y\rangle = &\int_{\Gamma_1} i\Delta z(x)\frac{\partial\overline{z(x)}}{\partial\nu}d\Gamma - \int_{\Gamma_0} i\Delta z(x)\frac{\partial\overline{z(x)}}{\partial\nu}d\Gamma - i\int_{\Omega_1} |\Delta z(x)|^2\, dx+ \\ &\int_{\Omega_2} \nabla u(x) \cdot \nabla\overline{y(x)}\, dx + \int_{\Gamma_2} \frac{\partial y(x)}{\partial\nu}\overline{u(x)}d\Gamma + \int_{\Gamma_0} \frac{\partial y(x)}{\partial\nu}\overline{u(x)}d\Gamma- \\ &\int_{\Omega_2} \nabla y(x) \cdot \nabla\overline{u(x)}dx.\end{aligned} \tag{5.19}$$

It follows from (5.19) and $(5.11) - (5.14)$ that

$$\begin{aligned}\langle AY, Y\rangle =&2i\Im\{\int_{\Omega_2} \nabla u(x) \cdot \nabla\overline{y(x)}\, dx\} - i\int_{\Omega_1} |\Delta z(x)|^2\, dx- \\ &2i\Re\int_{\Gamma_0} \Delta z(x)\frac{\partial\overline{z(x)}}{\partial\nu}d\Gamma - \mu\int_{\Gamma_2} |u(x)|^2\, d\Gamma.\end{aligned}$$

Hence

$$\Re\,\langle AY, Y\rangle = -\mu\int_{\Gamma_2} |u(x)|^2\, d\Gamma \tag{5.20}$$

which shows that A is dissipative.

Now we show that $\lambda I - A$ is onto for some $\lambda > 0$. Given $(f, g, h)^T \in \mathcal{H}$, we seek $Y = (z, y, u)^T \in D(A)$ such that

$$(\lambda I - A)Y = (f, g, h)^T \tag{5.21}$$

or equivalently

$$\begin{aligned}
&\lambda z(x) - i\Delta z(x) = f(x) && x \in \Omega_1, && (5.22)\\
&\lambda y(x) - u(x) = g(x) && x \in \Omega_2, && (5.23)\\
&\lambda u(x) - \Delta y(x) = h(x) && x \in \Omega_2, && (5.24)\\
&z(x) = 0 && x \in \Gamma_1, && (5.25)\\
&z(x) = y(x) && x \in \Gamma_0, && (5.26)\\
&\frac{\partial z(x)}{\partial \nu} = \frac{\partial y(x)}{\partial \nu} && x \in \Gamma_0, && (5.27)\\
&\frac{\partial y(x)}{\partial \nu} = -\mu u(x) && x \in \Gamma_2. && (5.28)
\end{aligned}$$

From (5.23)-(5.24), we have

$$\lambda^2 y(x) - \Delta y(x) = h(x) + \lambda g(x). \tag{5.29}$$

Let $(\varphi, \psi) \in \mathcal{V}$. Then, multiplying (5.22) by $(1-i)\overline{\varphi}$ (resp. (5.29) by $(1+i)\overline{\psi}$) and integrating in Ω_1(resp. in Ω_2), we obtain

$$\begin{aligned}
&(1-i)\lambda \int_{\Omega_1} z(x)\overline{\varphi(x)}dx + (1+i)\int_{\Omega_1} \nabla z(x) \cdot \nabla\overline{\varphi(x)}dx + (1+i)\int_{\Gamma_0} \frac{\partial z(x)}{\partial \nu}\overline{\varphi(x)}d\Gamma\\
&= (1-i)\int_{\Omega_1} f(x)\overline{\varphi(x)}dx,
\end{aligned} \tag{5.30}$$

$$\begin{aligned}
&(1+i)\lambda^2 \int_{\Omega_2} y(x)\overline{\psi(x)}dx + (1+i)\int_{\Omega_2} \nabla y(x) \cdot \nabla\overline{\psi(x)}dx - (1+i)\int_{\Gamma_0} \frac{\partial y(x)}{\partial \nu}\overline{\psi(x)}d\Gamma\\
&+ (1+i)\mu\int_{\Gamma_2} u(x)\overline{\psi(x)}d\Gamma = (1+i)\int_{\Omega_2} (h(x) + \lambda g(x))\overline{\psi(x)}dx.
\end{aligned} \tag{5.31}$$

Summing up (5.30) and (5.31), and recalling the transmission conditions (5.7) and (5.8), yields

$$\Lambda((z, y), (\varphi, \psi)) = \mathcal{F}(\varphi, \psi), \tag{5.32}$$

where

$$\begin{aligned}
\Lambda((z, y), (\varphi, \psi)) = {} & (1-i)\lambda \int_{\Omega_1} z(x)\overline{\varphi(x)}dx + (1+i)\int_{\Omega_1} \nabla z \cdot \nabla\overline{\varphi(x)}dx +\\
& (1+i)\lambda^2 \int_{\Omega_2} y(x)\overline{\psi(x)}dx + (1+i)\int_{\Omega_2} \nabla y \cdot \nabla\overline{\psi(x)}dx + \mu\lambda(1+i)\int_{\Gamma_2} y(x)\overline{\psi(x)}d\Gamma,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}(\varphi, \psi) = {} & (1-i)\int_{\Omega_1} f(x)\overline{\varphi}(x)dx + (1+i)\int_{\Omega_2} (h(x) + \lambda g(x))\overline{\psi}(x)dx +\\
& (1+i)\int_{\Gamma_2} \mu g(x)\overline{\psi}(x)d\Gamma.
\end{aligned}$$

Λ is a continuous bilinear form on $\mathcal{V}$ and $\mathcal{F}$ is a continuous linear form on $\mathcal{V}$. Moreover

$$\Re\Lambda((z, y), (z, y)) \geq min\{1, \lambda, \mu\lambda, \lambda^2\} \left\|(z, y)\right\|_{\mathcal{V}}^2.$$

By Lax-Milgram Theorem, for all $(f, g, h) \in \mathcal{H}$, problem (5.32) has a unique solution $(z, y) \in \mathcal{V}$.

If we consider in (5.30) (resp. (5.31)) $\varphi \in \mathcal{D}(\Omega_1)(resp.\psi \in \mathcal{D}(\Omega_2))$, then (z, y) is a solution of (5.22), (5.29) in the sense of distribution, and from the regularity of f, g and h, we have $z \in H^1_{\Gamma_1}(\Omega_1) \cap H^2(\Omega_1), \Delta z \in H^1_{\Gamma_1}(\Omega_1), y \in H^2(\Omega_2), u \in H^1(\Omega_2)$. Using Green's theorem in (5.32) and exploiting (5.22) and (5.29), we obtain for all $(\varphi, \psi) \in \mathcal{V}$

$$\int_{\Gamma_0} \frac{\partial z(x)}{\partial \nu}\overline{\varphi(x)}dx = \int_{\Gamma_0} \frac{\partial y(x)}{\partial \nu}\overline{\psi(x)}dx, \tag{5.33}$$

$$\int_{\Gamma_2} (\frac{\partial y(x)}{\partial \nu} + \mu\lambda y(x))\overline{\psi(x)}dx = \mu \int_{\Gamma_2} g(x)\overline{\psi(x)}dx, \tag{5.34}$$

from which follows

$$\begin{aligned} &\frac{\partial z(x)}{\partial \nu} = \frac{\partial y(x)}{\partial \nu} \qquad \text{on } \Gamma_0, \\ &\frac{\partial y(x)}{\partial \nu} + \mu\lambda y(x) = \mu g(x) \qquad \text{on } \Gamma_2. \end{aligned} \tag{5.35}$$

(5.35) together with (5.23) yields

$$\frac{\partial y(x)}{\partial \nu} = -\mu u(x) \qquad \text{on } \Gamma_2.$$

Moreover, (5.22), (5.23) and (5.26) and combined with the fact that $(f, g)^T \in \mathcal{V}$, imply that

$$u(x) = i\Delta z(x) \qquad \text{on } \Gamma_0.$$

So, we have found $(z, y, u) \in D(A)$ for which (5.21) holds and consequently $\lambda I - A$ is onto. Therefore A is maximal dissipative. Thus, by the Lumer-Phillips Theorem (see for instance [65], Theorem 1.4.3), A generates a contraction C_0- semigroup on $\mathcal{H}$. ☐

5.3 Proof of the main result

By Theorem 1.3.4, it is sufficient to prove (1.4) and (1.5).

Lemma 5.3.1. *The imaginary axis is a subset of the resolvent set of A, that is* $i\mathbb{R} \subset \rho(A)$.

Proof. The resolvent of A is compact by Sobolev embedding theorem. Threfore the spectrum of A consists only of the eigenvalues of A. Assume $\lambda = i\beta$, with $\beta \in \mathbb{R}^*$, is an eigenvalue of A and let $W = (z, y, u)^T \in D(A)$ be a corresponding eigenvector. Thus

$$AW = i\beta W \tag{5.36}$$

or equivalently

$$i\Delta z(x) = i\beta z(x) \qquad \text{in } \Omega_1, \tag{5.37}$$

$$u(x) = i\beta y(x) \qquad \text{in } \Omega_2, \tag{5.38}$$

$$\Delta y(x) = i\beta u(x) \qquad \text{in } \Omega_2, \tag{5.39}$$

$$z(x) = 0 \qquad \text{on } \Gamma_1, \tag{5.40}$$

$$z(x) = y(x) \qquad \text{on } \Gamma_0, \tag{5.41}$$

$$\frac{\partial z(x)}{\partial \nu} = \frac{\partial y(x)}{\partial \nu} \qquad \text{on } \Gamma_0, \tag{5.42}$$

$$\frac{\partial y(x)}{\partial \nu} = -\mu u(x) \qquad \text{on } \Gamma_2. \tag{5.43}$$

Taking the inner product of (5.36) with W and using the equality (5.20), we get

$$u(x) = 0 \quad \text{on } \Gamma_2,$$

which together with (5.43) and (5.38) yields

$$\frac{\partial y(x)}{\partial \nu} = y(x) = 0 \quad \text{on } \Gamma_2.$$

Hence y satisfies

$$\begin{cases} -\beta^2 y(x) - \Delta y(x) = 0 & \text{in} \quad \Omega_2, \\ y(x) = \frac{\partial y(x)}{\partial \nu} = 0 & \text{on} \quad \Gamma_2. \end{cases}$$

By the unique continuation principle (see for example [33], [74] and [43]), $y = 0$ in Ω_2. Consequently z satisfies

$$\begin{cases} -\Delta z(x) + \beta z(x) = 0 & \text{in} \quad \Omega_1, \\ z(x) = 0 & \text{on} \quad \Gamma_1, \\ z(x) = \frac{\partial z(x)}{\partial \nu} = 0 & \text{on} \quad \Gamma_0, \end{cases}$$

and again by the unique continuation principle $z = 0$ in Ω_1. Therefore $W = (0,0,0)^T$ and this contradicts that W is an eigenvector of A. ☐

Lemma 5.3.2. *The resolvent operator of A satisfies (1.5).*

Proof. Suppose that (1.5) is not satisfied. Then by the Banach-Steinhaus Theorem ([12]), there exists a sequence of real numbers β_k with $|\beta_k| \to +\infty$ and a sequence of vectors $W_k = (z_k, y_k, u_k) \in D(A)$ with $\|W_k\|_{\mathcal{H}} = 1$ such that

$$\lim_{k \to +\infty} \|(i\beta_k - A)W_k\|_{\mathcal{H}} = 0$$

that is

$$i\beta_k z_k - i\Delta z_k = f_k \to 0 \qquad \text{in } H^1_{\Gamma_1}(\Omega_1), \tag{5.44}$$

$$i\beta_k y_k - u_k = g_k \to 0 \qquad \text{in } H^1(\Omega_2), \tag{5.45}$$

$$i\beta_k u_k - \Delta y_k = h_k \to 0 \qquad \text{in } L^2(\Omega_2), \tag{5.46}$$

$$z_k(x) = 0 \qquad \text{on } \Gamma_1, \tag{5.47}$$

$$z_k(x) = y_k(x) \qquad \text{on } \Gamma_0, \tag{5.48}$$

$$\frac{\partial z_k(x)}{\partial \nu} = \frac{\partial y_k(x)}{\partial \nu} \qquad \text{on } \Gamma_0, \tag{5.49}$$

$$\frac{\partial y_k(x)}{\partial \nu} = -\mu u_k(x) \qquad \text{on } \Gamma_2. \tag{5.50}$$

Since

$$|\Re\langle (i\beta_k - A)W_k, W_k\rangle| \leq \|(i\beta_k - A)W_k\|_{\mathcal{H}}$$

and

$$\Re\langle (i\beta_k - A)W_k, W_k\rangle = \mu \int_{\Gamma_2} |u_k(x)|^2 \, d\Gamma$$

then

$$u_k \to 0 \text{ in } L^2(\Gamma_2). \tag{5.51}$$

Consequently, combining (5.45) and (5.51) and applying trace theory yields

$$y_k \to 0 \text{ in } L^2(\Gamma_2). \tag{5.52}$$

We split the remaining part of the proof into four steps.

Step 1. From (5.44) and the assumption $\|W_k\|_{\mathcal{H}} = 1$, we have

$$\lim_{k\to+\infty} \left\{\Re\langle\beta_k z_k, m\cdot\nabla z_k\rangle_{L^2(\Omega_1)} - \Re\langle\Delta z_k, m\cdot\nabla z_k\rangle_{L^2(\Omega_1)}\right\} = 0.$$

Applying the identities (5.75) and (5.76) in the Appendix D with $\mathcal{O} = \Omega_1, \mathcal{S} = \Gamma_0\cup\Gamma_1, \eta = \beta_k, w = z_k, h = m, H(x) = I_n$, we obtain after using the boundary condition (5.47)

$$\begin{aligned}&\lim_{k\to+\infty}\left\{-\frac{1}{2}\beta_k\int_{\Gamma_0}|z_k(x)|^2 m(x)\cdot\nu(x)d\Gamma + \Re\int_{\Gamma_0}\frac{\partial z_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{z_k(x)}d\Gamma - \right.\\ &\frac{1}{2}\int_{\Gamma_0}|\nabla z_k(x)|^2 m(x)\cdot\nu(x)d\Gamma - \frac{1}{2}\int_{\Gamma_1}|\nabla z_k(x)|^2 m(x)\cdot\nu(x)d\Gamma - \\ &\left.\frac{n}{2}\int_{\Omega_1}(\beta_k|z_k(x)|^2 + |\nabla z_k(x)|^2)dx + \int_{\Omega_1}|\nabla z_k(x)|^2 dx\right\} = 0\end{aligned}\tag{5.53}$$

and

$$\lim_{k\to+\infty}\left\{\int_{\Omega_1}(\beta_k|z_k(x)|^2 + |\nabla z_k(x)|^2)dx\right\} = \lim_{k\to+\infty}\left\{-\Re\int_{\Gamma_0}\frac{\partial z_k(x)}{\partial\nu}\overline{z_k(x)}d\Gamma\right\}.\tag{5.54}$$

Inserting (5.54) into (5.53), we get

$$\begin{aligned}&\lim_{k\to+\infty}\left\{-\frac{1}{2}\beta_k\int_{\Gamma_0}|z_k(x)|^2 m(x)\cdot\nu(x)d\Gamma + \Re\int_{\Gamma_0}\frac{\partial z_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{z_k(x)}d\Gamma - \right.\\ &\frac{1}{2}\int_{\Gamma_0}|\nabla z_k(x)|^2 m(x)\cdot\nu(x)d\Gamma - \frac{1}{2}\int_{\Gamma_1}|\nabla z_k(x)|^2 m(x)\cdot\nu(x)d\Gamma + \\ &\left.\frac{n}{2}\Re\int_{\Gamma_0}\frac{\partial z_k(x)}{\partial\nu}\overline{z_k(x)}d\Gamma + \int_{\Omega_1}|\nabla z_k(x)|^2 dx\right\} = 0.\end{aligned}\tag{5.55}$$

Step 2. We have from (5.45), (5.46) and the fact that $\|W_k\|_{\mathcal{H}} = 1$,

$$\lim_{k\to+\infty}\left\{\Re\langle u_k, i\beta_k m\cdot\nabla y_k\rangle_{L^2(\Omega_2)} - \Re\langle u_k, m\cdot\nabla u_k\rangle_{L^2(\Omega_2)}\right\} = 0,\tag{5.56}$$

$$\lim_{k\to+\infty}\left\{\Re\langle i\beta_k u_k, m\cdot\nabla y_k\rangle_{L^2(\Omega_2)} - \Re\langle\Delta y_k, m\cdot\nabla y_k\rangle_{L^2(\Omega_2)}\right\} = 0.\tag{5.57}$$

Combining (5.56) with (5.57) yields

$$\lim_{k\to+\infty}\left\{-\Re\langle u_k, m\cdot\nabla u_k\rangle_{L^2(\Omega_2)} - \Re\langle\Delta y_k, m\cdot\nabla y_k\rangle_{L^2(\Omega_2)}\right\} = 0.\tag{5.58}$$

Using Green's theorem and the identity (4.81) in the Appendix D with $\mathcal{O} = \Omega_2, \mathcal{S} = \Gamma_0\cup\Gamma_2, w = y_k$, $h = m$ and $H(x) = I_n$, we get

$$\begin{aligned}&\lim_{k\to+\infty}\left\{-\frac{1}{2}\int_{\Gamma_2}|u_k(x)|^2 m(x)\cdot\nu(x)d\Gamma - \frac{1}{2}\int_{\Gamma_0}|u_k(x)|^2 m(x)\cdot\nu(x)d\Gamma - \right.\\ &\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{y_k(x)}d\Gamma - \Re\int_{\Gamma_0}\frac{\partial y_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{y_k(x)}d\Gamma + \int_{\Omega_2}|\nabla y_k(x)|^2 dx + \\ &\frac{1}{2}\int_{\Gamma_2}|\nabla y_k(x)|^2 m(x)\cdot\nu(x)d\Gamma + \frac{1}{2}\int_{\Gamma_0}|\nabla y_k(x)|^2 m(x)\cdot\nu(x)d\Gamma + \\ &\left.\frac{n}{2}\int_{\Omega_2}\{|u_k(x)|^2 - |\nabla y_k(x)|^2\}dx\right\} = 0.\end{aligned}\tag{5.59}$$

Step 3. Multiplying (5.45) (resp. (5.46)) by u_k (resp. y_k), yields

$$\lim_{k\to+\infty}\left\{\Re\langle i\beta_k y_k, u_k\rangle_{L^2(\Omega_2)} - \Re\langle u_k, u_k\rangle_{L^2(\Omega_2)}\right\} = 0, \tag{5.60}$$

$$\lim_{k\to+\infty}\left\{\Re\langle i\beta_k u_k, y_k\rangle_{L^2(\Omega_2)} - \Re\langle \Delta y_k, y_k\rangle_{L^2(\Omega_2)}\right\} = 0. \tag{5.61}$$

the identities (5.60) and (5.61) imply

$$\lim_{k\to+\infty}\left\{-\Re\langle u_k, u_k\rangle_{L^2(\Omega_2)} - \Re\langle \Delta y_k, y_k\rangle_{L^2(\Omega_2)}\right\} = 0$$

from which we obtain via Green's theorem

$$\begin{aligned}&\lim_{k\to+\infty}\left\{\int_{\Omega_2}(|u_k(x)|^2 - |\nabla y_k(x)|^2)dx\right\}\\ &= \lim_{k\to+\infty}\left\{-\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}\overline{y_k(x)}d\Gamma - \Re\int_{\Gamma_0}\frac{\partial y_k(x)}{\partial\nu}\overline{y_k(x)}d\Gamma\right\}.\end{aligned} \tag{5.62}$$

Inserting (5.62) into (5.59), gives

$$\begin{aligned}&\lim_{k\to+\infty}\left\{-\frac{1}{2}\int_{\Gamma_2}|u_k(x)|^2\, m\cdot\nu d\Gamma - \frac{1}{2}\int_{\Gamma_0}|u_k(x)|^2\, m\cdot\nu d\Gamma -\right.\\ &\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{y_k(x)}d\Gamma - \Re\int_{\Gamma_0}\frac{\partial y_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{y_k(x)}d\Gamma + \int_{\Omega_2}|\nabla y_k(x)|^2\, dx +\\ &\frac{1}{2}\int_{\Gamma_2}|\nabla y_k(x)|^2\, m(x)\cdot\nu(x)d\Gamma + \frac{1}{2}\int_{\Gamma_0}|\nabla y_k(x)|^2\, m(x)\cdot\nu(x)d\Gamma -\\ &\left.\frac{n}{2}\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}\overline{y_k(x)}d\Gamma - \frac{n}{2}\Re\int_{\Gamma_0}\frac{\partial y_k(x)}{\partial\nu}\overline{y_k(x)}d\Gamma\right\} = 0.\end{aligned} \tag{5.63}$$

Step 4. Summing up (5.55) and (5.63), results in after using the boundary conditions (5.48) and (5.49)

$$\begin{aligned}&\lim_{k\to+\infty}\left\{-\frac{1}{2}\beta_k\int_{\Gamma_0}|z_k(x)|^2\, m(x)\cdot\nu(x)d\Gamma - \frac{1}{2}\int_{\Gamma_0}|u_k(x)|^2\, m(x)\cdot\nu(x)d\Gamma -\right.\\ &\frac{1}{2}\int_{\Gamma_1}|\nabla z_k(x)|^2\, m(x)\cdot\nu(x)d\Gamma + \int_{\Omega_1}|\nabla z_k(x)|^2\, dx - \frac{1}{2}\int_{\Gamma_2}|u_k(x)|^2\, m(x)\cdot\nu(x)d\Gamma -\\ &\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{y_k(x)}d\Gamma + \frac{1}{2}\int_{\Gamma_2}|\nabla y_k(x)|^2\, m(x)\cdot\nu(x)d\Gamma + \int_{\Omega_2}|\nabla y_k(x)|^2\, dx -\\ &\frac{n}{2}\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}\overline{y_k(x)}d\Gamma + \frac{1}{2}\int_{\Gamma_0}|\nabla y_k(x)|^2\, m(x)\cdot\nu(x)d\Gamma -\\ &\Re\int_{\Gamma_0}\frac{\partial y_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{y_k(x)}d\Gamma + \Re\int_{\Gamma_0}\frac{\partial z_k(x)}{\partial\nu}m\cdot\nabla\overline{z_k(x)}d\Gamma -\\ &\left.\frac{1}{2}\int_{\Gamma_0}|\nabla z_k(x)|^2\, m(x)\cdot\nu(x)d\Gamma\right\} = 0.\end{aligned}$$

Thus

$$\begin{aligned}
&\underset{k\to+\infty}{lim}\left\{\int_{\Omega_1}|\nabla z_k(x)|^2\,dx+\int_{\Omega_2}|\nabla y_k(x)|^2\,dx\right\}\\
&=\underset{k\to+\infty}{lim}\left\{\frac{1}{2}\beta_k\int_{\Gamma_0}|z_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\frac{1}{2}\int_{\Gamma_0}|u_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\right.\\
&\frac{1}{2}\int_{\Gamma_1}|\nabla z_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\frac{1}{2}\int_{\Gamma_2}|u_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\\
&\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{y_k(x)}d\Gamma-\frac{1}{2}\int_{\Gamma_2}|\nabla y_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\\
&\frac{n}{2}\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}\overline{y_k(x)}d\Gamma-\frac{1}{2}\int_{\Gamma_0}|\nabla y_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\\
&\Re\int_{\Gamma_0}\frac{\partial y_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{y_k(x)}d\Gamma-\Re\int_{\Gamma_0}\frac{\partial z_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{z_k(x)}d\Gamma+\\
&\left.\frac{1}{2}\int_{\Gamma_0}|\nabla z_k(x)|^2\,m(x).\nu(x)d\Gamma\right\}.
\end{aligned}\tag{5.64}$$

We conclude from the boundary conditions (5.47) – (5.49) that

$$\nabla z_k(x)=\frac{\partial z_k(x)}{\partial\nu}\nu(x),\qquad\text{on }\Gamma_1,\tag{5.65}$$

and

$$\begin{aligned}
\nabla(y_k(x)-z_k(x))&=\frac{\partial(y_k(x)-z_k(x))}{\partial\nu}\nu(x)\quad\text{on }\Gamma_0,\\
&=0\quad\text{on }\Gamma_0.
\end{aligned}\tag{5.66}$$

Then

$$|\nabla y_k(x)|^2=|\nabla z_k(x)|^2\qquad\text{on }\Gamma_0,$$

so on Γ_0,

$$\begin{aligned}
&\Re\left(\frac{\partial y_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{y_k(x)}\right)-\Re\left(\frac{\partial z_k(x)}{\partial\nu}m(x)\cdot\nabla\overline{z_k(x)}\right)\\
&-\frac{1}{2}|\nabla y_k(x)|^2\,m(x)\cdot\nu(x)+\frac{1}{2}|\nabla z_k(x)|^2\,m(x)\cdot\nu(x)=0.
\end{aligned}\tag{5.67}$$

Inserting (5.67) into (5.64) and recalling (5.45) and (5.48) yields

$$\begin{aligned}
&\underset{k\to+\infty}{lim}\left\{\int_{\Omega_1}|\nabla z_k(x)|^2\,dx+\int_{\Omega_2}|\nabla y_k(x)|^2\,dx\right\}\\
&=\underset{k\to+\infty}{lim}\left\{\frac{1}{2}\beta_k(1+\beta_k)\int_{\Gamma_0}|z_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\right.\\
&\frac{1}{2}\int_{\Gamma_1}|\nabla z_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\\
&\frac{1}{2}\int_{\Gamma_2}|u_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\Re\int_{\Gamma_2}\frac{\partial y_k}{\partial\nu}(x)m(x)\cdot\nabla\overline{y_k(x)}d\Gamma-\\
&\left.\frac{1}{2}\int_{\Gamma_2}|\nabla y_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\frac{n}{2}\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}\overline{y_k(x)}d\Gamma\right\}.
\end{aligned}\tag{5.68}$$

Applying the Cauchy-Schwarz and the Young inequalities to the fourth integral on the right-hand side of (5.68) and recalling assumption (5.16), we get

$$\begin{aligned}&\lim_{k\to+\infty}\left\{\int_{\Omega_1}|\nabla z_k(x)|^2\,dx+\int_{\Omega_2}|\nabla y_k(x)|^2\,dx\right\}\\&\leq\lim_{k\to+\infty}\left\{\frac{1}{2}\beta_k(1+\beta_k)\int_{\Gamma_0}|z_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\right.\\&\frac{1}{2}\int_{\Gamma_1}|\nabla z_k(x)|^2\,m(x)\cdot\nu(x)d\Gamma+\\&\left.\frac{C}{2}\int_{\Gamma_2}|u_k(x)|^2\,d\Gamma+\frac{C^2}{2\gamma}\int_{\Gamma_2}\left|\frac{\partial y_k(x)}{\partial\nu}\right|^2d\Gamma+\frac{n}{2}\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}\overline{y_k(x)}d\Gamma\right\}\end{aligned}\tag{5.69}$$

where

$$C=\sup_{x\in\overline{\Omega}_2}|m(x)|$$

Since $\beta_k(1+\beta_k)$ is positive for $|\beta_k|$ large, then it follows from (5.50), (5.69) and assumption (5.17) that

$$\begin{aligned}&\lim_{k\to+\infty}\left\{\int_{\Omega_1}|\nabla z_k(x)|^2\,dx+\int_{\Omega_2}|\nabla y_k(x)|^2\,dx\right\}\\&\leq\lim_{k\to+\infty}\left\{\frac{1}{2}(C+(\frac{C^2}{\gamma}+\frac{n}{2})\mu^2)\int_{\Gamma_2}|u_k(x)|^2\,d\Gamma+\frac{n}{4}\int_{\Gamma_2}|y_k(x)|^2\,d\Gamma\right\}.\end{aligned}\tag{5.70}$$

By (5.70), (5.51) and (5.52), we conclude that

$$\lim_{k\to+\infty}\{\|\nabla z_k(x)\|^2_{L^2(\Omega_1)}+\|\nabla y_k(x)\|^2_{L^2(\Omega_1)}\}=0.\tag{5.71}$$

From (5.54) and (5.62), we have

$$\begin{aligned}&\lim_{k\to+\infty}\left\{\int_{\Omega_2}(|u_k(x)|^2-|\nabla y_k(x)|^2)dx+\Re\int_{\Gamma_2}\frac{\partial y_k(x)}{\partial\nu}\overline{y_k(x)}d\Gamma\right\}\\&=\lim_{k\to+\infty}\left\{\int_{\Omega_1}(\beta_k\,|z_k(x)|^2+|\nabla z_k(x)|^2)dx\right\}.\end{aligned}\tag{5.72}$$

Employing (6.36), (5.52) and (5.71) in (5.72), we obtain

$$\lim_{k\to+\infty}\|u_k\|^2_{L^2(\Omega_2)}=\lim_{k\to+\infty}\beta_k\,\|z_k\|^2_{L^2(\Omega_1)}\,.\tag{5.73}$$

Now we apply Lemma 5.4.2 of the Appendix D to the function z_k to get

$$|\beta_k|\,\|z_k\|^2_{L^2(\Omega_1)}\leq\mathcal{C}\{\|\nabla z_k\|^2_{L^2(\Omega_1)}+\|f_k\|^2_{L^2(\Omega_1)}\}$$

which together with (5.44) and (5.71) gives

$$\lim_{k\to+\infty}|\beta_k|\,\|z_k\|^2_{L^2(\Omega_1)}=0$$

and consequently

$$\lim_{k\to+\infty}\|u_k\|^2_{L^2(\Omega_2)}=0\tag{5.74}$$

(5.71) and (5.74) show that $\lim_{k\to+\infty}\|W_k\|_{\mathcal{H}}=0$. This contradicts the assumption that $\|W_k\|_{\mathcal{H}}=1$. $\square$

5.4 Appendix D

.

Lemma 5.4.1. *Let $\mathcal{O}$ be an open bounded domain of $\mathbb{R}^n$ with a smooth boundary $\mathcal{S}$. Let $h(.) \in (C^2(\overline{\mathcal{O}}))^n$ be any real vector field. If $w \in H^2(\mathcal{O})$, then the following equalities hold*
(i)

$$\begin{aligned}&\Re\int_{\mathcal{O}}(\eta w(x)-\Delta w(x))h(x)\cdot\nabla\overline{w(x)}dx = \frac{\eta}{2}\int_{\mathcal{S}}|w(x)|^2\, h(x)\cdot\nu(x)d\mathcal{S}-\\ &\frac{1}{2}\int_{\mathcal{O}}\{\eta\,|w(x)|^2+|\nabla w(x)|^2\}divh(x)dx-\Re\int_{\mathcal{S}}\frac{\partial w(x)}{\partial\nu}h(x)\cdot\nabla\overline{w(x)}d\mathcal{S}+\\ &\Re\int_{\mathcal{O}}H(x)\nabla w(x)\cdot\nabla\overline{w(x)}dx+\frac{1}{2}\int_{\mathcal{S}}|\nabla w(x)|^2\, h(x)\cdot\nu(x)d\mathcal{S}\end{aligned}\tag{5.75}$$

where $H(.)$ is the Jacobian matrix of h.
(ii)

$$\begin{aligned}&\Re\int_{\mathcal{O}}(\eta w(x)-\Delta w(x))\overline{w(x)}divh(x)dx = \int_{\mathcal{O}}\{\eta\,|w(x)|^2+|\nabla w(x)|^2\}divh(x)-\\ &\Re\int_{\mathcal{S}}\frac{\partial w(x)}{\partial\nu}\overline{w(x)}divh(x)d\mathcal{S}+\Re\int_{\mathcal{O}}\overline{w(x)}\nabla w(x)\cdot\nabla(divh(x))dx.\end{aligned}\tag{5.76}$$

Proof. (i) We multiply $\eta w(x)-\Delta w(x)$ by $2h(x)\cdot\nabla\overline{w(x)}$ and integrate over $\mathcal{O}$;

$$\begin{aligned}&2\Re\int_{\mathcal{O}}(\eta w(x)-\Delta w(x))h(x)\cdot\nabla\overline{w(x)}dx = 2\Re\int_{\mathcal{O}}\eta w(x)h(x)\cdot\nabla\overline{w(x)}dx-\\ &2\Re\int_{\mathcal{O}}\Delta w(x)h(x)\cdot\nabla\overline{w(x)}dx.\end{aligned}\tag{5.77}$$

For the first integral on the right-hand side of (5.77), we have after using Green's theorem

$$\begin{aligned}2\Re\int_{\mathcal{O}}\eta w(x)h(x)\cdot\nabla\overline{w(x)}dx &= \eta\int_{\mathcal{O}}h(x)\cdot\nabla(|w(x)|^2)dx\\ &= \eta\int_{\mathcal{S}}|w(x)|^2\, h(x)\cdot\nu(x)d\mathcal{S}-\eta\int_{\mathcal{O}}|w(x)|^2\, divh(x)dx.\end{aligned}\tag{5.78}$$

For the second integral on the right of (5.77), we have also from Green's theorem,

$$\begin{aligned}&2\Re\int_{\mathcal{O}}\Delta w(x)h(x)\cdot\nabla\overline{w(x)}dx = 2\Re\int_{\mathcal{S}}\frac{\partial w(x)}{\partial\nu}h(x)\cdot\nabla\overline{w(x)}d\mathcal{S}-\\ &2\Re\int_{\mathcal{O}}\nabla w(x)\cdot\nabla(h(x)\cdot\nabla\overline{w(x)})dx.\end{aligned}\tag{5.79}$$

Applying the identity

$$2\Re\{\nabla w(x)\cdot\nabla(h(x)\cdot\nabla\overline{w(x)})\} = 2\Re\{H(x)\nabla w(x)\cdot\nabla\overline{w(x)}\}+h(x)\cdot\nabla(|\nabla w(x)|^2)$$

to the last integral at the right of (5.79), we find

$$\begin{aligned}&2\Re\int_{\mathcal{O}}\Delta w(x)h(x)\cdot\nabla\overline{w(x)}dx = 2\Re\int_{\mathcal{S}}\frac{\partial w(x,t)}{\partial\nu}h(x)\cdot\nabla\overline{w(x)}d\mathcal{S}-\\ &2\Re\int_{\mathcal{O}}H(x)\nabla w(x)\cdot\nabla\overline{w(x)})dx-\int_{\mathcal{O}}h(x)\cdot\nabla(|\nabla w(x)|^2)dx.\end{aligned}$$

Another use of Green's theorem yields

$$\begin{aligned}&2\Re\int_{\mathcal{O}}\Delta w(x)h(x)\cdot\nabla\overline{w(x)}dx = 2\Re\int_{\mathcal{S}}\frac{\partial w(x)}{\partial\nu}h(x)\cdot\nabla\overline{w(x)}d\mathcal{S}-\\&2\Re\int_{\mathcal{O}}H(x)\nabla w(x)\cdot\nabla\overline{w(x)}dx-\int_{\mathcal{S}}\left|\nabla w(x)\right|^2 h(x)\cdot\nu(x)d\mathcal{S}+\\&\int_{\mathcal{O}}\left|\nabla w(x)\right|^2 divh(x)dx.\end{aligned}\tag{5.80}$$

Combining (5.78) and (5.80), we obtain (5.75).
(ii) Identity (5.76) is proved by multiplying $\eta w(x)-\Delta w(x)$ by $\overline{w(x)}divh(x)$ and integrating by parts over $\mathcal{O}$. □

Lemma 5.4.2. *Let $w\in H^1(\Omega)$ satisfy the following elliptic problem*

$$\begin{aligned}&\eta w-\Delta w = w^*\in L^2(\Omega_1),\\&w=0\ \ on\ \Gamma_1,\\&w(x)=g\in H^{\frac{1}{2}}(\Gamma_0).\end{aligned}$$

Then

$$|\eta|\,\|w\|^2_{L^2(\Omega_1)}\le\mathcal{C}\{\|\nabla w\|^2_{L^2(\Omega)}+\|w^*\|^2_{L^2(\Omega)}\}.\tag{5.81}$$

for some constant $\mathcal{C}>0$.

Proof. This lemma is proved in [6] (Lemma 5.2.) but for completeness we give the details here. We define $w_1\in H^1(\Omega_1)$ to be the solution of the following boundary value problem

$$\begin{aligned}\Delta w_1(x)&=\Delta w(x)+w^*(x)\text{ in }\Omega_1,\\ w_1(x)&=0\text{ on }\partial\Omega_1.\end{aligned}\tag{5.82}$$

Then

$$\langle\Delta w_1,w_1\rangle_{L^2(\Omega_1)}=\langle\Delta w+w^*,w_1\rangle_{L^2(\Omega_1)}.\tag{5.83}$$

Applying Green's theorem to both sides of (5.83) and using the boundary condition (5.82), we obtain

$$\|\nabla w_1\|^2_{L^2(\Omega_1)}=\langle\nabla w,\nabla w_1\rangle_{L^2(\Omega_1)}+\langle w^*,w_1\rangle_{L^2(\Omega_1)}.\tag{5.84}$$

From (5.84) together with the Cauchy-Schwarz and the Poincaré inequalities, we have

$$\|\nabla w_1\|^2_{L^2(\Omega_1)}\le\|\nabla w\|_{L^2(\Omega_1)}\|\nabla w_1\|_{L^2(\Omega_1)}+\mathcal{C}\,\|w^*\|_{L^2(\Omega_1)}\|\nabla w_1\|_{L^2(\Omega_1)}.\tag{5.85}$$

The estimate (5.85) combined with Young's inequality gives

$$\|\nabla w_1\|^2_{L^2(\Omega_1)}\le\mathcal{C}\{\|\nabla w\|^2_{L^2(\Omega_1)}+\|w^*\|^2_{L^2(\Omega_1)}\}.\tag{5.86}$$

Since $\Delta w_1=\eta w$, we have

$$|\eta|\,\|w\|_{H^{-1}(\Omega_1)}=\|\Delta w_1\|_{H^{-1}(\Omega_1)}\le\mathcal{C}\,\|\nabla w_1\|_{L^2(\Omega_1)}.\tag{5.87}$$

after using Poincar's inequality. On the other hand, by interpolation, we have

$$\begin{aligned}|\eta|\,\|w\|^2_{L^2(\Omega_1)}&=|\eta|\,\|w\|_{[H^1(\Omega_1),H^{-1}(\Omega_1)]}\\&\le\mathcal{C}|\eta|\,\|w\|_{H^{-1}(\Omega_1)}\|w\|_{H^1(\Omega_1)}.\end{aligned}\tag{5.88}$$

The interpolation inequality (5.88) along with the estimate (5.87) and Poincaré's inequality yields

$$|\eta|\,\|w\|^2_{L^2(\Omega_1)}\le\mathcal{C}\,\|\nabla w_1\|_{L^2(\Omega_1)}\|\nabla w\|_{L^2(\Omega_1)}.\tag{5.89}$$

Consequently, combining (5.89) and (5.86) and using the Young inequality we arrive at the desired estimate (5.81). □

Chapter 6

Energy decay estimates for the transmission Schrödinger wave equation with distributed damping

6.1 Introduction and system description

We consider the Schrödinger equation in Ω_1 coupled with the wave equation in Ω_2,

$$\partial_t z(x,t) - i\Delta z(x,t) + \alpha z(x,t) = 0 \qquad \text{in } \Omega_1 \times (0,+\infty), \tag{6.1}$$

$$\partial_t^2 y(x,t) - \Delta y(x,t) + \mu \partial_t y(x,t) = 0 \qquad \text{in } \Omega_2 \times (0,+\infty), \tag{6.2}$$

$$z(x,0) = z_0(x) \qquad \text{in } \Omega_1, \tag{6.3}$$

$$y(x,0) = y_0(x), \partial_t y(x,0) = y_1(x) \qquad \text{in } \Omega_2, \tag{6.4}$$

$$z(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{6.5}$$

$$y(x,t) = 0 \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{6.6}$$

$$z(x,t) = y(x,t), \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{6.7}$$

$$\frac{\partial z(x,t)}{\partial \nu} = \frac{\partial y(x,t)}{\partial \nu} \qquad \text{on } \Gamma_0 \times (0,+\infty), \tag{6.8}$$

where

- $\Omega, \Omega_1, \Omega_2, \Gamma, \Gamma_0, \Gamma_1$ and Γ_2 are as in Chapter 2,
- α and μ are positive constants,
- z_0, y_0, y_1 are the initial data which belong to suitable spaces.

The aim of this chapter is to study the asymptotic behavior of the solutions of (6.1)-(6.8). Similar problems have been investigated in the case of one space dimension in [50], [51], [75] and [81] . In [51], the authors adopted an approach based on spectral analysis to establish a strong stability result for the system (6.1)-(6.8) with $\alpha = 0$ and $\mu > 0$. Spectral methods have also been used in [50] to study the stability of the system (6.1)-(6.8) with $\alpha = 0$ and the wave equation is subject to a distributed Kelvin-Voigt damping. In the authors used the Green function to obtain an estimate for the resolvent of the operator of the system studied in [51] from which an improvement of the stability result of [51] is deduced. Wang et al [81] proved by a Riesz basis approach that the system decribed in [51] with a wave equation subject also to a negative displacement is exponentially stable. Here in agreement with [81] we obtain an exponential stability result in general space dimension when $\alpha = 0$ and $\mu > 0$. However when $\alpha > 0$ and $\mu = 0$. the system is polynomially but not exponentially stable.

6.2 Well-posedness

Set

$$H^1_{\Gamma_2}(\Omega_2) = \{f \in H^1(\Omega_2) : f = 0 \text{ on } \Gamma_2\}$$

and consider system (6.1)-(6.8) in the Hilbert space

$$\mathcal{H} = L^2(\Omega_1) \times H^1_{\Gamma_2}(\Omega_2) \times L^2(\Omega_2)$$

with the inner product induced norm

$$\|(f_1, f_2, f_3)\|^2_{\mathcal{H}} = \int_{\Omega_1} |f_1(x)|^2 \, dx + \int_{\Omega_2} \{|\nabla f_2(x)|^2 + |f_3(x)|^2\} dx \tag{6.9}$$

The operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ generating the dynamics described by (6.1)-(6.8) is given by

$$A(z, y, u)^T = (i\Delta z - \alpha z, u, \Delta y - \mu u)^T \tag{6.10}$$

$$D(A) = \{(z, y, u)^T \in \mathcal{H}; z \in H^2(\Omega_1), y \in H^2(\Omega_2), u \in H^1_{\Gamma_2}(\Omega_2) \text{ satisfying (6.12)-(6.13) below}\} \tag{6.11}$$

$$i\frac{\partial z}{\partial \nu} = \frac{\partial y}{\partial \nu} \text{ on } \Gamma_0, \tag{6.12}$$

$$u = z \text{ on } \Gamma_0, \tag{6.13}$$

Note that for $(z, y, u)^T \in D(A)$, we have by trace theory the following boundary regularity:

- $\Delta z|_{\partial\Omega_1} \in H^{1/2}(\partial\Omega_1)$,
- $\frac{\partial z}{\partial \nu}|_{\partial\Omega_1} \in H^{1/2}(\partial\Omega_1)$,
- $\frac{\partial y}{\partial \nu}|_{\partial\Omega_2} \in H^{1/2}(\partial\Omega_2)$,
- $u \in H^{1/2}(\partial\Omega_2)$.

Then system (6.1)-(6.8) can be re-written as an abstract Cauchy problem in $\mathcal{H}$ as follows

$$\begin{cases} \frac{d}{dt}Y(t) = AY(t) \\ Y(0) = Y_0 \end{cases} \tag{6.14}$$

where

$$Y(t) = (z(.,t), y(.,t), \partial_t y(.,t))^T \text{ and } Y_0 = Y(0) = (z_0, y_0, y_1)^T$$

Proposition 6.2.1. *The operator A defined by (*$(6.10)-(6.11)$ *is maximal dissipative. Hence it generates a contraction* C_0-*semigroup on* $\mathcal{H}$*. Consequently, for every* $Y_0 \in \mathcal{H}$*, problem* (6.14) *has a unique solution* Y *whose regularity depends on the initial datum* Y_0 *as follows:*

$$\begin{aligned} Y(.) &\in C([0, +\infty); \mathcal{H}) \textit{ if } Y_0 \in \mathcal{H}, \\ Y(.) &\in C([0, +\infty); D(A)) \cap C^1([0, +\infty); \mathcal{H}) \textit{ if } Y_0 \in D(A). \end{aligned}$$

Proof. Let $Y = (z, y, u)^T \in D(A)$. Then

$$\langle AY, Y\rangle = \int_{\Omega_1} (i\Delta z(x))\overline{z(x)} dx - \alpha \int_{\Omega_1} | z(x)|^2 \, dx + \int_{\Omega_2} \nabla u(x).\nabla\overline{y(x)} \, dx + \int_{\Omega_2} \Delta y(x)\overline{u(x)} dx - \mu \int_{\Omega_2} | u(x)|^2 \, dx \tag{6.15}$$

Applying Green's theorem to the first and the third integral on the right-hand side of (6.15) and using the fact that the normal vector on Γ_0 is oriented towards the interior of Ω_2, we obtain

$$\langle AY,Y\rangle = \int_{\Gamma_1} i\frac{\partial z(x)}{\partial \nu}\overline{z(x)}d\Gamma - \int_{\Gamma_0} i\frac{\partial z(x)}{\partial \nu}\overline{z(x)}d\Gamma - i\int_{\Omega_1} |\nabla z(x)|^2\,dx - \alpha\int_{\Omega_1} |\,z(x)|^2\,dx +$$
$$\int_{\Omega_2} \nabla u(x).\nabla\overline{y(x)}\,dx + \int_{\Gamma_2} \frac{\partial y(x)}{\partial \nu}\overline{u(x)}d\Gamma + \int_{\Gamma_0} \frac{\partial y(x)}{\partial \nu}\overline{u(x)}d\Gamma - \int_{\Omega_2} \nabla y(x).\nabla\overline{u(x)}\,dx -$$
$$\mu\int_{\Omega_2} |\,u(x)|^2\,dx \tag{6.16}$$

(6.16) together with (6.12) – (6.13) yields

$$\langle AY,Y\rangle \;=\; 2i\Im\{\int_{\Omega_2} \nabla u(x).\nabla\overline{y(x)}\,dx\} - i\int_{\Omega_1} |\nabla z(x)|^2\,dx - \alpha\int_{\Omega_1} |\,z(x)|^2\,dx - \mu\int_{\Omega_2} |u(x)|^2\,dx$$

Hence

$$\Re\,\langle AY,Y\rangle = -\alpha\int_{\Omega_1} |\,z(x)|^2\,dx - \mu\int_{\Omega_2} |u(x)|^2\,dx \tag{6.17}$$

which shows that A is dissipative.

Now we show that A is onto. For this purpose, we fix $(f,g,h)^T \in \mathcal{H}$, and we look for $Y=(z,y,u)^T \in D(A)$ such that

$$AY = (f,g,h)^T \tag{6.18}$$

We have $g \in H^1_{\Gamma_2}(\Omega_2)$ and $z = g$ on Γ_0. We solve the following system

$$i\Delta z(x) - \alpha z(x) = f(x) \qquad x\in\Omega_1, \tag{6.19}$$
$$z(x) = 0 \qquad x\in\Gamma_1,$$
$$z(x) = g(x) \qquad x\in\Gamma_0. \tag{6.20}$$

Obviously, $z\in H^1_{\Gamma_1}(\Omega_1)$ and $\frac{\partial z}{\partial \nu}\in H^{-1/2}(\partial\Omega_1)$.
Now, we consider the boundary value problem

$$\Delta y(x) = \mu g(x) + h(x) \qquad x\in\Omega_{2,} \tag{6.21}$$
$$\frac{\partial y(x)}{\partial \nu} = i\frac{\partial z(x)}{\partial \nu} \qquad x\in\Gamma_0, \tag{6.22}$$
$$y(x) = 0 \qquad x\in\Gamma_2. \tag{6.23}$$

The variational formulation of problem (6.21)-(6.23) is to find $y\in H^1_{\Gamma_2}(\Omega_2)$ such that

$$\Lambda(y,\varphi) = \mathcal{F}(\varphi) \tag{6.24}$$

where

$$\Lambda(y,\varphi) = \int_{\Omega_2} \nabla y(x).\nabla\overline{\varphi(x)}dx$$

and

$$\mathcal{F}(\varphi) = \int_{\Omega_2} (h(x)+\alpha g(x))\overline{\varphi(x)}dx - \int_{\Gamma_0} \frac{\partial z(x)}{\partial \nu}\overline{\varphi(x)}d\Gamma$$

Λ is a continuous bilinear form on $H^1_{\Gamma_2}(\Omega_2)$ and $\mathcal{F}$ is a continuous linear form on $H^1_{\Gamma_2}(\Omega_2)$. Moreover

$$\Re\Lambda(y,y) \geq \|y\|^2_{H^1_{\Gamma_2}(\Omega_2)}$$

By Lax-Milgram Theorem, for all $h\in H^1_{\Gamma_2}(\Omega_2)$, problem (6.24) has a unique solution $y\in H^1_{\Gamma_2}(\Omega_2)$.
If we consider in (6.25) $\varphi\in\mathcal{D}(\Omega_2)$, then y is a solution of (6.21), in the sense of distribution, and from

the regularity of h, we have $y \in H^2(\Omega_2) \cap H^1_{\Gamma_2}(\Omega_2)$. Using Green's formula in (6.24) and exploiting (6.21), we obtain for all $\varphi \in H^1_{\Gamma_2}(\Omega_2)$

$$\int_{\Gamma_0} \frac{\partial z(x)}{\partial \nu} \overline{\varphi(x)} dx = \int_{\Gamma_0} \frac{\partial y(x)}{\partial \nu} \overline{\varphi(x)} dx$$

from which follows

$$\frac{\partial y(x)}{\partial \nu} = i \frac{\partial z(x)}{\partial \nu} \qquad \text{on } \Gamma_0, \tag{6.25}$$

Moreover, (6.19), (6.21) and (6.20)imply that

$$u(x) = z(x) \qquad \text{on } \Gamma_0.$$

So, we have found $(z, y, u) \in D(A)$ for which (6.18) holds and consequently A is onto, so $0 \in \rho(A)$. Therefore by the contraction principle we obtain $R(\lambda I - A) = \mathcal{H}$ for $\lambda > 0$ sufficiently small Thus, by the Lumer-Phillips Theorem (see for instance [65], Theorem 1.4.3), A generates a contraction C_0- semigroup on $\mathcal{H}$. ☐

6.3 Exponential stability

In this section, we establish an exponential stability result for system (6.1)-(6.8) in the case $\alpha = 0$ and $\mu > 0$.

Theorem 6.3.1. *Let $\alpha = 0$ and $\mu > 0$. Then there exist constants $M \geq 1$ and $\omega > 0$ such that*

$$E(t) \leq M e^{-\omega t} E(0) \qquad \textit{for all } t \geq 0$$

for all solutions of (6.1)-(6.8) with $(z_0, y_0, y_1) \in \mathcal{H}$., where $E(t)$ is the energy of a solution (y, z) of (6.1)-(6.8) at time t defined by

$$E(t) = \frac{1}{2} \int_{\Omega_1} |\ z(x,t)|^2 \, dx + \frac{1}{2} \int_{\Omega_2} \{ |\partial_t y(x,t)|^2 + |\nabla y(x,t)|^2 \} dx$$

To prove this result, we use Theorem 1.3.4.

Lemma 6.3.2. *The operator A satisfies condition (1.4), that is $i\mathbb{R} \subset \rho(A)$.*

Proof. By Sobolev embedding Theorem, A has a compact resolvent. As a consequence, its spectrum consists only of eigenvalues of A. Assume $\lambda = i\beta$, with $\beta \in \mathbb{R}^*$, is an eigenvalue of A and let $W = (z, y, u)^T \in D(A)$ be a corresponding eigenvector. Thus

$$AW = i\beta W \tag{6.26}$$

or equivalently

$$\begin{aligned} i\Delta z(x) &= i\beta z(x) && \text{in } \Omega_1, \\ u(x) &= i\beta y(x) && \text{in } \Omega_2, \qquad (6.27) \\ \Delta y(x) - \mu u(x) &= i\beta u(x) && \text{in } \Omega_2, \end{aligned}$$

$$\begin{aligned} z(x) &= 0 && \text{on } \Gamma_1, \\ z(x) &= u(x) && \text{on } \Gamma_0, \qquad (6.28) \\ i\frac{\partial z(x)}{\partial \nu} &= \frac{\partial y(x)}{\partial \nu} && \text{on } \Gamma_0, \\ y(x) &= 0 && \text{on } \Gamma_2. \end{aligned}$$

Taking the inner product of (6.26) with W and using the equality (6.17), we get

$$u(x) = 0 \text{ in } \Omega_2,$$

which together with (6.27) and (6.28) yields

$$y(x) = 0 \quad \text{in } \Omega_2.$$

and

$$z(x) = 0 \quad \text{on } \Gamma_0.$$

Consequently $z(.)$ satisfies

$$\begin{cases} -\Delta z(x) + \beta z(x) = 0 & \text{in} \quad \Omega_1, \\ z(x) = 0 & \text{on} \quad \Gamma_1, \\ z(x) = \frac{\partial z(x)}{\partial \nu} = 0 & \text{on} \quad \Gamma_0. \end{cases}$$

and by the unique continuation principle ([33], [74], [43]).

$$z(x) = 0 \text{ in } \Omega_1.$$

Therefore $W = (0,0,0)^T$ and this contradicts that W is an eigenvector of A. □

Lemma 6.3.3. *The resolvent operator of A satisfies condition (1.5).*

Proof. Suppose that (1.5) is not satisfied. Then by the Banach-Steinhaus Theorem, there exists a sequence of real numbers $|\beta_k| \to +\infty$ and a sequence of vectors $W_k = (z_k, y_k, u_k) \in D(A)$ with $\|W_k\| = 1$ such that

$$\underset{k\to+\infty}{lim} \|(i\beta_k - A)W_k\|_{\mathcal{H}} = 0$$

that is

$$i\beta_k z_k - i\Delta z_k = f_k \to 0 \quad \text{in } L^2(\Omega_1), \tag{6.29}$$

$$i\beta_k y_k - u_k = g_k \to 0 \quad \text{in } H^1_{\Gamma_2}(\Omega_2), \tag{6.30}$$

$$(i\beta_k + \mu)u_k - \Delta y_k = h_k \to 0 \quad \text{in } L^2(\Omega_2), \tag{6.31}$$

$$z_k = 0 \quad \text{on } \Gamma_1, \tag{6.32}$$

$$z_k = u_k \quad \text{on } \Gamma_0, \tag{6.33}$$

$$\frac{\partial z_k}{\partial \nu} = \frac{\partial y_k}{\partial \nu} \quad \text{on } \Gamma_0, \tag{6.34}$$

$$y_k = 0 \quad \text{on } \Gamma_2. \tag{6.35}$$

Since

$$|\Re\langle (i\beta_k - A)W_k, W_k \rangle| \leq \|(i\beta_k - A)W_k\|_{\mathcal{H}}$$

and

$$\Re\langle (i\beta_k - A)W_k, W_k \rangle = -\mu \int_{\Omega_2} |u_k(x)|^2 \, d\Gamma,$$

then

$$u_k \to 0 \text{ in } L^2(\Omega_2) \tag{6.36}$$

By (6.30) and (6.31), we have

$$\|\beta_k y_k\|^2_{L^2(\Omega_2)} \leq (\|u_k\|_{L^2(\Omega_2)} + \|g_k\|_{L^2(\Omega_2)})^2, \tag{6.37}$$

and

$$\left\|\beta_k^{-1} \Delta y_k\right\|^2_{L^2(\Omega_2)} \leq ((1+\mu)\,\|u_k\|_{L^2(\Omega_2)} + \|h_k\|_{L^2(\Omega_2)})^2. \tag{6.38}$$

Combining (6.36),(6.37) and (6.38) yields

$$\underset{k\to+\infty}{lim} \|\beta_k y_k\|_{L^2(\Omega_2)} = \underset{k\to+\infty}{lim} \left\|\beta_k^{-1}\Delta y_k\right\|_{L^2(\Omega_2)} = 0. \tag{6.39}$$

On the other hand, by classical theory of elliptic boundary value problem, there exists a positive constant $C > 0$ such that

$$\|y_k\|_{H^2(\Omega_2)} \leq C(\|\Delta y_k\|_{L^2(\Omega_2)}) + \|y_k\|_{L^2(\Omega_2)}). \tag{6.40}$$

(6.39) together with (6.40) implies that

$$\underset{k\to+\infty}{lim} \left\|\beta_k^{-1} y_k\right\|_{H^2(\Omega_2)} = 0.$$

and by interpolation, we have

$$\|y_k\|^2_{H^1_{\Gamma_2}(\Omega_2)} \leq C \|\beta_k y_k\|_{L^2(\Omega_2)} \left\|\beta_k^{-1} y_k\right\|_{H^2(\Omega_2)}$$

Therefore

$$\underset{k\to+\infty}{lim} \|\nabla y_k\|_{(L^2(\Omega_2))^n} = 0. \tag{6.41}$$

and consequently

$$\underset{k\to+\infty}{lim} \|y_k\|_{(L^2(\partial\Omega_2))} = 0.$$

Now let $h \in (C^1(\Omega))^n$ be a vector field satisfying $h = \nu$ on $\partial\Omega_2$. We have from (6.30) and (6.31)

$$\underset{k\to+\infty}{lim} \{\langle u_k, 2i\beta_k h \cdot \nabla y_k\rangle_{L^2(\Omega_1)} - \langle u_k, 2h \cdot \nabla y_k\rangle_{L^2(\Omega_1)}\} = 0$$

and

$$\underset{k\to+\infty}{lim} \{\langle i\beta_k u_k, 2h \cdot \nabla y_k\rangle_{L^2(\Omega_1)} + \mu\langle u_k, 2h \cdot \nabla y_k\rangle - \langle \Delta y_k, 2h \cdot \nabla y_k\rangle_{L^2(\Omega_1)}\} = 0$$

Hence

$$\underset{k\to+\infty}{lim} \{\langle -\langle u_k, 2h \cdot \nabla y_k\rangle_{L^2(\Omega_1)} + \mu\langle u_k, 2h \cdot \nabla y_k\rangle - \langle \Delta y_k, 2h \cdot \nabla y_k\rangle_{L^2(\Omega_1)}\} = 0$$

By Green's theorem, we have

$$\begin{aligned}&\underset{k\to+\infty}{lim} \{\int_{\partial\Omega_2} |u_k(x)|^2 \, d\Gamma - \int_{\Omega_2} |u_k(x)|^2 \, divh(x)dx + 2\Re \int_{\partial\Omega_2} \frac{\partial y_k(x)}{\partial\nu} h(x) \cdot \nabla\overline{y_k(x)}dx - \\ &2\Re \int_{\Omega_2} H(x)\nabla y_k(x) \cdot \nabla\overline{y}_k(x)dx - \int_{\partial\Omega_2} |\nabla y_k(x)^2 \, |h(x).\nu(x)d\Gamma + \int_{\Omega_2} |\nabla y_k(x)|^2 \, divh(x)dx\} = 0.\end{aligned} \tag{6.42}$$

From (6.36) and (6.41), we deduce that

$$\begin{aligned}&\underset{k\to+\infty}{lim} \int_{\Omega_2} |u_k(x)|^2 \, divh(x)dx = 0 \\ &\underset{k\to+\infty}{lim} 2\Re \int_{\Omega_2} H(x)\nabla y_k(x) \cdot \nabla\overline{y}_k(x)dx = \underset{k\to+\infty}{lim} \int_{\Omega_2} |\nabla y_k(x)|^2 \, divh(x)dx = 0 \\ &\underset{k\to+\infty}{lim} \{2\Re \int_{\partial\Omega_2} \frac{\partial y_k(x)}{\partial\nu} h(x) \cdot \nabla\overline{y_k(x)}d\Gamma - \int_{\partial\Omega_2} |\nabla y_k(x)^2 \, |h(x).\nu(x)d\Gamma\} = \underset{k\to+\infty}{lim} \int_{\partial\Omega_2} \left|\frac{\partial y_k(x)}{\partial\nu}\right|^2 h(x).\nu(x)d\Gamma\end{aligned} \tag{6.43}$$

Inserting (6.43) into (6.42) yields

$$\underset{k\to+\infty}{lim} \{\int_{\partial\Omega_2} |u_k(x)|^2 \, d\Gamma + \int_{\partial\Omega_2} \left|\frac{\partial y_k(x)}{\partial\nu}\right|^2 d\Gamma\} = 0. \tag{6.44}$$

(6.44) together with (6.33) and (6.34) implies that

$$\lim_{k\to+\infty}\int_{\partial\Omega_1}|z_k(x)|^2\,d\Gamma=\int_{\Gamma_0}\left|\frac{\partial z_k(x)}{\partial\nu}\right|^2 d\Gamma=0. \tag{6.45}$$

By elliptic regularity we have for system (6.29), (6.32), (6.33) and (6.34), $z_k\in H_2(\Omega_1)\cap H^1_{\Gamma_1}(\Omega_1)$. Because of (6.45), we deduce from the unique continuation theorem for elliptic operators, see for example [74], Corollary 15.2.2, that $z_k\to 0$ in $H^1_{\Gamma_1}(\Omega_1)$ and consequently

$$z_k\to 0 \quad \text{in } L^2(\Omega_1). \tag{6.46}$$

Therefore, by (6.36), (6.41) and (6.46), $\lim\limits_{k\to+\infty}\|W_k\|^2_{\mathcal{H}}=0$ and this contradicts the assumption that $\|W_k\|_{\mathcal{H}}=1$. □

6.4 Lack of exponential stability

Now let us consider the case where $\alpha>0$ and $\mu=0$. We show that the solution of our system is not expoentially stable. The proof follows an idea from ([56], Theorem 3.5).

Theorem 6.4.1. *For $\alpha>0$ and $\mu=0$, the system (6.1)-(6.8) is not exponentially stable.*

Proof. We consider the closed subspace

$$\mathcal{H}_0=\{0\}\times H^1_{\Gamma_2}(\Omega_2)\times L^2(\Omega_2)$$

of $\mathcal{H}$. On $\mathcal{H}_0$, we consider the operator

$$\tilde{A}:D(\tilde{A})\subset\mathcal{H}_0\to\mathcal{H}_0$$

defined by

$$\begin{aligned}&\tilde{A}(z,y,u)^T=(z,u,\Delta y)^T,\\ &D(\tilde{A}):=\{0\}\times(H^2(\Omega_2)\cap H^1_0(\Omega_2))\times H^1_0(\Omega_2)\end{aligned}$$

Let $(z,y,\partial_t y)$ be the solution of (6.1)-(6.8) and $(0,\tilde{y},\partial_t\tilde{y})$ be the solution of the problem

$$\partial_t^2\tilde{y}-\Delta\tilde{y}=0 \quad \text{in } \Omega_2\times(0,+\infty), \tag{6.47}$$

$$\tilde{y}(x,0)=y_0(x),\quad \partial_t\tilde{y}(x,0)=y_1(x)\quad \text{in } \Omega_2, \tag{6.48}$$

$$\tilde{y}(x,t)=0\quad \text{in } .\partial\Omega_2\times(0,+\infty) \tag{6.49}$$

It is well-known that the problem (6.47)-(6.49) is energy conserving, i.e.

$$\int_{\Omega_2}\{|\nabla\tilde{y}(x,t)|^2+|\partial_t\tilde{y}(x,t)|^2\}dx=\int_{\Omega_2}\{|\nabla y_0(x)|^2+|y_1(x)|^2\}dx$$

Hence the semigroup $\tilde{S}(t)$ associated with (6.47)-(6.49) and defined on the Hilbert space

$$\mathcal{H}_0=\{0\}\times H^1_{\Gamma_2}(\Omega_2)\times L^2(\Omega_2)$$

by

$$\tilde{S}(t)(0,y_0,y_1)=(0,\tilde{y}(t),\partial_t\tilde{y}(t))$$

is a unitary group. Thus, its essential spectral radius equals to one.
We will show that $S(t)-\tilde{S}(t):\mathcal{H}_0\to\mathcal{H}$ is compact, where $S(t)$ is the C_0-semigroup generated by A. It

is enough to prove that $S(t)-\tilde{S}(t):\mathcal{H}_1\to\mathcal{H}$ is compact for some dense subspace $\mathcal{H}_1$ of $\mathcal{H}_0$.
Set

$$\mathcal{H}_1 := \{0\}\times\mathcal{D}(\Omega_2)\times\mathcal{D}(\Omega_2)$$

Then $\mathcal{H}_1$ is dense in $\mathcal{H}$ and $\mathcal{H}_1\subset D(A)\cap D(\tilde{A})$.
For $Y_0\in\mathcal{H}_1$, consider for $t\geq 0$

$$F(t)=\frac{1}{2}\left\|S(t)Y_0-\tilde{S}(t)Y_0\right\|^2_{\mathcal{H}}$$

Let $(z,y,\partial_t y)^T := S(t)Y_0$ and $(0,\tilde{y},\partial_t\tilde{y})$:= $\tilde{S}(t)Y_0$. Then

$$\begin{aligned}&\frac{d}{dt}F(t)=\Re<\frac{d}{dt}(S(t)Y_0-\tilde{S}(t)Y_0),S(t)Y_0-\tilde{S}(t)Y_0>\\&=\Re<AS(t)Y_0-\tilde{A}\tilde{S}(t)Y_0),S(t)Y_0-\tilde{S}(t)Y_0>\\&=\Re<AS(t)Y_0,S(t)Y_0>+\Re<\tilde{A}\tilde{S}(t)Y_0),\tilde{S}(t)Y_0>-\Re<AS(t)Y_0,\tilde{S}(t)Y_0>-\Re<\tilde{A}\tilde{S}(t)Y_0,S(t)Y_0>\end{aligned}$$

From (6.17), we have

$$\Re<AS(t)Y_0,S(t)Y_0>=-\alpha\int_{\Omega_1}|z(x)|^2\,dx$$

and since $\tilde{S}(t)$ is a unitary group, $\Re<\tilde{A}\tilde{S}(t)Y_0),\tilde{S}(t)Y_0>=0$

Therefore

$$\frac{d}{dt}F(t)=-\Re<AS(t)Y_0,\tilde{S}(t)Y_0>-\Re<\tilde{A}\tilde{S}(t)Y_0,S(t)Y_0>$$

Moreover

$$\begin{aligned}&<AS(t)Y_0,\tilde{S}(t)Y_0>+<\tilde{A}\tilde{S}(t)Y_0,S(t)Y_0>=\int_{\Omega_2}\nabla\partial_t y(x,t)\cdot\nabla\bar{\tilde{y}}(x,t)dx+\int_{\Omega_2}\Delta y(x,t)\partial_t\bar{\tilde{y}}(x,t)dx+\\&\int_{\Omega_2}\nabla\partial_t\tilde{y}(x,t)\cdot\nabla\bar{y}(x,t)dx+\int_{\Omega_2}\Delta\tilde{y}(x,t)\partial_t\bar{y}(x,t)dx\end{aligned}\tag{6.50}$$

Applying Green's Theorem to the second and the fourth term on the right-hand side of (6.50), we obtain

$$\begin{aligned}&<AS(t)Y_0,\tilde{S}(t)Y_0>+<\tilde{A}\tilde{S}(t)Y_0,S(t)Y_0>=\int_{\Omega_2}\nabla\partial_t y(x,t)\cdot\nabla\bar{\tilde{y}}(x,t)dx-\int_{\Omega_2}\nabla y(x,t)\cdot\nabla\partial_t\bar{\tilde{y}}(x,t)dx+\\&\int_{\Omega_2}\nabla\partial_t\tilde{y}(x,t)\cdot\nabla\bar{y}(x,t)dx-\int_{\Omega_2}\nabla\tilde{y}(x,t)\cdot\nabla\partial_t\bar{y}(x,t)dx+\int_{\Gamma_0}\frac{\partial\tilde{y}(x,t)}{\partial\nu}\partial_t\bar{y}(x,t)d\Gamma\end{aligned}\tag{6.51}$$

where we used $y=0$ on Γ_2 and $\tilde{y}=0$ on $\partial\Omega_2$.
Inserting (6.51) into (6.4) yields

$$\frac{d}{dt}F(t)=-\alpha\int_{\Omega_1}|z(x)|^2\,dx-\int_{\Gamma_0}\frac{\partial\tilde{y}(x,t)}{\partial\nu}\partial_t\bar{y}(x,t)d\Gamma$$

This imlpies that

$$F(t)+\alpha\int_0^t\int_{\Omega_1}|z(x,t)|^2\,dxdt=-\Re\int_0^t\int_{\Gamma_0}\frac{\partial\tilde{y}(x,t)}{\partial\nu}\partial_t\bar{y}(x,t)d\Gamma dt\tag{6.52}$$

Let $(Y_0^k)_{k\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}_1$, and let $Y^k(t)=S(t)Y_0^k$ and $\tilde{Y}^k(t)=\tilde{S}(t)Y_0^k$. Since $Y^k\in C([0,+\infty),D(\tilde{A}))$, the sequence $(\frac{\partial\tilde{y}^k}{\partial\nu})_{k\in\mathbb{N}}\subset L^2([0,t],L^2(\Gamma_0))$ is uniformly bounded. Therefore there exists a subsequence denoted also by $(\tilde{y}^k)_{k\in\mathbb{N}}$ such that $(\frac{\partial\tilde{y}^k}{\partial\nu})_{k\in\mathbb{N}}$ converges weakly in $L^2([0,t],L^2(\Gamma_0))$.
Moreover, because $Y^k\in C^1([0,+\infty),\mathcal{H}))$ the sequences $(z^k)_{k\in\mathbb{N}}\subset L^2([0,t];H^2(\Omega_1))$ and $(z_t^k)_{k\in\mathbb{N}}\subset L^2([0,t];L^2(\Omega_1))$ are both uniformly bounded. By the Aubin-Lions theorem ([11]), there exists a subsequence of $(z^k)_{k\in\mathbb{N}}$ still denoted by $(z^k)_{k\in\mathbb{N}}$ such that $(z^k)_{k\in\mathbb{N}}\subset L^2([0,t];H^1_{\Gamma_1}(\Omega_1))$ converges. Using the

trace theorem we conclude that $(z^k|_{\Gamma 0})_{k\in\mathbb{N}} \subset L^2([0,t];L^2(\Gamma_0)$ and is convergent.
For $k,l \in \mathbb{N}$, set

$$F^{kl}(t) = \frac{1}{2}\left\|S(t)(Y_0^k - Y_0^l) - \tilde{S}(t)(Y_0^k - Y_0^l)\right\|_{\mathcal{H}}^2$$

By (6.52), we have

$$F^{kl}(t) \leq |\langle \frac{\partial \tilde{y}^{kl}(x,t)}{\partial \nu}, \partial_t y^{kl}(x,t)\rangle_{L^2([0,t];L^2(\Gamma_0)}| \to 0 \ (k,l \to +\infty), \tag{6.53}$$

where $Y^{kl}(t) = S(t)(Y_0^k - Y_0^l)$ and $\tilde{Y}(t) = \tilde{S}(t)(Y_0^k - Y_0^l)$ for $k,l \in \mathbb{N}$. (6.53) implies that $(S(t) - \tilde{S}(t))Y_0^k$ is a Cauchy sequence in $\mathcal{H}$ and thus convergent. Therefore, the operator $S(t) - \tilde{S}(t) : \mathcal{H}_1 \to \mathcal{H}$ is compact and so is $S(t) - \tilde{S}(t) : \mathcal{H}_0 \to \mathcal{H}$.
Since $r_{ess}(\tilde{S}(t)) = 1$, Theorem 1.3.6 (Theorem 3.3 in [56]) implies that $r_{ess}(S(t)) = 1$, and thus $(S(t))_{t\geq 0}$ is not exponentially stable.

6.5 Polynomial stability

Theorem 6.5.1. *Assume that there exists $x^0 \in \mathbb{R}^n$ such that for $m(x) = x - x^0$*

$$m(x)\cdot \nu(x) \leq 0 \text{ on } \partial\Omega_2 \tag{6.54}$$

Then there exists a constant $M > 0$ such that for all $Y_0 \in D(A)$ the corresponding solution to (6.1)-(6.8) with $\mu = 0$ and $\alpha > 0$satisfies the following estimate

$$\|Y(t)\|_{\mathcal{H}} \leq \frac{M}{\sqrt{t}}\|Y_0\|_{D(A)} \text{ for all } t > 0$$

Theorem 6.5.1 follows from Theorem 1.3.5 once we have verified that the hypothesis (1.4) and (1.6) holds for the operator A.

Lemma 6.5.2. *The spectrum of A contains no points on the imaginary axis, that is $i\mathbb{R} \subset \rho(A)$.*

Proof. By Sobolev embedding Theorem, A has a compact resolvent. As a consequence, its spectrum consists only of eigenvalues of A. Assume $\lambda = i\beta$, with $\beta \in \mathbb{R}^*$, is an eigenvalue of A and let $W = (z,y,u)^T \in D(A)$ be a corresponding eigenvector. Thus

$$AW = i\beta W \tag{6.55}$$

or equivalently

$$\begin{aligned}
i\Delta z(x) - \alpha z(x) &= i\beta z(x) && \text{in } \Omega_1,\\
u(x) &= i\beta y(x) && \text{in } \Omega_2,\\
\Delta y(x) &= i\beta u(x) && \text{in } \Omega_2,\\
z(x) &= 0 && \text{on } \Gamma_1,\\
z(x) &= u(x) && \text{on } \Gamma_0,\\
i\frac{\partial z(x)}{\partial \nu} &= \frac{\partial y(x)}{\partial \nu} && \text{on } \Gamma_0,\\
y(x) &= 0 && \text{on } \Gamma_2.
\end{aligned} \tag{6.56}$$

Taking the inner product of (6.55) with W and using the equality (6.17), we get

$$z(x) = 0 \text{ in } \Omega_1,$$

which together with (6.56) yields

$$u(x) = 0 \quad \text{on } \Gamma_0.$$

Consequently y satisfies

$$\begin{cases} -\Delta y(x) - \beta^2 y(x) = 0 & \text{in} \quad \Omega_1, \\ y(x) = 0 & \text{on} \quad \Gamma_2, \\ y(x) = \frac{\partial y(x)}{\partial \nu} = 0 & \text{on} \quad \Gamma_0. \end{cases}$$

and by the unique continuation principle

$$y(x) = 0 \text{ in } \Omega_2 \tag{6.57}$$

Therefore $W = (0,0,0)^T$ and this contradicts that W is an eigenvector of A. □

Lemma 6.5.3. *The resolvent operator of A satisfies condition (1.6).*

Proof. Suppose that (1.6) is not satisfied. Then by the Banach-Steinhaus Theorem, there exists a sequence of real numbers $\beta_k \to +\infty$ and a sequence of vectors $W_k = (z_k, y_k, u_k) \in D(A)$ with $\|W_k\| = 1$ such that

$$\lim_{k \to +\infty} \left\| \beta_k^2 (i\beta_k - A) W_k \right\|_{\mathcal{H}} = 0. \tag{6.58}$$

It then follows from (6.58) as $k \to +\infty$ that

$$\beta_k (i\beta_k z_k - i\Delta z_k + \alpha z_k) = f_k \to 0 \quad \text{ in } L^2(\Omega_1), \tag{6.59}$$

$$\beta_k (i\beta_k y_k - u_k) = g_k \to 0 \quad \text{ in } H^1_{\Gamma_2}(\Omega_2), \tag{6.60}$$

$$\beta_k ((i\beta_k - \Delta y_k) = h_k \to 0 \quad \text{in } L^2(\Omega_2), \tag{6.61}$$

Furthermore by (6.58) and the fact that

$$\Re \langle \beta_k^2 (i\beta_k I - A) W_k, W_k \rangle = \beta_k^2 \alpha \left\| z_k \right\|^2_{L^2(\Omega_1)} \tag{6.62}$$

we have

$$\beta_k^2 \alpha \left\| z_k \right\|^2_{L^2(\Omega_1)} \to 0. \tag{6.63}$$

This leads by (6.59) to

$$\Delta z_k \to 0 \quad \text{in } L^2(\Omega_1). \tag{6.64}$$

By the theory of elliptic boundary value problems ([23]), there exists a constant $C > 0$ such that

$$\|z_k\|_{H^2(\Omega_1)} \le C(\|\Delta z_k\|_{L^2(\Omega_1)} + \|z_k\|_{L^2(\Omega_1)}). \tag{6.65}$$

On the other hand by interpolation ([45]) we have

$$\|z_k\|^2_{H^1_{\Gamma_1}(\Omega_1)} \le C \left\| z_k \right\|_{L^2(\Omega_1)} \left\| z_k \right\|_{H^2(\Omega_1)}. \tag{6.66}$$

From (6.63)-(6.66), we find

$$\|z_k\|_{H^1_{\Gamma_1}(\Omega_1)} \to 0 \tag{6.67}$$

By the trace theorem and (6.67), we see that

$$\|z_k\|_{L^2(\Gamma_0)} \to 0. \tag{6.68}$$

Moreover, we conclude from (6.64) and (6.67) that (see [45], [35])

$$\left\| \frac{\partial z_k(x)}{\partial \nu} \right\|_{H^{-1}(\partial\Omega_1)} \to 0. \tag{6.69}$$

Therefore

$$\|u_k\|_{L^2(\Gamma_0)} \to 0, \|y_k\|_{L^2(\Gamma_0)} \to 0. \tag{6.70}$$

$$\left\|\frac{\partial y_k(x)}{\partial \nu}\right\|_{H^{-1}(\Gamma_0)} \to 0. \tag{6.71}$$

We split the remaining part of the proof into two steps.

Step 1.

We multiply (6.60) (resp. (6.61) by u_k (resp. y_k) and integrate over Ω_2 to find

$$\underset{k\to+\infty}{lim} \Re\langle i\beta_k y_k, u_k\rangle_{L^2(\Omega_2)} - \Re\langle u_k, u_k\rangle_{L^2(\Omega_2)} = 0, \tag{6.72}$$

$$\underset{k\to+\infty}{lim} \Re\langle i\beta_k u_k, u_k\rangle_{L^2(\Omega_2)} - \Re\langle \Delta y_k, k_k\rangle_{L^2(\Omega_2)} = 0, \tag{6.73}$$

Applying Green's theorem and using the boundary condition $y_k = 0$ on Γ_2 gives

$$\underset{k\to+\infty}{lim} \int_{\Omega_2} (|u_k(x)|^2 - |\nabla y_k(x)|^2)dx = \underset{k\to+\infty}{lim} \{-\Re \int_{\Gamma_0} \frac{\partial y_k(x)}{\partial \nu}\overline{y_k(x)}d\Gamma\} \tag{6.74}$$

from which follows via (6.68) and (6.69)

$$\underset{k\to+\infty}{lim} \|u_k\|^2_{L^2(\Omega_2)} = \underset{k\to+\infty}{lim} \|\nabla y_k|^2_{L^2(\Omega_2)} \tag{6.75}$$

Step 2.

We have from (6.60), (6.61) and the fact that $\|W_k\|_{\mathcal{H}} = 1$,

$$\underset{k\to+\infty}{lim} \left\{\Re\langle u_k, i\beta_k m\cdot\nabla y_k\rangle_{L^2(\Omega_2)} - \Re\langle u_k, m\cdot\nabla u_k\rangle_{L^2(\Omega_2)}\right\} = 0, \tag{6.76}$$

$$\underset{k\to+\infty}{lim} \left\{\Re\langle i\beta_k u_k, m\cdot\nabla y_k\rangle_{L^2(\Omega_2)} - \Re\langle \Delta y_k, m\cdot\nabla y_k\rangle_{L^2(\Omega_2)}\right\} = 0. \tag{6.77}$$

Combining (6.76) with (6.77) yields

$$\underset{k\to+\infty}{lim} \left\{-\Re\langle u_k, m\cdot\nabla u_k\rangle_{L^2(\Omega_2)} - \Re\langle \Delta y_k, m\cdot\nabla y_k\rangle_{L^2(\Omega_2)}\right\} = 0. \tag{6.78}$$

Using Green's theorem and the identity (4.81) in the Appendix C of Chapter 5 with $\mathcal{O} = \Omega_2, \mathcal{S} = \Gamma_0 \cup \Gamma_2, w = y_k$, $h = m$ and $H(x) = I_n$, we get

$$\underset{k\to+\infty}{lim} \left\{-\frac{1}{2}\int_{\Gamma_2} |u_k(x)|^2 m(x)\cdot\nu(x)d\Gamma - \frac{1}{2}\int_{\Gamma_0} |u_k(x)|^2 m(x)\cdot\nu(x)d\Gamma - \right.$$

$$\Re\int_{\Gamma_2} \frac{\partial y_k(x)}{\partial \nu} m(x)\cdot\nabla\overline{y_k(x)}d\Gamma - \Re\int_{\Gamma_0} \frac{\partial y_k(x)}{\partial \nu} m(x)\cdot\nabla\overline{y_k(x)}d\Gamma + \int_{\Omega_2} |\nabla y_k(x)|^2 dx +$$

$$\frac{1}{2}\int_{\Gamma_2} |\nabla y_k(x)|^2 m(x)\cdot\nu(x)d\Gamma + \frac{1}{2}\int_{\Gamma_0} |\nabla y_k(x)|^2 m(x)\cdot\nu(x)d\Gamma +$$

$$\left.\frac{n}{2}\int_{\Omega_2} \{|u_k(x)|^2 - |\nabla y_k(x)|^2\}dx\right\} = 0.$$

Using the boundary condition (6.35) and the limits (6.70) yields

$$\begin{aligned} \underset{k\to+\infty}{lim} &\left\{-\frac{1}{2}\int_{\Gamma_2} |u_k(x)|^2 m(x)\cdot\nu(x)d\Gamma - \Re\int_{\Gamma_2} \left|\frac{\partial y_k(x)}{\partial \nu}\right|^2 m(x)\cdot\nu(x)d\Gamma - \right. \\ &\left. \Re\int_{\Gamma_0} \left|\frac{\partial y_k(x)}{\partial \nu}\right|^2 m(x)\cdot\nu(x)d\Gamma + \int_{\Omega_2} |\nabla y_k(x)|^2 dx\right\} = 0. \end{aligned} \tag{6.79}$$

Recalling assumption (6.54), we obtain by (6.75) and (6.79)

$$\underset{k\to+\infty}{lim} \|u_k\|_{L^2(\Omega_2)} = \underset{k\to+\infty}{lim} \|\nabla y_k\|_{L^2(\Omega_2)} = 0 \tag{6.80}$$

Identities (6.63) and (6.80) contradict the assumption that $\|W_k\|_{\mathcal{H}} = 1$ for all $k \in \mathbb{N}$.

Conclusion

In this thesis we have established stability results for

- The transmission Schrödinger equation with discrete time delay in the Neumann boundary feedback,
- The transmission Schrödinger equation with time-varying delay in the Neumann boundary feedback,
- The transmission wave equation with distributed delay in the Neumann boundary feedback,
- The transmission Schrödingerr/wave equation with a dissipative boundary feedback acting on the wave equation through the Neumann boundary condition,
- The transmission Schrödinger/wave equation with a dissipative feedback acting either on the wave equation or on the Schrödinger equation.

The approach we adopted uses one or more of the following tools:

- Multipliers technique
- Lyapunov functionals
- Frequency domain method

There are several extensions of the results obtained in this thesis. For example the following questions can be considered for future work:

- Stability of the transmission Schrödinger/wave equation with a boundary feedback acting on the Schrödinger equation.
- Stability of the transmission Schrödinger/wave equation subject to a delayed feedback acting either on the Schrödinger equation or wave equation.
- Stability of the transmission Schrödinger/wave equation subject to a non linear (boundary or internal) feedback acting on one of the equations.

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