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Abstract

In this thesis, we deal with nonlinear integral equations using generalized contractions. Roughly speaking, we use different approaches based on fixed point theory to provide the existence of solutions for many integral equations in various functional spaces.

Résumé

Dans cette thèse, nous traitons d'équations intégrales non linéaires, en utilisant des contractions généralisées. De plus, nous utilisons différentes approches basées sur la théorie du point fixe pour démontrer l'existence de solutions à de nombreuses équations intégrales dans divers espaces fonctionnels.

ملخص

في هذه الأطروحة، نتعامل مع المعادلات التكاملية غير الخطية باستخدام الاختصارات المعممة. بشكل تقريبي، نستخدم أساليب مختلفة تعتمد على نظرية النقطة الثابتة لاثبات وجود حلول للعديد من المعادلات التكاملية في الفضاءات الوظيفية المختلفة.

Contents

Introduction

In this thesis, we deal with nonlinear integral equations using generalized contractions. Roughly speaking, we use different approaches based on the fixed point theory to prove the solvability of solutions for the aforementioned integral equations. More precisely, we prove many results through Wardowski-contractions in the context of b-metric spaces, generalized expansive mapping of Krasnosel'skii type, measure of noncompactness and generalized contractions in the bounded variation space.

The starting point in the introduction of the setting of b-metric space is due to Bakhtin [7] (see for instance Czerwik [18] as well). He generalized the standard metric space by supposing a weaker assumption than the triangle inequality. Further, several investigations regarding the concept of b-metric theory were established over these last years (see, e.g., [12, 13, 14, 16, 21, 23, 29, 35, 37, 39, 24, 44, 47, 52].

A powerful extension of the *Banach fixed point* [8] was given by Wardowski [59]. Its contribution is mainly based on an suitable function F, namely, *F-contraction*. The author established a new fixed point result. Later on, many interesting generalizations concerning Wardowski's fixed point have been stated and proved in different directions by lot of mathematicians. For more details and explanations dealing with F-contractions mappings, the following refernce [34] and references therein may be consulted. Contractive conditions including rational expressions can be found in [3, 27, 41, 42, 43] and references therein.

Schauder, Krasnosel'skii fixed point theorems and the measure of noncompactness play huge roles in nonlinear functional analysis and especially in the solvability of various nonlinear integral equations.

This thesis is organized as follows:

In the chapter 2, we introduce new kinds of Dass-Gupta-contractions. The aforementioned contractions mappings are consequently utilized to establish new fixed point theorems in the context of b-metric spaces. At the end of this chapter, we provide some applications, our obtained results are therefore used to discuss the existence of solutions for various nonlinear integral equations.

The main goal of the chapter 3 is to prove a fixed point result of Krasnosel'skii type based on a suitable generalization of expansive mappings. Some slight improvements of the works [54] and [61] are given.

Chapter 4 is devoted to prove the existence of some integral equations using the the measure of noncompactness. In this chapter, we introduce the notion of Proinov contraction of Darbo-type through the measure of noncompactness μ and we prove the existence of the fixed point in some bounded, closed and convex subset of a Banach space X.

In the chapter 5, we propose a contribution for solving nonlinear integral equations in the space of bounded variation. Roughly speaking, we utilize the composition operator and the notion of Henstock-Kurzweil integrals on an unbounded interval and their properties to establish some existence results of nonlinear Hammerstein integral equations.

Chapter 1

Contents

Preliminaries

1.1 Notations

In this chapter, we use the symbol $\mathbb{R} =]-\infty, +\infty[$ is commonly used to indicate real numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of every one of the natural numbers.

In the subsequent sections, we will review the primary findings and foundational concepts that are necessary to establish the results we will present in the future. Unless explicitly specified, let X stand for any set that is not empty. An operator $T : X \to X$ initialized at $x_0 \in X$ possesses a Picard sequence defined as $x_n = Tx_{n-1} = T^n x_0$ for whatever $n \in \mathbb{N}$, in which T^n represents the n-iterates of T.

1.2 Metric space

Herein, initially, by defining a metric space.

Definition 1.2.1. Let X have for any set that is not empty. A function $d : X \times X \to [0, \infty)$ is considered to be a metric if it implies the following conditions, for any $x, y, z \in X$:
(b_1) If and only if $x = y$, then $d(x, y) = 0$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, z) \leq [d(x, y) + d(y, z)]$.
A metric space consists of (X, d).

In the lemma below, we establish a different proof to the one given in [45].

Lemme 1.2.1. *Have (X, d) stand for a metric space and $\{x_n\}$ for a sequence in X which include:*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If If the sequence of $\{x_n\}$ does not satisfy the Cauchy criterion in the metric space (X, d), then there exists a positive value ε, as well as two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers that is why $n(k) > m(k) > k$, and the resulting equalities are true:

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right)$$
$$= \lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)}\right) = \lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \varepsilon^+.$$

Proof. If $\{x_n\}$ does not satisfy the Cauchy sequence criterion and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, then there exist positive real numbers ε and sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ consisting of positive integers which means for each $k \geq 1$, $n(k)$ is the smallest index including $n(k) > m(k) > k$, $d\left(x_{m(k)}, x_{n(k)}\right) > \varepsilon$ and $d\left(x_{m(k)}, x_{n(k)-1}\right) \leq \varepsilon$.

By employing the triangle inequality, we can establish the following:

$$\varepsilon < d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right)$$
$$\leq \varepsilon + d\left(x_{n(k)-1}, x_{n(k)}\right).$$

Taking $k \to \infty$, one can get

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon^+.$$

On the other hand, through again the triangle inequality and $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we obtain

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)}\right) &\leq d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)}, x_{n(k)+1}\right) \\ &\leq d\left(x_{m(k)}, x_{n(k)}\right) + 2d\left(x_{n(k)}, x_{n(k)+1}\right). \end{aligned}$$

Taking the limit as $k \to \infty$, we arrive at

$$d\left(x_{m(k)}, x_{n(k)+1}\right) \to \varepsilon^+.$$

In the same manner, we obtain also

$$d\left(x_{n(k)}, x_{m(k)+1}\right) \to \varepsilon^+.$$

The triangle inequality can be readily observed, indicating that

$$\left|d\left(x_{n(k)+1}, x_{m(k)+1}\right) - d\left(x_{m(k)}, x_{n(k)}\right)\right| \leq d\left(x_{n(k)}, x_{n(k)+1}\right) + d\left(x_{m(k)}, x_{m(k)+1}\right).$$

This implies

$$\lim_{k\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon^+.$$

□

1.2.1 Reminder on b-metric settings

Definition 1.2.2. ([7] and [19]) Consider a nonempty set, denoted by X, and let s be a value greater than or equal to 1. A b-metric function $\sigma : X \times X \to [0, \infty)$ is presented by the following presumptions; For each x, y *and*, z that belong to the set X,
(b_1) $\sigma\left(x, y\right) = 0$ *iff* $x = y$;
(b_2) $\sigma\left(x, y\right) = \sigma\left(y, x\right)$;
(b_3) $\sigma\left(x, z\right) \leq s\left[\sigma\left(x, y\right) + \sigma\left(y, z\right)\right]$.
The pair (X, σ) is referred to as a b-metric space for coefficient (constant) $s \geq 1$.

In regard to definition 1.2.2, it is observed that the metric space can be classified as a b-metric space with a coefficient of $s = 1$. However, it should be noted that the reverse is not necessarily valid in the broader context, as indicated by references (see [1, 7, 24]). Moreover, it can be observed that the class of standard metric spaces is encompassed within the larger set of b-metric spaces.

In the sequels, we provide some classical examples.

Example 1.2.1. ([52]) Let's Take (X, d) be a metric space be considered a metric space, and let $\sigma_d : X \times X \to [0, \infty)$ satisfy the condition:

$$\sigma_d\left(x, y\right) = \left(d\left(x, y\right)\right)^p \quad \text{for all } x, y \in X,$$

with $p > 1$. So, (X, σ_d) is a b-metric space where $s = 2^{p-1}$ is the coefficient.

Next, we remind some topological properties in b-metric spaces.

Definition 1.2.3. ([13, 14, 16]) Consider a b-metric space (X, σ) at the coefficient $s \geq 1$. So a sequence $\{x_n\}$ in X is designated as:
(a) convergent *iff* there is a $x \in X$ for which $\lim_{n\to+\infty} \sigma(x_n, x) = 0$, i.e., $\lim_{n\to+\infty} x_n = x$;
(b) Cauchy *iff* $\lim_{n,m\to+\infty} \sigma(x_n, x_m) = 0$.

Definition 1.2.4. ([13, 14, 16]) A b-metric space (X,σ) is deemed complete when the convergence of every Cauchy sequence in X is ensured.

The lemma presented herein is crucial for establishing the validity of one of our principal findings.

Lemme 1.2.2. *([53]) Let us consider a b-metric space (X,d) with a coefficient $s \geq 1$, where X is a set and $\{x_n\}$ is a sequence in X. It is assumed that*

$$\{d(x_n, x_{n+1})\} \in \bigcup \left\{O\left(n^{-\gamma}\right) : \gamma > 1 + \log_2 s\right\}.$$

Consequently, $\{x_n\}$ is a Cauchy sequence.

According to the findings of [4], it has been established that the b-metric does not exhibit continuity. The subsequent illustration serves to underscore this assertion.

Example 1.2.2. ([24]) As an $X = [0,\infty)$, the function is established as $\sigma : X \times X \to [0,\infty)$

$$\sigma\left(x,y\right) = \begin{cases} d\left(x,y\right), & xy \neq 0, \\ 4d\left(x,y\right), & xy = 0, \end{cases}$$

with $d\left(x,y\right) = \left|x-y\right|$. After that, we get

1. (X,σ) is a complete b-metric $(s=4)$;
2. The function σ does not satisfy the metric properties on the set X;
3. The continuity in each variable for the function σ is not satisfied.

The next lemma plays a vital part to correct the lack of the continuation for the b-metric.

Lemme 1.2.3. *([44]) We define a sequence $\{x_n\}$ in X satisfying the next if and only if (X,σ) is a b-metric space where the coefficient $s \geq 1$:*

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0.$$

If the sequence $\{x_n\}$ does not satisfy the Cauchy criterion in the sense of Lemma 1.2.1, so the following holds

$$\varepsilon \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon;$$

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)+1}\right) \leq s^2\varepsilon;$$

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)}\right) \leq s^2\varepsilon;$$

$$\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)+1}, x_{n(k)+1}\right) \leq s^3\varepsilon.$$

In the next part, we revise important fixed point results for b-metric spaces.

Theorem 1.2.1. *Let (E,σ) represent a complete b-metric space to a constant b-metric space where the constant $s \geq 1$. Consider an operator $T : E \to E$ that verifies*

$$\sigma(Tu, Tv) \leq k\sigma(u, v),$$

For any elements u, v belonging to the set E, and for some value $k \in [0,1)$. It is acknowledged that T possesses a solitary fixed point u^* within the set E. Furthermore, it can be observed that the sequence $\{T^n u\}$ converges towards this fixed point for every u within the set E.

Theorem 1.2.2. *([47]) Let (E,σ) be a complete b-metric space with constant $s \geq 1$ and $T : E \to E$ be an operator. Suppose that there exist $\alpha, \beta \in [0,1)$, where $\alpha s + \beta < 1$ such that the following is satisfied*

$$\sigma(Tu, Tv) \leq \alpha\sigma(u, v) + \beta\frac{\sigma(v, Tv)(1+\sigma(u, Tu))}{1+\sigma(u, v)},$$

for every pair of elements $u, v \in E$. Let T has a unique fixed point $u \in E$ and the sequence $\{T^n u\}$ converges to this fixed point.

1.2.2 Fixed point results regarding F-contractions

The concept of F-contraction has been initiated by Wardowski [59] in 2012 as follows:

Definition 1.2.5. Let (X,d) be a metric space. An operator $T : X \to X$ is said to be an F-contraction if there exist $\tau > 0$ and $F \in \mathscr{F}$ such as for all $u, v \in X$,

$$d(Tu, Tv) > 0 \Rightarrow \tau + F(d(Tu, Tv)) \leq F(d(u, v)) \tag{1.1}$$

with $\mathscr{F}$ is the entire set of all functions $F : (0,\infty) \to \mathbb{R}$ such that the below condition is satisfied:
(F_1) F is strictly increasing;
(F_2) For every positive number sequence $\{\alpha_n\}$, we consist of:

$$\lim_{n\to\infty} \alpha_n = 0 \textit{ iff } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F_3) $\exists k \in (0,1)$ satisfying $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Example 1.2.3. ([59]) Let $\alpha \in (0,\infty)$. The following functions $F_1(\alpha) = \ln\alpha$, $F_2(\alpha) = \ln\alpha + \alpha$, $F_3(\alpha) = \frac{-1}{\sqrt{\alpha}}$ and $F_4(\alpha) = \ln\left(\alpha^2 + \alpha\right)$ belong to the family $\mathscr{F}$.

Remark 1.2.1. If we take, in (1.1), $F(\alpha) = \ln\alpha$; a Banach contraction can be easily deduced.

The next result is established by Wardowski.

Theorem 1.2.3. *([59]) Consider (E,d) the complete metric space denoted, and let $T : E \to E$ be an F-contraction. For every element $x_0 \in E$ the sequence $\left\{T^n x_0\right\}$ converges to x^*. Furthermore, it can be noticed the value of x^* is unique.*

Later on, Wardowski [60], in 2018, generalized the class $\mathscr{F}$ as follows.

Definition 1.2.6. ([60]) Let (X,d) be a metric space. An operator $T:X\to X$ is characterized by (χ,F)-contraction if there exist $F:(0,\infty)\to\mathbb{R}$ and $\chi:(0,\infty)\to(0,\infty)$ thus, the next conditions are valid:

1. F satisfies (F_1);
2. (F_2'): $\lim_{t\to 0^+} F(t)=-\infty$;
3. (H_0): $\lim\inf_{t\to\varepsilon^+}\chi(t)>0$ for all $\varepsilon\geq 0$;
4. $\chi(d(x,y))+F(d(Tx,Ty))\leq F(d(x,y))$ for all $x,y\in X$ where $Tx\neq Ty$.

Example 1.2.4. Let $F_1,F_2:(0,\infty)\to\mathbb{R}$ be given by : $F_1(t)=\ln(t+1)$ and $F_2(t)=-\dfrac{1}{t+1}$ for all $t\in(0,\infty)$. It is obvious that F_1 and F_2 belonging to the class $\mathscr{F}$ but do not fulfill condition (F_2').

In addition, Wardowski [60] established the next outcome

Theorem 1.2.4. *([60]) Let (X,d) be a complete metric space. So every (χ,F)-contraction mapping admits a unique fixed point.*

We give below a detailed and slightly different proof to the one established in [58].

Proposition 1.2.1. *Let a complete metric space (X,d) and $T:X\to X$ be an operator. If there exist a nondecreasing function $F:(0,\infty)\to\mathbb{R}$ and τ satisfies (H_0) such that for all $u,v\in X$,*

$$d(Tu,Tv)>0\Rightarrow\tau(d(Tu,Tv))+F(d(Tu,Tv))\leq F(d(u,v)),\tag{1.2}$$

Then for every $x_0\in X$ the sequence $\{T^n x_0\}$ converges to x^*. Moreover x^* is unique.

Proof. First, we prove the existence of a fixed point in T. Let $x_0\in X$ and $\{x_n\}$ be the Picard sequence with the starting point x_0. If there exist $n_0\in\mathbb{N}_0$, as a result, $x_{n_0}=x_{n_0+1}$, then x_{n_0} is the fixed point of T. If $x_n\neq x_{n+1}$, for all $n\in\mathbb{N}_0$, we get

$$d_n:=d(x_n,x_{n+1})=d(Tx_{n-1},Tx_n)>0,\quad\text{for all } n\in\mathbb{N}.\tag{1.3}$$

When we use the contractive inequality (2.47), where $x=x_{n-1}$ and $y=x_n$, we get for all $n\in\mathbb{N}$

$$\tau(d(x_n,x_{n+1}))+F(d(x_n,x_{n+1}))\leq F(d(x_{n-1},x_n))\tag{1.4}$$

Since F is monotonically increasing and by the fact that $\tau(t)>0,\forall t>0$, one gets

$$d_n<d_{n-1},\quad\text{for all } n\in\mathbb{N}.\tag{1.5}$$

It establishes that $\{\sigma_n\}$ is a strictly decreasing sequence below. So, there is $l\geq 0$ that gives

$$\lim_{n\to\infty} d_n=l^+.$$

Now, we are going to show that $l=0$. By contradiction, we assume that $l>0$. Since F is nondeceasing, F doesn't go down, it has a right limit, i.e.,

$$\lim_{t\to r^+} F(t)=F(r+0)=F(r^+),\quad\text{for all } r\in(0,\infty).\tag{1.6}$$

Having in mind (1.6) and letting $n\to\infty$ in (1.4), one gets

$$\begin{aligned}\liminf_{t\to l^+}\tau(t) &\le \liminf_{n\to\infty}\tau(d_n)\\ &\le \lim_{n\to\infty}\left(F(d_{n-1})-F(d_n)\right)\\ &= F\left(l^+\right)-F\left(l^+\right)\\ &=0,\end{aligned}$$

which a contradiction. Hence,

$$\lim_{n\to\infty} d_n = 0^+. \tag{1.7}$$

Next, it will be established that $\{x_n\}$ is Cauchy. Consider that $\{x_n\}$ is not Cauchy.

By applying (2.47) to $x=x_{m(k)}$ and $y=x_{n(k)}$, we find

$$\tau\left(d\left(Tx_{m(k)},Tx_{n(k)}\right)\right)+F\left(d\left(Tx_{m(k)},Tx_{n(k)}\right)\right)\le F\left(d\left(x_{m(k)},x_{n(k)}\right)\right).$$

By Lemma 1.2.1 with (1.7) and taking the limit inferior as $k\to\infty$, we get

$$\begin{aligned}\lim\inf_{t\to\varepsilon^+}\tau(t) &\le \lim\inf_{k\to\infty}\tau\left(d\left(x_{m(k)+1},x_{n(k)+1}\right)\right)\\ &\le \lim\inf_{k\to\infty}\left[F\left(d\left(x_{m(k)},x_{n(k)}\right)\right)-F\left(d\left(x_{m(k)+1},x_{n(k)+1}\right)\right)\right]\\ &\le F\left(\varepsilon^+\right)-F\left(\varepsilon^+\right)=0,\end{aligned}$$

which contradicts the fact that $\lim\inf_{t\to\varepsilon^+}\tau(t)>0$.

In summary, it can be concluded that $\{x_n\}$ is a Cauchy sequence. Due to the completeness of the metric space (X,d), the sequence $\{x_n\}$ converges to a certain point x^* in X, .i.e.,

$$\lim_{n\to\infty} d\left(x_n,x^*\right)=0.$$

Given that T is a continuous function, we can readily obtain

$$Tx^*=T\left(\lim_{n\to\infty}x_n\right)=\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}x_{n+1}=x^*$$

Therefore, x^* is a fixed point of T.

The part about being unique is easy to get, as shown in [58]. □

Lukács and Kajántó [40] studied the following class of functions in 2018:

Let $\mathbb{F}^*$ be the family of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying (F_1) and (F_3).

Definition 1.2.7. Let $s\ge 1$ and $\tau>0$. The condition that $F\in\mathbb{F}^*$ belongs to $\mathscr{F}_{s,\tau}$ means that F satisfies the following assumption

$(\mathrm{F}_{s,\tau})$ Suppose that $\inf F=-\infty$ and $x,y,z\in(0,\infty)$ such that $\tau+F(sx)\le F\left(y\right)$ and $\tau+F\left(sy\right)\le F(z)$ then

$$\tau+F\left(s^2x\right)\le F\left(sy\right).$$

Theorem 1.2.5. *Suppose that (X,σ) is a complete b-metric space with constant $s\ge1$ and let $T:X\to X$ be an operator. If there exist $\tau>0$ and $F\in\mathscr{F}_{s,\tau}$ such that for all $x,y\in X$, $\sigma\left(Tx,Ty\right)>0$ yields*

$$(F)\qquad \tau+F\left(s\sigma\left(Tx,Ty\right)\right)\le F\left(\sigma\left(x,y\right)\right),$$

then T has a unique fixed point x^* and for every $x_0\in X$ the sequence $\left\{T^nx_0\right\}$ converges to x^*.

1.3 Background on measure of noncompactness

Henceforth, X is a Banach space and D is a nonempty subset of X. We denote by $\overline{Y}$ and $co(D)$ the closure of Y and the convex hull of D, respectively. In addition, we denote by $B(X)$ and $RC(X)$ the set of all nonempty bounded subsets of X and all relatively compact subsets of X, respectively.

Definition 1.3.1. ([9]) A mapping $\mu : B(X) \longrightarrow [0,\infty)$ is called a measure of noncompactness in X if the following holds

1) $\ker\mu = \{A \in B(X) : \mu(A) = 0\} \neq \emptyset$ with $\ker\mu \subseteq RC(X)$.

2) $Z \subseteq Y \Longrightarrow \mu(Z) \leqslant \mu(Y)$.

3) $\mu(\overline{Z}) = \mu(Z)$.

4) $\mu(co(A)) = \mu(A)$.

5) $\mu(\lambda Z + (1-\lambda)Y) \leqslant \lambda\mu(Z) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

6) If the sequence $\{D_n\}$ of closed sets starting from $B(X)$ is such as $D_{n+1} \subseteq D_n$ for all $n \in \mathbb{N}$ and $\lim_{n\longrightarrow\infty} \mu(D_n) = 0$, we have a nonempty set $D_\infty := \bigcap_{n=1}^{\infty} D_n$.

Remark 1.3.1. Let us observe that $D_\infty := \bigcap_{n=1}^{\infty} D_n$ belongs to $\ker\mu$ as such $\mu(D_\infty) \leq \mu(D_n)$ for all $n \in \mathbb{N}$ and we thus obtain $\mu(D_\infty) = 0$. Therefore, it follows that $D_\infty \in \ker\mu$.

Theorem of Schauder is a fundamental result needed for proving many theorems in integral equations theory. The aforementioned theorem is given below.

Henceforth, unless otherwise stated, D stands for a nonempty closed, bounded and convex subset of a Banach space E.

Theorem 1.3.1 (Schauder)**.** *If $L : D \to D$ is an operator on D. So L has a fixed point in D.*

Let $C(X)$ denote the space of all continuous functions on a compact metric sapace X. In $C(X)$ we always regard the distance between functions f and g in $C(X)$ to be

$$d(f,g) = \max\{|f(x) - g(x)| : x \in X\}.$$

(a) $F \subset C(X)$ is bounded means that there exists a positive constant $M < \infty$ where $|f(x)| \leq M$ for each $x \in X$ and each $f \in F$, and
(b) $F \subset C(X)$ is equicontinuous means that: for every $\varepsilon > 0$ there exists $\delta > 0$ (depending upon only on ε) such that for $x, y \in X$:

$$d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \quad \forall f \in F.$$

Theorem 1.3.2. *(Ascoli-Arzela) Let X be a compact set. A subset F of $C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.*

Theorem 1.3.3. *(Krasnosel'skii) If $L : D \to E$ is a contraction and $S : M \to E$ is a continuous compact mapping with $L(D) + S(D) \subset D$, then $L + S$ admits a fixed point.*

1.4 Some fixed point theorems by the measure of noncompactness

1.4.1 Darbo fixed point

Definition 1.4.1. An operator $S : D \to D$ is called μ-contraction if there exists a constant $k \in (0,1)$ satisfying

$$\mu(SX) \le k\mu(X), \tag{1.8}$$

where X is a subset of D.

Theorem of Darbo is stated as follows (refer to [9]).

Theorem 1.4.1. *Let $S : D \to D$ be a continuous self-operator. Consider that S is a μ-contraction. So, S admits a fixed point in D.*

1.4.2 F-Darbo contractions

Definition 1.4.2. ([57]) A continuous operator S on D is known as a F-contract of Darbo-type if there exists F and τ that satisfies (F_2) and (H_0), respectively and such that

$$\tau(\mu(X)) + F(\mu(SX)) \leqslant F(\mu(X)) \quad \text{for any} \quad X \subset M \quad \text{with} \quad \mu(X), \mu(SX) > 0. \tag{1.9}$$

Consistent with [57, Remark 7.3], we have

Theorem 1.4.2. *([57]) Assume that S is an F-contraction of Darbo-type, so S admits a fixed point in D.*

Remark 1.4.1. By analysing the proof of the above theorem, we have observed that hypothesis (H_0) cannot be assumed because $\{\mu(M_n)\}$ is only non-increasing and therefore we do not necessarily have

$$\lim_{n\to\infty} \mu(M_n) = r^+.$$

Actually, the suitable hypothesis (stronger than (H_0)) to make Theorem 1.4.2 valid is given by

$$\text{for every } r > 0, \quad \liminf_{t\to r} \tau(t) > 0. \tag{H_1}$$

1.4.3 θ-contractions

Let Ω be the set of all functions $\rho : (0,\infty) \to (1,\infty)$ such that the next assumption holds:

(θ_2): For any sequence $\{t_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} \rho(t_n) = 1 \Leftrightarrow \lim_{n\to\infty} t_n = 0.$$

The authors in [32] derived the theorem below.

Theorem 1.4.3. *Let $S : D \to D$ be a self-continuous operator that satisfies for all subset Z of D*

$$\rho\left(\mu(SZ)\right) \le \left[\rho\left(\mu(Z)\right)\right]^k, \quad with \quad \mu(TZ), \mu(TZ) > 0,$$

where $\rho \in \Omega$ and $k \in (0,1)$. There, T admits at least one fixed point.

The following lemmata play an important role in the sequel (see [46])

Lemme 1.4.1. *Let $\psi : (0,\infty) \to \mathbb{R}$. So the following conditions are equivalent:*

1) $\inf_{t>r} \psi(t) > -\infty$ *for any* $r > 0$;

2) $\lim_{n\to\infty} \psi(t_n) = -\infty$ *implies* $\lim_{n\to\infty} t_n = 0$

Lemme 1.4.2. *Let $\varphi : (0,\infty) \to (0,\infty)$. So the following two conditions are equivalent:*

1) If $\lim_{n\to\infty} \varphi(t_n) = 0$ *for a bounded sequence* $\{t_n\}$, *then* $\lim t_n = 0$.

2) $\liminf_{t\to r} \varphi(t) > 0$ *for any* $r > 0$.

From the above lemma, it is easy to deduce the following result

Lemme 1.4.3. *Let $\beta : (0,\infty) \to (0,1)$. So the next two conditions are equivalent:*

1) If $\lim_{n\to\infty} \beta(t_n) = 1$ *for a bounded sequence* $\{t_n\}$, *then* $\lim t_n = 0$.

2) $\limsup_{t\to r} \beta(t) < 1$ *for every* $r > 0$.

Remark 1.4.2. In view of Lemma 1.4.3, Proposition 1.2.1 remains valid if we consider $\tau(t) = -ln\beta(t)$, where β the function defined in Lemma 1.4.3 and satisfying either conditions 1) or 2).

1.5 The space of bounded variation *BV*

In this section we remind many relevant properties and advanced results dealing with the space of bounded variation ([15], [33]).

1.5.1 General properties

Definition 1.5.1. $f : [a,b] \to \mathbb{R}$ represents a given function and Γ represents a given partition of $[a,b]$ given by

$$\Gamma = \{a = x_0 < x_1 < ... < x_n = b\}.$$

Let us put

$$V_\Gamma(f) = \sum_{k=0}^{n-1} \left| f(x_{k+1}) - f(x_k) \right|$$

and

$$V_a^b(f)) := var(f;[a,b]) := \sup_\Gamma V_\Gamma(f),$$

so the supremum is caught over all the partitions of $[a,b]$. The quantity $V_a^b(f)$ is defined the total variation (in the sense of Jordan) on $[a,b]$ of f over $[a,b]$.

Definition 1.5.2. A function $f : [a,b] \to \mathbb{R}$ is called to be of bounded variation on $[a,b]$ if $V_a^b(f) < \infty$ and we write $f \in BV([a,b])$.

Theorem 1.5.1. *On $[a,b]$, we have f a given non-decreasing function, so $f \in BV([a,b])$ with*

$$V_a^b(f) = f(b) - f(a).$$

Theorem 1.5.2. *If $f \in BV([a,b])$, then there exist two nondecreasing functions g and h such that $f = g - h$.*

Theorem 1.5.3. *If $f \in BV([a,b])$ and $x \in [a,b]$, so the function*

$$L(x) = V_a^x(f)$$

is a nondecreasing function.

1.5.2 BV space on a unbounded interval

Proposition 1.5.1. *For all $f : [a,+\infty) \to \mathbb{R}$, we find*

$$V_a^{+\infty}(f) = var(f;[a,+\infty)) = \lim_{x\to+\infty} V_a^x(f).$$

Proposition 1.5.2. *Let $h \in BV([a,+\infty))$. Then $\lim_{x\to\infty} h(x)$ exists and is finite.*

Proposition 1.5.3. *Let $h \in BV([a,+\infty))$. The space $BV([a,+\infty))$ endowed with the norm*

$$\|h\|_{BV} := |h(a)| + V_a^{+\infty}(h)$$

is a Banach space.

1.5.3 Linear operators in BV spaces

In this part, let us recall some linear operators on BV spaces and their properties.

I. Multiplication operators

Given a function $r : [a,b] \to \mathbb{R}$, we define the linear operator m_r on $BV([a,b])$ as follows

$$m_r(x)(t) := r(t)\,x(t), \quad t \in [a,b], \quad x \in BV([a,b]).$$

The operator m_r is said the multiplication operator by the function r.

Theorem 1.5.4. *If $r \in BV([a,b])$, then m_r maps $BV([a,b])$ into itself, i.e., $m_r(BV) \subseteq BV$.*

II. Fredholm operators

Given a function $f : [a,b] \times [a,b] \to \mathbb{R}$ and we denote by K the linear integral defined as follows:

$$K(x)(t) = \int_a^b f(t,s)\,x(s)\,ds, \quad t \in [a,b], \quad x \in BV([a,b]).$$

The operator K is presented the Fredholm operator. Next, we recall the following results concerning the operator K.

Theorem 1.5.5. *Assume that the next conditions hold:*

- *1. For every $t \in [a,b] : f(t,.)$ is measurable and $f(a,.) \in \mathbb{L}^1$;*
- $\exists M \in \mathbb{L}^1, \forall s \in [a,b]$ *:*

$$var\left(f(.,s)\right) \leq M(s).$$

Then, the operator K maps $BV([a,b])$ into itself i.e., ($K(BV) \subseteq BV$) and is bounded.

Theorem 1.5.6. *Assume that the subsequent condition is verified:*

$$\forall t \in [a,b] : f(t,.) \in \mathbb{L}^1 \quad (C).$$

So, the above two conditions are equivalent.

(A) f satisfies the following condition:

$$\exists L > 0, \forall \theta \in [a,b] : var\left(\int_a^\theta f(.,s)\, ds\right) \leq L;$$

(B) $K(BV) \subseteq BV$ and is K is bounded.

Remark 1.5.1. Under the hypotheses of the theorem below, we can deduce that

$$\|K\|_{\mathscr{L}((BV))} \leq 2L + \left\|f(a,.)\right\|_{\mathbb{L}^1}.$$

III. Volterra operators

Given a function $g : [a,b] \times [a,b] \to \mathbb{R}$ and we denote by V the linear integral defined as:

$$V(x)(t) = \int_a^t g(t,s)\, x(s)\, ds, \quad t \in [a,b], \quad x \in BV([a,b]).$$

The operator V is defined the Volterra operator. The corresponding properties for the operator V can be deduced from the ones given for operator K with

$$f(t,s) = \begin{cases} g(t,s), & \text{if } a \leq s < t \leq b, \\ 0, & \text{if } a \leq t < s \leq b. \end{cases}$$

Due the particular form of the operator V, we find the next result.

Theorem 1.5.7. *Assume that condition (C) is satisfied for the function g. Moreover, assume that following condition holds*

$$\forall s \in [a,b] : \quad \left|f(s,s)\right| + var\left(f(.,s);[s,1]\right) \leq H(s),$$

where $h \in \mathbb{L}^p$, $p \in [1,\infty)$. Then, for $p \in (1,\infty)$, the operator V maps the space $\mathbb{L}^{\frac{p}{p-1}}$ into the space BV and is bounded. In the case $p = 1$, the operator $V : BV \to BV$ is compact.

1.5.4 Nonlinear operators in BV spaces

In this part, we introduce some nonlinear operators in $BV([a,b])$ space.

I. Composition operators

For some functions $x:[a,b]\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$, the non-linear operator is defined on $BV([a,b])$ by

$$C_h(x)(t) := h(x(t)), \quad t\in[a,b], \quad x\in BV([a,b]).$$

The operator C_h is called the composition (or autonomous) operator generated by the function h.

Theorem 1.5.8. *The next two conditions are equivalent*
1. $h:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz function($h\in Lip_{loc}(\mathbb{R})$), i.e.,

$$\forall R>0, \exists L_R>0, \forall p,q\in[-R,R]: \quad \left|h(p)-h(q)\right| \le L_R\left|p-q.\right|$$

2. The operator C_h maps $BV([a,b])$ into itself, i.e., $C_h(BV)\subseteq BV$.

II. Superposition operators

For some functions $x:[a,b]\to\mathbb{R}$ and $h:[a,b]\times\mathbb{R}\to\mathbb{R}$, we define the nonlinear operator on $BV([a,b])$ by

$$S_h(x)(t) := h(t,x(t)), \quad t\in[a,b], \quad x\in BV([a,b]).$$

The operator S_h is said the superposition (or nonautonomous) operator generated by the function h.

Theorem 1.5.9. *The two conditions below are equivalent.*
1. The mapping $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{aligned} \forall R &> 0, \exists K_R>0, \forall\{t_0,\dots,t_m\} \text{ (a partition of } [a,b]), \\ \forall p_0,\dots,p_n\in[-R,R]: \sum_{k=1}^{n}\left|p_k-p_{k-1}\right| &\le K_R \Rightarrow \sum_{k=1}^{n}\left|h(t_k,p_k)-h(t_{k-1},p_k)\right| \le K_R \\ \text{and} \quad \sum_{k=1}^{n}\left|h(t_{k-1},p_k)-h(t_{k-1},p_{k-1})\right| &\le K_R. \end{aligned}$$

2. The operator S_h maps $BV([a,b])$ into itself, (i.e., $S_h(BV)\subseteq BV$) and is bounded.

Remark 1.5.2. If $h\in Lip_{loc}([a,b]\times\mathbb{R})$, then h satisfies condition (H).

1.6 Henstock-Kurzweil integral

This section is devoted to state the main results concerning Henstock-Kurzweil integrals (see [55],[56]).

Definition 1.6.1. Let be a closed interval I in $\overline{\mathbb{R}}$ and Let $\{I_j\}_{j=1}^{n}$ be a partition of I. A tagged partition of I is a finite collection of pairs $\{(t_j,I_j)\}_{j=1}^{n}$, where $t_j\in I_j$ for each j.

Definition 1.6.2. We propose the function $h : I \to \mathbb{R}$ and $P = \{(t_j, I_j)\}_{j=1}^{n}$ a tagged partition of I. The Riemann sum of h with respect to P is the quantity given as follows

$$R(h,P) := \sum_{j=1}^{n} h(t_j)\, l(I_j),$$

where $l(I_j)$ represents the length of the interval I_j.

Definition 1.6.3. A gauge on I is a function $\rho : I \to (a,b) \subset \overline{\mathbb{R}}$ with $t \in \rho(t)$ for any $t \in I$ with $\rho(t)$ is bounded for finite t. A tagged partition $P = \{(t_j, I_j)\}_{j=1}^{n}$ of I is called ρ-fine if $I_j \subset \rho(t_j)$ for $j = 1, ..., n$.

Definition 1.6.4. Let I be a closed interval in $\overline{\mathbb{R}}$ and $h : I \to \mathbb{R}$ be a given function. h is colled to be Henstock-Kurzweil -integrable on I if there exists $B \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists a gauge ρ on I where

$$|R(h,P) - B| < \varepsilon$$

with P is a ρ-fine of I. Henceforth, we denote by

$$B = \int_I h(x)\, dx$$

the Henstock-Kurzweil integral of h on I, or HK -integral for short.

Theorem 1.6.1. *Let I be a closed interval in $\overline{\mathbb{R}}$ and $h : I \to \mathbb{R}$ be HK -integrable function. Then h is HK -integrable on any closed subinterval of I.*

Theorem 1.6.2. *Let $g : [a, +\infty] \to \mathbb{R}$ be HK -integrable function on each closed and bounded subinterval of $[a, \infty]$. Then HK -integrability of g on $[a, \infty]$ is equivalent to the existence of the limit*

$$L = \lim_{b \to +\infty} \int_a^b g(x)\, dx.$$

In addition, if L exists, one has

$$\int_a^{+\infty} g(x)\, dx = \lim_{b \to +\infty} \int_a^b g(x)\, dx.$$

Theorem 1.6.3. *Let $g : I \to \mathbb{R}$ be HK -integrable function on each closed interval I of $\overline{\mathbb{R}}$. So, the function $G : I \to \mathbb{R}$ defined by*

$$G(x) = \int_a^x g(s)\, ds,$$

where $a \in I$ is a.e. differentiable and

$$G' = g \quad a.e.$$

In the sequel, we introduce the suitable norm (Alexiewicz norm) for the functions h which are HK -integrable on some closed interval $I \subset \overline{\mathbb{R}}$, as follows

$$\|h\|_I^A := \sup_{J \subset I} \left| \int_J h(s)\, ds \right|,$$

where the supremum is taken over closed intervals belonging in the of domain of h.

Now, we make use the following Hölder-type inequality dealing with the HK-sense.

Theorem 1.6.4. *Let h be HK -integrable function on each closed interval I of* $\overline{\mathbb{R}}$ *and* $g \in BV(I \cap \mathbb{R})$. *Then*

$$\left|\int_I h(s)\, g(s)\, ds\right| \leq \left|\int_I h(s)\, ds\right| \inf_{I\cap\mathbb{R}} \left|g(s)\, ds\right| + \|h\|_I^A\, var\left(g; I \cap \mathbb{R}\right)$$

1.7 Denjoy-Perron-type integral

In this part, we remind some properties and new theorems of Denjoy-Perron integral (see [20], [28]).

Definition 1.7.1. On $S \subset I$, let's call the function $g : I \to \mathbb{R}$ by AC^* if for every ε there exists an $\delta > 0$ such as for every all family $\{[c_j, d_j]\}_{j=1}^n$ of pairwise disjoint compact intervals in S where for

$$\sum_{j=1}^{n} \left(c_j - d_j\right) < \delta,$$

the following holds

$$\sum_{j=1}^{n} \sup\left\{\left|g(x) - g\left(y\right)\right| : \quad x, y \in [c_j, d_j]\right\} < \varepsilon.$$

Definition 1.7.2. Let's call the function $g : I \to \mathbb{R}$ by a generalized absolute continuity in the narrow sense on $S \subset I$, or ACG^* for short, if g is continuous on S and S can be expressed as a countable union sets such that g is AC^* on each of them.

Definition 1.7.3. Let $[c, d]$ be a bounded interval in $\mathbb{R}$ and $g : [c, d] \to \mathbb{R}$. the function g is said to be Denjoy-Perron integrable on $[c, d]$, if there exists an ACG^* function $G : [c, d] \to \mathbb{R}$ where

$$G' = g \quad \text{a.e.}$$

Remark 1.7.1. The function G defined above is called the Denjoy-Perron integral of g and

$$G(d) - G(c) = (DP) \int_c^d g(x)\, dx$$

is called the definite Denjoy-Perron integral of g over $[c, d]$.

Henceforth, we will define as $DP([c, d])$ the vector space of all functions DP-integrable.

1.7.1 Connection between HK-integral and DP-integral

Theorem 1.7.1. *Let* $[c, d]$ *be a bounded interval in* $\mathbb{R}$ *and* $g : [c, d] \to \mathbb{R}$. *The function g is HK-integrable on* $[c, d]$ *if and if only it on* $[c, d]$ *we have DP-integrable and*

$$(HK) \int_c^d g(x)\, dx = (DP) \int_c^d g(x)\, dx.$$

In the sequel, we deal with the notion of DP-integrability on unbounded intervals.

Definition 1.7.4. Let $g : [a, +\infty] \to \mathbb{R}$. If g is DP-integrable on each $[a, x]$, where $x > a$ and

$$\lim_{x \to +\infty} (DP) \int_a^x g(x)\, dx$$

exists, we say that g is DP-integrable on $[a, +\infty]$ and we have

$$(DP) \int_a^{+\infty} g(x)\, dx = \lim_{x \to +\infty} (DP) \int_a^x g(x)\, dx.$$

Theorem 1.7.2. *Let $g : [a, +\infty] \to \mathbb{R}$. Then function g is HK-integrable on $[a, +\infty]$ if and if only it is DP-integrable on $[a, +\infty]$ and*

$$(HK) \int_a^{+\infty} g(x)\, dx = (DP) \int_a^{+\infty} g(x)\, dx.$$

Chapter 2

Contents

New fixed point theorems of Dass-Gupta (D-G)-type in b-metric spaces

Abstract

This chapter is devoted to solve some nonlinear integral equations using new fixed point theorems of Dass-Gupta-type in the framework of b-metric spaces.

NB: This chapter is entirely extracted from our paper [62] that is submitted to the scientific authorities of Batna 2 University in order to validate the doctoral thesis defense file.

2.1 Main results

As of right now, if T only has one fixed point, x^*, and the Picard sequence $\{T^n x\}$ converges to x^* for any $x \in X$, then $T : X \to X$ is referred to as a *Picard operator.*

First, we recall the original fixed point theorem of Dass-Gupta.

Theorem 2.1.1. *Consider a complete metric space (X,d) with an operator $T : X \to X$. Let us consider the existence of two real numbers, denoted as α and β, belonging to the interval $[0,1)$ such that* $\alpha + \beta < 1$.

$$d(Tu, Tv) \le \alpha(u,v) + \beta \frac{d(v,Tv)(1 + d(u,Tu))}{1 + d(u,v)}$$

for any $u, v \in X$. So T is a Picard operator.

In the following discussion, we shall denote $\mathscr{L}$ as a set of all functions χ mapping from the open interval $(0,\infty)$ to itself, subject to the after set of conditions:

$$\liminf_{t \to \eta^+} \chi(t) > 0 \quad \text{for all } \eta > 0. \tag{H}$$

Example 2.1.1. **(**[24] and [50]**)**
(a) Assume $\chi > 0$ is a constant real number and $\chi_1(t) = \chi$ for each $t \in (0,\infty)$. Then $\chi_1 \in \mathscr{L}$.
(b) Let $\chi_2(t) = \delta t$ for all $t \in (0,\infty)$, where $\delta > 0$. Then $\chi_2 \in \mathscr{L}$.
(c) Let $\chi_3(t) = e^t$ for all $t \in (0,\infty)$. Then $\chi_3 \in \mathscr{L}$.

Definition 2.1.1. Let (X,σ) be a b-metric space where the constant s greater than 1. The operator $T : X \to X$ is called a (χ, F)*-Dass-Gupta-contraction of type* (A) if there exist two functions $F : (0,\infty) \to \mathbb{R}$ and $\chi : (0,\infty) \to (0,\infty)$ such as for any $x, y \in X$ with $\sigma(Tx, Ty) > 0$, the next condition holds

$$\chi(\sigma(x,y)) + F(\sigma(Tx,Ty)) \le F(m(x,y)), \tag{2.1}$$

with

$$m(x,y) = \max\left\{\sigma(x,y), \frac{\sigma(y,Ty)(1 + \sigma(x,Tx))}{1 + \sigma(x,y)}\right\}.$$

Remark 2.1.1. If T is a (χ,F) If we consider the A-type contraction proposed by Dass-Gupta, with F being a non-decreasing function, we can determine

$$\sigma(Tx,Ty) < m(x,y) \tag{2.2}$$

for any $x, y \in X$ and we have $Tx \ne Ty$.

The following is our initial fixed-point result.

Theorem 2.1.2. *Let (X,σ) be a complete b-metric space with constant $s \ge 1$ and $T : X \to X$ a (χ,F)-Dass-Gupta-contraction of type (A). Suppose:*
(H_1) *F is non-decreasing;*
(H_2) $\chi \in \mathscr{L}$;
(H_3) $\exists\, k \in \left(0, \frac{1}{1+\log_2 s}\right)$ *satisfactory* $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.
So T is a Picard operator.

Proof. The Picard sequence $\{x_n\}$ is considered for an arbitrary point $x_0 \in X$ as the initial point. If there is a natural number n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} can be considered as the fixed point of the transformation T. If $x_n \neq x_{n+1}$ for any n belonging to the set of non-negative integers, then one obtains

$$\sigma_n := \sigma(x_n, x_{n+1}) = \sigma(Tx_{n-1}, Tx_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, by applying (2.1) where $x = x_{n-1}$ and $y = x_n$. Thus, we obtain

$$\begin{aligned} &\chi(\sigma(x_{n-1}, x_n)) + F(\sigma(x_n, x_{n+1})) \\ &\leq F\left(\max\left\{\sigma(x_{n-1}, x_n), \frac{\sigma(x_n, x_{n+1})(1+\sigma(x_{n-1}, x_n))}{1+\sigma(x_{n-1}, x_n)}\right\}\right) \\ &= F(\max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}) \end{aligned} \tag{2.3}$$

for every $n \in \mathbb{N}$.

As a result, (2.3) takes the form

$$\chi(\sigma_{n-1}) + F(\sigma_n) \leq F(\max\{\sigma_{n-1}, \sigma_n\}). \tag{2.4}$$

If there exists $m \in \mathbb{N}$ then

$$\max\{\sigma_{m-1}, \sigma_m\} = \sigma_m.$$

Then from (2.4), we infer that

$$F(\sigma_m) < \chi(\sigma_{m-1}) + F(\sigma_m) \leq F(\sigma_m),$$

a discrepancy. Therefore, for every $n \in \mathbb{N}$,

$$\max\{\sigma_{n-1}, \sigma_n\} = \sigma_{n-1}. \tag{2.5}$$

In view of (2.4), (2.5) and the monotonicity of F, we get

$$\sigma_n < \sigma_{n-1} \quad \text{for every } n \in \mathbb{N}.$$

Hence $\{\sigma_n\}$ is a strictly decreasing sequence. Then, there exists $\sigma \geq 0$ as

$$\lim_{n\to\infty} \sigma_n = \sigma^+. \tag{2.6}$$

First, we show that $\sigma = 0$. Suppose that $\sigma > 0$.

Furthermore, via (2.4) and (2.5), we find the next chain of inequalities

$$\begin{aligned} F(\sigma_n) &\leq F(\sigma_{n-1}) - \chi(\sigma_{n-1}) \\ &\leq F(\sigma_{n-2}) - \chi(\sigma_{n-2}) - \chi(\sigma_{n-1}) \\ &\leq \cdots \\ &\leq F(\sigma_0) - \sum_{i=0}^{n-1} \chi(\sigma_i). \end{aligned} \tag{2.7}$$

Through assumption (H_2) and (2.6), there exist $n_1 \in \mathbb{N}$ and $\mu > 0$ such that

$$\chi(\sigma_n) \geq \mu \quad \text{for all } n \geq n_1.$$

In order to express inequality (2.7), use the format below.

$$\begin{aligned} F(\sigma_n) &\leq F(\sigma_0) - \sum_{i=0}^{n_1-1} \chi(\sigma_i) - \sum_{i=n_1}^{n-1} \chi(\sigma_i) \\ &\leq F(\sigma_0) - \sum_{i=n_1}^{n-1} \chi(\sigma_i) \\ &\leq F(\sigma_0) - \sum_{i=n_1}^{n-1} \mu \\ &= F(\sigma_0) - (n-n_1)\mu \end{aligned} \tag{2.8}$$

for all $n \geq n_1$.

By using (2.6)and (2.8), along with the property of monotonicity of function F, we can derive the following result:

$$F(\sigma) \leq F(\sigma_n) \leq F(\sigma_0) - (n-n_1)\mu \quad \text{for all } n \geq n_1. \tag{2.9}$$

Taking the limit as n approaches infinity in (2.9), the result is

$$F(\sigma) \leq -\infty,$$

The statement presents a contradiction. Therefore, it follows that $\sigma = 0$, indicating that

$$\lim_{n\to\infty} \sigma_n = 0^+. \tag{2.10}$$

Next, by (H_3) and (2.10), there exists a value $k \in \left(0, \frac{1}{1+\log_2 s}\right)$ that is so

$$\lim_{n\to\infty} \sigma_n^k F(\sigma_n) = 0. \tag{2.11}$$

By (2.8), we obtain

$$0 \leq \sigma_n^k (n-n_1)\mu \leq \sigma_n^k F(\sigma_0) - \sigma_n^k F(\sigma_n) \quad \text{for all } n \geq n_1. \tag{2.12}$$

Owing to (2.10), (2.11) and (2.12), we get

$$\lim_{n\to\infty} \sigma_n^k (n-n_1)\mu = 0,$$

which implies

$$\lim_{n\to\infty} n\sigma_n^k = 0.$$

Therefore, that there is $n_2 \in \mathbb{N}$ in which case

$$\sigma_n \leq n^{-\frac{1}{k}} \quad \text{for every } n \geq n_2.$$

Hence,

$$\{\sigma_n\} = \{\sigma(x_n, x_{n+1})\} \in O(n^{-\frac{1}{k}}). \tag{2.13}$$

Since $\frac{1}{k} > 1+\log_2 s$ in (2.13), Lemma 1.2.2 allow us to infer that $\{x_n\}$ can be deduced to be a Cauchy sequence. It therefore results that the sequence $\{x_n\}$, is complete from (X, σ), It can be inferred that the sequence converges to a certain element, x^* in X, which can be denoted as

$$\lim_{n\to\infty} \sigma\left(x_n, x^*\right) = 0. \tag{2.14}$$

Next, we will prove that x^* is a fixed point of T, i.e., $Tx^* = x^*$. Assume $\sigma\left(x^*, Tx^*\right) > 0$. Given (2.14), it can be concluded that there is there a natural number n_3 that corresponds to

$$\sigma\left(x_n, x^*\right) \leq \frac{\sigma\left(x^*, Tx^*\right)}{2s}, \quad \forall n \geq n_3. \tag{2.15}$$

Alternatively, by employing the expression (b_3), we find

$$\sigma\left(x^*, Tx^*\right) \leq s\sigma\left(x^*, Tx_n\right) + s\sigma\left(Tx_n, Tx^*\right). \tag{2.16}$$

Utilizing (2.15), the inequality (2.16) gives

$$\begin{aligned}
\sigma\left(Tx_n, Tx^*\right) &\geq \frac{1}{s}\left(\sigma\left(x^*, Tx^*\right) - s\sigma\left(x^*, Tx_n\right)\right) \\
&= \frac{1}{s}\sigma\left(x^*, Tx^*\right) - \sigma\left(x^*, x_{n+1}\right) \\
&\geq \frac{\sigma\left(x^*, Tx^*\right)}{2s} > 0,
\end{aligned} \tag{2.17}$$

for all $n \geq n_3$. By (2.17), (2.2) is utilized in (2.2), where $x = x^*$ and $y = x_n$. Therefore, the formula labeled as (2.16) transforms into

$$\begin{aligned}
\sigma\left(x^*, Tx^*\right) &\leq s\sigma\left(x^*, Tx_n\right) + s\sigma\left(Tx_n, Tx^*\right). \\
&= s\sigma\left(x^*, Tx_n\right) + s\sigma\left(Tx^*, Tx_n\right). \\
&< s\sigma\left(x^*, Tx_n\right) + sm\left(x^*, x_n\right) \\
&= s\sigma\left(x^*, x_{n+1}\right) + s\max\left\{\sigma\left(x^*, x_n\right), \sigma_n \frac{1+\sigma\left(x^*, Tx^*\right)}{1+\sigma\left(x^*, x_n\right)}\right\}
\end{aligned} \tag{2.18}$$

for every $n \geq n_3$.

Tending to infinit with n in (2.18) and utilizing (2.10) and (2.14) in conjunction, we discover

$$\sigma\left(x^*, Tx^*\right) \leq 0,$$

a contradiction. As a result, $x*$ is a fixed point of T, where $Tx* = x*$.

In conclusion, we demonstrate the uniqueness of T's fixed point. Consider x^* and y^* to be two distinct fixed points of T, i.e., $Tx^* = x^* \neq y^* = Ty^*$. Therefore

$$\sigma\left(Tx^*, Ty^*\right) = \sigma\left(x^*, y^*\right) > 0. \tag{2.19}$$

Based on (2.19), the contractive inequality (2.1) can be derived by substituting $x = x^*$ and $y = y^*$, and we have

$$\chi\left(\sigma\left(x^*, y^*\right)\right) + F\left(\sigma\left(x^*, y^*\right)\right) \leq F(\sigma\left(x^*, y^*\right)),$$

a contradiction with $\chi\left(\sigma\left(x^*, y^*\right)\right) > 0$. Hence, we deduce that $x^* = y^*$ and the demonstration has been completed. □

Remark 2.1.2. Assuming the value of $s = 1$, then assumption (H_3) coincides with (F_3).

Theorem 2.1.2 reduces to the theorem below in the case when $s = 1$.

Corollary 2.1.1. *Consider a complete metric space (X, d), where X is the set and d is the metric. Let T be a self-mapping on X. Let us consider the hypothetical scenario in which there is a nondecreasing function $F : (0,\infty) \to \mathbb{R}$ that satisfies condition (F_3). Additionally, let $\chi \in \mathscr{L}$ be a variable that belongs to the set $\mathscr{L}$. Under these assumptions, for any x and y in the set X such that the distance between Tx and Ty is greater than zero, the following condition holds:*

$$\chi\left(d\left(x,y\right)\right) + F\left(d\left(Tx,Ty\right)\right) \leq F\left(m_d\left(x,y\right)\right),$$

where

$$m_d\left(x,y\right) = \max\left\{d\left(x,y\right), \frac{d\left(y,Ty\right)\left(1+d\left(x,Tx\right)\right)}{1+d\left(x,y\right)}\right\}.$$

So T is a Picard operator.

Corollary 2.1.2. *Consider a complete b-metric space (X,σ) with coefficient $s \geq 1$. Let $T : X \to X$ be a mapping. Let us consider the existence of two functions, denoted as $F : (0,\infty) \to \mathbb{R}$ and $\chi : (0,\infty) \to (0,\infty)$. These functions are subject to the condition that for any x and y belonging to a set X, where $\sigma\left(Tx,Ty\right) > 0$, the following inequality holds:*

$$\chi\left(\sigma\left(x,y\right)\right) + F\left(\sigma\left(Tx,Ty\right)\right) \leq F\left(\sigma\left(x,y\right)\right). \tag{2.20}$$

Furthermore, assume that (H_1), (H_2) and (H_3) are satisfied. So T is a Picard operator.

Proof. Obvious from $\sigma\left(x,y\right) \leq m(x,y)$. □

Remark 2.1.3. Corollary 2.1.2 greatly extend and improve Theorem 1.2.5.

Taking $F(t) = \ln(t)$ and $\chi(t) = \chi$ in Theorem 2.1.2, for some $\chi > 0$, we recover the following result.

Corollary 2.1.3. *([42]) Consider a complete b-metric space (X,σ) with coefficient $s \geq 1$. Let $T : X \to X$ be a mapping that satisfies*

$$\sigma\left(Tx,Ty\right) \leq \lambda m\left(x,y\right) \tag{2.21}$$

for all $x, y \in X$, with $\lambda \in [0,1)$. So T is a Picard operator.

Remark 2.1.4. (1) Explicitly, $\lambda = e^{-\chi} \in [0,1)$ in Corollary 2.1.3.

(2) Corollary 2.1.3 is reduces to Theorem 1.2.1 .

Next, we deduce the corollary below.

Corollary 2.1.4. ***(Dass-Gupta fixed point in b-metric spaces).*** *Consider a complete b-metric space (X,σ) with a constant $s \geq 1$. Let $T : X \to X$ be an operator. Let us consider the existence of two real numbers, α and β, both belonging to the interval $[0,1)$, such that their sum is less than 1. These numbers are chosen such that the following condition holds for all elements x and y in the set X,*

$$\sigma\left(Tx,Ty\right) \leq M\left(x,y\right),$$

where

$$M\left(x,y\right)=\alpha\sigma\left(x,y\right)+\beta\frac{\sigma\left(y,Ty\right)\left(1+\sigma\left(x,Tx\right)\right)}{1+\sigma\left(x,y\right)}. \tag{2.22}$$

So T is a Picard operator.

Proof. Obvious with $M(x,y)\le\lambda m(x,y)$ and $\lambda=\alpha+\beta$, where $\alpha,\beta\in[0,1)$ and $\alpha+\beta<1$. □

Corollary 2.1.5. *Consider a complete b-metric space (X,σ) with a constant $s\ge1$. Let $T:X\to X$ be a mapping. Let us consider the existence of two functions $F:(0,\infty)\to\mathbb{R}$ and $\chi:(0,\infty)\to(0,\infty)$. These functions satisfy the condition that for all $x,y\in X$ with $\sigma\left(Tx,Ty\right)>0$, the following condition holds*

$$\chi\left(\sigma\left(x,y\right)\right)+F\left(\sigma\left(Tx,Ty\right)\right)\le F\left(M\left(x,y\right)\right),$$

where $M\left(x,y\right)$ is given by (2.22) and $\alpha,\beta\in[0,1)$ such that $\alpha+\beta<1$. In addition, we assume that F is nondecreasing. So T is a Picard operator.

Proof. As F is nondecreasing, we get

$$\sigma\left(Tx,Ty\right)\le M\left(x,y\right)$$

for any x and y belonging to the set X, where Tx is equal to Ty. Given that the inequality remains valid when $Tx=Ty$, it can be deduced that the result is evident based on the Corollary 2.1.4. □

Remark 2.1.5. If the sum of α and β is equal to 1, it is not possible to deduce the conclusion of Corollary 2.1.5.

Example 2.1.2. Let us consider

$$X=\left\{x_n=\frac{n\left(n+1\right)}{2},\ n\in\mathbb{N}\right\}$$

and the mapping $\sigma:X\times X\to[0,\infty)$ is defined as $\sigma\left(x,y\right)=\left(x-y\right)^2$. Based on the information provided in Example 1.2.1, it can be inferred that the pair (X,σ) constitutes a complete b-metric space with a coefficient value of $s=2$. Define the mapping $T:X\to X$ as follows:

$$Tx=\begin{cases}x_{n-1}, & \text{if } x=x_n, n\ge2,\\ x_1, & \text{if } x=x_1.\end{cases}$$

It is evident that the function T does not fulfill the contractive condition (2.21) as stated in Corollary 2.1.3 due to properties

$$\lim_{n\to\infty}\frac{\sigma\left(Tx_n,Tx_1\right)}{m\left(x_n,x_1\right)}=\lim_{n\to\infty}\frac{\left(x_{n-1}-x_1\right)^2}{\left(x_n-x_1\right)^2}=\lim_{n\to\infty}\frac{\left(n^2-n-2\right)^2}{\left(n^2+n-2\right)^2}=1.$$

Thus, Corollary 2.1.3 can not be applied.

Now, we prove that T is a $\left(\chi,F\right)$-Dass-Gupta-contraction of type (A). Next, let us remark that for any $n,k\in\mathbb{N}$,

$$\sigma\left(Tx_{n+k},Tx_n\right)>0\Leftrightarrow\left(k\ge2\wedge n=1\right)\vee\left(k\in\mathbb{N}\wedge n>1\right).$$

Case 1. For any $k \geq 2$ and $n = 1$, we get

$$\begin{aligned}
\sigma(Tx_{k+1}, Tx_1) - m(x_{k+1}, x_1) &\leq \sigma(x_k, x_1) - \sigma(x_{k+1}, x_1) \\
&= (x_k - x_1)^2 - (x_{k+1} - x_1)^2 \\
&= \left(\frac{k^2 + k - 2}{2}\right)^2 - \left(\frac{k^2 + 3k}{2}\right)^2 \\
&= -(k+1)\left(k^2 + 2k - 1\right) \\
&\leq -21 \leq -\frac{1}{\sigma(x_{k+1}, x_1) + 1}.
\end{aligned}$$

Case 2. For any $k \in \mathbb{N}$ and $n > 1$, we obtain

$$\begin{aligned}
\sigma(Tx_{n+k}, Tx_n) - m(x_{n+k}, x_n) &\leq \sigma(x_{n+k-1}, x_{n-1}) - \sigma(x_{n+k}, x_n) \\
&= (x_{n+k-1} - x_{n-1})^2 - (x_{n+k} - x_n)^2 \\
&= \frac{k^2}{4}\left((2n + k - 1)^2 - (2n + k + 1)^2\right) \\
&= -k^2(2n + k) \\
&\leq -5 \leq -\frac{1}{\sigma(x_{n+k}, x_n) + 1}.
\end{aligned}$$

Based on the aforementioned cases, it can be observed that T is a (χ, F)-Dass-Gupta-contraction of type (A) with $\chi(t) = \dfrac{1}{1+t}$ and $F(t) = t$ for all $t > 0$. Moreover, all the conditions outlined in Theorem 2.1.2 have been satisfied. Therefore, the operator T possesses a distinct and singular fixed point denoted as $x^* = x_1 = 1$. Furthermore, it should be noted that the function F does not fulfill assumption (F_2').

Remark 2.1.6. Through Example 2.1.2, Corollary 2.1.3 greatly generalize Theorem 2.1.2.

Example 2.1.3. Let us consider

$$X = \left\{ x_n = \frac{1}{2^{\frac{n-1}{2}}\sqrt{n}}, n \in \mathbb{N} \right\} \cup \{0\}$$

and $\sigma : X \times X \to [0, \infty)$ be the function given by $\sigma(x, y) = (x - y)^2$.

Define $T : X \to X$

$$Tx = \begin{cases} x_{n+1}, & \text{if } x = x_n,\ n \in \mathbb{N}, \\ 0, & \text{if } x = 0. \end{cases}$$

Now, we are going to establish that Theorem 1.2.5 is not applicable.

Assuming that all the conditions stated in Theorem 1.2.5 are satisfied. Therefore, there exists a nondecreasing function $F : (0, \infty) \to \mathbb{R}$ and a positive real number $\tau > 0$ such that for any x equal to x_n for some $n \in \mathbb{N}$ and y equal to 0, the following holds:

$$\tau + F(2\sigma(Tx_n, T0)) \leq F(\sigma(x_n, 0)),$$

which leads to

$$\tau + F\left(2x_{n+1}^2\right) \leq F\left(x_n^2\right) \quad \text{for all } n \in \mathbb{N}. \tag{2.23}$$

Putting $\alpha_n = x_{n+1}^2$. Hence we get

$$\tau + F\left(2^n \alpha_n\right) \leq F\left(2^{n-1} \alpha_{n-1}\right) \quad \forall n \in \mathbb{N}.$$

Setting $r_n = 2^n \alpha_n$, one gets

$$\sum_{n=1}^{p} \tau \leq \sum_{n=1}^{p} \left[F\left(r_{n-1}\right) - F\left(r_n\right)\right], \quad p \in \mathbb{N},$$

which implies

$$\tau \leq \frac{1}{p}\left(F\left(r_0\right) - F\left(r_p\right)\right), \quad p \in \mathbb{N},$$

or, equivalently

$$\tau \leq \frac{1}{p}\left(F(1) - F\left(\frac{1}{p+1}\right)\right). \tag{2.24}$$

According to condition (F_3), there exists a value k within the interval $(0,1)$ such thatAccording to condition (F_3), there exists a value k within the interval $k \in (0,1)$ that corresponds to

$$\lim_{p \to \infty} \frac{1}{(p+1)^k} F\left(\frac{1}{p+1}\right) = 0.$$

Consequently,

$$\lim_{p \to \infty} \frac{1}{p} F\left(\frac{1}{p+1}\right) = \lim_{p \to \infty} \frac{(p+1)^k}{p} \frac{1}{(p+1)^k} F\left(\frac{1}{p+1}\right) = 0.$$

Letting $p \to \infty$ in (2.24), we obtain $\tau \leq 0$, which is a contradiction. Hence, inequality (F) from Theorem 1.2.5 is not verified.

In this section, we will demonstrate that the mapping T satisfies the properties of a (χ, F)-Dass-Gupta-contraction of type (A). In this analysis, we shall consider the following case study:

Case 1. If $x = x_n, n \in \mathbb{N}$ with $y = 0$, we get

$$\begin{aligned}
\sqrt{\sigma(Tx_n, T0)} - \sqrt{\sigma(x_n, 0)} &= x_{n+1} - x_n \\
&= \sqrt{\frac{n}{2(n+1)}} x_n - x_n \\
&\leq \left(\frac{1}{\sqrt{2}} - 1\right) x_n \\
&= \left(\frac{1}{\sqrt{2}} - 1\right) \sqrt{\sigma(x_n, 0)}.
\end{aligned}$$

Case 2. If $x = x_{n+k}$ with $y = x_n$ for any $n, k \in \mathbb{N}$. Given that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence that is decreasing in nature, the following holds:

$$\sqrt{\sigma\left(Tx_{n+k}, Tx_n\right)} - \sqrt{\sigma\left(x_{n+k}, x_n\right)} = |x_{n+k+1} - x_{n+1}| - |x_{n+k} - x_n|$$

$$= x_{n+1} - x_n - x_{n+k+1} + x_{n+k}$$

$$= \left(\sqrt{\frac{n}{2(n+1)}} - 1\right) x_n - \left(\sqrt{\frac{n+k}{2(n+k+1)}} - 1\right) x_{n+k}$$

$$\leq \left(\sqrt{\frac{n}{2(n+1)}} - 1\right) (x_n - x_{n+k})$$

$$\leq \left(\frac{1}{\sqrt{2}} - 1\right) |x_{n+k} - x_n|$$

$$= \left(\frac{1}{\sqrt{2}} - 1\right) \sqrt{\sigma\left(x_{n+k}, x_n\right)}.$$

Based on the aforementioned cases, it can be observed that T exhibits characteristics of a (χ, F)-Dass-Gupta-contraction of type (A), where $\chi(t) = \left(1 - \frac{1}{\sqrt{2}}\right)\sqrt{t}$ and $F(t) = \sqrt{t}$ for all $t > 0$. Moreover, it can be observed that all the conditions stated in Corollary 2.1.2 are verified. Therefore, the operator T possesses only one fixed point denoted as $x^* = 0$. It can be observed that the function F does not verify assumption (F_2').

Inspired by Remark 2.1.5, we will try to analyse Corollary 2.1.5 whenever $\alpha + \beta = 1$.

Definition 2.1.2. Consider a b-metric space (X, σ) where a constant $s \geq 1$. A mapping $T : X \to X$ is defined as a (χ, F) mapping if it verifies the next conditions: the contraction of type (B), known as Dass-Gupta contraction, is characterized by the existence of a nondecreasing function $F : (0, \infty) \to \mathbb{R}$ and $\chi \in \mathcal{L}$. This contraction satisfies the next assumption for all $x, y \in X$ with $d(Tx, Ty) > 0$:

$$\chi(\sigma(x, y)) + F(\sigma(Tx, Ty)) \leq F\left(\mathcal{R}_{\alpha,\beta}(x, y)\right), \tag{2.25}$$

where

$$\mathcal{R}_{\alpha,\beta}(x, y) = \alpha\sigma(x, y) + \beta\frac{\sigma(y, Ty)(1 + \sigma(x, Tx))}{1 + \sigma(x, y)} \tag{2.26}$$

with $\alpha, \beta \geq 0$.

Remark 2.1.7. Immediately we get

$$\sigma(Tx, Ty) < \mathcal{R}_{\alpha,\beta}(x, y) \tag{2.27}$$

for every $x, y \in X$ where $Tx \neq Ty$.

Given a real number s such that $s \geq 1$, we consider the following set

$$\mathcal{B}_{\alpha,\beta} = \left\{(\alpha, \beta) \in \mathbb{R}^2 : 0 < \alpha < \frac{1}{s^2}, 0 < \beta < 1, \alpha + \beta = 1\right\}.$$

Next, we will prove the theorem below.

Theorem 2.1.3. *Let (X,σ) be a pair consisting of a set X and a σ. Consider a complete b-metric space X with a constant $s \geq 1$. Let $T : X \to X$ be a mapping that satisfies the conditions (χ, F). The contraction of Dass-Gupta, specifically of type (B), Let (α, β) be an element of the set $\mathscr{B}_{\alpha,\beta}$. So T is a Picard operator.*

Proof. Let us make the assumption, without any loss of generality, that for every natural number n, the value of x_n is not equal to the value of x_{n+1}. Therefore,

$$\sigma_n = \sigma(x_n, x_{n+1}) = \sigma(Tx_{n-1}, Tx_n) > 0 \quad \text{for all } n \in \mathbb{N}.$$

Therefore, by utilizing (2.25) with the substitution $x = x_{n-1}$ and $y = x_n$, we derive

$$\chi(\sigma_{n-1}) + F(\sigma_n) \leq F(\alpha\sigma_{n-1} + \beta\sigma_n) \quad \text{for all } n \in \mathbb{N}. \tag{2.28}$$

By virtue of the property of monotonicity possessed by the function F, we obtain

$$\sigma_n < \alpha\sigma_{n-1} + \beta\sigma_n \quad \text{for every } n \in \mathbb{N}. \tag{2.29}$$

Under assumptions of our theorem, (2.29) turns into

$$\sigma_n < \sigma_{n-1} \quad \text{for every } n \in \mathbb{N}. \tag{2.30}$$

Therefore, there exists a non-negative value $\sigma \geq 0$ which means

$$\lim_{n\to\infty} \sigma_n = \sigma^+.$$

Next, we prove that $\sigma = 0$. Assume on the contrary, $\sigma > 0$. As F is nondecreasing, we find

$$\lim_{t\to r^+} F(t) = F(r+0) = F(r^+) \quad \text{for every } r \in (0,\infty). \tag{2.31}$$

Substituting (2.30) into (2.28) using the monotonicity of F with $\alpha + \beta = 1$, one gets

$$\chi(\sigma_{n-1}) + F(\sigma_n) \leq F(\sigma_{n-1}) \quad \text{for all } n \in \mathbb{N}. \tag{2.32}$$

Bearing in mind (2.31), we obtain

$$\begin{aligned}
\liminf_{t\to\sigma^+} \chi(t) &\leq \liminf_{n\to\infty} \chi(\sigma_{n-1}) \\
&\leq \lim_{n\to\infty} (F(\sigma_{n-1}) - F(\sigma_n)) \\
&= F(\sigma^+) - F(\sigma^+) \\
&= 0,
\end{aligned}$$

which contradicts (H). Hence,

$$\lim_{n\to\infty} \sigma_n = 0^+. \tag{2.33}$$

We'll demonstrate that $\{x_n\}$ is a Cauchy sequence. However, $\{x_n\}$ is not Cauchy. By (2.33) and Lemma 1.2.3, there exists $\varepsilon > 0$ and two sequences $\{m(k)\}, \{n(k)\}$ of positive integers that match

$$\varepsilon \leq \liminf_{k\to\infty} \sigma(x_{m(k)}, x_{n(k)}) \leq \limsup_{k\to\infty} \sigma(x_{m(k)}, x_{n(k)}) \leq s\varepsilon.$$

Then, there exists $k_1 \in \mathbb{N}$ such that $\{\sigma(x_{m(k)}, x_{n(k)})\}$ is bounded for all $k \geq k_1$ and thus it has a convergent subsequence. Consequently, there exist a real number μ and a subsequence $\{k(j)\}_{j\geq k_1}$ of $\{k\}_{k\geq k_1}$ that satisfies

$$\lim_{j\to\infty} \sigma\left(x_{m(k(j))}, x_{n(k(j))}\right) = \mu, \tag{2.34}$$

where

$$0 < \varepsilon \leq \liminf_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq \mu \leq \limsup_{k\to\infty} \sigma\left(x_{m(k)}, x_{n(k)}\right) \leq s\varepsilon. \tag{2.35}$$

Via the fact that $\alpha < \dfrac{1}{s^2}$ with $\mu > 0$, the equation below

$$\beta s^2 t^2 + 2s^2 t - \mu\left(1 - \alpha s^2\right) = 0, \quad t \in \mathbb{R}, \tag{2.36}$$

has a positive root, noted q_s.

As $q_s > 0$, (2.33) yields that there exist $j_1 \geq k_1, j_2 \geq k_1$ satisfying

$$\begin{aligned} \sigma_{m(k(j))} &= \sigma\left(x_{m(k(j))}, x_{m(k(j))+1}\right) \leq q_s \quad \text{for all } j \geq j_1, \\ \sigma_{n(k(j))} &= \sigma\left(x_{n(k(j))}, x_{n(k(j))+1}\right) \leq q_s \quad \text{for all } j \geq j_2. \end{aligned} \tag{2.37}$$

Using (2.34) with $q_s > 0$, we infer that there exists $j_3 \geq k_1$ satisfying

$$\sigma\left(x_{m(k(j))}, x_{n(k(j))}\right) \leq \mu + q_s \quad \text{for all } j \geq j_3. \tag{2.38}$$

Besides, by (b_3), we get

$$\begin{aligned} &\sigma\left(x_{m(k(j))}, x_{n(k(j))}\right) \\ &\leq s\sigma\left(x_{m(k(j))}, x_{m(k(j))+1}\right) + s\sigma\left(x_{m(k(j))+1}, x_{n(k(j))}\right) \\ &\leq s\sigma\left(x_{m(k(j))}, x_{m(k(j))+1}\right) + s^2\sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) \\ &+ s^2\sigma\left(x_{n(k(j))}, x_{n(k(j))+1}\right) \\ &= s\sigma_{m(k(j))} + s^2\sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) + s^2\sigma_{n(k(j))} \end{aligned}$$

for all $j \geq k_1$.

This implies

$$\begin{aligned} &\sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) \\ &\geq \frac{1}{s^2}\left(\sigma\left(x_{m(k(j))}, x_{n(k(j))}\right) - s\sigma_{m(k(j))} - s^2\sigma_{n(k(j))}\right) \end{aligned} \tag{2.39}$$

for all $j \geq k_1$.

By taking limit inferior as $j \to \infty$ in (2.39) with (2.33) and (2.34), we arrive at

$$\liminf_{j\to\infty} \sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) \geq \frac{\mu}{s^2}. \tag{2.40}$$

Therefore, in view of (2.40) and $q_s > 0$, there exists $j_4 \geq k_1$ such that

$$\sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right) > \frac{\mu}{s^2} - q_s \quad \text{for all } j \geq j_4. \tag{2.41}$$

Hence

$$\frac{\mu}{s^2} - q_s \geq \frac{\mu}{s^2}\left(1-\alpha s^2\right) - q_s = \beta q_s^2 + q_s > 0. \tag{2.42}$$

Consequently, (2.41) and (2.42) imply that

$$\sigma\left(Tx_{m(k(j))}, Tx_{n(k(j))}\right) > 0 \quad \text{for all } j \geq j_4. \tag{2.43}$$

Putting $u_j = \sigma\left(x_{m(k(j))}, x_{n(k(j))}\right)$ and $v_j = \sigma\left(x_{m(k(j))+1}, x_{n(k(j))+1}\right)$, then by (2.43), inequality (2.25) with $x = x_{m(k(j))}$ and $y = x_{n(k(j))}$ turns into

$$\chi\left(u_j\right) + F\left(v_j\right) \leq F\left(\alpha u_j + \beta \sigma_{n(k(j))}\left(\frac{1+\sigma_{m(k(j))}}{1+u_j}\right)\right) \quad \text{for all } j \geq j_4. \tag{2.44}$$

We set $N = \max\{j_1, j_2, j_3, j_4\}$. Then, using (2.37), (2.38), (2.41), (2.42) and (2.44), we obtain

$$\begin{aligned}
\chi\left(u_j\right) + F\left(\frac{\mu}{s^2} - q_s\right) &\leq F\left(\alpha\left(\mu + q_s\right) + \beta q_s\left(1+q_s\right)\right) \\
&= F\left(\alpha\mu + q_s + \beta q_s^2\right) \\
&= F\left(\alpha\mu + \frac{\mu}{s^2}\left(1-\alpha s^2\right) - q_s\right) \\
&= F\left(\frac{\mu}{s^2} - q_s\right)
\end{aligned}$$

For all values of j greater than or equal to N, this statement contradicts the given information.

As (X, σ) is complete b-metric space, there exists x^* in X such that

$$\lim_{n\to\infty} \sigma\left(x_n, x^*\right) = 0. \tag{2.45}$$

Similarly as in the proof of Theorem 2.1.2, we derive that there exists $n_4 \in \mathbb{N}$ such that for all $n \geq n_4$,

$$\sigma\left(Tx_n, Tx^*\right) > 0. \tag{2.46}$$

By utilizing equation (2.46), one can apply inequality (2.27) in the case where x is equal to x^* and y is equal to x_n. Therefore, we have

$$\begin{aligned}
\sigma\left(x^*, Tx^*\right) &\leq s\sigma\left(x^*, Tx_n\right) + s\sigma\left(Tx_n, Tx^*\right) \\
&= s\sigma\left(x^*, Tx_n\right) + s\sigma\left(Tx^*, Tx_n\right) \\
&< s\sigma\left(x^*, Tx_n\right) + s\mathscr{R}_{\alpha,\beta}\left(x^*, x_n\right) \\
&= s\sigma\left(x^*, x_{n+1}\right) + s\alpha\sigma\left(x^*, x_n\right) + s\beta\sigma_n\frac{1+\sigma\left(x^*, Tx^*\right)}{1+\sigma\left(x^*, x_n\right)}
\end{aligned} \tag{2.47}$$

for all $n \geq n_4$.

Taking the limit as n approaches infinity in (2.47) and combining it with (2.14) and (1.7), we obtain

$$\sigma\left(x^*, Tx^*\right) \leq 0,$$

which is a contradiction. Therefore, x^* is a fixed point of T. Let us consider the scenario where x^* and y^* represent two distinct fixed points of the transformation T. Then

$$\sigma\left(Tx^*, Ty^*\right) = \sigma\left(x^*, y^*\right) > 0. \tag{2.48}$$

From (2.48), we deduce that

$$\begin{aligned} \chi\left(\sigma\left(x^*, y^*\right)\right) + F\left(\sigma\left(x^*, y^*\right)\right) &\leq F(\alpha\sigma\left(x^*, y^*\right)) \\ &\leq F(\sigma\left(x^*, y^*\right)), \end{aligned}$$

a contradiction. Therefore, we have proved the uniqueness of the fixed point of T. □

Remark 2.1.8. Assumptions (F_2'), (H_3) and the strictness of the monotonicity of F are not needed in Theorem 2.1.3.

Taking $\chi(t) = \ln(t + \alpha + 1)$ (with $0 < \alpha < \frac{1}{s^2}$) and $F(t) = \ln(t)$, we deduce the following corollary.

Corollary 2.1.6. *Consider a complete b-metric space (X, σ), where $s \geq 1$ is a constant and let $T: X \to X$ be a mapping*

$$\sigma\left(Tx, Ty\right) \leq \frac{\mathscr{R}_{\alpha,\beta}\left(x, y\right)}{\sigma\left(x, y\right) + \alpha + 1},$$

where $\left(\alpha, \beta\right) \in \mathscr{B}_{\alpha,\beta}$. So T is a Picard operator.

Example 2.1.4. Let $X = [\frac{1}{3}, 5]$ and the function $d: X \times X \to [0, \infty)$ provided by

$$d\left(x, y\right) = \begin{cases} \max\left\{x, y\right\}, & x \neq y, \\ 0, & x = y. \end{cases}$$

For all x and y belonging to the set X. Consider the metric space (X, d), which we assert to be complete (see [30]).

Define $\sigma: X \times X \to [0, \infty)$ as follows

$$\sigma\left(x, y\right) = \left(d\left(x, y\right)\right)^2 \quad \text{for } x, y \in X.$$

The function $T: X \to X$ is defined as follows:

$$Tx = \begin{cases} \frac{1}{3}, & x \in [\frac{1}{3}, 3) \cup (3, 5], \\ \frac{2}{3}, & x = 3. \end{cases}$$

First, we observe that

$$\sigma(Tx,Ty)=\frac{4}{9}>0 \Leftrightarrow \left[\left(x=3\wedge y\in[\frac{1}{3},3)\cup(3,5]\right)\vee\left(y=3\wedge x\in[\frac{1}{3},3)\cup(3,5]\right)\right].$$

For $x,y\in X$, designate

$$\mathscr{R}_{\frac{1}{8},\frac{7}{8}}(x,y)=\frac{1}{8}\sigma(x,y)+\frac{7}{8}\frac{\sigma(y,Ty)(1+\sigma(x,Tx))}{1+\sigma(x,y)}.$$

In both two cases $\left[\left(x=3\wedge y\in[\frac{1}{3},3)\cup(3,5]\right)\vee\left(y=3\wedge x\in[\frac{1}{3},3)\cup(3,5]\right)\right]$, we easily obtain

$$\mathscr{R}_{\frac{1}{8},\frac{7}{8}}(x,y)\geq\frac{1}{8}\sigma(x,y)=\frac{1}{8}\left(\max\{x,y\}\right)^2\geq\frac{9}{8}.$$

Further, using the inequality

$$h+\frac{1}{h}\geq 2 \quad \text{for all } h>0,$$

we obtain

$$\begin{aligned}\frac{\sigma(x,y)}{26}+\frac{(-1)^q-1}{2(\sigma(Tx,Ty))^2}+\frac{((-1)^q+1)\sigma(Tx,Ty)}{2}&\leq\frac{\sigma(x,y)}{26}+\sigma(Tx,Ty)\\&\leq\frac{25}{26}+\frac{4}{9}\leq 2\\&\leq\mathscr{R}_{\frac{1}{8},\frac{7}{8}}(x,y)+\frac{1}{\mathscr{R}_{\frac{1}{8},\frac{7}{8}}(x,y)}\end{aligned} \tag{2.49}$$

From $\sigma(Tx,Ty)=\frac{4}{9}<1$ and $\mathscr{R}_{\frac{1}{8},\frac{7}{8}}(x,y)\geq\frac{9}{8}>1$, one can consider $\chi:(0,\infty)\to(0,\infty)$ given by $\chi(t)=\frac{t}{26}$ and definition of $F:(0,\infty)\to\mathbb{R}$

$$F(t)=\begin{cases}\frac{(-1)^q-1}{2t^2}+\frac{((-1)^q+1)t}{2}, & \text{if } 0<t\leq 1, \quad q\in\mathbb{N}_0,\\ t+\frac{1}{t}, & \text{if } t>1.\end{cases}$$

Therefore, T is a (χ,F)-Dass-Gupta-contraction of type (B) and all the conditions of Theorem 2.1.3 are fulfilled for $\alpha=\frac{1}{8}$ and $\beta=\frac{7}{8}$. Hence, T has a fixed point x^*.

2.2 Applications

2.2.1 Nonlinear Fredholm integral equation in a space endowed with a complete b-metric

Let $W=\mathscr{C}([0,1];[0,\infty))$ and the function $d:W\times W\to[0,\infty)$ defined by

$$d_\infty(x,y)=\sup_{t\in[0,1]}\left|x(t)-y(t)\right| \quad \text{for every } x,y\in W.$$

For some $p > 1$, we define

$$\sigma_\infty(x, y) = (d_\infty(x, y))^p = \sup_{t\in[0,1]} |x(t) - y(t)|^p \quad \text{for every } x, y \in W. \tag{2.50}$$

According to Example 1.2.1, the pair (W, σ_∞) can be classified as a $s = 2^{p-1}$A b-metric space that is complete.

We are going now to discuss the solvability of a particular nonlinear Fredholm integral equation given by:

$$u(t) = \int_0^1 G(t, r) f(r, u(r))\,\mathrm{d}r, \quad u \in W,\ t \in [0, 1], \tag{2.51}$$

where $f : [0, 1] \times [0, \infty) \to \mathbb{R}$ and

$$G(t, r) = \begin{cases} t(1-r), & 0 \le t \le r \le 1, \\ r(1-t), & 0 \le r \le t \le 1. \end{cases}$$

Theorem 2.2.1. *Let us consider an important assumption:*
(A) *For any $r \in [0, 1]$ and for any $z, w \in [0, \infty)$,*

$$|f(r, z) - f(r, w)| \le \mu e^r |z - w|,$$

where

$$\mu := \frac{4}{\left(\frac{1}{p} + \frac{1}{p-1}\right)^{\frac{1}{p}} e}. \tag{2.52}$$

Then (2.51) *admits a unique solution in W.*

Proof. Let $T : (W, \sigma_\infty) \to (W, \sigma_\infty)$ be the mapping defined as follows

$$(Tu)(t) = \int_0^1 G(t, r) f(r, u(r))\,\mathrm{d}r, \quad u \in W,\ t \in [0, 1].$$

Setting

$$q := \frac{p}{p-1} > 1, \quad \text{i.e.,} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Suppose now that $u, v \in W$ where $Tu \neq Tv$ and $t \in [0, 1]$. By Hölder inequality with condition (A), we find

$$\begin{aligned} |(Tu)(t) - (Tv)(t)|^p &\le \left(\int_0^1 G(t, r)^q\,\mathrm{d}r\right)^{\frac{p}{q}} \int_0^1 |f(r, u(r)) - f(r, v(r))|^p\,\mathrm{d}r \\ &\le \mu^p \left(\sup_{t\in[0,1]} \int_0^1 G(t, r)^q\,\mathrm{d}r\right)^{\frac{p}{q}} \int_0^1 e^{pr} |u(r) - v(r)|^p\,\mathrm{d}r \\ &\le \mu^p \sigma_\infty(u, v) \left(\sup_{t\in[0,1]} \int_0^1 G(t, r)^q\,\mathrm{d}r\right)^{\frac{p}{q}} \int_0^1 e^{pr}\,\mathrm{d}r \\ &= \frac{\mu^p \sigma_\infty(u, v)}{p} (e^p - 1) \left(\sup_{t\in[0,1]} \int_0^1 G(t, r)^q\,\mathrm{d}r\right)^{\frac{p}{q}}. \end{aligned} \tag{2.53}$$

A straightforward calculation, we arrive at

$$\int_0^1 G(t,r)^q \, \mathrm{d}r = \frac{t^q (1-t)^q}{q+1}$$

and so

$$\sup_{t\in[0,1]} \int_0^1 G(t,r)^q \, \mathrm{d}r = \frac{1}{q+1} \frac{1}{2^{2q}} = \frac{1}{\frac{p}{p-1}+1} \frac{1}{2^{2\frac{p}{p-1}}}.$$

Hence, (2.53) turns into

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)|^p &\le \frac{\mu^p \sigma_\infty(u,v)}{p} \left(e^p - 1\right) \left[\frac{1}{\dfrac{p}{p-1}+1} \frac{1}{2^{\frac{2p}{p-1}}} \right]^{p-1} \\
&= \frac{\mu^p \sigma_\infty(u,v)}{p} \left(e^p - 1\right) \frac{1}{\left(\dfrac{p}{p-1}+1\right)^{p-1}} \frac{1}{2^{2p}} \\
&\le \frac{1}{2^{2p}} \frac{\mu^p \sigma_\infty(u,v)}{p} \left(e^p - 1\right) \frac{1}{\left(\dfrac{p}{p-1}+1\right)^{p-1}} \\
&\le \frac{\sigma_\infty(u,v)(e^p-1)}{e^p \left(\dfrac{p}{p-1}+1\right)} \frac{1}{\left(\dfrac{p}{p-1}+1\right)^{p-1}} \\
&\le \sigma_\infty(u,v) \left(1 - e^{-p}\right) \\
&= \sigma_\infty(u,v) - \sigma_\infty(u,v) e^{-p}.
\end{aligned}$$

Therefore

$$\sigma_\infty(Tu, Tv) \le \sigma_\infty(u,v) - \sigma_\infty(u,v) e^{-p},$$

or, equivalently,

$$\sigma_\infty(u,v) e^{-p} + \sigma_\infty(Tu, Tv) \le \sigma_\infty(u,v).$$

As a result, for any $t > 0$ $F(t) = t$ and $\chi(t) = e^{-p} t$ satisfy all of the conditions of Corollary 2.1.2 . As a result, T possesses a special fixed point, u^* in $\mathscr{C}([0,1];[0,\infty))$. □

Example 2.2.1. Let us consider $r \in [0,1]$, $u \in W$ and μ given by (2.52). Obviously, assumption (A) is satisfied for the following function

$$f(r,u) = \frac{\mu e^r u}{1+u},$$

satisfies .

2.2.2 Nonlinear Volterra integral equation in a space endowed with a complete b-metric

This section deals with the nonlinear Volterra integral equation as follows:

$$x(t) = h(t) + \int_0^t G(t,r) f(r, x(r)) \, \mathrm{d}r, \quad t \in I, \tag{2.54}$$

with $I=[0,\lambda]$ and $\lambda>0$, $f: I\times\mathbb{R}\to\mathbb{R}$, $G: I\times I\to\mathbb{R}$ and $h: I\to\mathbb{R}$ are functions.

Let $X=\mathscr{C}(I;\mathbb{R})$ be the space of all continuous functions $x: I\to\mathbb{R}$. For every $x\in\mathscr{C}(I;\mathbb{R})$ and fixed arbitrary $\tau>0$, we introduce the following norm

$$\|x\|_\tau=\sup_{t\in I} e^{-\tau t}|x(t)|.$$

From [49], the space $(X,\|.\|_\tau)$ is a space of Banach . Therefore, X endowed with the metric d_τ defined by

$$d_\tau(x,y)=\sup_{t\in I} e^{-\tau t}\left|x(t)-y(t)\right| \quad \text{for every } x,y\in X,$$

with the given space is a metric space that satisfies all the necessary conditions for completeness.

Next, we define

$$\sigma_\tau(x,y)=\left(d_\tau(x,y)\right)^2=\sup_{t\in I} e^{-2\tau t}(x(t)-y(t))^2 \quad \text{for all } x,y\in X. \tag{2.55}$$

We immediately have, (X,σ_τ) is a complete $s=2$-metric space.

Theorem 2.2.2. *Assume that the given conditions are fulfilled:*

(A_1) *h is a continuous function;*

(A_2) *G is a continuous function and there exist $\tau>0$ and $K>0$ satisfying*

$$\sup_{t\in I}\int_0^t |G(t,r)|\, e^{\tau(r-t)}\mathrm{d}r\le K; \tag{2.56}$$

(A_3) *the function f is continuous and there exists a constant $\alpha\in(0,\frac{1}{4})$ that satisfies for any $r\in I$ and for any $z,w\in\mathbb{R}$,*

$$\left|f(r,z)-f(r,w)\right|\le\frac{\sqrt{\alpha}\,|z-w|}{K\sqrt{1+\alpha+(z-w)^2}}. \tag{2.57}$$

So the integral equation (2.54) *admits a unique solution in X.*

Proof. Let $T:(X,\sigma_\tau)\to(X,\sigma_\tau)$ be the mapping defined as follows:

$$(Tx)(t)=h(t)+\int_0^t G(t,r)\,f(r,x(r))\,\mathrm{d}r, \quad x\in X,\ t\in I.$$

Let $x, y \in X$ with $Tx \neq Ty$. By assumptions (A_2) and (A_3), one can get

$$\begin{aligned}\left|(Tx)(t)-\left(Ty\right)(t)\right| &\le \int_0^t |G(t,r)|\left|f(r,x(r))-f\left(r,y(r)\right)\right|\mathrm{d}r\\ &\le \frac{\sqrt{\alpha}}{K}\int_0^t |G(t,r)|\frac{\left|x(r)-y(r)\right|}{\sqrt{1+\alpha+\left(x(r)-y(r)\right)^2}}\mathrm{d}r\\ &\le \frac{\sqrt{\alpha}}{K}\int_0^t |G(t,r)|\frac{\left|x(r)-y(r)\right|e^{-\tau r}e^{\tau r}}{\sqrt{1+\alpha+\left(x(r)-y(r)\right)^2 e^{-2\tau r}}}\mathrm{d}r\\ &\le \frac{\sqrt{\alpha}}{K}\frac{d_\tau\left(x,y\right)}{\sqrt{1+\alpha+\sigma_\tau\left(x,y\right)}}\int_0^t |G(t,r)|e^{\tau r}\mathrm{d}r\\ &\le \frac{\sqrt{\alpha}}{K}\frac{d_\tau\left(x,y\right)e^{\tau t}}{\sqrt{1+\alpha+\sigma_\tau\left(x,y\right)}}\sup_{t\in I}\int_0^t |G(t,r)|e^{\tau(r-t)}\mathrm{d}r\\ &\le \frac{\sqrt{\alpha}d_\tau\left(x,y\right)e^{\tau t}}{\sqrt{1+\alpha+\sigma_\tau\left(x,y\right)}}.\end{aligned}$$

This leads to

$$((Tx)(t)-\left(Ty\right)(t))^2 e^{-2\tau t} \le \frac{\alpha\sigma_\tau\left(x,y\right)}{1+\alpha+\sigma_\tau\left(x,y\right)}.$$

Hence

$$\sigma\left(Tx,Ty\right) \le \frac{\alpha\sigma_\tau\left(x,y\right)}{1+\alpha+\sigma_\tau\left(x,y\right)}, \tag{2.58}$$

which further implies that

$$\sigma_\tau\left(Tx,Ty\right) \le \frac{\mathscr{R}^\tau_{\alpha,\beta}\left(x,y\right)}{\sigma_\tau\left(x,y\right)+\alpha+1},$$

where

$$\mathscr{R}^\tau_{\alpha,\beta}\left(x,y\right) = \alpha\sigma_\tau\left(x,y\right)+\beta\frac{\sigma_\tau\left(y,Ty\right)(1+\sigma_\tau(x,Tx))}{1+\sigma_\tau\left(x,y\right)}$$

with $\beta \in (0,1)$ satisfies $\alpha+\beta=1$.

Therefore, the equation labeled as (2.54) admits only one solution in the function space $X = \mathscr{C}(I;\mathbb{R})$. □

Example 2.2.2. In this study, we will examine the initial value problem as presented in the reference [51].

$$\begin{cases} \dfrac{\mathrm{d}^2u}{\mathrm{d}t^2}+\dfrac{k}{m}\dfrac{\mathrm{d}u}{\mathrm{d}t} = f(t,u(t)), \quad t\in I=[0,2\pi],\\ u(0)=0,\ u'(0)=a, \end{cases} \tag{2.59}$$

with $k>0$ and $a\in\mathbb{R}$.

The choice of the function f as stated by

$$f(t,u) = \frac{1}{8\pi^2}\max\{\sin(t),\frac{|u|}{1+u^2}\},\ t\in I,\ u\in\mathscr{C}(I;\mathbb{R}).$$

Let us observe that problem (2.59) is equivalent to solve the next integral equation .

$$u(t) = \int_0^t G(t,r)\, f(r,u(r))\,\mathrm{d}r, \quad t \in I \tag{2.60}$$

where $G : I \times I \to \mathbb{R}$ is the function of Green given by

$$G(t,r) = \begin{cases} (t-r)\, e^{\tau(t-r)}, & 0 \le r \le t \le 2\pi, \\ 0, & 0 \le t \le r \le 2\pi, \end{cases}$$

where $\tau > 0$.

First, assumption (A_1) is immediately holds with $h(t) = 0$ for every $t \in I$.

Next, since

$$\sup_{t\in I} \int_0^t |G(t,r)|\, e^{\tau(r-t)}\mathrm{d}r = \sup_{t\in I} \frac{t^2}{2} = 2\pi^2,$$

we deduce that hypothesis (A_2) holds with $K = 2\pi^2$.

It remains to check the condition (A_3). Let $z, w \in \mathbb{R}$ and utilizing the following inequality (refer to [38])

$$|\max\{a,b\} - \max\{c,d\}| \le \max\{|a-c|, |b-d|\},\ a,b,c,d \in \mathbb{R},$$

we get

$$\begin{aligned}
\left| f(r,z) - f(r,w) \right| &\le \frac{1}{8\pi^2} \left| \max\{\sin(r), \frac{|z|}{1+z^2}\} - \max\{\sin(r), \frac{|w|}{1+w^2}\} \right| \\
&\le \frac{1}{8\pi^2} \max\{0, \left| \frac{|z|}{1+z^2} - \frac{|w|}{1+w^2} \right| \} \\
&= \frac{1}{8\pi^2} \left| \frac{|z|}{1+z^2} - \frac{|w|}{1+w^2} \right| \\
&\le \frac{1}{8\pi^2} \frac{|z-w|\,|1-|zw||}{\left(1+z^2\right)\left(1+w^2\right)} \\
&= \frac{1}{8\pi^2} \frac{|z-w|\,|1-|zw||}{\sqrt{\left(1+z^2\right)\left(1+w^2\right)}\sqrt{\left(1+z^2\right)\left(1+w^2\right)}} \\
&\le \frac{1}{8\pi^2} \frac{|z-w|\,|1-|zw||}{\sqrt{(1+zw)^2}\sqrt{1+\frac{1}{2}(z-w)^2}} \\
&\le \frac{1}{2\pi^2\sqrt{8}} \frac{|z-w|}{\sqrt{2+(z-w)^2}} \\
&\le \frac{1}{2\pi^2\sqrt{8}} \frac{|z-w|}{\sqrt{\frac{9}{8}+(z-w)^2}}.
\end{aligned}$$

Hence, assumption (A_3) holds with $\alpha = \frac{1}{8}$ and $K = 2\pi^2$.

Chapter 3

Contents

Integral equations via Krasnosel'skii fixed point-type

Abstract

This chapter allows us to establish a new theorem concerning a fixed point result for a generalized expansive mappings. The aforementioned results looks like Krasnosel'skii fixed point. More precisely, we propose a slight improvement of the work [54] and we give an application to rational integral equations to illustrate the validity of our results.

3.1 Preliminaries and auxiliary results

In this section, we would like to draw attention to several crucial tools and present supporting evidence that will be utilized in the subsequent analysis.

Let us consider the set $\mathbb{S}$, which comprises all functions $\beta : (0,\infty) \to (0,1)$ that adhere to the aforementioned assumption:

$$\limsup_{t\to r^+} \beta(t) < 1, \quad \text{for all } r > 0$$

Proposition 3.1.1. *Let $\beta \in \mathbb{S}$. Let (E,d) be a complete metric space and $S : E \to E$ a self-map that fulfills*

$$d\left(Sx,Sy\right) \leq \frac{d\left(x,y\right)}{1-d\left(x,y\right)\ln\left(\beta\left(d\left(Sx,Sy\right)\right)\right)}. \tag{3.1}$$

This convergence property holds regardless of the initial point chosen in the set E. The uniqueness of the fixed point x^* on S guarantees that the sequence $\left\{S^n x_0\right\}_{n\in\mathbb{N}}$ will always converge to the same value, providing a stable and predictable behavior for the system.

Proof. Applying Proposition 1.2.1 with $F(t) = -\frac{1}{t}$ and $\tau(t) = -\ln\left(\beta(t)\right)$, it is easy to get (3.1). Indeed, we have

$$\begin{aligned}
\liminf_{t\to\eta^+} \tau(t) &= \liminf_{t\to\eta^+} \left(-\ln\left(\beta(t)\right)\right) \\
&= -\limsup_{t\to\eta^+} \left(\ln\left(\beta(t)\right)\right) \\
&= -\ln\left(\limsup_{t\to\eta^+} \beta(t)\right) > 0
\end{aligned}$$

□

For the sake of convenience, we would like to bring to mind an additional lemma (refer to Lemma 1.4.3 in Chapter 1).

Lemme 3.1.1. *Let $\beta : (0,\infty) \to (0,1)$. The two conditions stated below are equivalent:*

1) *If $\lim_{n\to\infty} \beta(t_n) = 1$ for a bounded sequence $\{t_n\}$, then $\lim t_n = 0$.*
2) $\limsup_{t\to r} \beta(t) < 1$ *for every $r > 0$.*

Remark 3.1.1. In view of previous lemma, Proposition 3.1.1 remains valid if the function β satisfying either conditions 1) or 2).

3.2 Krasnosel'skii fixed point-type

The objective of this section is to investigate the presence and singularity of the fixed point for various types of generalized expansive mappings. In other words, we generalize and improve some results existing in the paper [54].

For the given metric space (X,d) and subset $M \subset X$, the expansive mappings $T: M \to E$, i.e., satisfying the following state

$$d\left(Tx,Ty\right) \geq hd\left(x,y\right)$$

For every of $x,y \in M$ and some $h>1$.

The authors in [54] , proposed the following condition

(A) : T is injective and exists $H>0$ such that for any $x,y \in M$, $H>0$ holds. $x \neq y$, the subsequent holds:

$$H \leq \frac{1}{d\left(x,y\right)} - \frac{1}{d\left(Tx,Ty\right)}. \tag{3.2}$$

Let $\mathbb{L}$ be the family of all functions $\beta : (0,\infty) \to (0,1)$ satisfying the following properties:

(L_1) For a bounded sequence $\{t_n\}$, we obtain

$$\lim_{n\to\infty} \beta\left(t_n\right) = 1 \Leftrightarrow \lim t_n = 0.$$

(L_2)

$$\limsup_{t\to\infty} \beta\left(t\right) < 1.$$

In our work, we propose the following assumption

$\left(A^*\right)$: Let T be an injective function and let β be an element of the set $\mathbb{L}$. For all x and y in the set M, where x is not equal to y, the subsequent condition holds:

$$\beta(d(x,y)) \geq e^{\frac{1}{d(Tx,Ty)} - \frac{1}{d(x,y)}}. \tag{3.3}$$

Remark 3.2.1. Taking $\beta(t) = e^{-H}$ with $H>0$ in (3.3), we obtain (3.2).

Theorem 3.2.1. *Let M be a nonempty closed subset of a complete metric space (X,d). If a mapping $T: M \to E$ satisfies the condition $\left(A^*\right)$ and M is a subset of $T(M)$, then the mapping T possesses a single fixed point.*

Proof. The condition denoted as $\left(A^*\right)$ is indicative of the existence of the inverse $T^{-1} : T(M) \to M$ for the mapping T. Taking $a,b \in T(M)$ such that $a \neq b$, one can find $x,y \in M, x \neq y$ with $Tx = a, Ty = b$. By $\left(A^*\right)$, there exists $\beta \in \mathbb{L}$ such that

$$\beta(d(x,y)) \geq e^{\frac{1}{d(Tx,Ty)} - \frac{1}{d(x,y)}}, \tag{3.4}$$

or, equivalent to

$$-\ln(\beta\left(d\left(T^{-1}a, T^{-1}b\right)\right)) \leq \frac{1}{d(T^{-1}a, T^{-1}b)} - \frac{1}{d\left(a,b\right)}. \tag{3.5}$$

Due to the fact that $M \subset T(M)$, the above inequality leads to

$$d\left(Sa,Sb\right) \leq \frac{d\left(a,b\right)}{1 - d\left(a,b\right)\ln\left(\beta\left(d\left(Sa,Sb\right)\right)\right)}, \quad \forall a,b \in M. \tag{3.6}$$

with $S = T^{-1}$

Since M is closed, we obtain that M is a complete subspace of E. Applying Proposition 3.1.1, we deduce the existence of a unique $x^* \in M$ and hence $Tx^* = x^*$. □

Example 3.2.1. Let $c > 1$ and consider $E = [0, c]$ equipped with the Euclidean metric. Let $\mu \in]1, c]$, $M = [0, 1]$ and $Tx = \mu x$, $x \in M$. Then T satisfies $\left(A^*\right)$ and $M \subset T(M)$.

It is evident that the mapping $T : M \to E$ is well-defined due to the fact that $\mu \in]1, c]$ and T is injective. Also, we get, $M = [0, 1] \subset [0, \mu] = T(M)$.

In contrast, it holds true that for every pair of distinct elements x and y belonging to the set M;

$$\begin{aligned} \frac{1}{|Tx - Ty|} - \frac{1}{|x - y|} &= \frac{1}{\mu |x - y|} - \frac{1}{|x - y|} \\ &= \left(\frac{1}{\mu} - 1\right) \frac{1}{|x - y|} \\ &\leq \left(\frac{1}{\mu} - 1\right) |x - y|. \end{aligned}$$

The last inequality holds through the fact that $|x - y| \leq 1$.

Therefore, we acquire

$$\beta(|x - y|) \geq e^{\left(\frac{1}{|Tx - Ty|} - \frac{1}{|x - y|}\right)},$$

where $\beta : (0, \infty) \to (0, 1)$ is a function defined as $\beta(t) = e^{Ct}$, where $C = \left(\frac{1}{\mu} - 1\right) < 0$. It is evident that the element β belongs to the set $\mathbb{L}$. Therefore, the assumption denoted as A^* is fulfilled.

Remark 3.2.2. Consider a metric space (E, d) and a non-empty subset M of E. If the linear transformation $T : M \to E$ satisfies the condition $\left(A^*\right)$, then the set M is bounded.

Proof. First, we make the assumption that M is unbounded. In this case, we can conclude that the supremum of the distance function over all pairs (x, y) in M is equal to infinity, denoted as $\sup_{(x,y) \in M} d(x, y) = \infty$. Consequently, there exist two sequences (x_n) and (y_n) contained in M, where x_n is not equal to y_n, such that:

$$\lim_{n \to \infty} d(x_n, y_n) = \infty,$$

that is

$$\lim_{n \to \infty} \frac{1}{d(x_n, y_n)} = 0.$$

Next, since T is injective, $Tx_n \neq Ty_n$ for all n and hence, by $\left(A^*\right)$, one has:

$$\begin{aligned} \ln(\beta(d(x_n, y_n))) &> \ln(\beta(d(x_n, y_n))) - \frac{1}{d(Tx_n, Ty_n)} \\ &\geq -\frac{1}{d(x_n, y_n)}, \quad \forall n, \end{aligned}$$

or, equivalently,

$$\beta(d(x_n, y_n)) > e^{-\frac{1}{d(x_n, y_n)}}, \quad \forall n.$$

By defining the upper limit as $nrightarrowinfty$, we gain

$$\limsup_{t\to\infty}\left(\beta(t)\right) \geq \limsup_{n\to\infty}\beta\left(d\left(x_n, y_n\right)\right) \geq 1,$$

a contradiction. □

Proposition 3.2.1. *Let us Consider a normed space* $(E, \|.\|)$, $\phi \neq M \subset M$ *and let* $T : M \to E$ *verify* (A^*) *then the inverse :* $(I-T)^{-1} : (I-T)(M) \to M$ *exists and is continuous.*

Proof. Let x and y be elements of the set M, where $x \neq y$. We will denote B as the difference between the identity matrix I and the matrix T. Via (A^*), we get for $\beta \in \mathbb{L}$

$$1 + \|x-y\| \ln(\beta(\|x-y\|)) \geq \frac{\|x-y\|}{\|Tx-Ty\|} > 0, \tag{3.7}$$

However, it is important to consider the alternative perspective:

$$0 < \|Tx-Ty\| \leqslant \|Bx-By\| + \|x-y\|.$$

Combining (3.7) and the last inequality, we deduce

$$\frac{1}{\|Bx-By\|+\|x-y\|} \leq \frac{1}{\|Tx-Ty\|} \leq \frac{1+\|x-y\|\ln(\beta(\|x-y\|))}{\|x-y\|}.$$

Again, through (3.7), we obtain

$$\|Bx-By\| \geq \frac{-\|x-y\|^2 \ln(\beta(\|x-y\|))}{1+\|x-y\|\ln(\beta(\|x-y\|))}.$$

Therefore for all $x, y \in M$, one gets

$$\|Bx-By\| \geq -\|x-y\|^2 \ln(\beta(\|x-y\|)). \tag{3.8}$$

F is immediately injective ; otherwise, for some $x \neq y$ and $Fx = Fy$, we get

$$-\|x-y\|^2 \ln\left(\beta\left(\|x-y\|\right)\right) \leq 0,$$

a contradiction.

Consequently, B is invertible. Moreover, through the fact that $\beta \in \mathbb{L}$, (3.8) and for every $x, y \in (I-T)(M)$, it can be observed that there is $C > 0$ that is so

$$\|B^{-1}x - B^{-1}y\|^2 \leq C\|x-y\|, \tag{3.9}$$

which ensures the the continuity of B^{-1}. □

Theorem 3.2.2. *Consider a nonempty closed convex subset* M *of a Banach space* E. *Let* T *and* S *be mappings from* M *to* E *that satisfy the following conditions:*

1. *S is continuous and the set $S(M)$ is contained within a compact subset of the space E.*

2. *T satisfies* (A^*)

3. *for every* $u \in S(M)$*, the inclusion* $M \subset u + T(M)$ *holds.*

Subsequently, there exists an element z *belonging to the set* M *such that the equation* $Sz + Tz = z$ *holds true.*

Proof.
First, let us observe that for any $u \in S(K)$ the mapping $T + u$, due to (2) and (3), satisfies: all the assumptions of Theorem 3.2.1. Thus, the equation

$$Ty + u = y \tag{3.10}$$

has a unique solution $y \in M$. Let ω a function such that for any $u \in S(M)$ which corresponds $y \in M$ and the above equation holds. Therefore for every $u \in S(M)$ we have:

$$T(\omega(u)) + u = \omega(u) \tag{3.11}$$

Consider $u_1, u_2 \in S(M)$.Without loss of generality, it can be assumed that $\omega(u_1) \neq \omega(u_2)$. From the fact that T satisfies (A^*) and since $\beta \in \mathbb{L}$, we have

$$-\ln(\beta(\|u_1 - u_2\|)) \leq \frac{1}{\|\omega(u_1) - \omega(u_2)\|} - \frac{1}{\|T(\omega(u_1)) - T(\omega(u_2))\|} \tag{3.12}$$

Hence,

$$\|\omega(u_1) - \omega(u_2)\| \leq \frac{\|T(\omega(u_1)) - T(\omega(u_2))\|}{1 - \ln(\beta(\|u_1 - u_2\|))\,\|T(\omega(u_1)) - T(\omega(u_2))\|}. \tag{3.13}$$

Further, in view of $T(\omega(u)) + u = \omega(u)$, the following holds:

$$\|T(\omega(u_1)) - T(\omega(u_2))\| \leq \|\omega(u_1) - \omega(u_2)\| + \|u_1 - u_2\|. \tag{3.14}$$

Thus, we obtain

$$\begin{aligned}
\|\omega((u_1)) - \omega(u_2)\| &\leq \frac{\|\omega(u_1) - \omega((u_2))\| + \|u_1 - u_2\|}{1 - \ln(\beta(\|u_1 - u_2\|))\,\|\omega(u_1) - \omega(u_2)\| - \ln(\beta(\|u_1 - u_2\|))\,\|u_1 - u_2\|} \\
&= \frac{\|\omega(u_1) - \omega(u_2)\|}{1 - \ln(\beta(\|u_1 - u_2\|))\,\|\omega(u_1) - \omega(u_2)\| - \ln(\beta(\|u_1 - u_2\|))\,\|u_1 - u_2\|} \\
&+ \frac{\|u_1 - u_2\|}{1 - \ln(\beta(\|u_1 - u_2\|))\,\|\omega(u_1) - \omega(u_2)\| - \ln(\beta(\|u_1 - u_2\|))\,\|u_1 - u_2\|} \\
&\leq \frac{\|\omega(u_1) - \omega(u_2)\|}{1 - \ln(\beta(\|u_1 - u_2\|))\,\|\omega(u_1) - \omega(u_2)\|} + \frac{\|u_1 - u_2\|}{1 - \ln(\beta(\|u_1 - u_2\|))\,\|u_1 - u_2\|}
\end{aligned}$$

Consequently, we get

$$\frac{-\ln(\beta(\|u_1 - u_2\|))\,\|\omega(u_1) - \omega(u_2)\|^2}{1 - \ln(\beta(\|u_1 - u_2\|))\,\|\omega(u_1) - \omega(u_2)\|} \leq \frac{\|u_1 - u_2\|}{1 - \ln(\beta(\|u_1 - u_2\|))\,\|u_1 - u_2\|}, \tag{3.15}$$

which leads to have $\|u_1 - u_2\| \to 0$ implies $\|\omega(u_1) - \omega(u_2)\| \to 0$, and therefore ω is continuous on $S(M)$. By the continuity of S, consequently, it can be inferred that the mapping denoted as $\omega S : M \to M$ exhibits continuity. Moreover, $(\omega S)(K)$ resides in a compact subset of E. Through to Schauder's fixed point theorem, there exists $z \in M$ such that $\omega(S(z)) = z$. Through $T(\omega(u)) + u = \omega(u)$ one has

$$T(\omega(S(z))) + S(z) = \omega(S(z)), \tag{3.16}$$

which allows us to conclude that $Tz + Sz = z$. □

3.3 Nonlinear integral equation of rational-type

Herein, we deal with the following integral equation including delay rational type.

$$u(t) = \frac{a(t)\, u(t-r)}{1 - b(t)\, u(t-r)} + \int_{-\infty}^{t} k(t-s)\, h(u(s))\, ds, \tag{3.17}$$

where $r > 0$ is fixed, the mappings $a, b : \mathbb{R} \to \mathbb{R}$ are continuous and periodic with period $T > 0$, $h, k : \mathbb{R} \to \mathbb{R}$ are continuous. In this analysis, we will examine the following assumptions.

(H_1)

$$b_m = \min_{t\in[0,T]} b(t) > 0, b_M = \max_{t\in[0,T]} b(t) < 1;$$

(H_2)

$$a(t) \geq (rb(t)+1)^2 + 2rb(t)$$

for all $t \in \mathbb{R}$, where

$$0 < r := \frac{b_m}{a_M} \leq 1$$

and

$$a_M := \max_{t\in[0,T]} a(t);$$

(H_3)

$$L := \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} |k(t-s)|\, ds < \infty$$

and

$$d := \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \left|k'(t-s)\right| ds < \infty$$

for all $t \in \mathbb{R}$;

(H_4)

$$LM_h \leq \frac{ra_M + r^2 b_M - r}{1 - rb_M},$$

where

$$M_h := \max_{|t|\leq r} h(t).$$

Let us consider the Banach space E of all continuous periodic self-mappings on $\mathbb{R}$ with period $T > 0$, For any $u \in E$, we denote by $\|u\|$ the supremum norm over $[0, T]$, i.e.,

$$\|u\| = \sup_{t\in[0,T]} |u(t)|.$$

Let M denote a subset of E that is defined by

$$M := \{u \in E; \|u\| \leq r\},$$

where r is from condition (H_2).

Theorem 3.3.1. *If the conditions $(H_1)-(H_4)$ are fulfilled then equation has a $T-$periodic solution.*

Proof. For $u \in M$, we set

$$(Tu)(t) = \frac{a(t)\,u(t-r)}{1-b(t)\,u(t-r)}$$

and

$$(Su)(t) = \int_{-\infty}^{t} k(t-s)\,h(u(s))\,ds.$$

Using (H_1), we immediately have $1-b(t)\,u(t-r) > 0$ for all $t \in \mathbb{R}$ and $u \in M$. The problem is equivalent to finding a fixed point of the equation

$$u = Tu + Su.$$

Let us consider $u, v \in M$ with $Tu \neq Tv$. Let $t \in [0,T]$, then we get

$$\begin{aligned}
&\frac{1}{|u(t)-v(t)|} - \frac{1}{|(Tu)(t+r)-(Tv)(t+r)|}\\
=\ &\frac{1}{|u(t)-v(t)|} - \frac{(1-b(t+r)\,u(t))\,(1-b(t+r)\,v(t))}{a(t+r)\,|u(t)-v(t)|}\\
=\ &\frac{b(t+r)}{a(t+r)}\,\frac{\frac{a(t+r)-1}{b(t+r)} + u(t)+v(t)-b(t+r)\,u(t)\,v(t)}{|u(t)-v(t)|}
\end{aligned}$$

Utilizing (H_2), we obtain

$$\begin{aligned}
&\frac{1}{|u(t)-v(t)|} - \frac{1}{|(Tu)(t+r)-(Tv)(t+r)|}\\
\geq\ & r\,\frac{r^2 b(t+r) - b(t+r)\,u(t)\,v(t) + 4r + u(t) + v(t)}{|u(t)-v(t)|}\\
\geq\ & r\,\frac{4r + u^2(t) + v^2(t)}{|u(t)-v(t)|}\\
\geq\ & \frac{4r^2}{|u(t)-v(t)|} \geq \|u-v\|.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\frac{\|Tu-Tv\|}{1+\|u-v\|\,\|Tu-Tv\|} &\geq \frac{|(Tu)(t+r)-(Tv)(t+r)|}{1+\|u-v\|\,|(Tu)(t+r)-(Tv)(t+r)|}\\
&\geq |u(t)-v(t)|
\end{aligned}$$

for all $t \in [0,T]$. Consequently, we find

$$\frac{\|Tu-Tv\|}{1+\|u-v\|\,\|Tu-Tv\|} \geq \|u-v\|.$$

Let us remark that T is one-to-one mapping on M. So, $\left(A^*\right)$ is satisfied for $\beta(t) = e^{-t}$.

Conversely and in order to check that (1) of Theorem 3.2.2, we follow the same method developed in [61] and we demonstrate that S is fully continuous. We only highlight the main points of the proof.

Let $z \in M$ then $\|z\| \leq r$ and

$$\|(Sz)(t)\| \leq \left\| \int_{-\infty}^{t} k(t-s)\, h(z(s))\, ds \right\| \leq LM_h.$$

First, a basic change of variable allows us to get :

$$\begin{aligned} (Sz)(t+T) &= \int_{-\infty}^{t+T} k(t+T-s)\, h(z(s))\, ds \\ &= \int_{-\infty}^{t} k(t-s)\, h(z(s))\, ds = (Sz)(t) \end{aligned}$$

From above, we infer that the operator S maps K into E and $S(M)$ is uniformly bounded.

Next, for $z \in M$, differentiating $(Sz)(t)$ with respect the variable t (using Leibniz formula), we obtain

$$(Sz)'(t) = k(0)\, h(z(t)) + \int_{-\infty}^{t} k'(t-s)\, h(z(s))\, ds,$$

which yields

$$\left\|(Sz)'\right\| \leq (|k(0)| + d)\, M_h.$$

Through the mean value theorem, the above inequality implies that $S(M)$ is an equicontinuous subset of E. Then using the Ascoli-Arzela Theorem, we obtain that S is a compact mapping.

On the other hand, as h is a continuous function, we have

$$\|Sz\| \leq C(M)\, \|z\|,$$

where $C(M)$ is postive constant which depends on M. Thus, S is continuous. Consequently, $S : M \to E$ is completely continuous and hence the condition (1) of Theorem 3.2.2 is satisfied.

Finally, we are going to prove that inclusion (3) in Theorem 3.2.2 holds. Let $z \in S(M)$. Hence, there exists $w \in M$ with $z(t) = (Sw)(t)$ for all $t \in [0, T]$. Take $u \in M$. In view of (H_4), we get

$$\begin{aligned} \|u - z\| &\leq \|u\| + \|Sw\| \\ &\leq r + LM_h \\ &\leq \frac{r a_M}{1 - r b_M}, \end{aligned}$$

which indicates

$$u - z \in B(0, R),$$

where

$$B(0, R) := \left\{ y \in E; \left\| y \right\| \leq R \right\}$$

with

$$R := \frac{r a_M}{1 - r b_M}.$$

Initially, we note that for every $y \in M$,

$$\|Ty\| \leq R,$$

which yields that $y \in B(0, R)$.

On the other hand, taking any $y \in E$ such that $\|y\| \leq R$, and setting

$$\theta(t) = \frac{y(t+r)}{a(t+r) + b(t+r)\, y(t+r)}$$

for $t \in [0, T]$, we get $\|\theta\| \leq R$ and

$$(T\theta)(t) = \frac{a(t)\theta(t-r)}{1 - b(t)\theta(t-r)} = y(t).$$

Hence $\theta \in T(M)$. Therefore

$$T(M) = B(0, R).$$

This yields $u - z \in T(M)$. Thus the condition (3) of Theorem 3.2.2 is satisfied and the proof is completed. □

Chapter 4

Contents

Proinov contraction of Darbo-type

Abstract

In this chapter, a generalization of Darbo fixed point using the Proinov contraction and the measure of noncompactness is provided. Our results furnish some improvements of some results existing in the literature.

NB: This chapter is a join work with master's students **Hadjer Boudjemaa and Amel Laiche** from Khenchela university and which will be submitted in a near futur.

4.1 Proinov contraction of Darbo-type

In this section, we present the concept of Proinov contraction of Darbo-type utilizing the measure of noncompactness μ. Additionally, we establish the existence of a fixed point within a bounded, closed, and convex subset of a Banach space X.

In the following discussion, unless explicitly specified, the symbol D denotes a nonempty closed, bounded, and convex subset of a Banach space X.

Definition 4.1.1. A self-operator that operates continuously. The mapping $T : D \to D$ is considered a Proinov contraction of Darbo-type if there exist two mappings $\varphi, \psi : (0,\infty) \to \mathbb{R}$ such that for any nonempty subset Z of D, the following condition is satisfied:

$$\psi\left(\mu(TZ)\right) \leq \varphi\left(\mu(Z)\right) \quad \text{with} \quad \mu(Z), \mu(TZ) > 0. \tag{4.1}$$

4.2 Proinov-Darbo fixed point

Theorem 4.2.1. *Let us consider a continuous Proinov contraction $T : D \to D$ of Darbo-type. We assume that at least one of the following three conditions is satisfied:*

(i)

$$\limsup_{t\to r} \varphi(t) < \liminf_{t\to r} \psi(t) \quad \textit{for all} \quad r > 0;$$

(ii) *$\varphi(t) < \psi(t)$ for any $t > 0$, ψ nondecreasing and*

$$\limsup_{t\to r^+} \varphi(t) < \psi\left(r^+\right) \quad \textit{for all} \quad r > 0;$$

(iii) *the expression $\varphi(t) < \psi(t)$ if and only if $t > 0$ and the following conditions hold:*

(a)

$$\inf_{t>r} \psi(t) > -\infty \quad \textit{for all} \quad r > 0,$$

(b) *If $\{\psi(t_n)\}$ and $\{\varphi(t_n)\}$ are two convergent sequences with the same limit for a bounded sequence $\{t_n\}$, then $t_n \to 0$ as $n \to \infty$.*

Thus, T admits a minimum of one fixed point in D.

Proof. We shall proceed to establish a sequence denoted as $\{D_n\}$, where the initial term is defined as $D_0 = D$ and

$$D_n = \overline{co(TD_{n-1})}, \quad \text{for } n \geq 1.$$

We are going to prove that

$$D_{n+1} \subseteq D_n \quad \text{and } TD_n \subseteq D_n \quad \text{for any } n \in \mathbb{N}.$$

For the first inclusion, we use the mathematical induction. From the fact that $D_0 = D$ and D is convex and closed with $T : D \to D$, we obtain

$$D_1 = \overline{co(TD_0)} = \overline{co(TD)} \subseteq D = D_0.$$

Now suppose that $D_n \subseteq D_{n-1}$ for $n \geq 1$. It follows that

$$D_{n+1} = \overline{co(TD_n)} \subseteq \overline{co(TD_{n-1})} = D_n$$

and we are done.

From the first inclusion $D_{n+1} \subseteq D_n$, we immediately get

$$TD_n \subseteq \overline{co(TD_n)} = D_{n+1} \subseteq D_n.$$

Hence the second inclusion is proved.

Next, we distinguish two cases:

Case 1. If $n_0 \in \mathbb{N}$ is a non-negative integer, then $\mu\left(D_{n_0}\right) = 0$. As a result, D_{n_0} is a condensed set in X. The theorem of Schauder states that since $TD_{n_0} \subseteq D_{n_0}$, T has a fixed point in $D_{n_0} \subseteq D$.

Case 2. If for any $n \in \mathbb{N}$, $\mu(D_n) > 0$. Based on the properties of the noncompactness measure and the contractive inequality (4.1), we deduce that

$$\begin{aligned} \psi\left(\mu(D_{n+1})\right) &= \psi\left(\mu\left(\overline{co(TD_n)}\right)\right) \\ &= \psi\left(\mu(TD_n)\right) \leq \varphi\left(\mu(D_n)\right), \quad \text{for any } n \in \mathbb{N}. \end{aligned} \tag{4.2}$$

In contrast, it can be observed that $0 \leq \mu(D_{n+1}) \leq \mu(D_n)$, thereby indicating that the sequence $\{\mu(D_n)\}$ exhibits a decreasing trend and is bounded from below. Therefore, there exists a non-negative real number r such that

$$\lim_{n\to\infty} \mu(D_n) = r. \tag{4.3}$$

Now let us prove that $r = 0$. Assume $r > 0$.

Suppose that condition (i) holds. From (4.2) and (4.3), we get

$$\begin{aligned} \liminf_{t\to r} \psi(t) &\leq \liminf_{n\to\infty} \psi\left(\mu(D_n)\right) \\ &\leq \limsup_{n\to\infty} \varphi\left(\mu(D_n)\right) \\ &\leq \limsup_{t\to r} \varphi(t), \end{aligned}$$

a contradiction. Hence $r = 0$ and $\lim_{n\to\infty} \mu(D_n) = 0$.

If condition (ii) holds. In view of the fact that $\varphi(t) < \psi(t)$ for any $t > 0$, the monotonicity of ψ and (4.2), we obtain that $\{\mu(D_n)\}$ is a strictly decreasing sequence and boubded below. Therefore, there exists a non-negative real number r that is such

$$\lim_{n\to\infty} \mu(D_n) = r^+.$$

Again from (4.3), we have

$$\begin{aligned} \psi\left(r^+\right) &= \lim_{n\to\infty} \psi\left(\mu(D_n)\right) \\ &\leq \limsup_{n\to\infty} \varphi\left(\mu(D_n)\right) \\ &\leq \limsup_{t\to r^+} \varphi(t), \end{aligned}$$

a contraction. Therefore $\lim_{n\to\infty} \mu(D_n) = 0$.

Now assume that condition (iii) holds. In this case, we consider the following subcases.

Subcase 1. If $\{\psi(\alpha_n)\}$ is not bounded below, where $\alpha_n := \mu(D_n)$. It follows that

$$\lim_{n\to\infty} \psi(\alpha_n) = -\infty.$$

Due to Lemma 1.4.1, we obtain $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \mu(D_n) = 0$.

Subscase 2. If $\{\psi(\alpha_n)\}$ is bounded below. Utilizing the fact that $\varphi(t) < \psi(t)$ for any $t > 0$, we get

$$\psi(\alpha_{n+1}) \leq \varphi(\alpha_n) < \psi(\alpha_n).$$

This implies that $\{\psi(\alpha_n)\}$ is strictly decreasing. Then, $\{\psi(\alpha_n)\}$ is a convergent sequence and therefore $\{\varphi(\alpha_n)\}$ is also a convergent sequence with the same limit. Consequently, we obtain

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \mu(D_n) = 0.$$

In conclusion, in each case, we have obtained that $\lim_{n\to\infty} \mu(D_n) = 0$. It follows from property (6) in Definition 1.3.1, that

$$D_\infty = \bigcap_{n=1}^{\infty} D_n$$

is nonempty and compact subset. Furthermore, we have

$$\begin{aligned} TD_\infty &= T\left(\bigcap_{n=1}^{\infty} D_n\right) \subseteq \bigcap_{n=1}^{\infty} TD_n \\ &\subseteq \bigcap_{n=1}^{\infty} D_n = D_\infty. \end{aligned}$$

In conclusion, the existence of a fixed point in the subset $D_\infty \subset D$ for the operator $T : D_\infty \to D_\infty$ can be established by virtue of Schauder's theorem. The proof of the theorem is now concluded. □

Remark 4.2.1. Taking $\psi = Id$ and $\varphi(t) = kt$, with $0 \leq k < 1$, we recover Theorem 1.4.1.

Setting $\psi(t) = t$ in Theorem 4.2.1, we obtain the following result (Boyd-Wong's type):

Corollary 4.2.1. *Assume that $T : D \to D$ be a continuous operator verifying*

$$\mu(TZ) \leq \varphi\big(\mu(Z)\big) \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0.$$

Moreover, we assume that the following assumption holds:

(i) *$\varphi(t) < t$ for any $t > 0$ and*

$$\limsup_{t\to r^+} \varphi(t) < r \quad \textit{for every} \quad r > 0.$$

So T admits at least one fixed point in D.

Corollary 4.2.2. *Assume that $T : D \to D$ be an operator and $\varphi : (0,\infty) \to \mathbb{R}$ a nondecreasing and upper semicontinuous from the right with $\varphi(t) < t$ for any $t > 0$ and satisfying*

$$\|Tx - Ty\| \leq \varphi(\|x-y\|) \quad \textit{for } Tx \neq Ty \textit{ and } x, y \in X.$$

The set T possesses at least one fixed point within the domain D.

Proof. Let $\mu : B(X) \to [0,\infty)$ be a set quantity given by

$$\mu(Z) = diamZ = \sup\{\|x-y\|, x, y \in Z\}.$$

$diamZ$ (the diameter of X) is evidently a measure of noncompactness in the space X. Taking into account the assumptions of the Corollary, we get

$$\sup_{x,y\in Z} \|Tx - Ty\| \leq \sup_{x,y\in Z} \varphi(\|x-y\|) \leq \varphi\left(\sup_{x,y\in Z} \|x-y\|\right),$$

which implies that

$$\mu(TZ) \leq \varphi(\mu(Z)), \quad \text{with } \mu(TZ) > 0.$$

In addition, φ is right continuous since φ is nondecreasing and upper semicontinuous from the right. Hence, we obtain

$$\limsup_{t\to r^+} \varphi(t) = \varphi(r) < r; \quad \text{for every } r > 0.$$

Consequently, all the hypotheses outlined in Corollary 4.2.1 are fulfilled, thereby implying that T possesses at least one fixed point within the set M. □

Let $Delta$ represent a set of functions $\beta : (0,\infty) \to (0,1)$ meeting the following criteria:

$$\limsup_{t\to r^+} \beta(t) < 1 \quad \text{for any} \quad r > 0,$$

Let $\beta \in \Delta$. Taking $\varphi(t) = \beta(t)\psi(t)$ in Theorem 4.2.1, we get the result below (Geraghty's-type).

Corollary 4.2.3. *Assume that $T : D \to D$ be a continuous operator that satisfies*

$$\psi(\mu(TZ)) \leq \beta(\mu(Z))\psi(\mu(Z)) \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0.$$

It can be concluded that the operator T possesses at least one fixed point within the domain D.

Let $\alpha : (0,\infty) \to (0,1)$ and $\gamma : (0,\infty) \to (0,\infty)$. In Theorem 4.2.1, when $\varphi(t) = \alpha(t)\gamma(t)$, we obtain the following result.

Corollary 4.2.4. *Suppose that $T : D \to D$ be a continuous operator satisfying*

$$\psi(\mu(TZ)) \leq \alpha(\mu(Z))\gamma(\mu(Z)) \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0,$$

where ψ is nondecreasing, γ is a right continuous function with $\gamma(t) < \psi(t)$, for all $t > 0$ and

$$\limsup_{t\to r^+} \alpha(t) < \frac{\psi(r^+)}{\gamma(r^+)} \quad \textit{for any} \quad r > 0. \tag{4.4}$$

So T admits at least one fixed point in D.

Remark 4.2.2. Corollary 4.2.3 is an improvement of [6, Corollary 3.10]. Indeed, the condition $\beta \in \Delta$ is replaced by the weaker condition (4.4) since $\dfrac{\psi(r^+)}{\gamma(r^+)} > 1$. Moreover, the continuity of γ is weaken to the continuity from the right.

Corollary 4.2.5. *Assume that $T : D \to D$ be a continuous Proinov contraction of Darbo-type where $\varphi(t) < \psi(t)$ for any $t > 0$. Furthermore, if at least one of the following conditions holds:*

(i) *ψ is lower semicontinuous and φ is upper semicontinuous;*

(ii) *ψ nondecreasing and φ is upper semicontinuous from the right.*

So T admits at least one fixed point in D.

Proof. Assume first that (i) holds. Then, we get

$$\limsup_{t\to r} \varphi(t) \le \varphi(r) < \psi(r) \le \liminf_{t\to r} \psi(t) \quad \text{for every } r > 0.$$

If (ii) holds. In this case, we have

$$\limsup_{t\to r^+} \varphi(t) \le \varphi(r) < \psi(r) \le \psi\left(r^+\right) \quad \text{for every } r > 0;$$

□

Setting $\varphi = \psi - \tau$, where $\tau : (0,\infty) \to (0,\infty)$ in Theorem 4.2.1, we obtain the following result

Corollary 4.2.6. *Assume that $T : D \to D$ be a continuous operator satisfying*

$$\psi\left(\mu(TZ)\right) \le \psi\left(\mu(Z)\right) - \tau\left(\mu(Z)\right) \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0.$$

Moreover, we assume that at least one of the three following conditions holds:

(i)

$$\liminf_{t\to r} \tau(t) > \limsup_{t\to r} \psi(t) - \liminf_{t\to r} \psi(t) \quad \textit{for every} \quad r > 0;$$

(ii) *ψ nondecreasing and $\tau \in \mathscr{L}$;*

(iii) *The following conditions are satisfied:*

(a)

$$\inf_{t>r} \psi(t) > -\infty \quad \textit{for every} \quad r > 0,$$

(b) *If $\lim_{n\to\infty} \tau(t_n) = 0$ for a bounded sequence $\{t_n\}$, then $\lim t_n = 0$.*

So T admits at least one fixed point in D.

Remark 4.2.3. Combining Corollary 4.2.6 with Lemmas 1.4.1 and 1.4.2, assumption (iii) is replaced by the two following conditions:

(a') $\lim_{n\to\infty} \psi(t_n) = -\infty$ implies $\lim_{n\to\infty} t_n = 0$.

(b') τ satisfies (H_1).

Remark 4.2.4. It follows from the above remark that Corollary 4.2.6 is a generalization and an improvement of Theorem 1.4.2.

Corollary 4.2.7. *Let $T : D \to D$ be a continuous operator such that there exist $\theta : (0,\infty) \to (1,\infty)$ and for every subset Z of M*

$$\theta\left(\mu(TZ)\right) \leq \left[\theta\left(\mu(Z)\right)\right]^{\chi(\mu(Z))} \quad \textit{with} \quad \mu(Z), \mu(TZ) > 0,$$

Moreover, we assume that the following conditions hold:

(a)

$$\inf_{t>r} \theta(t) > 1 \quad \textit{for every} \quad r > 0,$$

(b) *If* $\lim_{n\to\infty} \chi(t_n) = 1$ *for a bounded sequence* $\{t_n\}$, *then* $\lim t_n = 0$.

So T admits at least one fixed point in D.

Proof. The result derived from Corollary 4.2.6 with $\psi(t) = \ln\ln\theta(t)$ and $\tau(t) = -\ln\chi(t)$. □

Remark 4.2.5. Combining Remark 4.2.3 with Corollary 4.2.7, assumptions (a) and (b) are replaced by the two following conditions:

(a') $\lim_{n\to\infty} \theta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

(b') $\chi \in \Delta$.

Remark 4.2.6. It follows from the above remark that Corollary 4.2.7 is a generalization and an improvement of Theorem 1.4.3.

4.3 An application to a nonlinear integral equation

In this part, we discuss the existence of various integral equations using the results obtained in the previous chapter.

4.3.1 Measure of noncompactness in $BC(\mathbb{R}^+)$

In this part, we focus on the existence of solution for the the integral equation of Volterra type:

$$u(t) = f(t, u(t)) + g(t, u(t)) \int_0^t K(t, s, u(s))ds, \quad t \in \mathbb{R}^+ = [0, \infty), \tag{4.5}$$

where $K : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $f, g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are suitable functions.

Roughly speaking, we utilize our obtain results in the previous section to prove at least one solution of (4.5) in the space $BC(\mathbb{R}^+)$ consisting of all bounded and continuous real functions defined on $\mathbb{R}^+$. The space $BC(\mathbb{R}^+)$ is equipped with the standard supremum norm given as follows:

$$\| u \| = \sup\left\{| u(t) | : \quad t \in \mathbb{R}^+\right\}.$$

Now, let us define the measure of noncompactness in the space $BC(\mathbb{R}^+)$ (see [10]). Let us consider Z be a nonempty, bounded subset of $BC(\mathbb{R}^+)$ and a positive number $R > 0$. For $u \in Z$ and $\varepsilon > 0$, let us denote $\rho^R(u,\varepsilon)$ the modulus of continuity of the functional u on the interval $[0,R]$, that is

$$\rho^R(u,\varepsilon) = \sup\{| u(t) - u(s) | : t,s \in [0,R], | t-s | \leq \varepsilon\}.$$

Moreover, we put

$$\rho^R(Z,\varepsilon) = sup\left\{\rho^R(u,\varepsilon) : u \in Z\right\},$$

$$\rho_0^R(Z) = \lim_{\epsilon\to 0}\rho^R(Z,\varepsilon),$$

$$\rho_0(Z) = \lim_{L\to\infty}\rho_0^R(Z).$$

In addition, for a fixed number $t \in \mathbb{R}^+$, we denote

$$Z(t) = \{u(t) : u \in Z\}.$$

Henceforth, we can define the measure of noncompactness μ on $B(BC(\mathbb{R}^+))$, as follows:

$$\mu(Z) = \rho_0(Z) + \limsup_{t\to+\infty} diam Z(t), \tag{4.6}$$

where $diam Z(t)$ is the diameter of $Z(t)$.

4.3.2 Integral equations via Proinov contractions

In this section we use Corollary 4.2.5 to establish the existence of the integral equation (4.5).

Let Φ be the set of all nondecreasing functions $\eta : (0,\infty) \to (0,\infty)$ that satisfies the following conditions:

(a)

$$\eta(t) < t \quad \text{for any } t > 0;$$

(b) η is superadditive, i.e.,

$$\eta(t) + \eta(s) \leq \eta(t+s) \quad \text{for } t,s \in \mathbb{R}^+.$$

In the sequel, we consider equation (4.5) under the following assumptions:

(1) f and g are continuous functions such that both $t \to f(t,0)$ and $t \to g(t,0)$ are elements of the space $BC\left(\mathbb{R}^+\right)$;

(2) There exists an upper semicontinuous function $\varphi \in \Phi$ and a constant $k \in \left(0,\frac{1}{2}\right)$ such that for any $x,y \in \mathbb{R}$ and $t \in \mathbb{R}^+$, we have

$$\left|f(t,x) - f\left(t,y\right)\right| \leq k\eta\left(\left|x-y\right|\right);$$

(3) Additionally, there exists a continuous function $p : \mathbb{R}^+ \to \mathbb{R}^+$ with respect to each of $x,y \in \mathbb{R}$ and $t \in \mathbb{R}^+$, we have

$$\left|g(t,x) - g\left(t,y\right)\right| \leq p(t)\eta\left(\left|x-y\right|\right);$$

(4) The function K is continuous, and there also exist continuous functions $h, m : \mathbb{R}^+ \to \mathbb{R}^+$ so that

$$\lim_{t\to\infty} h(t) \int_0^t m(s)\,ds = 0,$$

$$\lim_{t\to\infty} p(t)\,h(t) \int_0^t m(s)\,ds = 0$$

and

$$|K(t,s,x)| \le h(t)\,m(s) \quad \text{for all } x \in \mathbb{R} \text{ and } t,s \in \mathbb{R}^+;$$

(5)

$$p(t)\,h(t) \int_0^t m(s)\,ds \le k \quad \text{for any } t \in \mathbb{R}^+;$$

(6) Additionally, there exists a positive solution l_0 for the given inequality

$$\eta(l) + A \le l, \quad l > 0,$$

where

$$A = \sup_{t\ge 0}\left\{\left|f(t,0)\right| + \left|g(t,0)\right| h(t) \int_0^t m(s)\,ds\right\}.$$

Theorem 4.3.1. *Under assumptions* (1) – (6), *integral equation* (4.5) *admits at least one solution* $u = u(t)$ *in* $BC(\mathbb{R}^+)$.

Proof. Let's contemplate the operator on the space $BC(\mathbb{R}^+)$ as follows

$$(Tu)(t) = f(t,u(t)) + g(t,u(t)) \int_0^t K(t,s,u(s))\,ds, \quad t \in \mathbb{R}^+, u \in BC(\mathbb{R}^+).$$

Because of the conditions that were established, we easily observe that Tu is continuous on $\mathbb{R}^+$ with every function $u \in BC(\mathbb{R}^+)$, then T is well- defined.

Further, through our assumptions, we get the following inequality

$$\begin{aligned}
|(Tu)(t)| &\le \left|f(t,u(t))\right| + \left|g(t,u(t))\right| \int_0^t |K(t,s,u(s))|\,ds \\
&\le \left|f(t,u(t))\right| + \left|g(t,u(t))\right| h(t) \int_0^t m(s)\,ds \\
&\le \left|f(t,u(t)) - f(t,0)\right| + \left|f(t,0)\right| \\
&\quad + \left|g(t,u(t)) - g(t,0)\right| h(t) \int_0^t m(s)\,ds + \left|g(t,0)\right| h(t) \int_0^t m(s)\,ds \\
&\le k\eta(|u(t)|) + \left|f(t,0)\right| + \eta(|u(t)|)\,p(t)\,h(t) \int_0^t m(s)\,ds + \left|g(t,0)\right| h(t) \int_0^t m(s)\,ds \\
&\le 2k\eta(|u(t)|) + \left|f(t,0)\right| + \left|g(t,0)\right| h(t) \int_0^t m(s)\,ds \\
&\le \eta(|u(t)|) + A.
\end{aligned}$$

As φ is nondecreasing, we obtain

$$\|Tu\| \le \eta(\|u\|) + A. \tag{4.7}$$

Since $A < \infty$ (by assumptions (1) and (4)), we infer that Tu is bounded on $\mathbb{R}^+$. Hence, T maps the space $BC\left(\mathbb{R}^+\right)$ into itself. In addition, $T\left(B_{l_0}\right) \subseteq B_{l_0}$ where

$$B_{l_0} = \left\{u \in BC\left(\mathbb{R}^+\right), \quad \|x - y\| \leq l_0\right\}.$$

Indeed, in view of (4.7) and assumption (6), we deduce

$$\|Tu\| \leq \eta(l) + A \leq l.$$

Now, we discuss the continuity of the operator T on B_{l_0}. Let us fix an arbitrary $\varepsilon > 0$. and take $u, v \in B_{l_0}$ such that $\|u - v\| \leq \varepsilon$. Therefore, for $t \in \mathbb{R}^+$, we get

$$\begin{aligned}
|(Tu)(t) - (Tv)(t)| &\leq \left|f(t,u(t)) - f(t,v(t))\right| \\
&\quad + \left|g(t,u(t))\int_0^t K(t,s,u(s))\,ds - g(t,v(t))\int_0^t K(t,s,u(s))\,ds\right| \\
&\quad + \left|g(t,v(t))\int_0^t K(t,s,u(s))\,ds - g(t,v(t))\int_0^t K(t,s,v(s))\,ds\right| \\
&\leq k\eta(|u(t) - v(t)|) + \left|g(t,u(t)) - g(t,v(t))\right|\int_0^t |K(t,s,u(s))|\,ds \\
&\quad + \left|g(t,v(t))\right|\int_0^t |K(t,s,u(s)) - K(t,s,v(s))|\,ds \\
&\leq k\eta(|u(t) - v(t)|) + \varphi(|u(t) - v(t)|)\,p(t)\,h(t)\int_0^t m(s)\,ds \\
&\quad + \left[\left|g(t,v(t)) - g(t,0)\right| + \left|g(t,0)\right|\right]\int_0^t |K(t,s,u(s)) - K(t,s,v(s))|\,ds \\
&\leq 2k\eta(|u(t) - v(t)|) + \left[p(t)\,\eta(|u(t)|) + \left|g(t,0)\right|\right]\int_0^t |K(t,s,u(s)) - K(t,s,v(s))|\,ds \\
&\leq \eta(\varepsilon) + 2\left[p(t)\,\eta(l_0) + \left|g(t,0)\right|\right]h(t)\int_0^t m(s)\,ds. \qquad (4.8)
\end{aligned}$$

Taking into account assumptions (1) and (4), we infer that there exists $R > 0$ such that for $t \geq R$, the following inequalities hold

$$2p(t)\,\eta(l_0)\,h(t)\int_0^t m(s)\,ds \leq \frac{\varepsilon}{2}$$

and

$$2\left|g(t,0)\right|h(t)\int_0^t m(s)\,ds \leq Bh(t)\int_0^t m(s)\,ds \leq \frac{\varepsilon}{2},$$

where

$$B := \sup_{t\geq 0}\left\{\left|g(t,0)\right|\right\} < \infty.$$

Consequently, we deduce

$$|(Tu)(t) - (Tv)(t)| \leq \eta(\varepsilon) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq 2\varepsilon \quad \text{for all } t \geq R.$$

Next, we discuss the case $0 \leq t \leq R$. In this case, we consider the quantity

$$\rho^{L}(K,\varepsilon) = \sup\left\{\left|K(t,s,x) - K\left(t,s,y\right)\right| : t,s \in [0,R], x,y \in [-l_0, l_0], \left|x-y\right| \leq \varepsilon\right\}.$$

From the fact that $K(t,s,x)$ is uniformly continuous on the $[0,R] \times [0,R] \times [-l_0, l_0]$, we deduce that

$$\lim_{\varepsilon \to 0} \rho^{R}(K,\varepsilon) = 0.$$

Coming back to (4.8), we obtain for $t \in [0,L]$

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \eta(\varepsilon) + \left[p(t)\eta(|u(t)|) + \left|g(t,0)\right|\right] \int_0^R \rho^{R}(K,\varepsilon)\,ds \\ &\leq \eta(\varepsilon) + \left[\eta(l_0)C_1 + C_2\right] R\rho^{R}(K,\varepsilon), \end{aligned}$$

where

$$C_1 := \sup_{t \leq R}\left\{p(t)\right\} < \infty$$

and

$$C_2 := \sup_{t \leq R}\left\{\left|g(t,0)\right|\right\} < \infty.$$

Hence, we conclude that T is continuous on the closed ball B_{l_0}.

Let's take a nonempty set $Z \subset B_{l_0}$. Then, for $u,v \in Z$ and for $t \in \mathbb{R}^+$ is a fixed value and following the same steps as those used in (4.8), one gets

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \eta(|u(t) - v(t)|) + 2p(t)\eta(l_0)h(t)\int_0^t m(s)\,ds \\ &\quad + 2\left|g(t,0)\right| h(t)\int_0^t m(s)\,ds. \end{aligned}$$

Hence, the above estimate turns into

$$\begin{aligned} diam(TZ)(t) &\leq \eta(diam(Z)(t)) + 2p(t)\varphi(l_0)h(t)\int_0^t m(s)\,ds \\ &\quad + 2\left|g(t,0)\right| h(t)\int_0^t m(s)\,ds. \end{aligned}$$

Using assumption (4) and the upper semicontinuity of the function η, we find

$$\limsup_{t \to \infty} diam(TZ)(t) \leq \eta\left(\limsup_{t \to \infty} diam(Z)(t)\right). \tag{4.9}$$

On the other hand, let us fix $\varepsilon > 0$ and $R > 0$ and take arbitrarily $t,s \in [0,R]$ so that $|t-s| \leq \varepsilon$. It is assumed, without loss of generality, that $s < t$. Ainsi, pour $u \in Z$, et en se basant sur les hypothÃÍses dÃľveloppÃľes, nous obtenons.

$$\begin{aligned}
|(Tu)(t)-(Tu)(s)| &\leq \left|f(t,u(t))-f(s,u(s))\right| \\
&\leq \left|g(t,u(t))\int_0^t K(t,\tau,u(\tau))\,d\tau - g(s,u(s))\int_0^t K(t,\tau,u(\tau))\,d\tau\right| \\
&\quad + \left|g(s,u(s))\int_0^t K(t,\tau,u(\tau))\,d\tau - g(s,u(s))\int_0^s K(s,\tau,u(\tau))\,d\tau\right| \\
&\leq \left|f(t,u(t))-f(t,u(s))\right| + \left|f(t,u(s))-f(s,u(s))\right| \\
&\quad + \left|g(t,u(t))-g(s,u(s))\right|\int_0^t |K(t,\tau,u(\tau))|\,d\tau \\
&\quad + \left|g(s,u(s))\right|\left|\int_0^t K(t,\tau,u(\tau))\,d\tau - \int_0^s K(s,\tau,u(\tau))\,d\tau\right| \\
&\leq k\eta(|u(t)-u(s)|) + \rho_{l_0}^R(f,\varepsilon) \\
&\quad + \left[\left|g(t,u(t))-g(t,u(s))\right| + \left|g(t,u(s))-g(s,u(s))\right|\right] h(t)\int_0^t m(\tau)\,d\tau \\
&\quad + \left[\left|g(s,u(s))-g(s,0)\right| + \left|g(s,0)\right|\right] \\
&\quad \times \left[\int_s^t |K(t,\tau,u(\tau))|\,d\tau + \int_0^s |K(t,\tau,u(\tau))-K(s,\tau,u(\tau))|\,d\tau\right],
\end{aligned}$$

where

$$\rho_{l_0}^R(f,\varepsilon) = \sup\left\{\left|f(t,x)-f(s,x)\right| : t,s\in[0,R],\, x\in[-l_0,l_0],\, |t-s|\leq\varepsilon\right\}.$$

It follows that

$$\begin{aligned}
|(Tu)(t)-(Tu)(s)| &\leq k\eta(|u(t)-u(s)|) + \rho_{l_0}^R(f,\varepsilon) \\
&\quad + \left[p(s)\eta(|u(t)-u(s)|) + \rho_{l_0}^R(g,\varepsilon)\right] h(t)\int_0^t m(\tau)\,d\tau \\
&\quad + \left[p(s)\eta(|u(s)|) + \left|g(s,0)\right|\right] \\
&\quad \times \left[\int_s^t |K(t,\tau,u(\tau))|\,d\tau + \int_0^s |K(t,\tau,u(\tau))-K(s,\tau,u(\tau))|\,d\tau\right],
\end{aligned}$$

where

$$\rho_{l_0}^R(g,\varepsilon) = \sup\left\{\left|g(t,x)-g(s,x)\right| : t,s\in[0,R],\, x\in[-l_0,l_0],\, |t-s|\leq\varepsilon\right\}.$$

This yields that

$$\begin{aligned}
|(Tu)(t)-(Tu)(s)| &\leq 2k\eta(|u(t)-u(s)|) + \rho_{l_0}^R(f,\varepsilon) + \rho_{l_0}^R(g,\varepsilon)\,h(t)\int_0^t m(\tau)\,d\tau \\
&\quad + \left[p(s)\eta(l_0) + \left|g(s,0)\right|\right] h(t)\int_s^t m(\tau)\,d\tau \\
&\quad + \left[p(s)\eta(l_0) + \left|g(s,0)\right|\right] R\rho_{r_0}^R(K,\varepsilon),
\end{aligned} \tag{4.10}$$

where

$$\rho_{l_0}^R(K,\varepsilon) = \sup\{|K(t,\tau,x)-K(s,\tau,x)| : t,s,\tau\in[0,R],\, x\in[-l_0,l_0],\, |t-s|\leq\varepsilon\}.$$

Hence, we find

$$\begin{aligned}\rho^R(TX,\varepsilon) &\leq \eta\left(\rho^R(X,\varepsilon)\right)+\rho^R_{l_0}\left(f,\varepsilon\right)+R\rho^R_{l_0}\left(g,\varepsilon\right)D_1\\ &+\varepsilon\left[p(s)\eta(l_0)+\left|g(s,0)\right|\right]D_1\\ &+L\rho^R_{l_0}(K,\varepsilon)D_2,\end{aligned}$$

where

$$D_1 := \sup_{0\leq\tau,t\leq R}\{h(t)\,m(\tau)\}<\infty$$

and

$$D_2 := \sup_{0\leq t\leq R}\left\{p(s)\eta(l_0)+\left|g(s,0)\right|\right\}<\infty.$$

Moreover, due to the uniform continuity of functions, $f \ \ and \ \ g$ on $[0,R]\times[-l_0,l_0]$ as well as the function K on $[0,R]\times[0,R]\times[-l_0,l_0]$, we infer that

$$\lim_{\varepsilon\to0}\rho^R_{l_0}\left(f,\varepsilon\right)=0,\quad \lim_{\varepsilon\to0}\rho^R_{l_0}\left(g,\varepsilon\right)=0,\quad \text{and}\ \lim_{\varepsilon\to0}\rho^R_{l_0}(K,\varepsilon)=0. \tag{4.11}$$

In view of (4.10) and (4.11), it follows that

$$\rho^R_0(TZ)\leq\lim_{\varepsilon\to0}\eta\left(\rho^R(Z,\varepsilon)\right).$$

Again, through the upper semicontinuity of η, we get

$$\rho^R_0(TZ)\leq\eta\left(\rho^R_0(Z)\right),$$

which yields

$$\rho_0(TZ)\leq\eta(\omega_0(Z)). \tag{4.12}$$

Combining (4.9), (4.12) and the superadditivity of η, we get

$$\rho_0(TZ)+\limsup_{t\to\infty} diam(TZ)(t)\leq\eta\left(\rho_0(Z)+\limsup_{t\to\infty} diam(Z)(t)\right),$$

or, equivalently

$$\mu(TZ)\leq\eta\left(\mu(Z)\right).$$

Hence, due to Corollary 4.2.5-(i) with $\psi(t)=t$. In conclusion, it can be inferred that the operator T possesses at least one fixed point within the ball B_{l_0}, thus establishing the completion of our argument. □

Remark 4.3.1. Based on the research paper referenced as [2], it is possible to substitute the hypothesis regarding the superadditivity of the function φ with the following alternative hypothesis:

"η is a concave function."

4.4 Example

In this study, we shall examine the functional integral equation presented below

$$u(t)=\frac{t}{2(1+t)}\ln(1+|u(t)|)+\frac{e^{-t}}{1+t^2}\ln(1+|u(t)|)\int_0^t\frac{se^{-t}\sin x(s)}{1+|x(s)|}ds,\quad t\in\mathbb{R}^+. \tag{4.13}$$

Let us observe that

$$f(t,x)=\frac{t}{2(1+t)}\ln(1+|x|),$$
$$g(t,x)=\frac{e^{-t}}{1+t^2}\ln(1+|x|)$$

and

$$K(t,s,x)=\frac{se^{-t}\sin x}{1+|x|}.$$

It is evident that the integral equation (4.13) can be identified as a specific instance of the integral the formula (4.5). Indeed, the functions f, g, and K exhibit continuity over their respective domains. Also, if we take

$$\eta(t)=\ln(1+t),$$

we have $\varphi(t)<t$ for every $t>0$ and φ nondecreasing and concave on $\mathbb{R}^+$. In addition, for arbitrary $x,y\in\mathbb{R}$ such that $|x|\le|y|$ (or $|y|\le|x|$) and we get

$$|f(t,x)-f(t,y)|\le\frac{1}{2}\ln(1+|x-y|)=k\eta(|x-y|),$$

where $k=\frac{1}{2}$.

Additionally, there is

$$|g(t,x)-g(t,y)|\le p(t)\ln(1+|x-y|),$$

where $p(t)=\frac{e^{-t}}{1+t^2}$ and

$$|K(t,s,x)|\le e^{-t}s.$$

Hence, we can take $h(t)=e^{-t}$ and $m(s)=s$. Therefore, we acquire

$$\lim_{t\to\infty}h(t)\int_0^t m(s)\,ds=\lim_{t\to\infty}e^{-t}\int_0^t s\,ds=0,$$
$$\lim_{t\to\infty}p(t)h(t)\int_0^t m(s)\,ds=\lim_{t\to\infty}\frac{e^{-2t}}{1+t^2}\int_0^t s\,ds=0$$

and

$$p(t)h(t)\int_0^t m(s)\,ds=\frac{e^{-2t}}{1+t^2}\frac{t^2}{2}\le\frac{1}{2}=k\quad\text{for any } t\in\mathbb{R}^+.$$

Moreover, a positive solution l_0 is furnished for the inequality

$$\ln(1+l)+A\le l,$$

since

$$A=\sup_{t\ge0}\left\{f(t,0)+|g(t,0)|h(t)\int_0^t m(s)\,ds\right\}=0<\infty.$$

Chapter 5

Contents

Hammerstein integral equation via Henstock-Kurzweil integrals

Abstract

This chapter aims to contribute to the solvability of BV solutions of Hammerstein integral equations formulated in terms of the Henstock-Kurzweil integral on unbounded intervals.

5.1 Introduction

Henceforth, all the integrals are taken in the Henstock-Kurzweil sense unless stated otherwise. Let us study the following Hammerstein nonlinear integral equation:

$$u(t) = v(t) + \mu p(t) \int_0^{\infty} f(t,r)\, g(u(r))\, \mathrm{d}r, \quad t \in [0,+\infty], \rho \in \mathbb{R}. \tag{5.1}$$

The problem which consists to solve the integral equation of (5.1) in the space $BV([0,+\infty))$ was stated and proved in the paper [15] in the case where $p(t) = 1$. The authors in the work [15] considered the integral equation (5.1) with the following form

$$u = v + \rho G\left(C_g(u)\right),$$

where C_g is the composition operator generated by the function g, $|\rho| < \mu$ for some $\mu > 0$ and K the operator defined by

$$G(u)(t) = \int_0^{+\infty} f(t,r)\, u(r)\, dr, \quad t \in [0,\infty). \tag{5.2}$$

5.2 Main result

At this part, we establish the ability to be solved of the integral equation (5.1) in the space $BV([0,+\infty))$. We may write (5.1) as the following operator equation:

$$u = v + \rho p K\left(C_g(u)\right),$$

where $p \in BV([0,+\infty))$.

Denote by $B_R^{BV} := B(0,R)$ a certain closed ball of radius $R > 0$ in the space $BV([0,\infty))$. Let us now consider the following assumptions

a) $v : [0,\infty) \to \mathbb{R}$ is a BV-function;

b) $g : \mathbb{R} \to \mathbb{R}$ acts in $BV([0,\infty))$ and as an any given value $R > 0$, , it has a corresponding value $M_R > 0$ so that:
$$\left\|C_g(u) - C_g(w)\right\|_{BV} \le M_R \|u - w\|_{BV};$$

c) $f : [0,\infty) \times [0,\infty] \to \mathbb{R}$ is the function such that $f(t,.)$ is HK-integrable on $[0,\infty]$ and satisfies the following conditions:

H_1) $V_0^{+\infty}\left(\int_0^{+\infty} f(.,r)\, dr\right) < +\infty$;

H_2) There exists postive numbers a and b such that for any partition $\mathscr{P} = \{0 = t_0 < ... < t_n = t\}$, we have
$$\sum_{j=1}^{n}\left(\|f(t_j,.)\|_{[0,\infty)}^{A} + 2\,\|f(t_{j-1},.)\|_{[0,\infty)}^{A}\right) < a < +\infty$$
and
$$\lim_{t\to+\infty}\ \sup_{0=t_0<...<t_n=t} \sum_{j=1}^{n}\left|\int_0^{+\infty} f(t_{j-1},r)\, dr\right| = b < +\infty.$$

Theorem 5.2.1. *Assume that conditions a), b) and c) are fulfilled. There are exists a positive value μ which means for any ρ satisfying $|\rho| < \mu$, the integral equation* (5.1) *has a unique solution in B_R^{BV}.*

Proof. Choosing $\mu > 0$ such that

$$R > \|v\|_{BV} + K\left(RM_R + |g(0)|\right) \tag{5.3}$$

and

$$M_R K < 1, \tag{5.4}$$

where

$$K := \mu \|p\|_{BV} \max\left\{\left(\|f(0,.)\|^A_{[0,\infty)} + a\right), \left(\left\|\int_0^{+\infty} f(.,r)\,dr\right\|_{BV} + b\right)\right\}. \tag{5.5}$$

Fix ρ such that $|\rho| < \mu$ and let us introduce

$$H(u)(t) := v(t) + \rho p(t) G(u)(t),$$

where

$$G(u)(t) = \int_0^{+\infty} f(t,r)\, g(u(r))\, ds, \quad t \in [0,\infty), \quad u \in B_R^{BV}.$$

Through the fact that $C_g(u)$ in $BV([0,\infty))$ and $p(t) f(t,.)$ is HK-integrable on $[0,\infty]$ for any $t \in [0,\infty)$, the function $p(t) f(t,.) C_g(u)(.)$ is HK-integrable on $[0,\infty]$. Then, H and G are well defined.

First, we show that H maps B_R^{BV} into itself. Let $u \in B_R^{BV}$, then we get

$$\|H(u)\|_{BV} \le \|v\|_{BV} + |\rho|\,|p(0)|\,|G(u)(0)| + |\rho|\, V_0^{+\infty}\left(pG(u)\right).$$

First, we remind that

$$V_0^{+\infty}\left(pG(u)\right) = \lim_{t\to+\infty} V_0^t\left(pG(u)\right).$$

Next, we have

$$\begin{aligned}
V_0^t\left(pG(u)\right) &= \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^n \left|p(t_j) G(u)(t_j) - p(t_{j-1}) G(u)(t_{j-1})\right| \\
&= \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^n \left|\int_0^{+\infty} \left[p(t_j) f(t_j,r) - p(t_{j-1}) f(t_{j-1},r)\right] g(u(r))\,dr\right| \\
&\le \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^n \left|p(t_j) \int_0^{+\infty} \left[f(t_j,r) - f(t_{j-1},r)\right] g(u(r))\,dr\right| \\
&+ \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^n \left|\int_0^{+\infty} \left[p(t_j) - p(t_{j-1})\right] f(t_{j-1},r)\, g(u(r))\,dr\right|.
\end{aligned}$$

In view of Theorem 1.6.4, we obtain

$$\begin{aligned}
V_0^t\left(pG(u)\right) &\le \|p\|_{BV} \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left| \int_0^{+\infty} \left[f\left(t_j,r\right)-f\left(t_{j-1},r\right)\right] g\left(u(r)\right) dr \right| \\
&+V_0^t\left(p\right) \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left| \int_0^{+\infty} f\left(t_{j-1},r\right) g\left(u(r)\right) dr \right| \\
&\le \|p\|_{BV} \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left\{ \left| \int_0^{+\infty} \left[f\left(t_j,r\right)-f\left(t_{j-1},r\right)\right] dr \right| \inf_{r\in[0,\infty)} \left|g\left(u(r)\right)\right| \right. \\
&\left. + \left\|f\left(t_j,.\right)-f\left(t_{j-1},.\right)\right\|_{[0,\infty)}^{A} V_0^{+\infty}\left(C_g(u)\right) \right\} \\
&+V_0^t\left(p\right) \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left\{ \left| \int_0^{+\infty} f\left(t_{j-1},r\right) dr \right| \inf_{r\in[0,\infty)} \left|g\left(u(r)\right)\right| \right. \\
&\left. + \left\|f\left(t_{j-1},.\right)\right\|_{[0,\infty)}^{A} V_0^{+\infty}\left(C_g(u)\right) \right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
V_0^{+\infty}\left(pG(u)\right) &\le \left[\|p\|_{BV} V_0^{+\infty}\left(\int_0^{+\infty} f(.,r)\, dr \right) \right. \\
&\left. +V_0^{+\infty}\left(p\right) \lim_{t\to+\infty} \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left| \int_0^{+\infty} f\left(t_{j-1},r\right) dr \right| \right] \inf_{r\in[0,\infty)} \left|g\left(u(r)\right)\right| \\
&+a\|p\|_{BV} V_0^{+\infty}\left(C_g(u)\right).
\end{aligned}$$

A different option is provided by

$$|G(u)(0)| \le \inf_{r\in[0,\infty)} \left|g\left(u(r)\right)\right| \left| \int_0^{+\infty} f(0,r)\, dr \right| + V_0^{+\infty}\left(C_g(u)\right) \|f(0,.)\|_{[0,\infty)}^{A}.$$

Consequently,

$$\begin{aligned}
\|H(u)\|_{BV} &\le \|v\|_{BV} + |\rho| \left|p(0)\right| \left\{ \inf_{r\in[0,\infty)} \left|g\left(u(r)\right)\right| \left| \int_0^{+\infty} f(0,r)\, dr \right| + V_0^{+\infty}\left(C_g(u)\right) \|f(0,.)\|_{[0,\infty)}^{A} \right\} \\
&+|\rho| \left\{ \|p\|_{BV} V_0^{+\infty}\left(\int_0^{+\infty} f(.,r)\, dr \right) \right. \\
&\left. +V_0^{+\infty}\left(p\right) \lim_{t\to+\infty} \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left| \int_0^{+\infty} f\left(t_{j-1},r\right) dr \right| \right] \inf_{r\in[0,\infty)} \left|g\left(u(r)\right)\right| \\
&+a|\rho| \|p\|_{BV} V_0^{+\infty}\left(C_g(u)\right).
\end{aligned}$$

The above inequality turns into

$$\begin{aligned}\|H(u)\|_{BV} &\leq \|v\|_{BV} + |\rho| \left\{ \left(|p(0)| \, \|f(0,.)\|^A_{[0,\infty)} + a \, \|p\|_{BV} \right) V_0^{+\infty} \left(C_g(u) \right) \right. \\ &\quad \left. + \|p\|_{BV} \, |g(u(0))| \left\| \int_0^{+\infty} f(.,r)\, dr \right\|_{BV} \right\} + b|\rho| \, \|p\|_{BV} \, |g(u(0))| .\end{aligned}$$

Therefore

$$\|H(u)\|_{BV} \leq \|v\|_{BV} + |\rho| \, \|p\|_{BV} \max \left\{ \left(\|f(0,.)\|^A_{[0,\infty)} + a \right), \left(\left\| \int_0^{+\infty} f(.,r)\, dr \right\|_{BV} + b \right) \right\} \|C_g(u)\|_{BV} .$$

Utilizing (5.3) and (5.5), we obtain

$$\|H(u)\|_{BV} \leq \|v\|_{BV} + \mu \, \|p\|_{BV} \max \left\{ \left(\|f(0,.)\|^A_{[0,\infty)} + a \right), \left(\left\| \int_0^{+\infty} f(.,r)\, dr \right\|_{BV} + b \right) \right\} \left(R M_R + |g(0)| \right) < R,$$

which allow us to deduce that H maps B_R^{BV} into itself.

Now, we prove that H is a contraction. For $u, w \in B_R^{BV}$ one gets

$$\|H(u) - H(w)\|_{BV} \leq |\rho| \left[|p(0)| \, |G(u)(0) - G(w)(0)| + V_0^{+\infty} \left(p(G(u) - G(w)) \right) \right] .$$

On one hand, we have

$$\begin{aligned}|G(u)(0) - G(w)(0)| &= \left| \int_0^{+\infty} f(0,r) \, |g(u(r)) - g(w(r))| \, dr \right| \\ &\leq \left| \int_0^{+\infty} f(0,r)\, dr \right| \inf_{r \in [0,\infty)} |g(u(r)) - g(w(r))| \\ &\quad + \|f(0,.)\|^A_{[0,\infty)} V_0^{+\infty} \left(C_g(u) - C_g(w) \right) .\end{aligned}$$

Alternatively, we receive

$$\begin{aligned}V_0^t \left(p(G(u) - G(w)) \right) &= \sup_{0 = t_0 < \ldots < t_n = t} \sum_{j=1}^n \left| p(t_j) (G(u) - G(w))(t_j) - p(t_{j-1}) (G(u) - G(w))(t_{j-1}) \right| \\ &\leq \sup_{0 = t_0 < \ldots < t_n = t} \sum_{j=1}^n \left| p(t_j) \int_0^{+\infty} \left[f(t_j, r) - f(t_{j-1}, r) \right] \left(g(u(r)) - g(w(r)) \right) dr \right| \\ &\quad + \sup_{0 = t_0 < \ldots < t_n = t} \sum_{j=1}^n \left| \int_0^{+\infty} \left[p(t_j) - p(t_{j-1}) \right] f(t_{j-1}, r) \left(g(u(r)) - g(w(r)) \right) dr \right| .\end{aligned}$$

Hence

$$
\begin{aligned}
V_0^t\left(p(G(u)-G(w))\right) &\le \|p\|_{BV} \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left| \int_0^{+\infty} \left[f\left(t_j,r\right)-f\left(t_{j-1},r\right)\right]\left(g\left(u\left(r\right)\right)-g\left(w\left(r\right)\right)\right) dr \right| \\
&+V_0^t\left(p\right) \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left| \int_0^{+\infty} f\left(t_{j-1},r\right)\left(g\left(u\left(r\right)\right)-g\left(w\left(r\right)\right)\right) dr \right| \\
&\le \|p\|_{BV} \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left\{ \left| \int_0^{+\infty} \left[f\left(t_j,r\right)-f\left(t_{j-1},r\right)\right] dr \right| \inf_{r\in[0,\infty)} \left|\left(g\left(u\left(r\right)\right)-g\left(w\left(r\right)\right)\right)\right| \right. \\
&\left. + \left\|f\left(t_j,.\right)-f\left(t_{j-1},.\right)\right\|_{[0,\infty)}^{A} V_0^{+\infty}\left(C_g\left(u\right)-C_g\left(w\right)\right) \right\} \\
&+V_0^t\left(p\right) \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left\{ \left| \int_0^{+\infty} f\left(t_{j-1},r\right) dr \right| \inf_{r\in[0,\infty)} \left|\left(g\left(u\left(r\right)\right)-g\left(w\left(r\right)\right)\right)\right| \right. \\
&\left. + \left\|f\left(t_{j-1},.\right)\right\|_{[0,\infty)}^{A} V_0^{+\infty}\left(C_g\left(u\right)-C_g\left(w\right)\right) \right\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
V_0^t\left(p(G(u)-G(w))\right) &\le \|p\|_{BV} \left\{ V_0^t\left(\int_0^{+\infty} f\left(.,r\right) dr\right) \inf_{r\in[0,\infty)} \left|\left(g\left(u\left(r\right)\right)-g\left(w\left(r\right)\right)\right)\right| \right. \\
&\left. + \left\|f\left(t_j,.\right)-f\left(t_{j-1},.\right)\right\|_{[0,\infty)}^{A} V_0^{+\infty}\left(C_g\left(u\right)-C_g\left(w\right)\right) \right\} \\
&+ \|p\|_{BV} \sup_{0=t_0<\ldots<t_n=t} \sum_{j=1}^{n} \left\{ \left| \int_0^{+\infty} f\left(t_{j-1},r\right) dr \right| \inf_{r\in[0,\infty)} \left|\left(g\left(u\left(r\right)\right)-g\left(w\left(r\right)\right)\right)\right| \right. \\
&\left. + \left\|f\left(t_{j-1},.\right)\right\|_{[0,\infty)}^{A} V_0^{+\infty}\left(C_g\left(u\right)-C_g\left(w\right)\right) \right\}.
\end{aligned}
$$

The above inequality yields

$$
\begin{aligned}
V_0^{+\infty}\left(p(G(u)-G(w))\right) &\le \|p\|_{BV} \left(V_0^{+\infty}\left(\int_0^{+\infty} f\left(.,r\right) dr\right) + b \right) \inf_{r\in[0,\infty)} \left|\left(g\left(u\left(r\right)\right)-g\left(w\left(r\right)\right)\right)\right| \\
&+a\|p\|_{BV} V_0^{+\infty}\left(C_g\left(u\right)-C_g\left(w\right)\right).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\|H(u)-H(w)\|_{BV} &\le |\rho| \, \|p\|_{BV} \left(\left\| \int_0^{+\infty} f\left(.,r\right) dr \right\|_{BV} + b \right) \left|\left(g\left(u\left(0\right)\right)-g\left(w\left(0\right)\right)\right)\right| \\
&+|\rho| \, \|p\|_{BV} \left(\left\|f\left(0,.\right)\right\|_{[0,\infty)}^{A} + a \right) V_0^{+\infty}\left(C_g\left(u\right)-C_g\left(w\right)\right).
\end{aligned}
$$

Hence

$$
\|H(u)-H(w)\|_{BV} \le |\rho| \, \|p\|_{BV} \max\left\{ \left(\left\|f\left(0,.\right)\right\|_{[0,\infty)}^{A} + a \right), \left(\left\| \int_0^{+\infty} f\left(.,r\right) dr \right\|_{BV} + b \right) \right\} \left\|C_g\left(u\right)-C_g\left(w\right)\right\|_{BV},
$$

which implies

$$\|H(u)-H(w)\|_{BV} \leq M_R \mu \left\| p \right\|_{BV} \max\left\{ \left(\| f(0,.)\|^{A}_{[0,\infty)} + a \right), \left(\left\| \int_0^{+\infty} f(.,r)\,dr \right\|_{BV} + b \right) \right\} \|u-w\|_{BV}.$$

Using (5.4) and (5.5), we infer that H is a contraction and thus H admits a unique fixed point in B_R^{BV}, which completes the proof. □

Conclusion

In this thesis, we have used different approaches based on fixed point theory to solve nonlinear integral equations in various functional spaces. More precisely, we have used: fixed point techniques in the framework of b-metric spaces, a version of Krasnosel'skii theorem based on a generalized expansive mapping, the measure of noncompactness via Proinov contraction and finally Henstock-Kurzweil integrals on an unbounded interval; to establish the solvability of solutions for many nonlinear integral equations.

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