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FEASIBLE INTERIOR POINT METHOD FOR LINEAR COMPLEMENTARITY PROBLEM

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In the name of Allah, the most beneficient, the most merciful

Speak these 5 lines to yourself every single day...

Allah is with me, I can do it,

I am the best, I am a winner,

Today is my day and most importantly

Work hard in silence; let success make the noise.

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I have the immense pleasure of dedicating this modest work to :

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GLOSSARY OF NOTATION

We will use the following notations throughout the work :

NOTATION :	MEANING :
$\mathbb{R}^n$	The n-dimensional real space.
$\mathbb{R}^n_+$	The n-dimensional nonnegative real space.
$\mathbb{R}^n_{++}$	The n-dimensional positive real space.
$\mathcal{M}_n$	The real square matrices of order n.
$\mathbb{S}_n$	The symmetric real square matrices of order n.
$\mathbb{S}^+_n$	The set of positive semidefinite symmetric matrices of order n.
$\mathbb{S}^{++}_n$	The set of positive definite symmetric matrices of order n.

TABLE 0.1: SPACES.

NOTATION :	MEANING :
(MP)	Mathematical programming.
(DP)	The dual of mathematical programming.
(LCP)	Standard linear complementarity problem.
(LO)	Linear optimization.
(DLO)	The dual of linear optimization.
(CQP)	The primal problem of the convex quadratic programming.
(DCQP)	The dual problem of the convex quadratic programming.
(HLCP)	Horizontal linear complementarity problem.
(AVE)	Absolute value equation.
(GAVE)	The generalized absolute value equation.
(ODE)	The ordinary differential equation.
(HE)	The hydrodynamic equations.

TABLE 0.2: PROBLEM CLASSES.

NOTATION :	MEANING :
u^t	The transpose of a vector u.
u_i	The i-th component of u.
$0_{\mathbb{R}^n}$	$(0, \dots, 0)^t$.
$\lvert u\rvert$	$(\lvert u_1\rvert, \dots, \lvert u_n\rvert)^t$.
$u \in \mathbb{R}^n_+$	$u_i \geq 0$ for all $1 \leq i \leq n$.
$u \in \mathbb{R}^n_{++}$	$u_i > 0$ for all $1 \leq i \leq n$.
uv	$(u_1v_1, \dots, u_nv_n)^t$ "*Hadamard product*".
e	$(1, \dots, 1)^t$.
$\dfrac{u}{v}$	$\left(\dfrac{u_1}{v_1}, \dots, \dfrac{u_n}{v_n}\right)^t$ with $v \neq 0_{\mathbb{R}^n}$.
$\sqrt{u}$	$(\sqrt{u_1}, \dots, \sqrt{u_n})^t$ with $u \geq 0$.
u^{-1}	$\left(\dfrac{1}{u_1}, \dots, \dfrac{1}{u_n}\right)^t$ with $u \neq 0_{\mathbb{R}^n}$.

TABLE 0.3: VECTORS.

NOTATION :	MEANING :
$\frac{\partial h}{\partial u_i}(u)$	The partial derivative of h.
$\nabla h(u)$	The gradient of h.
$\nabla_u h(u, v)$	The gradient of h with respect to u.
$\mathcal{L}(u, \lambda, \mu)$	The Lagrangian associated with (MP).
$\phi(\nu, s)$	A smoothing function.
$\psi(x)$	A kernel function.

TABLE 0.4: FUNCTIONS.

NOTATION :	MEANING :
$\lambda_i(T)$	The eigenvalues of $T \in \mathcal{M}_n$, $1 \leq i \leq n$.
$\lambda_{max}(T)$	The largest eigenvalues of $T \in \mathcal{M}_n$.
$\lambda_{min}(T)$	The smallest eigenvalues of $T \in \mathcal{M}_n$.
$\sigma_i(T)$	The singular value of $T \in \mathcal{M}_n$, $1 \leq i \leq n$.
$\sigma_{max}(T)$	The largest singular value of $T \in \mathcal{M}_n$.
$\sigma_{min}(T)$	The smallest singular value of $T \in \mathcal{M}_n$.
I	The identity matrix.
T^{-1}	The inverse of a matrix $T \in \mathcal{M}_n$.
T^t	The transposed matrix $T \in \mathcal{M}_n$.
$0_{\mathcal{M}_n}$	The null matrix of order n.
$T \in \mathbb{S}_n^+$	The matrix $T \in \mathcal{M}_n$ is positive semidefinite.
$T \in \mathbb{S}_n^{++}$	The matrix $T \in \mathcal{M}_n$ is positive definite.
$diag(u)$	The diagonal matrix of U with $U_{ii} = u_i$, $1 \leq i \leq n$.
$H_h(u)$	The Hessian matrix of h.
$J_h(u)$	The Jacobian matrix of h.
$J_{h_u}(\nu, u)$	The Jacobian matrix of h with respect to u.

TABLE 0.5: MATRICES.

NOTATION :	MEANING :
KKT	Karush-Kuhn-Tucker.
NE	Newton equations.
LS	The line search.
IPC	Interior-Point-Condition.
IPMs	Interior-point methods.
i.e.	That is to say.
resp	respectively.
e.g	For example.

TABLE 0.6: ABBREVIATIONS.

ABSTRACT

The purpose of this study is to solve the linear complementarity problem (*LCP*). To this end, we divided this work into two parts. In the first half, we provide an interior-point algorithm based on two novel parametric kernel functions. Proving that, under suitable assumptions, we guarantee the existence and uniqueness of the solution to the *LCP*. Afterwards, we show that a certain choice of the barrier degrees of our functions coincide with the best-known iteration bound for large-update methods. Finally, we offer some numerical results that prove the utility of the proposed algorithm. In the second part, we employ a smoothing-type algorithm and assuming a reasonable assumption, we can describe the *LCP* as an NP-hard absolute value equation (*AVE*). So we must rewrite *AVE* as a set of smooth equations and introduce two smoothing functions. Then, we show that the method is well-defined when the singular values of the matrix related to *AVE* exceed one and that it is convergent under the same assumption. We also demonstrate the numerical efficacy of this algorithm using these two functions.

KEYWORDS. Linear complementarity problem, interior-point method, kernel function, large-update methods, absolute value equation, smoothing-type algorithm, smoothing function.

RÉSUMÉ

Le but de cette étude est de résoudre le problème complémentaire linéaire (*PCL*). Pour cela, nous avons divisé ce travail en deux parties. Dans la première moitié, nous proposons un algorithme de point intérieur basé sur deux nouvelles fonctions paramétriques du noyau. Prouver que, sous des hypothèses appropriées, nous garantissons l'existence et l'unicité de la solution au *PCL*. Ensuite, nous montrons qu'un certain choix des degrés de barrière de nos fonctions coïncide avec l'itération la plus connue liée aux méthodes à grand pas. Enfin, nous proposons quelques résultats numériques qui prouvent l'utilité de l'algorithme proposé. Dans la deuxième partie, nous utilisons un algorithme de type lissage et en supposant une hypothèse raisonnable, nous pouvons décrire le *PCL* comme une équation de valeur absolue (*EVA*). Nous devons donc réécrire *EVA* sous la forme d'un ensemble d'équations lisses et introduire deux fonctions de lissage. Ensuite, nous montrons que la méthode est bien définie lorsque les valeurs singulières de la matrice relatif à *EVA* dépassent un et qu'elle est convergente sous la même hypothèse. Nous démontrons également l'efficacité numérique de cet algorithme en utilisant ces deux fonctions.

MOTS CLÉS. Problème complémentaire linéaire, méthode du point intérieur, fonction du noyau, méthodes de grand pas, équation de valeur absolue, algorithme de type lissage, fonction de lissage.

ملخص

الغرض من هذه الدراسة هو حل مشكلة التكامل الخطي. و تحقيقا لهذه الغاية، قسمنا هذا العمل إلى جزئين. في النصف الأول، نقدم خوارزمية نقطة داخلية تعتمد على دالتين جديدتين

للنواة. إثبات أننا، في ظل الافتراضات المناسبة، نضمن وجود وتفرد الحل لـمشكلة التكامل الخطي. بعد ذلك، نظهر أن اختيارا معينا لدرجات الحاجز لدوالنا يتزامن مع التكرار الأكثر شهرة المرتبط بطرق التحديث الكبيرة. وأخيرا، نقدم بعض النتائج العددية التي تثبت فائدة الخوارزمية المقترحة. في الجزء الثاني، قمنا باستخدام خوارزمية من نوع التنعيم وبتقديم افتراض معقول، يمكننا وصف مشكلة التكامل الخطي كمعادلة القيمة المطلقة. لذلك يجب علينا إعادة كتابة معادلة القيمة المطلقة كمجموعة من المعادلات السلسة وإدخال دالتين من دوال التنعيم. بعد ذلك، نوضح أن الطريقة تكون معرفة جيدا عندما تتجاوز القيم المفردة للمصفوفة المتعلقة بمعادلة القيمة المطلقة واحدا، وأنها متقاربة تحت نفس الافتراض. نوضح أيضا الفعالية العددية لهذه الخوارزمية باستخدام هاتين الدالتين.

الكلمات الدالة. مشكلة التكامل الخطي، طريقة النقطة الداخلية، دالة النواة، طرق التحديث الكبيرة، معادلة القيمة المطلقة، خوارزمية من نوع التنعيم، دالة التنعيم.

GENERAL INTRODUCTION

Operations research is a large field of mathematics that covers many aspects of minimization and optimization. In any case, we make every effort to produce the finest results possible. Every problem has a set of qualities that must be attained by the expected solutions, which can be either a decision problem or a problem improvement problem. All problems are defined as problems in order to find a better solution. The challenge of improvement consists of locating an effective solution that fulfills the maximum number of limitations while still achieving specific goals.

The linear complementarity problem (*LCP*) is a mathematical theory-rich inequality system. The first-order optimality requirements of a quadratic optimization problem are a well-known scenario in which linear complementarity issues can be encountered. In the mid-1960s, a systematic study of the broad form of complementarity problem started. Over the course of five decades, the topic has evolved into a highly productive specialty in the field of mathematical optimization. A rich mathematical theory, a slew of efficient solution methods, a slew of intriguing linkages to many disciplines, and a wide variety of relevant applications in geoscience, engineering, and economics are among the developments. Mathematicians (pure, applied and computational), computer scientists, engineers and economists have all contributed to the literature on complementarity problems. To tackle issue *LCP*, many methods have been presented. They may be based on pivoting techniques **[23]** and **[52]**, which frequently suffer from the combinatorial aspect of the problem, on interior point methods **[51]** or on nonsmooth. Other iterative approaches can be found as **[25]**.

The absolute value equation (*AVE*) was one of the most important issues that drew the attention of many scholars who were particularly interested in exploring ways to solve it and, before that, whether it has a solution or not. The importance of *AVE* is derived from its numerous applications in mathematics and applied sciences. The linear complementarity problem (*LCP*), which incorporates linear optimization (*LO*), convex quadratic programming (*CQP*), ordinary differential equation (*ODE*), and hydrodynamic equation (*HE*), may be solved using the *AVE*. In its general version, the absolute value equation is hence NP-hard. As a consequence, we have various results concerning the presence of a solution to *AVE*, which were presented by *Mangasarian et al.* **[58]** and *Rohn et al.* **[69]** as necessary requirements for the unique solvability of the particular form of absolute value equation. Because the *AVE* system is nonlinear, there are numerous numerical techniques in the literature for solving it. In 2004, *Rohn* **[68]** published the general form of the *AVE* and utilized the theorem of the alternatives to solve it. The research then revealed a number of methods, such as the reformulation of the *AVE* as a standard linear complementarity problem (see **[39]**, **[56]** and **[78]**), a concave minimization optimization method (see **[1]** and **[57]**), a generalized Newton method (see **[39]** and **[55]**), a smoothing Newton approach (see **[18]**, **[43]** and **[71]**), many iterative techniques like **[29]**, **[45]**, **[46]**, **[61]**, **[63]**, **[69]** and **[77]** . Several ways were also presented to solve the absolute value equations, which is still ongoing to this day.

On the other hand, we discovered that the interior-point method (*IPM*), which *Karmarkar* [44] introduced in 1984, is one of the most efficient numerical methods for solving large classes of optimization problems and is highly efficient in both theory and practice because it has polynomial complexity, can solve very large problems, and is of the Newton type (leads to digital efficiency) during our study of other problems. The essential difference between this approach and the others is that the former advances at the boundary of the realizable domain, while the *Karmarkar* method progresses entirely within the realizable domain. Several studies on this issue were undertaken at that time, resulting in about three thousand articles in a short period of time. The most important result is the primal-dual techniques, which are successful and easy to adapt to solve many types of problems (linear complementarity problems, quadratic problems, convex problems, ...). However, because the goal of this sort of study is usually to attain the greatest known complexity for large-update techniques, the authors relied on a new class of functions known as kernel functions to do so.

Roos et al. [70] initially described the primal-dual *IPM*s, which are based on the conventional logarithmic barrier function. *Peng et al.* [65] proposed the so-called self-regular proximity function in 2002. According to *Bai et al.* [6], some analytical approaches for assessing the difficulty of primal-dual *IPM*s were developed in 2004 based on a novel class of function called kernel functions with some simple conditions on them and their derivatives, from which this research yielded the most accurate findings for both small-update complexity, namely, $O(\sqrt{n}\log\frac{n}{\varepsilon})$ and large-update complexity, namely, $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$, wherein *Bai et al.* suggested a new kernel function with an exponential barrier term and introduced the first new kernel function with a trigonometric barrier term in the same reference. Then, very recently, this last class was embraced by all of the articles made by researchers to solve multiple challenges based on various kernel functions, some of which are listed below.

In 2008, *El Ghami et al.* [33] offered a parameterized kernel function with a logarithmic barrier term, while the following year, *Bai et al.* [8] introduced a parameterized kernel function with no logarithmic barrier term in the general case. In 2011, *Liu et al.* [54] proposed a novel type of simple kernel function that yields reasonable iteration bounds for primal-dual *IPM*s. *El Ghami et al.* [32] evaluated the first kernel function for linear optimization with a trigonometric barrier term given in [6] in 2012, and since then, research has focused on developing a new kernel function with a trigonometric barrier term to improve the complexity bound, and *Zhang* [79] published a novel kernel function for convex quadratic semi-definite optimization. *Peyghami et al.* [67] proposed a unique kernel function with an exponential trigonometric for linear optimization problems in 2014. *Achache* [2] investigated the complexity analysis of a new kernel function for semidefinite optimization in 2015, while *Li et al.* [53] introduced another trigonometric barrier function. *Bouafia et al.* [12] and [14] introduced two parameterized kernel functions in 2016 for interior-point procedures. The first has a trigonometric barrier term where the complexity bound was generalized and enhanced based on a new kernel function with trigonometric barrier terms derived in [53], while the second function generalizes the kernel function reported in **[8]**. In 2017, *Peyghami et al.* **[66]** proposed a new kernel function for convex quadratic semidefinite optimization problems. In 2018, *Bouafia et al.* **[13]** presented the first parameterized logarithmic kernel function, *Fathi-Hafshejani et al.* **[31]** presented a large-update primal-dual interior-point algorithm for linear optimization problems based on a new kernel function with a trigonometric growth term and a novel kernel function for primal-dual *IPM*s for linear complementarity problems was put out by *Djeffal et al.* **[28]**. In 2019, *Bounibane et al.* **[16]** expanded on linear optimization using a novel kernel function and *Moaberfardi et al.* **[60]** presented an *IPM* for lineair optimization based on a trigonometric kernel function. The next year, *Touil et al.* **[76]** introduced and offered a new nonlogarithmic kernel function based on the type of his barrier term, hyperbolic, while *Benterki et al.* **[15]** proposed a new parametric kernel function for convex quadratic programming based on the primal-dual *IPM*. *Guerdouh et al.* **[34]** introduced a unique parametrized kernel function with a hyperbolic barrier term in 2021 to provide an efficient primal-dual interior-point technique for linear optimization problems. *Benhadid et al.* **[11]** reported a complexity study of an interior-point approach for linear optimization that is based on a novel parametric kernel function with a double barrier term in 2022. With their unique barrier degree selections, the majority of these algorithms produce the best-known complexity iteration bounds.

Our objective in this study is to solve the linear complementarity problem using theoretical, algorithmic and numerical contributions. It is split into two halves. The first section focuses on interior-point techniques of the type central trajectory through a new kernel function, due to their numerical successes in comparison to other classes of interieur-point methods. In the second, we are interested in the smoothing-type method to solve our problem, but it is necessary to reformulate *LCP* into *AVE* then as a collection of smooth equations, as for the existence of the solution, we use an assumption of *Mangasarian* and our new theoretical results must be supported by effective numerical experiments on several examples. So, while we attempt to create a work that combines the most significant topic and two methodologies in recent times, we continue to offer three obvious questions.

- ► Is it possible to develop new kernel functions for solving the linear complementarity problem using the interior-point method ? Most critically, do these functions validate the well-known complexity iteration bounds ?
- ► Is it possible to solve the linear complementarity problem using any other new smoothing functions ?
- ► As for the numerical implementation, is it better for our functions that we propose in both methods ? Is it true that the solutions to *LCP* and *AVE* are equivalent ?

In response to the three points raised above, we developed this thesis and divided the material gleaned from our articles into three chapters, as follows :

The **FIRST CHAPTER** reminds us of the essential conceptions of frequent usage for the following : matrix analysis, basic differetial and topological notions, analyzing convexity, fundamental optimization tools such as mathematical programming, optimization algorithms and the Newton method are used to solve nonlinear systems. Then, the linear complementarity problem, the absolute value equation, relation between *LCP* and *AVE* and their solutions, the smoothing-type algorithm and interior-point methods.

We provided two new parameterized kernel functions in the **SECOND CHAPTER**, which significantly contribute to the development of a novel design for primal–dual interior-point algorithms to solve the linear complementarity problem. The first function lacks a logarithmic barrier component, whereas the second incorporates a new hyperbolic-logarithmic barrier term. Then, in order to analyze the complexity of our approach, we investigate some essential properties of the two new kernel functions. After that, using our two functions, we compute the iteration bounds for small- and large-update techniques and identify the values of barrier degrees that reach the best-known complexity constraints. The numerical findings are amazing, suggesting that our theoretical study is valid, as presented at the end of the chapter.

In the **THIRD CHAPTER**, we are interested in a smoothing-type approach for solving the absolute value equation which was obtained from the system *LCP*. The key contribution of this chapter is that we change the absolute value equations into a system of smooth equations, present the two smoothing functions, explore their characteristics, and illustrate their graphs. Then, assuming a suitable assumption, we describe the algorithm based on our two functions, provide its fundamental features, and demonstrate its well-definedness. Furthermore, we show that this last condition ensures the convergence of our method. We conclude this chapter with some numerical results from our investigations that employ the procedure to illustrate the effectiveness of our new functions.

Finally, we conclude our work with a general conclusion that summarizes the new findings in this thesis, as well as their uniqueness and implications for future studies and we provide an appendix with a comparison of the two algorithms and their solutions, followed by a bibliography.

1

BASIC CONCEPTS

INTRODUCTION

In this chapter, we review several fundamental ideas (definition, theorem, ...) that will be used in the subsequent chapters of this thesis. We review the key findings in matrix analysis, differential calculus, analysis convexe, fundamental optimization tools, linear complementarity problem, absolute value equation and some numerical methods are used to solve numerous types of optimization problems. We first identify the sets of :

- Real square matrices of order n by : $\mathcal{M}_n = \{T := (t_{ij}) \in \mathbb{R}^{nxn}\}$.
- Symmetric real square matrices of order n by : $\mathbb{S}_n = \{T \in \mathcal{M}_n / T^t = T\}$.

1.1 MATRIX ANALYSIS

We will start by going over some basic matrix concepts and notation.

EIGENVALUES & SINGULAR VALUES

PROPOSITION 1.1.1. *λ is an eigenvalue of the matrix $T \in \mathcal{M}_n$ if it is a roots of characteristic polynomial of T which given as follows :*

$$P_T(\lambda) = det(T - \lambda I),$$

where $det(.)$ called the determinant of an matrix and I denotes the identity matrix in $\mathcal{M}_n$. We call $u \in \mathbb{R}^n \backslash \{0\}$ an eigenvector of T associated to the eigenvalue λ if it achieves :

$$Tu = \lambda u.$$

REMARK 1.1.2. *The determinant of a matrix T is the product of its eigenvalues.*

PROPERTIES 1.1.3. *Let λ_{max} (resp. λ_{min}) be the maximal (resp. minimal) eigenvalue of T, then :*

(a) $\lambda_{min}(T) = \lambda_1(T) \leq \lambda_2(T) \leq \cdots \leq \lambda_n(T) = \lambda_{max}(T)$.

(b) *For all* $T, R \in \mathbb{S}_n$ *:*

(i) $\lambda_{min}(T+R) \geq \lambda_{min}(T) + \lambda_{min}(R)$.

(ii) $\lambda_{max}(T+R) \leq \lambda_{max}(T) + \lambda_{max}(R)$.

(iii) $\lambda_{min}(-T) = -\lambda_{max}(T)$ *and* $\lambda_{max}(-T) = -\lambda_{min}(T)$.

DEFINITION 1.1.4. *The singular values of a matrix* $T \in \mathcal{M}_n$ *are the square roots of the eigenvalues of symmetric matrix* T^tT*, such that :*

$$\sigma_i(T) = \sqrt{\lambda_i(T^tT)}, \qquad i = 1, \ldots, n.$$

REMARK 1.1.5. *If* $T \in \mathbb{S}_n$*, then we have :* $\sigma_i(T) = \lambda_i(T)$ *for* $i = 1, \ldots, n$.

PROPOSITION 1.1.6. *Suppose that* $T, R \in \mathcal{M}_n$*. Let* $\sigma_{min}(T)$ *denote the minimum singular value of* T *and* $\sigma_{max}(R)$ *denote the maximum singular value of* R*. Then, we have :*

$$\sigma_{min}(T) > \sigma_{max}(R) \quad \Leftrightarrow \quad \sigma_{min}(T^tT) > \sigma_{max}(R^tR).$$

DEFINITION 1.1.7. *A matrix* $T \in \mathcal{M}_n$ *is invertible (nonsingular) if there exists a matrix* R *such that* $TR = RT = I$ *where* R *denoted by* T^{-1}.

REMARK 1.1.8. (a) *If* $T \in \mathbb{S}_n$ *then all eigenvalues of* T *are real.*

(b) *The singular values of* $T \in \mathcal{M}_n$ *are always positive or null.*

(c) *The matrix* $T \in \mathcal{M}_n$ *is invertible* $\Leftrightarrow$ 0 *is not an eigenvalue of* T.

(d) *Let the matrix* $T \in \mathcal{M}_n$ *is invertible, then :*

(i) λ *is an eigenvalue of* $T \Leftrightarrow \lambda$ *is an eigenvalue of* T^t.

(ii) λ *is an eigenvalue of* $T \Leftrightarrow \lambda^{-1}$ *is an eigenvalue of* T^{-1}.

(iii) λ *is an eigenvalue of* $T \Leftrightarrow \lambda + \alpha$ *is an eigenvalue of* $T + \alpha I$.

SCALAR PRODUCT & NORM

DEFINITION 1.1.9. *The euclidean scalar product of two vectors* u *and* v *of* $\mathbb{R}^n$ *is defined by :*

$$\langle u, v \rangle = u^t v = \sum_{i=1}^{n} u_i v_i, \tag{1.1}$$

where the application $\langle ., . \rangle : \mathbb{R}^n \text{x}\, \mathbb{R}^n \longrightarrow \mathbb{R}$ *is verifies :*

(a) $\langle u, u \rangle \geq 0$ *and* $\langle u, u \rangle = 0 \Leftrightarrow x = 0_{\mathbb{R}^n}$,

(b) $\langle u, v \rangle = \langle v, u \rangle$,

(c) *the application* $\langle u, v \rangle$ *is linear.*

DEFINITION 1.1.10. *For* $u, v \in \mathbb{R}^n$*, we recall an application* $\|.\| : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ *a vectorial norm if it verifies the following :*

(a) $\|u\| \geq 0$ *and* $\|u\| = 0 \Leftrightarrow u = 0_{\mathbb{R}^n}$,

(b) $\|\alpha u\| = |\alpha|\|u\|, \quad \forall \alpha \in \mathbb{R}$,

(c) $\|u + v\| \leq \|u\| + \|v\|$.

REMARK 1.1.11. (a) *The euclidean norm associated with the scalar product* (1.1) *is defined by :*

$$\|u\|_2 = \sqrt{\langle u, u\rangle} = \left(\sum_{i=1}^{n} u_i^2\right)^{\frac{1}{2}}.$$

(b) *The Cauchy-Schwarz inequality given as follows :*

$$|\langle u, v\rangle| \leq \|u\|\|v\|, \quad \forall u, v \in \mathbb{R}^n.$$

(c) *For two vectors* u *and* v *in* $\mathbb{R}^n$*, if* $\langle u, v\rangle = 0$ *then* u *and* v *are orthogonal.*

Before presenting the next definition, we recall that the function trace $Tr : \mathcal{M}_n \to \mathbb{R}$ is given by the expression :

$$Tr(T) = \sum_{i=1}^{n} t_{ij}.$$

DEFINITION 1.1.12. *The usual scalar product of* $T \in \mathcal{M}_n$ *and* $R \in \mathcal{M}_n$ *is defined by :*

$$\langle T, R\rangle = Tr(T^t R) = \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij} r_{ij}. \tag{1.2}$$

REMARK 1.1.13. *The norm associated with the scalar product* (1.2) *is called the Frobenius norm and given by :*

$$\|T\|_F = \sqrt{Tr(T^t T)}.$$

DEFINITION 1.1.14. *An application* $\|.\| : \mathcal{M}_n \to \mathbb{R}$ *is said to be a matrix norm if it satisfies the following properties :*

(a) $\|T\| \geq 0$ *and* $\|T\| = 0 \Leftrightarrow T = 0_{\mathcal{M}_n}$,

(b) $\|\alpha T\| = |\alpha|\|T\|, \quad \forall \alpha \in \mathbb{R}$,

(c) $\|T + R\| \leq \|T\| + \|R\|$,

(d) $\|TR\| \leq \|T\|\|R\|$.

PROPOSITION 1.1.15. *For all* $T \in \mathcal{M}_n$*, we have :*

(a) $\|T\|_1 = \max\limits_{j=1,\ldots,n} \sum\limits_{i=1}^{n} |t_{ij}|$.

(b) $\|T\|_2 = \sqrt{\lambda_{max}(T^t T)}$ *(The spectral norm).*

(c) $\|T\|_\infty = \max\limits_{i=1,\ldots,n} \sum\limits_{j=1}^{n} |t_{ij}|$.

REMARK 1.1.16. (a) *Let* $T \in \mathcal{M}_n$*, then :*

$$\|T\|_2 \leq \|T\|_F \leq \sqrt{n}\|T\|_2, \quad \text{for all } n \geq 1.$$

(b) *If* $T \in \mathbb{S}_n$*, we have the following :*

$$\|T\|_F = \sqrt{Tr(T^2)} \quad \text{and} \quad \|T\|_2 = \lambda_{max}(T),$$

where $\lambda_{max}(T) = \max\limits_{\|u\|=1} \langle u, Tu\rangle$*, for all* $u \in \mathbb{R}^n$ *(Rayleigh-Ritz).*

ABSOLUTE VALUE

DEFINITION 1.1.17. *The absolute values of a matrix $T \in \mathcal{M}_n$ is given by :*

$$|T| = (|t_{ij}|)_{1\leq i,j\leq n}.$$

DEFINITION 1.1.18. *For the vector $u \in \mathbb{R}^n$. Then, the absolute values of u is given by :*

$$|u| = (|u_i|)_{1\leq i\leq n}.$$

DEFINITION 1.1.19. *For $u \in \mathbb{R}^n$, we define the two vectors u_+ and u_- as follow :*

$$u_+ = \max(u, 0) \quad \text{and} \quad u_- = \max(0, -u)$$

such that :

$$\begin{cases} u = u_+ - u_-, & (u_-, u_+) \in \mathbb{R}^{2n}_+ \\ |u| = u_+ + u_- \\ u_-^t u_+ = 0. \end{cases} \tag{1.3}$$

CLASSES OF MATRIX

The most significant matrix classes are recalled in this part.

DEFINITION 1.1.20. *$T \in \mathcal{M}_n$ is called a positive semi-definite matrix (i.e., $T \in \mathbb{S}^+_n$ or $T \succeq 0$) if :*

$$\langle Tu, u\rangle = u^t T u \geq 0, \quad \forall u \in \mathbb{R}^n.$$

DEFINITION 1.1.21. *$T \in \mathcal{M}_n$ is called a positive definite matrix (i.e., $T \in \mathbb{S}^{++}_n$ or $T \succ 0$) if :*

$$\langle Tu, u\rangle = u^t T u > 0, \quad \forall u \in \mathbb{R}^n \backslash \{0\}.$$

THEOREM 1.1.22. ([37]) *Let $T \in \mathbb{S}_n$, then the following propositions are equivalents :*

(a) *$T \succ 0$ (resp. $T \succeq 0$),*

(b) *$\lambda_i(T) > 0$ (resp. $\lambda_i(T) \geq 0$) $i = 1, ..., n$,*

(c) *There is a matrix $O \in \mathcal{M}_n$ such that $T = O^T O$*

(d) *All dominant principal minors of T are strictly positive (resp. positive).*

PROPERTIES 1.1.23. *If $T, R \in \mathbb{S}_n$, then :*

(a) *$T \succeq R \Leftrightarrow T - R \succeq 0$,*

(b) *$T + R \succeq R$,*

(c) *If $T - I \succeq 0$, then T is invertible and $I - T^{-1} \succeq 0$,*

(d) *If $R \succeq T \succ 0$, then R is invertible (i.e., $R \succ 0$) and $T^{-1} \succeq R^{-1}$.*

REMARK 1.1.24. *Let $T \in \mathbb{S}_n$, then $T \succ 0$ if and only if $T^{-1} \succ 0$.*

THEOREM 1.1.25. *Suppose that* $\boldsymbol{T} \succ 0$, *then the following matrix is invertible :*

$$\begin{pmatrix} \boldsymbol{T} & -\boldsymbol{I} \\ \boldsymbol{U}_+ & \boldsymbol{U}_- \end{pmatrix}$$

where $\boldsymbol{U}_- = \boldsymbol{diag}(\boldsymbol{u}_-) \in \boldsymbol{\mathcal{M}}_n$ *and* $\boldsymbol{U}_+ = \boldsymbol{diag}(\boldsymbol{u}_+) \in \boldsymbol{\mathcal{M}}_n$.

PROOF. Using the last equality in system (1.3) and decomposed of matrix $\boldsymbol{T}$ we get :

$$\boldsymbol{R} = \begin{pmatrix} t_{11} & \dots & \dots & t_{1n} & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\ t_{n1} & \dots & \dots & t_{nn} & 0 & \dots & \dots & -1 \\ 0 & \dots & \dots & 0 & (u_-)_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 0 & (u_-)_{n_1} & \dots & 0 \\ (u_+)_1 & \dots & \dots & 0 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & (u_+)_{n_2} & 0 & \dots & \dots & 0 \end{pmatrix}$$

where $\boldsymbol{R} \in \mathbb{R}^{2nx2n}$, $n_1 = \sup\{i : (u_-)_i > 0\}$ and $n_2 = \sup\{i : (u_+)_i > 0\}$. The determinant of this matrix is equal to :

$$\begin{aligned} det(\boldsymbol{R}) &= det \begin{pmatrix} t_{11}(u_-)_1 & \dots & t_{1n_1}(u_-)_{n_1} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ t_{n1}(u_-)_1 & \dots & t_{nn_1}(u_-)_{n_1} & 0 & \dots & 0 \\ 0 & 0 & 0 & (u_+)_1 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (u_+)_{n_2} \end{pmatrix} \\ &= \prod_{i=1}^{n_2} (u_+)_i \cdot \prod_{i=1}^{n_1} (u_-)_i \cdot det \begin{pmatrix} t_{11} & \dots & t_{1n_1} \\ \vdots & \ddots & \vdots \\ t_{n1} & \dots & t_{nn_1} \end{pmatrix} \end{aligned} \tag{1.4}$$

since all the principal minors of the matrix $\boldsymbol{T}$ are nonzeros and $\prod_{i=1}^{n_2} (u_+)_i \prod_{i=1}^{n_1} (u_-)_i$ is nonzero, we conclude that $det(\boldsymbol{R}) \neq 0$. □

EXAMPLE 1.1.26. *Let :*

$$T_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad (u_-)_1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \quad (u_+)_1 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad (u_-)_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad (u_+)_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

By using (**1.4**), *we can easily check that :*

$$det(R_1) = det \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & 4 & 0 & -1 \\ 0 & 0 & 5 & 0 \\ 0 & 4 & 0 & 0 \end{pmatrix} = 4 \cdot 5 \cdot det(1) = 20 \neq 0,$$

with $n_1 = n_2 = 1$ *and*

$$det(R_2) = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 0 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} = 1 \cdot 2 \cdot 1 \cdot det \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = 4 \neq 0$$

where $n_1 = 2$.

DEFINITION 1.1.27. *Let* $T \in \mathcal{M}_n$, *then the following propositions are equivalents :*

(a) T *is a* $\mathscr{P}_0$*-matrix,*

(b) $\forall i \in \{1, \ldots, n\}, \quad det(T_{ii}) \geq 0$,

(c) *for all* $0 \neq u \in \mathbb{R}^n$, *we can find an index* i *such that* $u_i \neq 0$ *and* $u_i(Tu)_i \geq 0$,

(d) $\forall i \in \{1, \ldots, n\}$, *the real eigenvalues of* T_{ii} *are positive.*

REMARK 1.1.28. *If* T *is positive semi-definite then* T *is* $\mathscr{P}_0$*-matrix.*

DEFINITION 1.1.29. *Let* $T \in \mathcal{M}_n$, *then the following propositions are equivalents :*

(a) T *is a* $\mathscr{P}$*-matrix,*

(b) $\forall i \in \{1, \ldots, n\}, \quad det(T_{ii}) > 0$,

(c) *any* $u \in \mathbb{R}^n$ *satisfying* $u(Tu) \leq 0$ *is zero,*

(d) $\forall i \in \{1, \ldots, n\}$, *the real eigenvalues of* T_{ii} *are strictly positive.*

REMARK 1.1.30. *If* T *is positive definite then* T *is* $\mathscr{P}$*-matrix.*

DEFINITION 1.1.31. *Let* $\kappa \geq 0$. $T \in \mathcal{M}_n$ *is called* $\mathscr{P}_*(\kappa)$*-matrix if for all* $u \in \mathbb{R}^n$, *we have :*

$$(1 + 4\kappa) \sum_{i \in I^+(u)} (u_i(Tu)_i) + \sum_{i \in I^-(u)} (u_i(Tu)_i) \geq 0, \tag{1.5}$$

where $I^+(u) = \{i : u_i(Tu)_i \geq 0\}$ *and* $I^-(u) = \{i : u_i(Tu)_i < 0\}$.

REMARK 1.1.32. *The relation (1.5) can also be written as follows :*

$$u^tTu \geq -4\kappa \sum_{i\in I^+(u)} (u_i(Tu)_i), \quad \forall u \in \mathbb{R}^n, \kappa \geq 0,$$

and from this last inequality it is clear that if $\kappa = 0$, then T is positive semi-definite matrix.

DEFINITION 1.1.33. *We say that a matrix $T \in \mathcal{M}_n$ is a $\mathscr{P}_*$-matrix if there exists a $\kappa > 0$ such that $T \in \mathscr{P}_*(\kappa)$. We note the set of $\mathscr{P}_*$-matrices by :*

$$\mathscr{P}_* := \bigcup_{\kappa>0} \mathscr{P}_*(\kappa).$$

REMARK 1.1.34. *According to the latest definitions, we can easily check that :*

$$\mathscr{P} \subsetneq \mathscr{P}_* \subsetneq \mathscr{P}_0.$$

1.2 BASIC DIFFERENTIAL AND TOPOLOGICAL NOTIONS

The term topology in mathematics refers to the study of the characteristics of functions, such as their continuity, limits and limits of sequences. The concept of differential allows the notion of derivative to be extended to functions of many variables.

OPEN & CLOSED SETS

DEFINITION 1.2.1. *A subset $\mathscr{C}$ of $\Omega \subset \mathbb{R}^n$ is said to be open if for all element $u \in \mathscr{C}$ there exists $r > 0$ such that $B(u, r) \subset \mathscr{C}$.*

DEFINITION 1.2.2. *A subset $\mathscr{C}$ of $\Omega \subset \mathbb{R}^n$ is said to be closed if its complementary is open.*

EXAMPLE 1.2.3. *The two sets $\emptyset$ and Ω are the only open-closed parts of Ω, the open ball $B(u, r)$ is an open of Ω and the closed ball $\overline{B}(u, r)$ is a closed of Ω.*

DEFINITION 1.2.4. *Let $\mathscr{C}$ a subset of $\Omega \subset \mathbb{R}^n$, the interior of $\mathscr{C}$ which noted $\mathtt{int}(\mathscr{C})$ is the set of elements $u \in \mathscr{C}$ for which there exists $r > 0$ such that $B(u, r) \subset \mathscr{C}$.*

DEFINITION 1.2.5. *Let $\mathscr{C}$ a subset of $\Omega \subset \mathbb{R}^n$, the adhesion (or the closure) of $\mathscr{C}$ is noted $\overline{\mathscr{C}}$ and given by :*

$$\overline{\mathscr{C}} = \{u \in \mathscr{C}, \forall r > 0, B(u, r) \cap \mathscr{C} = \emptyset\}.$$

ACCUMULATION POINT

DEFINITION 1.2.6. *Let $\mathscr{C}$ a subset of $\Omega \subset \mathbb{R}^n$. A vector u of Ω is called an accumulation point of $\mathscr{C}$ if u is in the closure of $\mathscr{C}\backslash\{u\}$ $\left(\text{i.e., } \overline{\mathscr{C}\backslash\{u\}}\right)$.*

REMARK 1.2.7. *An accumulation point of $\mathscr{C}$ is not necessarily an element of $\mathscr{C}$.*

EXAMPLE 1.2.8. *Let $\mathscr{C} =]0, 1[\cup \{2\}$, the set of accumulation points of $\mathscr{C}$ is $[0, 1]$.*

NOTION OF CONVERGENCE AND BORNITUDE OF A SEQUENCE

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of elements of $\Omega \subset \mathbb{R}^n$ and $u^* \in \Omega$.

DEFINITION 1.2.9. *The sequence $(u_k)_{k\in\mathbb{N}}$ converges to u^* if $\lim\limits_{k\to+\infty} \|u_k - u^*\| = 0$ and we can write $\lim\limits_{k\to+\infty} u_k = u^*$. In this case, we say that $(u_k)_{k\in\mathbb{N}}$ is convergent.*

REMARK 1.2.10. *A subset $\mathscr{C} \subset \Omega$ is said to be closed if for any sequence $(u_k)_{k\in\mathbb{N}}$ which satisfies $\lim\limits_{k\to+\infty} \|u_k - u^*\| = 0$ we have $u^* \in \mathscr{C}$.*

DEFINITION 1.2.11. *A sequence $(u_k)_{k\in\mathbb{N}}$ is bounded if it remains between two real fixed values σ and Σ. With mathematical notations, we have :*

$$\sigma \leq u_k \leq \Sigma, \quad \text{for any rank } k \in \mathbb{N}.$$

CONTINUITY

Let $\Omega \subset \mathbb{R}^n$ is an open set and the function $h : \Omega \to \mathbb{R}$.

DEFINITION 1.2.12. *We say that h is continuous in $u \in \Omega$ if and only if for any sequence $(u_k)_{k\in\mathbb{N}}$ of elements of Ω which converges to u^*, the sequence $(h(u_k))_{k\in\mathbb{N}}$ converges to $h(u^*)$. If h is continuous at any point of Ω, then h is continuous over all Ω.*

DEFINITION 1.2.13. *The partial derivative of h on point u with respect to the variable u_i is given by :*

$$\frac{\partial h}{\partial u_i}(u) = \frac{\partial h}{\partial e_i}(u) = \lim_{t\to 0} \frac{h(u + te_i) - h(u)}{t}, \quad \forall u \in \Omega,$$

where e_i is the i element of the canonical basis of $\mathbb{R}^n$.

DEFINITION 1.2.14. *The gradient of h on u is given by :*

$$\nabla h(u) = \left(\frac{\partial h}{\partial u_1}(u) \quad \frac{\partial h}{\partial u_2}(u) \quad \dots \quad \frac{\partial h}{\partial u_n}(u) \right)^t \in \mathbb{R}^n, \quad \forall u \in \Omega.$$

DEFINITION 1.2.15. *The Hessian matrix of h on $u \in \Omega$ is given by :*

$$H_h(u) = \nabla^2 h(u) = \begin{pmatrix} \frac{\partial^2 h}{\partial u_1^2}(u) & \dots & \frac{\partial^2 h}{\partial u_1 \partial u_n}(u) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 h}{\partial u_n \partial u_1}(u) & \dots & \frac{\partial^2 h}{\partial u_n^2}(u) \end{pmatrix} \in \mathcal{M}_n.$$

DEFINITION 1.2.16. *We say that h is of class $\mathcal{C}^k$ on Ω $\left(\text{i.e., } h \in \mathcal{C}^k(\Omega)\right)$ if all partial derivatives until order k exist and are continuous.*

PROPOSITION 1.2.17. *The critical point of the function h is the point denoted by u^* and check :*

$$\nabla h(u^*) = 0_{\mathbb{R}^n}.$$

REMARK 1.2.18. *(a) $\forall u \in \Omega$ and $\forall a \in \mathbb{R}^n$, we have* $\dfrac{\partial h}{\partial a}(u) = \langle \nabla h(u), a \rangle = a^t \nabla h(u)$.

(b) If $h \in \mathcal{C}^2(\Omega)$, then $H_h(u) \in \mathbb{S}_n$.

(c) If $h : \mathbb{R}^n \to \mathbb{R}^n$, the Jacobian matrix of h on u is given by :

$$J_h(u) = \begin{pmatrix} \dfrac{\partial h_1}{\partial u_1}(u) & \ldots & \dfrac{\partial h_1}{\partial u_n}(u) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 h_n}{\partial u_1}(u) & \ldots & \dfrac{\partial^2 h_n}{\partial u_n}(u) \end{pmatrix} \in \mathcal{M}_n.$$

1.3 CONVEX ANALYSIS

The concept of convexity, a field of mathematics that examines convex sets and functions, is introduced in this section. It has evolved significantly as a result of its interactions with optimization, where it adds unique qualities to the issues being researched.

CONVEX SETS

DEFINITION 1.3.1. *A subset $\mathscr{C}$ of $\mathbb{R}^n$ is said affine (i.e., linear) if :*

$$\alpha u + (1-\alpha)v \in \mathscr{C}, \quad \forall u, v \in \mathscr{C}, \quad \forall \alpha \in \mathbb{R}.$$

DEFINITION 1.3.2. *A subset $\mathscr{C}$ of $\mathbb{R}^n$ is said convex if :*

$$\alpha u + (1-\alpha)v \in \mathscr{C}, \quad \forall u, v \in \mathscr{C}, \quad \forall \alpha \in [0,1].$$

DEFINITION 1.3.3. *We call affine (i.e., linear) combination of Elements $u_1, \ldots, u_k$ of $\mathbb{R}^n$ any element of the form :*

$$u = \sum_{i=1}^{k} \alpha_i u_i, \quad \text{with} \quad \alpha_i \in \mathbb{R} \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i = 1.$$

DEFINITION 1.3.4. *Any linear combination satisfied :*

$$\sum_{i=1}^{k} \alpha_i u_i, \quad \text{where} \quad \alpha_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i = 1.$$

is called convex combination of k-vectors $u_1, \ldots, u_k$ of $\mathbb{R}^n$.

CONVEX FUNCTIONS

Let $\mathscr{C}$ be a convex subset and the function $h : \mathscr{C} \to \mathbb{R}$.

DEFINITION 1.3.5. *h is said convex on $\mathscr{C}$ if :*

$$h\left(\alpha u + (1-\alpha)v\right) \leq \alpha h(u) + (1-\alpha)h(v), \quad \forall u, v \in \mathscr{C}, \quad \forall \alpha \in [0,1]. \tag{1.6}$$

REMARK 1.3.6. *If $u = v$ or $\alpha = 0$, then the inequality (1.6) is trivial.*

DEFINITION 1.3.7. *h is said strictly convex on $\mathscr{C}$ if :*

$$h(\alpha u + (1-\alpha)v) < \alpha h(u) + (1-\alpha)h(v), \quad \forall u, v \in \mathscr{C}, \quad \forall \alpha \in]0,1[.$$

DEFINITION 1.3.8. *h is said Strongly convex (β-convex) on $\mathscr{C}$ if there exists $\beta > 0$ such that for all $u, v \in \mathscr{C}$ and for all $\alpha \in [0,1]$, we have :*

$$h(\alpha u + (1-\alpha)v) \leq \alpha h(u) + (1-\alpha)h(v) - \frac{\beta}{2}\alpha(1-\alpha)\|u-v\|^2.$$

We can clarify the implication between the three previous concepts in the following schema.

$$\boxed{\text{Strongly convex} \Rightarrow \text{Strictly convex} \Rightarrow \text{Convex.}}$$

REMARK 1.3.9. *The converses of the above implications are not true in general and to confirm this we suggest the following function $h(u) = Au + b$ with $A \in \mathcal{M}_n$ and $u, b \in \mathbb{R}^n$. Since h is linear then it is clear that it is convex but it is not strictly convex so it is not strongly convex.*

In general, it is difficult to check the convexity of a function using only the definition. To this end, we present the following results.

THEOREM 1.3.10. *If $h \in \mathcal{C}^2(\Omega)$, Ω an open and $\mathscr{C} \subset \Omega$ convex. The following conditions are equivalents :*

(a) h is convex on $\mathscr{C}$,

(b) $\forall u, v \in \mathscr{C}, \quad h(v) \geq h(u) + \langle \nabla h(u), v-u \rangle$,

(c) $\nabla^2 h(u)$ is positive semi-definite on $\mathscr{C}$.

PROPERTIES 1.3.11. *Let h a convex function on $\mathbb{R}$, we get :*

(a) If h is positive and admits two distinct zeros $u < v$, then h is zero on $[u, v]$.

(b) If h is upper bound, then h is constant.

(c) If h is increasing and not constant, then $\lim\limits_{u \to +\infty} h(u) = +\infty$.

DEFINITION 1.3.12. *h is said Concave on $\mathscr{C}$ if $-h$ convex on $\mathscr{C}$.*

REMARK 1.3.13. *The affine functions are both convex and concave.*

PROPERTIES 1.3.14. *For the two functions $h : \mathbb{R} \to \mathbb{R}$ and $l : \mathbb{R} \to \mathbb{R}$, we have :*

(a) If h and l are convex, then the funcion $h + l$ is convex.

(b) If h is convex, then the funcion αh is convex for all $\alpha > 0$.

(c) If h is convex, then the funcion e^h is convex.

(d) If h is convex and $h \geq 0$, then h^α is convex for all $\alpha > 1$.

(e) If l is convex and h is convex and increasing, then the composition funcion $l \circ h$ is also convex.

(f) If h is concave, then the funcion $\dfrac{1}{h}$ is convex.

EXAMPLE 1.3.15. *Let the function $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex, then the function $h(u+td)$ is convex with $d \in \mathbb{R}^n \backslash \{0\}$ and $t \in \mathbb{R}$ since for all $t_1, t_2 \in \mathbb{R}$ and $\alpha \in [0,1]$ we have :*

$$\begin{aligned} h(u+(\alpha t_1+(1-\alpha)t_2)d) &= h(u+\alpha t_1 d+(1-\alpha)t_2 d) \\ &= h(\alpha(u+t_1 d)+(1-\alpha)(u+t_2 d)) \\ &\leq \alpha h(u+t_1 d)+(1-\alpha)h(u+t_2 d). \end{aligned}$$

EXAMPLE 1.3.16. *$h(u) = \|u\|$, $u \in \mathbb{R}^n$ is convex.*

In fact, the two concepts convexity and convcavity can be also depicted by geometric views.

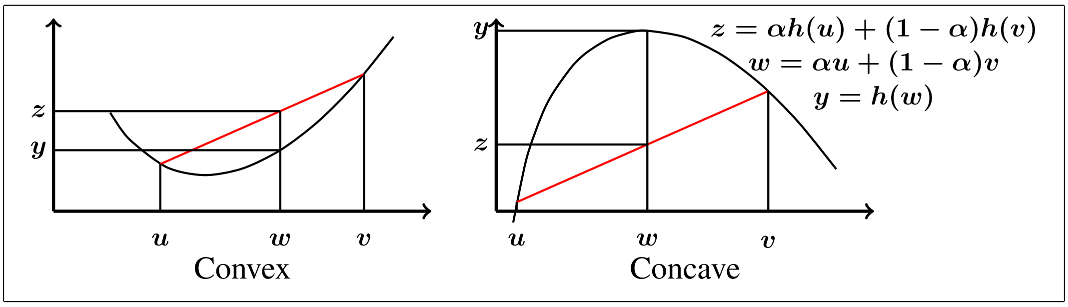

FIGURE 1.1: GRAPHS OF A CONVEX AND CONCAVE FUNCTIONS.

ELLIPTIC, COERCIVE, BARRIER & SMOOTHING FUNCTION

For the function $h : \mathbb{R}^n \to \mathbb{R}$, we have the following definitions.

DEFINITION 1.3.17. *We say that h is an elliptic function if it is of class $\mathcal{C}^1$ and if there exists $\beta > 0$ such that :*

$$\langle \nabla h(u) - \nabla h(v), u-v \rangle \geq \beta \|u-v\|^2, \quad \forall u, v \in \mathbb{R}^n.$$

DEFINITION 1.3.18. *We say that h is an coercive function on a convex set $\mathscr{C}$ if :*

$$\lim_{u \in \mathscr{C}, \|u\| \to +\infty} h(u) = +\infty.$$

DEFINITION 1.3.19. *If a continuous function h has a value on a point that grows to infinity as the point gets closer to the limit of the feasible domain of an optimization problem, then we say that h is a barrier function. These functions are used to substitute inequality restrictions with a more manageable penalizing term in the objective function.*

DEFINITION 1.3.20. *We say that h is a smoothing function if it is infinitely differentiable i.e., $h \in \mathcal{C}^\infty$.*

1.4 MATHEMATICAL PROGRAMMING

Numerical analysis encompasses a wide and diverse topic known as mathematical programming. It refers to mathematical models that are used to address issues, including framing an optimization issue in terms of an objective function $h : \mathbb{R}^n \to \mathbb{R}$ and restrictions. The general shape of a mathematical program may be summed up as follows :

$$\begin{cases} \min_{u} h(u) \\ u \in \mathscr{C} \end{cases} \tag{MP}$$

where h is continuously differentiable function and $\mathscr{C}$ is the set of feasible solutions presented as $\mathscr{C} = \{u \in \mathbb{R}^n : f_i(u) \leq 0, \ i = 1, \ldots, k \text{ and } g_j(u) = 0, \ j = 1, \ldots, m\}$ with $f_i(u)$ and $g_j(u)$ are scalar functions.

DEFINITION 1.4.1. *We said that the vector u^* is a feasible solution of (MP) if u^* satisfying the constraints (i.e., if $u^* \in \mathscr{C}$).*

Let $\mathscr{C} \subset \mathbb{R}^n$, $h : \mathscr{C} \to \mathbb{R}$ and $u^* \in \mathscr{C}$.

DEFINITION 1.4.2. *u^* is said to be a global minimum of h on $\mathscr{C}$ if :*

$$h(u^*) \leq h(u), \quad \forall u \in \mathscr{C}.$$

DEFINITION 1.4.3. *u^* is said to be a local minimum of h on $\mathscr{C}$ if there exists a neighborhood $\mathscr{D}$ of u^* such that :*

$$h(u^*) \leq h(u), \quad \forall u \in \mathscr{C} \cap \mathscr{D}.$$

The following figure helps to clarify the distinction between the local and global minimum.

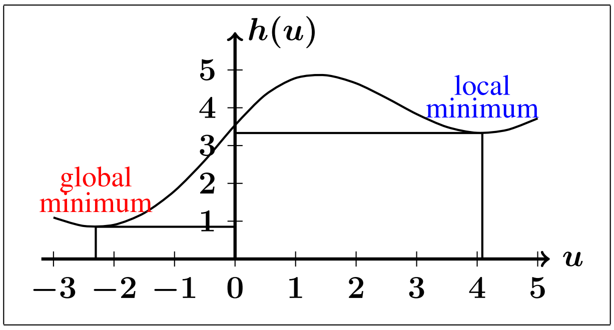

FIGURE 1.2: LOCAL AND GLOBAL MINIMUM FOR $\cos(u-1) + \tan^{-1}(u) + 3$.

DEFINITION 1.4.4. *For all $i \in [1, k]$, an inequality constraint f_i is said to be active (or saturated) at u^* if $f_i(u^*) = 0$.*

REMARK 1.4.5. *It is clear from the definition of the set of feasible solutions and* **DEFINITION 1.4.4** *that for all $j \in [1, m]$ the equality constraint g_j is active at any $u \in \mathscr{C}$.*

CLASSIFICATION OF MATHEMATICAL PROGRAMMING

Now, it is crucial for the researcher to categorize the (**MP**) and understand the procedures that must be taken to solve it before beginning any step. Because of convexity, differentiability, and linearity, the three key characteristics of the functions $\boldsymbol{h}$, $\boldsymbol{f}_i$ and $\boldsymbol{g}_j$ are what determine how (**MP**) is classified. which are displayed in the table below.

BRANCHES :	PROPERTIES :
Linear programming	$\boldsymbol{h}$, $\boldsymbol{f}_i$ and $\boldsymbol{g}_j$ are linear.
Convex programming	$\boldsymbol{h}$, $\boldsymbol{f}_i$ and $\boldsymbol{g}_j$ are convex.
Quadratic programming	$\boldsymbol{h}$ is quadratic and convex and $\mathscr{C}$ defined by linear equalities and inequalities
Integer (or discrete) programming	$\mathscr{C}$ is a discrete set (i.e., the variables are integers).

TABLE 1.1: THE CLASSIFICATION OF MATHEMATICAL PROGRAMMING.

THEOREM 1.4.6. *For a convex program, any local optimum is a global optimum.*

The complete resolution of (**MP**) is processed in the order of following points :

$$\boxed{\begin{array}{c}\text{Verify the existence (the uniqueness, if possible) of an optimal solution.}\\ \Downarrow \\ \text{Characterization of the solution.}\\ \Downarrow \\ \text{Development of algorithms to numerically calculate this solution.}\end{array}}$$

FIGURE 1.3: BASIC STEPS TO SOLVE A MATHEMATICAL PROGRAM.

MAIN RESULTS OF EXISTENCE

The three most popular theorems that ensure the existence and uniqueness of the solution to a mathematical programming problems are presented in this paragraph.

THEOREM 1.4.7. *(Weirstrass, [10]) If $\mathscr{C}$ is nonempty compact (closed and bounded) of $\mathbb{R}^n$ and if $\boldsymbol{h}$ is continuous on $\mathscr{C}$ then (**MP**) admits at least one optimal solution $\boldsymbol{u}^* \in \mathscr{C}$.*

THEOREM 1.4.8. *If $\mathscr{C}$ is nonempty closed of $\mathbb{R}^n$, $\boldsymbol{h}$ is continuous and coercive on $\mathscr{C}$ then (**MP**) admits at least one optimal solution.*

THEOREM 1.4.9. *If $\mathscr{C}$ is nonempty convex of $\mathbb{R}^n$, $\boldsymbol{h}$ is strictly convex on $\mathscr{C}$ then (**MP**) admits at most one optimal solution.*

REMARK 1.4.10. *The strict convexity does not ensure the existence of the solution but ensures its uniqueness.*

QUALIFICAION OF CONSTRAINTS

Before giving the optimal conditions of (**MP**), we require that the constraints satisfy certain criteria known as *qualification criteria*.

PROPOSITION 1.4.11. *A vector $u^* \in \mathscr{C}$ is said to be regular (i.e., the constraints are qualified at u^*) if the gradient components, corresponding to the saturated constraints in u^*, are linearly independent.*

PROPOSITION 1.4.12. *The constraint qualificaion is satisfied if at any vector of $\mathscr{C}$, if :*

(a) *All the constraints are affine.*

(b) *$\mathscr{C}$ is defined only by inequalities, which means that we have the following Slater criteria : $f_i(u)$ is convex for all $i \in [1, k]$ and there exists a vector u_0 such that $f_i(u_0) < 0$ ($int(\mathscr{C}) \neq \emptyset$).*

OPTIMALITY CONDITIONS

In this part, we first define the *Lagrangian* function, then list the *Karush-Kuhn-Tucker* conditions. The qualifying condition of the restrictions is assumed to be verified in what follows.

DEFINITION 1.4.13. *We call the function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ the Lagrangian associated with the mathematical program* (**MP**) *and is given by :*

$$\mathcal{L}(u, \lambda, \mu) = h(u) + \sum_{j=1}^{m} \lambda_j g_j(u) + \sum_{i=1}^{k} \mu_i f_i(u),$$

where λ_j and μ_i are reals called Lagrange multipliers such that $\lambda_j \in \mathbb{R}$ and $\mu_i \in \mathbb{R}_+$ for all $j \in [1, m]$ and $i \in [1, k]$, respectively.

Now, we suppose that the functions h, f_i and g_j are at least twice continuously differentiable and we present the gradient vector of *Lagrangian* in $u \in \mathbb{R}^n$:

$$\nabla_u \mathcal{L}(u, \lambda, \mu) = \nabla h(u) + \sum_{j=1}^{m} \lambda_j \nabla g_j(u) + \sum_{i=1}^{k} \mu_i \nabla f_i(u).$$

REMARK 1.4.14. *If u is a local minimum, then there exists a unique pair $(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R}^k$ that satisfies $\nabla_u \mathcal{L}(u, \lambda, \mu) = 0_{\mathbb{R}^n}$.*

Due to the vital role that duality plays in the analysis and resolution of mathematical programming, it is important to remember that the dual problem of (**MP**) is provided by :

$$\begin{cases} \sup\limits_{\lambda,\mu} \inf\limits_{u \in \mathscr{C}} \mathcal{L}(u, \lambda, \mu) \\ \nabla_u \mathcal{L}(u, \lambda, \mu) = 0_{\mathbb{R}^n} \\ \lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^k_+ \end{cases} \tag{DP}$$

THEOREM 1.4.15. *(Karush-Kuhn-Tucker (KKT), [62]) If u^* is a local optimal solution of (MP) satisfying one of the previous qualifying conditions, then there are multipliers $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^k_+$ such as :*

$$\begin{cases} \mathcal{L}(u^*, \lambda, \mu) = 0, & \textit{(optimality condition)} \\ \mu_i f_i(x^*) = 0, \quad i \in [1, k], & \textit{(condition of complementarity)} \\ g_j(x^*) = 0, \quad j \in [1, m] & \end{cases} \tag{KKT}$$

REMARK 1.4.16. (a) *If the constraints are not qualified in u^* the conditions of (KKT) do not apply (u^* may be optimal without verifying these conditions).*

(b) *If (MP) is convex, the conditions of (KKT) are both necessary and sufficient for u^* to be a global minimum.*

1.5 OPTIMIZATION ALGORITHM

The aforementioned information makes it abundantly evident that we must use an iterative approach, often expressed in the same way and referred to as an algorithm, in order to arrive at an optimal solution to (MP).

An algorithm is defined by an application called $Algo : \mathscr{C} \to \mathscr{C}$, where $\mathscr{C}$ is the set of feasible solutions. Defining an algorithm is nothing else than generating a sequence $(u_k)_{k\in\mathbb{N}}$ of $\mathscr{C}$. The generation of a sequence of elements of $\mathscr{C}$ given by the formula :

```
INPUT :
    u_0 ∈ C given ; k = 0.
ITERATION :
BEGIN :
    u_{k+1} = Algo(u_k) ;
    k = k + 1.
END.
```

FIGURE 1.4: THE GENERIC OPTIMIZATION ALGORITHM.

REMARK 1.5.1. *If we replace $\mathscr{C}$ by its interior, assuming that $int(\mathscr{C}) \neq \emptyset$, the algorithm is says an interior point algorithm.*

CONVERGENCE OF ALGORITHM

We will discuss several definitions about the convergence of $(u_k)_{k\in\mathbb{N}}$ and asymptotic notations.

DEFINITION 1.5.2. *Algorithm $Algo$ is said to be convergent if the sequence $(u_k)_{k\in\mathbb{N}}$ generated by the algorithm converges to a limit u^*.*

PROPOSITION 1.5.3. *A criteria for measuring the rate of convergence is the evolution of the error committed at each iteration : $e_k = \|u_k - u^*\|$.*

Let $(u_k)_{k\in\mathbb{N}}$ a sequence given by algorithm $Algo$ and converging to u^*.

PROPOSITION 1.5.4. *The speed of convergence of a sequence which based on the notions of comparison of functions in the neighborhood of $+\infty$ could be :*

(a) *If* $\exists\alpha \in [0,1[, \exists k_0 \in \mathbb{N}, \forall k \geq k_0, \|u_{k+1} - u^*\| \leq \alpha\|u_k - u^*\|$, *then u_k converges linearly to u^*.*

(b) *If there is a positive sequence α_k that converges to 0 such that $\|u_{k+1}-u^*\| \leq \alpha_k\|u_k-u^*\|$, then u_k converges superlinear to u^*.*

(c) *If* $\exists\alpha \in [0,1[, \exists k_0 \in \mathbb{N}, \forall k \geq k_0, \|u_{k+1}-u^*\| \leq \alpha\|u_k-u^*\|^\beta$, *then the convergence is of order β with $\beta > 1$.*

REMARK 1.5.5. *In the case $\beta = 2$, u_k converges quadratically to u^*.*

COMPLEXITY OF ALGORITHM

Let $h, l : \mathbb{N} \to \mathbb{R}^+$.

DEFINITION 1.5.6. *(O-notation) We note $h(n) = O(l(n))$ if and only if :*

$$\exists k \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n > n_0 : h(n) \leq kl(n).$$

DEFINITION 1.5.7. *(Θ-notation) We note $h(n) = \Theta(l(n))$ if and only if :*

$$\exists k_1, k_2 \in \mathbb{R}^+, \exists n_0 \in \mathbb{N}, \forall n > n_0 : k_1 l(n) \leq h(n) \leq k_2 l(n).$$

DEFINITION 1.5.8. *(o-notation) We note $h(n) = o(l(n))$ if and only if :*

$$\forall k \in \mathbb{R}, \exists n_0 \in \mathbb{N}, \forall n > n_0 : h(n) \leq kl(n).$$

DEFINITION 1.5.9. *The complexity of the algorithm $Algo$ is measured by how many simple operations it performs to solve a problem (e.g., assignment, comparisons, arithmetic operations,$\ldots$). If the number of operations required to solve the issue must be constrained by a polynomial P that depends on the problem's size $($i.e., $O(P(n)))$ then the method Algo is said to have polynomial complexity.*

NEWTON'S METHOD FOR SOLVING NONLINEAR SYSTEMS

Since the resolution of mathematical programming is equivalent to solving the Karush-Khun-Tucker optimality conditions, it is important to remember one of the most common approaches used to solve a system of nonlinear equations. This approach is known as Newton's method and it contributes mainly to finding the iterations of the algorithm $Algo$. The suggested approach is characterized by its speed of convergence near the solution. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous, differentiable function and let $J_h(u)$ be the Jacobian matrix of the function h. The principle of this method is consist of finding a zero of the function h of a variable $u \in \mathbb{R}^n$. Therefore, we consider the following nonlinear system $h(u) = 0_{\mathbb{R}^n}$ where $h(u) = (h_1(u), \ldots, h_n(u))^t$ with h_i is a nonlinear real function for $i = 1, \ldots, n$. Given an initial estimate u_0 of $\mathbb{R}^n$ we compute a sequence $(u_k)_{k\in\mathbb{N}}$ of trial solution. For $k = 0, \ldots, n-1$, we use the following formula :

$$u_{k+1} = u_k - J_h(u_k)^{-1}h(u_k),$$

If we want to facilitate the study, we will :

$$J_h(u_k)\Delta u_k = -h(u_k) \tag{1.7}$$

where Δu_k is the solution of (1.7) called the direction vector. The algorithm is given as follows :

INPUT :
  $u_0 \in \mathbb{R}^n$ given ; $\epsilon > 0$ stopping tolerance ; $k = 0$.
ITERATION :
BEGIN :
  WHILE ($\|h(u_k)\| > \epsilon$) **DO**
  BEGIN :
    Solve the system (1.7) to obtain Δu_k ;
    $u_{k+1} = u_k + \Delta u_k$;
    $k = k + 1$.
  END.
END.

FIGURE 1.5: NEWTON'S METHOD ALGORITHM.

The computation of $J_h(u)$ and solving the system (1.7) in the aforementioned method are crucial steps.

REMARK 1.5.10. (a) *If u_0 is sufficiently inside the set of solutions of h, then this sequence converges to a root of $h(u) = 0_{\mathbb{R}^n}$.*

(b) *If the function h is defined by $h(u) = Au + b$ (i.e., linear) with $A \in \mathcal{M}_n$ and $b \in \mathbb{R}^n$, then Newton's method amounts to solving the linear system $Au = b$. In effect, $J_h(u_k) = A$ for all $k \geq 0$ and the iterations give $Au_{k+1} = b$.*

1.6 LINEAR COMPLEMENTARITY PROBLEM

The linear complementarity problem, abbreviated as *LCP*, is a system of finitely many equalities in finitely many nonnegative variables along with a special equation that expresses the complementary relationship between the variables and corresponding equalities.

Mathematically and more specifically, in operational research and optimization, the standard linear complementarity problem is defined by the data of a matrix $M \in \mathcal{M}_n$ and a vector $q \in \mathbb{R}^n$. It consists in determining a vector $z \in \mathbb{R}^n$ whose components and those of $w \in \mathbb{R}^n$ are both positive and $\boldsymbol{z}$ and $\boldsymbol{w}$ are orthogonal for the *Euclidean* scalar product of $\mathbb{R}^n$. The general format of a standard linear complementarity problem is given as follows :

$$\begin{cases} \boldsymbol{w} = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{q}, \quad (\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{R}^{2n}_{+} \\ \boldsymbol{z}^t \boldsymbol{w} = 0 \end{cases} \tag{LCP}$$

where the first constraint represents feasibility and the second one is called a complementary condition, which is the key feature distinguishing the complementarity problem from a general equality system. The system (**LCP**) is often written concisely as follows :

$$0_{\mathbb{R}^n} \leq \boldsymbol{z} \perp \boldsymbol{M}\boldsymbol{z} + \boldsymbol{q} \geq 0_{\mathbb{R}^n}. \tag{1.8}$$

The standard linear complementarity problem is said to be linear because $\boldsymbol{z}$ occurs linearly in terms of left and right in (**1.8**), but in reality it is a nonlinear problem because of the last relation (**LCP**) between $\boldsymbol{z}$ and $\boldsymbol{w} = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{q}$. Thus, there is no linear relationship between $\boldsymbol{q}$ and the possible solutions to the *LCP*. (**LCP**) is also known as mathematical programming :

$$\begin{cases} \min \quad \boldsymbol{z}^t \boldsymbol{w} \\ \boldsymbol{w} - \boldsymbol{M}\boldsymbol{z} = \boldsymbol{q}, \quad (\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{R}^{2n}_{+} \end{cases}$$

The standard *LCP* is not an optimization problem, but it is well known that it includes many important mathematical problems such as linear optimization and convex quadratic programming problems. The reformulation of these two problems into standard *LCPs* is based on the *Karush-Kuhn-Tucker* (*KKT*) optimality conditions. To simplify the idea of transformation, we present the following two examples :

EXAMPLE 1.6.1. *(Linear optimization)*

We consider the linear optimization in the standard format as follows :

$$\begin{cases} \min\limits_{\tilde{\boldsymbol{z}}} \quad \boldsymbol{c}^t \tilde{\boldsymbol{z}} \\ \boldsymbol{A}\tilde{\boldsymbol{z}} = \boldsymbol{b}, \quad \tilde{\boldsymbol{z}} \in \mathbb{R}^n_{+} \end{cases} \tag{LO}$$

and the dual problem of (**LO**) *is given by :*

$$\begin{cases} \max\limits_{\boldsymbol{u}} \quad \boldsymbol{b}^t \boldsymbol{u} \\ \boldsymbol{A}^t \boldsymbol{u} + \boldsymbol{\lambda} = \boldsymbol{c}, \quad \boldsymbol{\lambda} \in \mathbb{R}^n_{+} \end{cases} \tag{DLO}$$

where $\tilde{\boldsymbol{z}}, \boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{u}, \boldsymbol{b} \in \mathbb{R}^m$ *and* $\boldsymbol{A} \in \mathbb{R}^{mxn}$. *The Karush-Kuhn-Tucker (KKT) conditions for* (**LO**) *and* (**DLO**) *are given by :*

$$\begin{cases} \lambda = A^t u + c, & (\tilde{z}, \lambda) \in \mathbb{R}_+^{2n} \\ \mu = b - A\tilde{z}, & (u, \mu) \in \mathbb{R}_+^{2m} \\ u^t \mu = 0, & \lambda^t \tilde{z} = 0 \end{cases}$$

The standard LCP associated with linear optimization is given as follows :

$$\begin{cases} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0_{\mathcal{M}_n} & A^t \\ -A & 0_{\mathcal{M}_m} \end{pmatrix} \begin{pmatrix} \tilde{z} \\ u \end{pmatrix} + \begin{pmatrix} c \\ b \end{pmatrix} \\ u^t \mu = 0, \quad \lambda^t \tilde{z} = 0 \end{cases}$$

now we consider $w = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$, $z = \begin{pmatrix} \tilde{z} \\ u \end{pmatrix}$, $M = \begin{pmatrix} 0_{\mathcal{M}_n} & A^t \\ -A & 0_{\mathcal{M}_m} \end{pmatrix}$ *and* $q = \begin{pmatrix} c \\ b \end{pmatrix}$.

EXAMPLE 1.6.2. *(Convex quadratic programming)*

The primal problem of the convex quadratic programming is given by :

$$\begin{cases} \min\limits_{\tilde{z}} \quad c^t \tilde{z} + \frac{1}{2} \tilde{z}^t Q \tilde{z} \\ A\tilde{z} = b, \quad \tilde{z} \in \mathbb{R}_+^n \end{cases} \tag{CQP}$$

the dual problem of (**CQP**) *is given by*

$$\begin{cases} \max\limits_{(\tilde{z}, u)} \quad b^t u - \frac{1}{2} \tilde{z}^t Q \tilde{z} \\ A^t u + \lambda - Q\tilde{z} = c, \quad \lambda \in \mathbb{R}_+^n \end{cases} \tag{DCQP}$$

where $\tilde{z}, c \in \mathbb{R}^n$, $u, b \in \mathbb{R}^m$, $A \in \mathbb{R}^{mxn}$ *and* $Q \in \mathcal{M}_n$ *a positive semidefinite matrix. The Karush-Kuhn-Tucker (KKT) conditions for* (**CQP**) *and* (**DCQP**) *are given by :*

$$\begin{cases} \lambda = Q\tilde{z} + A^t u + c, & (\tilde{z}, \lambda) \in \mathbb{R}_+^{2n} \\ \mu = b - A\tilde{z}, & (u, \mu) \in \mathbb{R}_+^{2m} \\ u^t \mu = 0, & \lambda^t \tilde{z} = 0 \end{cases}.$$

The above system can be transformed into a standard LCP of the form :

$$\begin{cases} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} Q & A^t \\ -A & 0_{\mathcal{M}_m} \end{pmatrix} \begin{pmatrix} \tilde{z} \\ u \end{pmatrix} + \begin{pmatrix} c \\ b \end{pmatrix} \\ u^t \mu = 0, \quad \lambda^t \tilde{z} = 0 \end{cases}$$

with $w = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$, $z = \begin{pmatrix} \tilde{z} \\ u \end{pmatrix}$, $M = \begin{pmatrix} Q & A^t \\ -A & 0_{\mathcal{M}_m} \end{pmatrix}$ *and* $q = \begin{pmatrix} c \\ b \end{pmatrix}$.

1.6.1 UNIQUE SOLVABILITY OF THE LCP

We started by introducing the subsequent sets :

$$\mathcal{F}_{LCP} = \{(z,w) \in \mathbb{R}^{2n}_{+} : w = Mz + q\},$$
$$\mathcal{F}^{*}_{LCP} = \{(z,w) \in \mathbb{R}^{2n}_{++} : w = Mz + q\},$$

and

$$Sol_{LCP} = \{(z,w) \in \mathcal{F}_{LCP} : z^t w = 0\}$$

where $\mathcal{F}_{LCP}$, $\mathcal{F}^{*}_{LCP}$ and Sol_{LCP} are the feasible set, the strictly feasible set of (**LCP**) and its solution set, respectively.

PROPOSITION 1.6.3. *The LCP is monotone (resp. strictly monotone) if the matrix $M \in \mathcal{M}_n$ is positive semi-definite (resp. positive definite) i.e.,*

$$w - Mz = 0_{\mathbb{R}^n} \Rightarrow z^t w \geq 0, \quad \forall z, w \in \mathbb{R}^n \text{ (resp. } z^t w > 0, \quad \forall z, w \in \mathbb{R}^n \backslash \{0\}). \tag{1.9}$$

REMARK 1.6.4. *If the LCP is monotone (M may not be symmetric), then the LCP is called a monotone linear complementarity problem (MLCP).*

We refer to the following theorem, which *Cottle*, *Pang* and *Stone* demonstrated, to show that *LCP* is solvable in a unique manner.

THEOREM 1.6.5. *(Theorem 3.3.7, [24]) A matrix $M \in \mathcal{M}_n$ is a $\mathscr{P}$-matrix if and only if the LCP has a unique solution for every $q \in \mathbb{R}^n$. In this case, the LCP with $\mathscr{P}$-matrix is denoted by $\mathscr{P}$-LCP.*

1.6.2 HORIZONTAL LINEAR COMPLEMENTARITY PROBLEM

The horizontal linear complementarity problem (*HLCP*) consists of finding a pair $(z,w) \in \mathbb{R}^n \times \mathbb{R}^n$ that achieves the system below with the given $N, M \in \mathcal{M}_n$ and $q \in \mathbb{R}^n$.

$$\begin{cases} Nw = Mz + q, \quad (z,w) \in \mathbb{R}^{2n}_{+} \\ z^t w = 0 \end{cases} \tag{HLCP}$$

REMARK 1.6.6. *The HLCP is reduced to a LCP if $N = I$ or N is invertible.*

We refer to the set of feasible points, the set of strictly feasible points and the set of solutions for *HLCP*, respectively, by :

$$\mathcal{F}_{HLCP} = \{(z,w) \in \mathbb{R}^{2n}_{+} : Nw = Mz + q\},$$
$$\mathcal{F}^{*}_{HLCP} = \{(z,w) \in \mathbb{R}^{2n}_{++} : Nw = Mz + q\}$$

and

$$Sol_{HLCP} = \{(z,w) \in \mathcal{F}_{HLCP} : z^t w = 0\},$$

PROPOSITION 1.6.7. *An HLCP is said to be monotone (resp. strictly monotone) if the pair of matrices $[N, M]$ satisfies :*

$$Nw - Mz = 0_{\mathbb{R}^n} \Rightarrow z^t w \geq 0, \quad \forall z, w \in \mathbb{R}^n \text{ (resp. } z^t w > 0, \quad \forall z, w \in \mathbb{R}^n \backslash \{0\}).$$

1.7 THE ABSOLUTE VALUE EQUATION

Let us consider the absolute value equation represented by (*AVE*), which consists of finding a vector u in real space $\mathbb{R}^n$ given as follows :

$$Au - |u| = b \tag{AVE}$$

where $A \in \mathcal{M}_n$, $b \in \mathbb{R}^n$ are given and $|u|$ denotes the vector with the absolute values of each component of u. (**AVE**) is a special form of *AVE* that can be obtained in the following way :

$$Au + B|u| = b \tag{GAVE}$$

where $B \in \mathcal{M}_n$ is a given non-zero matrix. If $B = 0_{\mathcal{M}_n}$, then the (**GAVE**) is equivalent to the classical system of linear equations :

$$Au = b.$$

The general form of *AVEs* was first introduced by *Rohn* in **[68]**, then investigated in a more general context by *Mangasarian* in **[56]**.

REMARK 1.7.1. *If we replace B by $-I$ we get the system* (**AVE**).

The following are some examples of problems that may be expressed as an absolute value equation.

EXAMPLE 1.7.2. *(Ordinary differential equation)*

The ordinary differential equation (ODE) is taken into consideration in the following manner :

$$\begin{cases} -\dfrac{d^2u}{dv^2}(v) - |u(v)| = f(v) \\ u(a) = u_{ini}, \quad u(b) = u_{fin}, \quad v \in [a, b] \end{cases} \tag{ODE}$$

we initially use the finite difference method to discretize the ODE. We obtain the following results after approximating the second order derivative using the second-order centred finite difference :

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} - |u_i| = f_i$$

where $u(v_i) = u_i$, $f(v_i) = f_i$ and $v_i = a + ih$, $\forall i = 1, \ldots, n$ with $h = \dfrac{b-a}{n+1}$. For $i = 0$ and $i = n+1$ we have $u_0 = u_{ini}$ and $u_{n+1} = u_{fin}$, respectively. Then, we obtain the following problem :

$$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} - \begin{pmatrix} |u_1| \\ |u_2| \\ \vdots \\ \vdots \\ |u_{n-1}| \\ |u_n| \end{pmatrix} = \begin{pmatrix} f_1 - u_{ini} \\ f_2 \\ \vdots \\ \vdots \\ f_{n-1} \\ f_n - u_{fin} \end{pmatrix}$$

which is equivalent to the absolute value equation shown below :

$$A_h u - |u| = b_h.$$

EXAMPLE 1.7.3. *(The hydrodynamic equations)*

The hydrodynamic equations (HE) are taken into account as follows :

$$Hu + \max(u, 0) = h \tag{HE}$$

where $H \in \mathcal{M}_n$, $h \in \mathbb{R}^n$ are given. We obtain the following equation by applying (1.3) :

$$Hu + \frac{1}{2}(u + |u|) = h$$

then we discover :

$$(2H + I)\, u + |u| = 2h$$

consequently, the HE can be transformed into an AVE with $A = -2H - I$ and $b = -2h$.

1.7.1 UNIQUE SOLVABILITY OF THE AVE

Mangasarian and Meyer **[58]** provided some results about the existence of *AVE* solutions, which we enumerate in the definitions that follow.

DEFINITION 1.7.4. *The system (AVE) is uniquely solvable for any $b \in \mathbb{R}^n$ if $\sigma_{min}(A) > 1$.*

DEFINITION 1.7.5. *If 1 is not an eigenvalue of A, the singular values of A are merely greater than or equal to 1 and $\{v : (A + I)v - b \geq 0, (A - I)v - b \geq 0\} \neq \emptyset$, then (AVE) is solvable.*

DEFINITION 1.7.6. *(AVE) is uniquely solvable for any b if $\|A^{-1}\| < 1$.*

DEFINITION 1.7.7. *If $b < 0$ and $\|A\| < \dfrac{\min_i |b_i|}{2\max_i |b_i|}$, then the system (AVE) has exactly 2^n distinct solutions each of which has no zero components and a different sign pattern.*

REMARK 1.7.8. *Recently, most writers have relied on* **DEFINITION 1.7.4** *to guarantee the special form of AVEs is uniquely solvable. This is exactly what we will follow throughout this thesis.*

1.8 RELATION BETWEEN AVE AND LCP

TRANSFORM AVE INTO LCP

This section will describe how, given a suitable assumption, we can write the absolute value equation (**AVE**) as a monotone linear complementarity problem (**LCP**) (this latter is similar to the one given by *Yong et al.* **[78]**). In order to achieve this, we presume that (**AVE**) meets the following requirement throughout this thesis :

ASSUMPTION 1. *The singular values of A exceed* 1.

REMARK 1.8.1. *When we want to ensure the unique solvability of* (**GAVE**). **ASSUMPTION 1** *generalizes to "The minimal singular value of the matrix A is strictly greater than the maximal singular value of the matrix B".*

PROPOSITION 1.8.2. *Under* **ASSUMPTION 1**, *the system* (**AVE**) *is written as* (**LCP**).

PROOF. It is possible to change the system (**AVE**) into the following system by using (**1.3**) :

$$\begin{cases} (A-I)u_+ - (A+I)u_- = b, \quad (u_-, u_+) \in \mathbb{R}^{2n}_+ \\ u_-^t u_+ = 0 \end{cases} \tag{1.10}$$

we first show that the matrix $(A - I)$ is nonsingular since, if not for a nonzero vector $s \in \mathbb{R}^n$, we have that $(A - I)s = 0$. So :

$$\sigma^2_{min}(A) = \lambda_{min}(A^t A) = \min_{\|s\|=1} s^t A^t A s \leq \|As\|^2 = \|s\|^2 = 1$$

this is in conflict with **ASSUMPTION 1**. Consequently, $(A - I)$ is invertible and the system (**1.10**) can be reduced to the following :

$$\begin{cases} u_+ = (A-I)^{-1}(A+I)u_- + (A-I)^{-1}b, \quad (u_-, u_+) \in \mathbb{R}^{2n}_+ \\ u_-^t u_+ = 0 \end{cases} \tag{1.11}$$

corresponding to (**LCP**), we put :

$$z = u_-, \quad w = u_+, \quad M = (A-I)^{-1}(A+I), \quad \text{and} \quad q = (A-I)^{-1}b. \tag{1.12}$$

and we obtain the desired result. □

REMARK 1.8.3. *Without any hypothesis, the absolute value equation* (**AVE**) *can be converted into the following HLCP :*

$$\begin{cases} (A-I)u_+ = (A+I)u_- + b, & (u_-, u_+) \in \mathbb{R}^{2n}_+ \\ u_-^t u_+ = 0 \end{cases}$$

with : $z = u_-, \quad w = u_+, \quad M = (A+I), \quad N = (A-I)$ *and* $q = (A-I)^{-1}b.$

TRANSFORM LCP INTO AVE

Furthermore, there is an equivalence between (**AVE**) and (**LCP**) under a suitable assumption. Therefore, we can rewrite any (**LCP**) which fulfills this assumption as an (**AVE**). Afterward, we apply the subsequent variable change :

$$w = \frac{1}{2}(|u| + u) \quad \text{and} \quad z = \frac{1}{2}(|u| - u) \tag{1.13}$$

where the first equality is obtained by adding the second and third equations of the system (**1.3**) and the second equality is obtained by deducing the same equations for the same system. When (**1.13**) is substituted for (**LCP**), the system shown below results :

$$\begin{cases} (M+I)u - (M-I)|u| = 2q, & (|u| - u, |u| + u) \in \mathbb{R}^{2n}_+ \\ (|u| - u)^t(|u| + u) = 0 \end{cases}$$

if $\lambda_i(M) \neq 1$ for all $i = 1, \ldots, n$ (i.e., $(M-I)$ is invertible), then the first equation in the above system becomes :

$$(M-I)^{-1}(M+I)u - |u| = 2(M-I)^{-1}q. \tag{1.14}$$

Using the formulas $A = (M-I)^{-1}(M+I)$ and $b = 2(M-I)^{-1}q$, we uncover an equivalence system which similar to the (**AVE**).

REMARK 1.8.4. *If the matrix $(M-N)$ is nonsingular, then the HLCP is an (**AVE**), with :*

$$A = (M-N)^{-1}(M+N), \quad b = 2(M-N)^{-1}q.$$

REMARK 1.8.5. *Since the standard LCP subsumes many mathematical programming problems, we draw the conclusion that the problems raised in **EXAMPLE 1.6.1** and **EXAMPLE 1.6.2** may be restated as (**AVE**) based on this final justification.*

HARDNESS OF THE AVE

A complexity class is a collection of issues having similar difficulty in complexity theory. These classes assist scientists in grouping issues based on how long $Algo$ takes to solve problems (the number of steps necessary may also be used to indicate how long it takes to verify the results). There are other classes available ; however, we are most interested in the following :

DEFINITION 1.8.6. *"Non-deterministic Polynomial-time" is the abbreviation for the* NP *class. It is a set of decision problems that can be solved in polynomial time by a non-deterministic computer. The solutions to* NP *issues are difficult to find, but they are simple to verify, and NP problems can be verified in polynomial time.*

DEFINITION 1.8.7. *An* NP*-hard class is a class of issues that is at least as difficult as the toughest problem in* NP*, and it is a class of problems in which every problem in NP can be reduced to* NP*-hard in polynomial time.* NP *does not encompass all* NP*-hard issues. Checking them takes a long time.*

At this point, we infer that solving the (**AVE**) is NP-hard by reducing it to the standard *LCP*, which subsumes numerous mathematical problems into the (**AVE**). *Mangasarian* confirms this in the next lemma.

LEMMA 1.8.8. *(Proposition* 2, **[56]***) Solving the* (**AVE**) *is* NP*-hard.*

1.8.1 EQUIVALENCE BETWEEN THE SOLUTIONS OF AVE AND LCP

In addition to the equivalence between the two systems (**AVE**) and (**LCP**), we present the following corollary, which shows the relation between their solutions.

COROLLARY 1.8.9. *The system* (**AVE**) *is uniquely solvable for any* $b \in \mathbb{R}^n$ *if and only if the system* (**LCP**) *is an* $\mathscr{P}$*-LCP.*

PROOF. We suppose that (**LCP**) is an $\mathscr{P}$*-LCP*. Due to **THEOREM 1.6.5**, the system (**1.11**) has a unique solution for any vector $(A-I)^{-1}b \in \mathbb{R}^n$ noted by (u_-^*, u_+^*) and satisfies the following :

$$\begin{cases} u_+^* = (A-I)^{-1}(A+I)u_-^* + (A-I)^{-1}b, & (u_-^*, u_+^*) \in \mathbb{R}_+^{2n} \\ {u_-^*}^t u_+^* = 0 \end{cases}$$

Now, we simplify the feasible constraint we obtain :

$$\begin{aligned} u_+^* = (A-I)^{-1}(A+I)u_-^* + (A-I)^{-1}b \quad &\Leftrightarrow \quad (A-I)u_+^* = (A+I)u_-^* + b \\ &\Leftrightarrow \quad (A-I)u_+^* - (A+I)u_-^* = b \\ &\Leftrightarrow \quad A(u_+^* - u_-^*) - (u_+^* + u_-^*) = b \end{aligned}$$

substituting in the above system and using (**1.3**), we get :

$$Au^* - |u^*| = b$$

hence, u^* is the unique solution of (**AVE**) for any $b \in \mathbb{R}^n$. □

REMARK 1.8.10. *From the results of* **SECTION 1.1***, the condition "*(**LCP**) *is an* $\mathscr{P}$*-LCP" used in* **COROLLARY 1.8.9** *can be reduced to "The matrix* $(A-I)^{-1}(A+I)$ *is positive definite".*

PROPOSITION 1.8.11. *Under* **ASSUMPTION 1***, the matrix* $M = (A-I)^{-1}(A+I)$ *is positive definite.*

PROOF. First, due to **PROPERTIES 1.1.3** and **ASSUMPTION 1** the matrix $A^tA - I$ is positive definite since :

$$\lambda_{min}(A^tA - I) \geq \lambda_{min}(A^tA) - \lambda_{max}(I) = \lambda_{min}(A^tA) - 1 > 0.$$

Using **REMARK 1.1.8**, the matrix $AA^t - I$ is also positive definite since the eigenvalues of the two matrices A^tA and AA^t are equal. So for all nonzero $s \in \mathbb{R}^n$, we get :

$$s^t(A+I)(A^t - I)s = s^t(AA^t - I)s > 0$$

letting $v = (A^t - I)s \neq 0_{\mathbb{R}^n}$, we obtain :

$$\begin{aligned} s^t(A+I)(A^t - I)s &= ((A^t - I)^{-1}v)^t(A+I)v \\ &= v^t(A - I)^{-1}(A+I)v \\ &= v^tMv \end{aligned}$$

thus, M is positive definite. □

REMARK 1.8.12. *Under* **ASSUMPTION 1**, (**1.11**) *is uniquely solvable for any* q.

Now, we present a theorem that shows the unique solvability of (**AVE**).

THEOREM 1.8.13. *If* **ASSUMPTION 1** *is satisfied, then* (**AVE**) *is uniquely solvable for every* b.

PROOF. Because **ASSUMPTION 1** is satisfied, the matrix M is positive definite (according to **PROPOSITION 1.8.11**). So, using **REMARK 1.1.30**, we obtain that M is a $\mathscr{P}$-matrix. Consequently, the *LCP* has a unique solution for every q from **THEOREM 1.6.5**. Hence, due to **COROLLARY 1.8.9**, we get that (**AVE**) is uniquely solvable for every $b \in \mathbb{R}^n$. □

REMARK 1.8.14. *If the matrix* $A = (M - I)^{-1}(M + I)$ *satisfy* **ASSUMPTION 1**, *the system* (**1.14**) *is uniquely solvable for any* $b = 2(M - I)^{-1}q$.

This part is concluded by a corollary, which, under the suitable assumption, ensures that the solutions of (**LCP**) and (**AVE**) are equivalent.

COROLLARY 1.8.15. *(Corollary 2.12, [3]) Under* **ASSUMPTION 1**, *the pair of vectors* (u_-^*, u_+^*) *is the unique solution of* (**LCP**) *if and only if the vector* $u^* = u_+^* - u_-^*$ *is the unique solution of* (**AVE**).

PROOF. **ASSUMPTION 1** ensures that (**LCP**) is a $\mathscr{P}$*-LCP* based on **THEOREM 1.6.5** and **COROLLARY 1.8.9**. Then, due to **THEOREM 1.8.13**, we get that (u_-^*, u_+^*) is the unique solution of (**LCP**) if and only if $u^* = u_+^* - u_-^*$ is the unique solution of (**AVE**). □

REMARK 1.8.16. *Under* **ASSUMPTION 1**, *the vector* u^* *is the unique solution of* (**1.14**) *if and only if the pair of vectors* $\left(\frac{1}{2}(|u^*| - u^*), \frac{1}{2}(|u^*| + u^*)\right)$ *is the unique solution of* (**LCP**).

1.9 SOME NUMERICAL METHODS

Several academics are particularly interested in researching ways to solve problems such as linear optimization, convex quadratic programming, linear complementarity, and absolute value equation, etc. In our research, we are interested in examining the two approaches described below, which are employed to solve the linear complementarity problem in the next chapters.

1.9.1 INTERIOR-POINT METHODS

In 1947, *Dantzig* **[26]** proposed a very effective technique for solving linear programming problems called the simplex method, and this method remained dominant for almost forty years, despite the efforts of many researchers who have written hundreds of books and published thousands of articles on the subject. But with the emergence of the important concepts of optimization and complexity theory in the 1970s, it meant that to solve the problem in a given number of operations bounded by a polynomial of the same size as the problem. Unfortunately, this property is not achieved by the simplex method, as *Klee and Minty* have shown in **[48]**, because the complexity of this method is exponential. In 1979, *Khachiyan* **[47]** implemented the first polynomial algorithm for linear programming called the ellipsoid algorithm. However, the proposed algorithm turned out to be completely inefficient in practice. In 1984, *Karmarkar* **[44]** developed a polynomial algorithm based on interior penalty methods, this algorithm was essentially different from the simplex method by the fact that, in the latter, one moves along the boundary of the realizable domain, whereas in the Karmarkar method, one progresses while remaining strictly inside the realizable domain. In that period, studies have been launched and went in depth on this subject, such that we record more than 3000 publications in a few years and the most important result is that there are four fundamental classes of interior-point methods, namely :

AFFINE METHODS

The affine methods were first introduced by *Dikin* **[27]** in 1967 (affine algorithms with small steps), but it was not until the appearance of the famous article by *Karmarkar* **[44]** in 1984 that it was shown that affine algorithms are interior point algorithms, i.e., algorithms that progress while remaining strictly inside the feasible domain by using an affine transformation and we replace the non-negativity constraint by an ellipsoid that contains the new iterate. The convergence of these methods has been the subject of many studies, but no polynomiality bound has been proven so far. The algorithm is characterized by its simplicity, efficiency and each iteration of the affine algorithm requires a number of operations of the order $O(n^3)$.

PROJECTIVE METHODS

Despite the projective methods that were proposed by *Karmarkar* [44], the implementation and the transformations necessary to solve a standard linear programming problem remained secret until 1985, when they were published by *Tomlin* [75] and *Shanno and Marsten* [72] in particular. This algorithm is an iterative procedure that consists of generating a sequence $\{u_k\}$ of points strictly feasible and verifying :

$$\begin{cases} u_0 = e \\ c^t u_k \leq \left(\exp \frac{-k}{5n}\right) c^t u_0, \quad \forall k \geq 1 \end{cases}$$

i.e., the sequence generated by the algorithm is satisfied :

$$c^t u_k \leq \left(\exp \frac{-k}{5n}\right) c^t u_0, \quad \forall k \geq 1$$

this is equivalent to :

$$n \log c^t u_k \leq n \log c^t u_0 - \frac{k}{5}, \quad \forall k \geq 1.$$

To pass from iteration k to iteration $k+1$ we used a projective transformation, which is defined as follows :

$$u \xmapsto{T_p} \widehat{u} = \frac{n(U_k)^{-1}u}{e^t(U_k)^{-1}u}$$

where U_k is a diagonal matrix whose components are the u_i for $i = 1, \ldots, k$. The projective transformation is bijective and the inverse transformation is given by :

$$(T_p)^{-1}(\widehat{u}) = \frac{nU_k\widehat{u}}{e^tU_k\widehat{u}}.$$

The projective method is more efficient than the simplex method, especially for large problems. In fact, there are several researchers who have been able to highlight the remarkable comparison between these two methods, such as those of *Ye and Kojima* , *Monma and Morton*, *Adler et al.* and many other researchers. The algorithm requires $O(nL)$ iterations and each iteration requires $O(n^3)$ arithmetic operations, hence the total complexity is $O(n^4L)$ operations. Karmarkar was even able to reduce it to $O(n^{3.5}L)$ operations using a partial update process.

POTENTIAL METHODS

The potential method is another variant of *Karmarkar*'s algorithm. The algorithms are distinguished by the fact that they retain the characteristics of the algorithm, in particular, the polynomial complexity bound and the ability to take large steps based on the potential function without the use of the projective change of variables. The algorithms of this method are simply steepest descent algorithms with affine conditioning applied to a potential function. The primal potential function that is defined by *Karmarkar* is given as follow :

$$f_{KP}(u, v_L) = q\log(c^t u - v_L) - \sum_{i=1}^{n} \log u_i$$

where v_L is a lower bound of the optimal value on the objective and $q = n + 1$. Its dual function is :

$$f_{KD}(z, v_U) = q\log(v_U - b^t z) - \sum_{i=1}^{n} \log w_i$$

where v_U is the upper bound of the optimal value. *Ye* and *Todd* [73] proposed the following primal-dual potential function :

$$f_{YT}(u, w) = q\log(u^t w) - \sum_{i=1}^{n} \log u_i - \sum_{i=1}^{n} \log w_i,$$

where $q = 2n$ in [74] and $q = n + \sqrt{n}$ in [73]. These functions have the primary role of measuring the progress of the algorithm (analysis and complexity), but they can also be used to find directions. The potential functions are not convex ; however, they admit a unique minimum for any bound v ($v = v_L$ or $v = v_U$) and for any $q > n$. This algorithm has (n^3L) operations for linear programming problems. They show that the use of the rule of "*Goldstein-Armijo*" to save the direction sought during the primal steps is sufficient to keep an average complexity of $(n^{2.5})$ operations for iteration ; they extend the dual step to apply the partial update, thus the total complexity of their algorithm is only $0(n^3L)$ operations.

CENTRAL PATH METHODS

They were introduced at the same time as potential reduction methods and developed by *Barnes* [9] in 1987. The central trajectory method was studied first by *Bayer* and *Lagarias*, then by *Meggido*, *Ben Daya and Shetly*, *Renegar*, *Gonzaga*, *Vaidya*, *Monteiro and Adler*, *Kojima et al.*, *Roos and Vial*, *Goldfarb and Liu*, etc. As its name indicates, this method is based on following a central trajectory. Depending on the case, this central trajectory is defined as the set of minimal auxiliary functions considered. This method consists of staying in a certain neighborhood of the central trajectory using iterations of Newton. There are two different approaches to central trajectory methods :

- ► Primal approach.
- ► Primal-dual approach.

Primal-dual methods are the most widely used in practice. They are efficient and can easily be extended to other types of problems (quadratic problems, convex problems, ...). The first algorithm of this type was developed by *Kojima et al.* [49]. This large-step algorithm has a complexity of the order of $O(nL)$ iterations. These same researchers have described another primal-dual algorithm, but with small steps, whose complexity is better and is of order $O(\sqrt{n}L)$ iterations. After intense research has been done in this field, which has given rise to a new type of primal-dual algorithm called the predictor-corrector algorithm.

1.9.2 SMOOTHING-TYPE ALGORITHM

Smoothing-type algorithms are one of the most significant approaches since they have been created for tackling numerous types of optimization problems (see, e.g., [17], [19], [21], [20], [30], [36], [40], [41]). The publications [71] and [43] recently examined a family of smoothing functions as well as a smoothing-type approach to solve the absolute value problem. The smoothing-type approach for solving the system primary idea is to reformulate the system (AVE) as a system of smoothing equations. More specifically, we define the function $\phi(\nu, s)$ for every $(\nu, s) \in \mathbb{R}^2$. Following that, we shall create the problem reformulation (AVE). In this regard, we define :

$$H(\nu, u) = \begin{pmatrix} \nu \\ Au - \Phi(\nu, u) - b \end{pmatrix} \text{ with } \Phi(\nu, u) = \begin{pmatrix} \phi(\nu, u_1) \\ \vdots \\ \phi(\nu, u_n) \end{pmatrix} \text{ and } \nu > 0.$$

Thereby, it is obvious that if $H(\nu, u) = 0$, then $\nu = 0$ and $u \in \mathbb{R}^n$ solves the system (AVE). Then, the suggested algorithm is well-defined under ASSUMPTION 1. They demonstrate, in particular, that the suggested method is globally convergent and the convergence rate is quadratica without any further assumptions.

2

Complexity analysis of an interior-point algorithm for linear complementarity problem based on two new kernel functions

Introduction

In this chapter, we propose two different parametric kernel functions to solve the linear complementarity problem (**LCP**) via primal–dual interior-point methods (*IPMs*). As a result, we will pay special attention to the (**LCP**), which is derived from the system (**AVE**) under **Assumption 1**, as mentioned in **Section 1.8** of the previous chapter. The significant point is that we rely on the two kernel functions, which will be crucial in selecting new search directions and calculating the distance between the given iteration and the center. In which, this chapter's results are taken from **Paper 1** and **Paper 3**.

Our goal in this work is to deal with the complexity analysis and the numerical implementation of a primal–dual interior point methods based on two new parametric kernel functions for the linear complementarity problem. The basic idea of our theoretical study is like all studies that recently adopted by the publications written by researchers to answer several difficulties based on various kernel functions (see, e.g., **[2]**, **[6]**, **[8]**, **[11]**, **[12]**, **[13]**, **[15]**, **[16]**, **[31]**, **[32]**, **[33]**, **[34]**, **[53]**, **[54]**, **[60]**, **[66]**, **[67]**, **[76]**, **[79]**), we will follow the approach introduced by *Bai et al.* **[6]** to compute the iteration bounds for large- and small-update methods.

This chapter is organized as follows : In **Section 2.1**, we first investigate the central-path of the linear complementarity problem, as well as the search directions and the proximity measure, before presenting the generic full-Newton step feasible interior-point algorithm for the linear complementarity problem. In **Section2.2**, we develop some basic properties of the two new kernel functions. In **Section2.3**, we analyze the complexity for large-update and small-update methods based on the two kernel functions and obtain the best known iteration bounds, namely, $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$ and $O(\sqrt{n}\log\frac{n}{\varepsilon})$, respectively, with a special choose of the barrier degree in the two cases. Finally, in the last section, we report numerical results with different values of the parameter r and θ to evaluate the efficiency of the algorithm which based on our two functions. First, for all $x > 0$ we shall consider the two following functions :

$$\psi_E(x) = \frac{x^2-1}{2} + \frac{e^{(r-1)(1-x)} + x^{-r+1} - 2}{2(r-1)}, \quad \forall r > 1 \tag{2.1}$$

and

$$\psi_H(x) = \frac{x^2-1}{2} + \frac{1}{2r}\left(\frac{\cosh^r(x^{-1}) - \cosh^r(1)}{\tanh(1)\cosh^r(1)x^r} - \log(x^r)\right), \quad \forall r \geq 4 \tag{2.2}$$

where r is called the barrier degree. We recall that the linear complementarity problem consists to find a couple of vectors $(z, w) \in \mathbb{R}^n \times \mathbb{R}^n$ such that :

$$\begin{cases} w = Mz + q, \quad (z, w) \in \mathbb{R}^{2n}_+ \\ z^t w = 0 \end{cases} \tag{2.3}$$

where the square matrix $M \in \mathcal{M}_n$ and the vector $q \in \mathbb{R}^n$ are given and the linear complementarity problem extracted from the system (**AVE**) consists to find a couple of vectors $(u_-, u_+) \in \mathbb{R}^n \times \mathbb{R}^n$ such that :

$$\begin{cases} u_+ = (A-I)^{-1}(A+I)u_- + (A-I)^{-1}b, \quad (u_-, u_+) \in \mathbb{R}^{2n}_+ \\ u^t_- u_+ = 0 \end{cases} \tag{2.4}$$

where $M = (A-I)^{-1}(A+I) \in \mathcal{M}_n$ is a square matrix, $q = (A-I)^{-1}b \in \mathbb{R}^n$, $z = u_- \in \mathbb{R}^n$ and $w = u_+ \in \mathbb{R}^n$ are vectors with $A \in \mathcal{M}_n$, $b \in \mathbb{R}^n$ are given and I is the identity matrix.

2.1 CENTRAL-PATH FOR LCP

This section will cover the novel search directions, the general interior-point algorithm for (2.3), the existence and uniqueness of the solution to (2.3), the generic interior-point algorithm and the proximity measure. In this chapter, we will presume that system (2.3) fulfills the following requirements :

ASSUMPTION 2. *The interior point condition (IPC) :*

$$\textit{there exists } (z_0, w_0) > 0 \quad \textit{such that :} \quad w_0 = Mz_0 + q,$$

in other words, the set $\mathcal{F}^*_{LCP}$ is non-empty set.

ASSUMPTION 3. *The LCP is monotone.*

REMARK 2.1.1. *According to* **ASSUMPTION 1** *and* **PROPOSITION 1.6.3**, *the matrix* $M = (A - I)^{-1}(A + I)$ *is positively definite (see* **PROPOSITION 1.8.11**). *This implies a strict monotonicity of LCP* (2.4), *i.e., the system* (2.4) *satisfy* **ASSUMPTION 3**.

2.1.1 EXISTENCE AND UNIQUENESS OF SOLUTION

We well know that under **ASSUMPTION 2** the system (2.3) has a solution since *Kojima et al.* [51] proved that the central path of the system (2.3) is well-defined (exists) if and only if the problem (2.3) satisfies the *IPC* and **ASSUMPTION 3** (which clearly reduced to **ASSUMPTION 1** according to **REMARK 2.1.1** for the system (2.4)) ensure the uniqueness of this solution. The following system of equations must be solved in order to solve system (2.3) :

$$\begin{cases} w = Mz + q, \quad (z, w) \in \mathbb{R}^{2n}_{++} \\ zw = 0 \end{cases} \tag{2.5}$$

The basic idea of the *IPMs* is to replace the complementary condition of the system (2.5) with the centring condition $zw = \mu e$ and we find the following system :

$$\begin{cases} w = Mz + q, \quad (z, w) \in \mathbb{R}^{2n}_{++} \\ zw = \mu e \end{cases} \tag{2.6}$$

where $\mu > 0$ denotes a positive parameter, $e = (1, ..., 1)^t \in \mathbb{R}^n$ denotes an all-one vector and the last equation of the system (2.6) leads us to the conclusion that $(z)_i(w)_i = \mu$ for all $i = 1, \ldots, n$. As a result, $z^t w$ equals $n\mu$. In [50], *Kojima et al.* demonstrated that the limit of μ-center when μ goes to zero exists and converges to (z, w) the solution of the system (2.3) (i.e., $\lim_{\mu \to 0}(z(\mu), w(\mu)) = (z, w)$). The new parameterized system (2.6) has a unique solution denoted by $(z(\mu), w(\mu))$ for all $\mu > 0$ and is called the μ-center of the system (2.3). The central-path of (2.3) is the name given to the set of μ-center.

2.1.2 NEW SEARCH DIRECTIONS

The *IPMs* are iterative methods, therefore in order to obtain new iterations (z^+, w^+) (which satisfy the *IPC*), we must locate the displacements $(\Delta z, \Delta w)$. In order to do this, we use Newton's method on the system (2.6) with fixed $\mu > 0$ to produce the following system :

$$\begin{cases} \Delta w = M\Delta z \\ w\Delta z + z\Delta w = \mu e - zw \end{cases} \tag{2.7}$$

where the first equality is obtained from $Mz + q - w = 0$ and the final equality is obtained by ignoring the quadratic component $\Delta z \Delta w$. It is now simple to verify that $(\Delta z, \Delta w)$ is non-zero if $(z, w) \neq (z(\mu), w(\mu))$. The monotonicity of the *LCP* states that the system (2.7) has a unique solution $(\Delta z, \Delta w)$ and this is due exactly to the fact that for any positive definite matrices $Z = diag(z) \in \mathcal{M}_n$ and $W = diag(w) \in \mathcal{M}_n$ the following matrix :

$$J = \begin{pmatrix} M & -I \\ W & Z \end{pmatrix}$$

is invertible (see **THEOREM 1.1.25** and **REMARK 1.8.11** for the system (**2.1.1**)). The scaling directions (d_z, d_w), as well as the scaled vector v, is defined below to make the analysis simpler.

$$d_z = \frac{v\Delta z}{z}, \quad d_w = \frac{v\Delta w}{w} \quad \text{and} \quad v = \sqrt{\frac{zw}{\mu}} \quad \text{with} \quad v^{-1} = \sqrt{\frac{\mu}{zw}} \tag{2.8}$$

hence, we have :

$$d_z + d_w = \frac{w\Delta z + z\Delta w}{\mu v} \quad \text{and} \quad d_z d_w = \frac{\Delta z \Delta w}{\mu} \tag{2.9}$$

For convenience, we present the next vector :

$$d = \sqrt{zw^{-1}} \quad \text{with} \quad d^{-1} = \sqrt{wz^{-1}}. \tag{2.10}$$

according to (2.8) and (2.10), it is not difficult to verify that :

$$d_z = \frac{d^{-1}\Delta z}{\sqrt{\mu}}, \quad d_w = \frac{d\Delta w}{\sqrt{\mu}} \quad \text{and} \quad v = \frac{d^{-1}z}{\sqrt{\mu}} = \frac{dw}{\sqrt{\mu}}. \tag{2.11}$$

The scaling directions (d_z, d_w) and the two vectors v and d can be used to describe the system (2.7). We discover the following linear system from (2.8), (2.9), (2.10) and (2.11) :

$$\begin{cases} d_w = \Lambda d_z \\ d_z + d_w = v^{-1} - v \end{cases} \tag{2.12}$$

where $\Lambda = DMD$ and $D = diag(d)$. Under our assumptions, the system (2.12) has a unique solution (d_z, d_w) for each $\mu > 0$ which can be used to compute $(\Delta z, \Delta w)$ via (2.8) or (2.11). By taking a step size α (which is defined by some line search rules) along the search direction, we construct a new pair (z^+, w^+) according to :

$$z^+ = z + \alpha\Delta z \quad \text{and} \quad w^+ = w + \alpha\Delta w. \tag{2.13}$$

This process is repeated until we find the iterates (z, w) that are close enough to $(z(\mu), w(\mu))$ and μ is small enough. At this stage, we have found the solution of the system (2.3).

2.1.3 GENERIC FEASIBLE INTERIOR-POINT ALGORITHM

INPUT :
An accuracy parameter $\varepsilon > 0$; a threshold parameter $\tau \geq 1$; an update parameter $0 < \theta < 1$; a strictly feasible point (z_0, w_0) and $\mu_0 > 0$ with $\mu_0 = \dfrac{(z_0)^t w_0}{n}$, $v_0 = \sqrt{\dfrac{z_0 w_0}{\mu}}$ such that $\Psi(v_0) \leq \tau$.
ITERATION :
BEGIN :
$z = z_0$; $w = w_0$; $\mu = \mu_0$; $v = v_0$;
WHILE $(n\mu \geq \varepsilon)$ **DO**
BEGIN :
update of μ and v ;
WHILE $(\Psi(v) > \tau)$ **DO**
BEGIN :
solve system (2.7) using (2.12) to obtain $(\Delta u_-, \Delta u_+)$;
determine a step size $\alpha > 0$;
$(z^+, w^+) = (z, w) + \alpha(\Delta z, \Delta w)$;
update v ;
END.
END.
END.

FIGURE 2.1: FEASIBLE INTERIOR-POINT ALGORITHM FOR SOLVING LCP.

The feasible interior-point algorithm used to solve the *LCP* works as follows : begins with a strictly feasible initial interior point (z_0, w_0) $((z_0, w_0) \in \mathscr{F}^*)$ in a $\tau-$neighborhood $(\Psi(v) \leq \tau)$ and set of parameters, the ones that change their value are the parameter $\mu > 0$ and the scaled vector v, which are fixed throughout the algorithm are the accuracy parameter ε is greater than zero, the proximity parameter $\tau \geq 1$ and barrier update $\theta \in]0, 1[$. In each outer iteration, we lowered to μ to $(1-\theta)\mu$ and v to $\dfrac{v}{\sqrt{1-\theta}}$. If $\Psi(v) > 0$, we begin the inner iteration by applying the system (2.7) and (2.12) to acquire the unique search directions $(\Delta z, \Delta w)$. Then we must calculate the value for step size α with respect to the current value of μ. So the only thing left is to find the next iteration (z^+, w^+) using (2.13). According to our assumptions, the system (LCP) has a unique solution, thus we continue the outer iteration until we discover a point that satisfy $n\mu < \varepsilon$ and $\Psi(v) \leq \tau$. This point is known as the optimal solution to *LCP* which in case it is (2.4) we can derive the optimal solution to (AVE) from it and (1.3).

REMARK 2.1.2.

- *The values τ , θ and α ought to be selected in a way that the algorithm is optimized in the sense that the minimum number of iterations necessary is needed by the algorithm.*
  - *In both theory and practice of IPMs, the barrier update value θ selection is crucial. The*

procedure is typically referred to as a large-update method if θ is a constant independent of the dimension of the issue problem n and a small-update method if θ relies on n.

- ► *Another important consideration in the study of the algorithm is the decision regarding the step size α. It must be carried out in a way that improves iterates' proximity to the current μ-center by an adequate quantity.*

► *An update of μ and a sequence of (one or more) inner iterations make up each outer iteration. The worst-case iteration limit for our algorithm is the overall number of inner iterations.*

2.1.4 PROXIMITY MEASURE

Any approximation of $(z(\mu), w(\mu))$ must be monitored for the algorithm's analysis by its quality. To that aim, we provide the next logarithmic barrier function $\Psi(v)$ in its classical form :

$$\Psi(z, w; \mu) = \Psi(v) = \sum_{i=1}^{n} \psi_R(v_i) = \sum_{i=1}^{n} \left(\frac{v_i^2 - 1}{2} - \log v_i \right) \tag{2.14}$$

where $\Psi(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ and $\psi_R(x)$ is twice differentiable, go to infinity if either $x \to 0$ or $x \to \infty$, strictly convex and minimal at $x = 1$, with $\psi_R(1) = 0$. We call the function $\psi_R(x)$ the kernel function of the barrier function $\Psi(v)$.

The right-hand side of the second equation in (2.12) easily equals minus the derivative of $\Psi(v)$. The system (2.12) can be rewritten as follows :

$$\begin{cases} d_w = \Lambda d_z \\ d_z + d_w = -\nabla\Psi(v) \end{cases} \tag{2.15}$$

moreover, the proximity measure $\delta(v) : \mathbb{R}^n_{++} \to \mathbb{R}_+$ is given by :

$$\delta(z, w; \mu) = \delta(v) = \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\|d_z + d_w\| \tag{2.16}$$

in order to make it obvious that if $(z, w) = (z(\mu), w(\mu))$, we have :

$$zw = \mu e \Leftrightarrow v = e \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow \Psi(v) = 0$$

if not, using (2.14) and (2.16) we get $\Psi(v) > 0$.

REMARK 2.1.3. *The barrier function $\Psi(v)$ in our algorithm is used to calculate the distance that the iterates (z, w) are from the μ-center $(z(\mu), w(\mu))$. The proximity measure $\delta(v)$ is used as a second proximity measure in the analysis of the algorithm. Our kernel functions inevitably decide both measures.*

2.2 GENERIC KERNEL FUNCTION

In this section, we give a detailed study of our two kernel functions by showing some basic properties that are essential in the complexity analysis. Before presenting our results, we mention some concepts that will be used in the subsequent analysis.

DEFINITION 2.2.1. *$\psi(x) : \mathbb{R}_{++} \to \mathbb{R}_{+}$ is a kernel function if ψ is twice differentiable and satisfies the following conditions :*

COND 1. $\psi(1) = \psi'(1) = 0$.

COND 2. $\psi''(x) > 0, \quad \forall x > 0$.

COND 3. $\lim_{x \to 0^+} \psi(x) = \lim_{x \to +\infty} \psi(x) = +\infty$.

From COND 1 and COND 2, the kernel function $\psi(x)$ is a nonnegative strictly convex function. Moreover, if $\psi(x)$ is twice differentiable, then it is determined by its second derivative as follows :

$$\psi(x) = \int_1^x \int_1^\xi \psi''(z)\, \mathrm{d}z\, \mathrm{d}\xi, \quad \forall x > 0. \tag{2.17}$$

and COND 3 expresses that $\psi(x)$ is coercive and has the barrier property.

REMARK 2.2.2. *The two functions (2.1) and (2.2) can be expressed as follows :*

$$\psi_i(x) = \frac{x^2 - 1}{2} + \psi_{ib}(x), \quad \textit{for } i = E, H \tag{2.18}$$

where $\frac{x^2-1}{2}$ is the so-called growth term and $\psi_{ib}(x)$ is the barrier term of our two kernel functions $\psi_i(x)$, respectively.

The important characteristics that demonstrate any kernel function's eligibility are shown in the lemma below.

LEMMA 2.2.3. *For all $x > 0$, $\psi(x)$ is eligible kernel function if check the following :*

$$x\psi''(x) + \psi'(x) > 0, \qquad x < 1. \tag{2.19}$$

$$x\psi''(x) - \psi'(x) > 0, \qquad x > 1. \tag{2.20}$$

$$\psi''(x) \textit{ is monotonically decreasing}, \quad x > 0. \tag{2.21}$$

$$2\psi''(x)^2 - \psi'(x)\psi'''(x) > 0, \qquad x < 1. \tag{2.22}$$

Note that (2.21) requires that $\psi(x)$ be three times differentiable, (2.19) and (2.22) are conditions on the barrier behavior of $\psi(x)$ and (2.20) concerns the growth behavior of $\psi(x)$. In the following, we present some relationships between eligibility conditions (the reader can refer to [6] for the proof).

COROLLARY 2.2.4. *(Lemma 2.4, [6]) If $\psi(x)$ satisfies (2.20) and (2.21), then :*

$$\psi''(x)\psi'(\eta x) - \eta\psi'(x)\psi''(\eta x) > 0, \quad x > 1, \quad \eta > 1. \tag{2.23}$$

LEMMA 2.2.5. *(Lemma 2.1.2, [64]) Let $\psi(x)$ be a twice differentiable function for $x > 0$. Then the following three properties are equivalent :*

$$\psi(\sqrt{x_1 x_2}) \leq \frac{\psi(x_1) + \psi(x_2)}{2}, \quad x_1, x_2 > 0. \tag{2.24}$$

$$x\psi''(x) + \psi'(x) \geq 0, \qquad x > 0. \tag{2.25}$$

$$\psi(e^{\xi}) \text{ is convex.} \tag{2.26}$$

LEMMA 2.2.6. *(Lemma 2.2, [6]) Let $\psi(x)$ be a twice differentiable function for $x > 0$. Then the following three properties are equivalent :*

$$\psi\left(\sqrt{\frac{x_1^2 + x_2^2}{2}}\right) \leq \frac{\psi(x_1) + \psi(x_2)}{2}, \quad x_1, x_2 > 0. \tag{2.27}$$

$$x\psi''(x) - \psi'(x) > 0, \qquad x > 0. \tag{2.28}$$

$$\psi(\sqrt{\xi}) \text{ is convex.} \tag{2.29}$$

LEMMA 2.2.7. *(Lemma 2.6, [6]) If the kernel function $\psi(x)$ satisfies (2.21), then :*

$$\frac{1}{2}\psi''(x)(x-1)^2 < \psi(x) < \frac{1}{2}\psi''(1)(x-1)^2 \text{ if } x > 1, \tag{2.30}$$

$$\frac{1}{2}\psi''(1)(x-1)^2 < \psi(x) < \frac{1}{2}\psi''(x)(x-1)^2 \text{ if } x < 1. \tag{2.31}$$

The following theorem gives an estimate of the effect of a μ-update on the value of $\Psi(v)$, which is valid for all kernel functions that satisfy (2.20) and (2.21).

THEOREM 2.2.8. *(Theorem 3.2, [6]) Let $\gamma(s) : [0, +\infty[\to [1, +\infty[$ be the inverse function of $\psi(x)$ for all $x \geq 1$. Then we have for any positive vector v and any $\beta \geq 1$ that :*

$$\Psi(\beta v) \leq n\psi\left(\beta\gamma\left(\frac{\Psi(v)}{n}\right)\right). \tag{2.32}$$

The characteristics of the two new kernel functions (2.1) and (2.2) are now ready to be provided.

2.2.1 FIRST KERNEL FUNCTION

In this part, we present the new parameterized kernel function (2.1). The proposed kernel function is neither a self-regular nor a logarithmic barrier function. This function is based on a kernel function introduced by *Bai et al.* [7], namely, $\psi_e(x) = \dfrac{x^2-1}{2} + \dfrac{e^{r(1-x)}-1}{r}$ for some $r \geq 1$, where $\psi_e(x)$ fails to have the first property of COND 3, and so we tried to modify it so that we can follow the analysis used in [6]. First, to investigate the behavior of the new kernel function (2.1), we present the following figure, which plotted for three different successive values of parameter $r = \{1.1, 2, 4\}$.

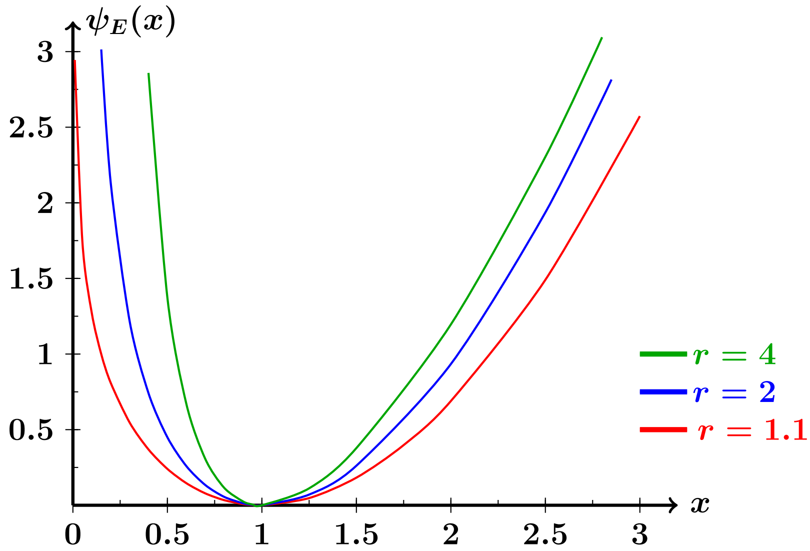

FIGURE 2.2: THE BEHAVIOR OF THE NEW KERNEL FUNCTION $\psi_E(x)$ FOR $r = \{1.1, 2, 4\}$.

REMARK 2.2.9. *As seen in* **FIGURE 2.2**, *our kernel function* (2.1) *verifies the three conditions of* **DEFINITION 2.2.1**.

The three first derivatives of the function (2.1) for all $x > 0$ are given as follows :

$$\psi_E'(x) = x - \frac{1}{2}\left(e^{(r-1)(1-x)} + \frac{1}{x^r}\right). \tag{2.33}$$

$$\psi_E''(x) = 1 + \frac{1}{2}\left((r-1)e^{(r-1)(1-x)} + \frac{r}{x^{r+1}}\right). \tag{2.34}$$

$$\psi_E'''(x) = -\frac{1}{2}\left((r-1)^2 e^{(r-1)(1-x)} + \frac{r(r+1)}{x^{r+2}}\right). \tag{2.35}$$

it follows that :

COND 1.

$$\psi_E(1) = \frac{1^2-1}{2} + \frac{e^{(r-1)(1-1)} + 1^{-r+1} - 2}{2(r-1)} = 0,$$
$$\psi_E'(1) = 1 - \frac{1}{2}\left(e^{(r-1)(1-1)} + \frac{1}{1^r}\right) = 0.$$

COND 2.

$$\psi_E''(x) = 1 + \frac{1}{2}\left((r-1)e^{(r-1)(1-x)} + \frac{r}{x^{r+1}}\right) > 0,$$

since $r > 1$ and $x > 0$.

COND 3.

$$\lim_{x\to 0^+}\psi_E(x) = \lim_{x\to 0^+}\frac{x^2-1}{2} + \frac{e^{(r-1)(1-x)} + x^{-r+1} - 2}{2(r-1)} = +\infty,$$
$$\lim_{x\to +\infty}\psi_E(x) = \lim_{x\to +\infty}\frac{x^2-1}{2} + \frac{e^{(r-1)(1-x)} + x^{-r+1} - 2}{2(r-1)} = +\infty$$

this implies that the function defined in (2.1) is a barrier kernel function. From **COND 1**, **COND 2** and **COND 3**, we conclude that the function defined in (2.1) is a barrier kernel function.

REMARK 2.2.10. *Note that by subtracting* (2.34), *we get :*

$$\psi''_E(x) \geq 1 > 0, \quad \textit{for all } x > 0. \tag{2.36}$$

The following lemma establishes that our new kernel function (2.1) is an eligible kernel function.

LEMMA 2.2.11. *Let $\psi_E(x)$ be as defined in* (2.1). *Then it was satisfies* (2.19), (2.20), (2.21) *and* (2.22).

PROOF. These inequalities result from some basic calculations involving (2.1), (2.33), (2.34) and (2.35).

$$(a)\quad x\psi''_E(x) + \psi'_E(x) = 2x + \frac{1}{2}\left(((r-1)x-1)\,e^{(r-1)(1-x)} + \frac{r-1}{x^r}\right) > 0,$$

for all $x \in]0,1[$.

$$(b)\quad x\psi''_E(x) - \psi'_E(x) = \frac{1}{2}\left(((r-1)x+1)\,e^{(r-1)(1-x)} + \frac{r+1}{x^r}\right) > 0, \quad x > 1.$$

(c) We can draw a conclusion from the fact that $x > 0$, $r > 0$ and $e^{(r-1)(1-x)} > 0$ that $\psi'''_E(x) < 0$.

$$\begin{aligned}(d)\quad 2\psi''_E(x)^2 - \psi'_E(x)\psi'''_E(x) = {} & 2 + \frac{(r-1)^2}{4}e^{2(r-1)(1-x)} + \frac{r^2-r}{4x^{2(r+1)}} + \frac{r^2+5r}{2x^{r+1}} + \\ & e^{(r-1)(1-x)}\left(\frac{x(r-1)^2}{4} + 2(r-1) - \frac{(r-1)^2}{4x^r} + \right.\\ & \left.\frac{(r-1)r}{x^{r+1}} - \frac{r(r+1)}{4x^{r+2}}\right) > 0, \quad 0 < x < 1,\end{aligned}$$

As a result, the kernel function (2.1) is efficient. □

REMARK 2.2.12. *From* **LEMMA 2.2.11**. *The function $\psi_E(x)$ defined by* (2.1) *satisfies the properties presented in* **COROLLARY 2.2.4**, **LEMMA 2.2.5**, **LEMMA 2.2.6** *and* **LEMMA 2.2.7**.

COROLLARY 2.2.13. *For $r > 1$, we have :*

$$\psi_E(x) \leq \frac{x^2-1}{2}.$$

PROOF. For all $x \geq 1$ and $r > 1$, we find :

$$\begin{aligned}\Upsilon(x) &= \psi_E(x) - \frac{x^2-1}{2} \\ &= \frac{e^{(r-1)(1-x)} + x^{-r+1} - 2}{2(r-1)} \\ &\leq 0.\end{aligned}$$

this second inequality arises from the fact that the function $\Upsilon(x)$ decreases monotonically with respect to x, yielding $\Upsilon(x) \leq \Upsilon(1) = 0$. Then for every $x \geq 1$, $\Upsilon(x) \leq 0$. □

In anticipation of later, we present some technical results of our new kernel function (2.1).

LEMMA 2.2.14. *For the kernel function (2.1), we have for all $r > 1$:*

$$\frac{1}{2}(x-1)^2 \leq \psi_E(x) \leq \frac{1}{2}\left(\psi_E'(x)\right)^2, \qquad x > 0. \tag{2.37}$$

$$\psi_E(x) \leq \frac{2r+1}{4}(x-1)^2, \quad x \geq 1. \tag{2.38}$$

PROOF. For (2.37). The left equality may be easily verified using (2.17) and (2.36) :

$$\psi_E(x) = \int_1^x \int_1^\xi \psi_E''(z)\,\mathrm{d}z\,\mathrm{d}\xi \geq \int_1^x \int_1^\xi \mathrm{d}z\,\mathrm{d}\xi = \frac{1}{2}(x-1)^2, \quad \forall x > 0.$$

we have the following for right equality :

$$\begin{aligned}\psi_E(x) = \int_1^x \int_1^\xi \psi_E''(z)\,\mathrm{d}z\,\mathrm{d}\xi &\leq \int_1^x \int_1^\xi \psi_E''(\xi)\psi_E''(z)\,\mathrm{d}z\,\mathrm{d}\xi \\ &= \int_1^x \psi_E''(\xi)\psi_E'(\xi)\,\mathrm{d}\xi = \frac{1}{2}\left(\psi_E'(x)\right)^2.\end{aligned}$$

For (2.38), we can deduce the following from **REMARK 2.2.12**, the right portion of the inequality (2.31) and the function (2.34) :

$$\begin{aligned}\psi_E(x) &\leq \frac{1}{2}\psi_E''(1)(x-1)^2 \\ &= \frac{1}{2}\left(r + \frac{1}{2}\right)(x-1)^2, \quad x \geq 1\end{aligned}$$

the evidence is now complete. □

Let us denote by $\gamma_E(s) : [0, +\infty[\to [1, +\infty[$ be the inverse function of $\psi_E(x)$ described in (2.1) for all $x \geq 1$ and by $\rho_E(s) : [0, +\infty[\to]0, 1]$ be the inverse function of $-\frac{1}{2}\psi_E'(x)$ for all $x \in]0, 1]$. The following are the outcomes :

LEMMA 2.2.15. *For our kernel function $\psi_E(x)$ with $r > 1$, we have :*

$$1 + \sqrt{\frac{4s}{2r+1}} \leq \gamma_E(s) \leq 1 + \sqrt{2s}. \tag{2.39}$$

$$\rho_E(s) \geq \frac{1}{(4s+2)^{\frac{1}{r}}}. \tag{2.40}$$

PROOF. For (2.39). As a result of the definition of γ_E, we have :

$$s = \psi_E(x), \quad x \geq 1 \Leftrightarrow \gamma_E(s) = x, \quad s \geq 0$$

using the inequality's left side (**2.37**), we obtain :

$$s = \psi_E(x) \geq \frac{1}{2}(x-1)^2 \Leftrightarrow x = \gamma_E(s) \leq 1 + \sqrt{2s}$$

as a result of (**2.38**), we get :

$$s = \psi_E(x) \leq \frac{2r+1}{4}(x-1)^2 \Leftrightarrow x = \gamma_E(s) \geq 1 + \sqrt{\frac{4s}{2r+1}}.$$

Using the definition of ρ_E for (**2.40**), we have :

$$s = -\frac{1}{2}\psi_E'(x), \quad 0 < x \leq 1 \Leftrightarrow \rho_E(s) = x, \quad s \geq 0.$$

(**2.33**) yields :

$$\begin{aligned} s = -\frac{1}{2}\psi_E'(x) \quad & \Leftrightarrow \quad s = -\frac{1}{2}\left(x - \frac{1}{2}\left(e^{(r-1)(1-x)} + \frac{1}{x^r}\right)\right) \\ & \Leftrightarrow \quad 2s + x = \frac{1}{2}\left(e^{(r-1)(1-x)} + \frac{1}{x^r}\right) \\ & \Leftrightarrow \quad \frac{1}{x^r} \leq 4s + 2 \\ & \Leftrightarrow \quad x \geq \frac{1}{(4s+2)^{\frac{1}{r}}} \end{aligned} \tag{2.41}$$

where the third inequality derives from $r > 1$, $x \leq 1$ and $\dfrac{e^{(r-1)(1-x)}}{2} > 0$. □

The lemma that follows demonstrates a relationship between (**2.16**) and $\Psi_E(v)$.

LEMMA 2.2.16. *Let $\delta(v)$ be defined as in (**2.16**). Then, we have :*

$$\delta(v) \geq \sqrt{\frac{\Psi_E(v)}{2}}.$$

PROOF. Due (**2.14**), (**2.16**) and (**2.37**), we have :

$$\begin{aligned} \Psi_E(v) &= \sum_{i=1}^{n} \psi_E(v_i) \\ &\leq \sum_{i=1}^{n} \frac{1}{2}\left(\psi_E'(v_i)\right)^2 \\ &= \frac{1}{2}\|\nabla\Psi_E(v)\|^2 \\ &= 2\delta^2(v) \end{aligned}$$

then, we obtain $\sqrt{2}\delta(v) \geq \sqrt{\Psi_E(v)}$. □

The following theorem assesses the effect of a μ-update on the value of $\Psi_E(v)$.

LEMMA 2.2.17. *If $\Psi_E(v) \leq \tau$ and $\beta = \dfrac{1}{\sqrt{1-\theta}}$. Then for all $r > 1$ we have :*

$$\Psi_E(v^+) \leq \frac{2r+1}{4(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2. \tag{2.42}$$

$$\Psi_E(v^+) \leq \frac{2\tau+n\theta+2\sqrt{2n\tau}}{2(1-\theta)}. \tag{2.43}$$

With $0 < \theta < 1$ and $v^+ = \dfrac{v}{\sqrt{1-\theta}}$.

PROOF. Due to **THEOREM 2.2.8** with $\Psi_E(v) \leq \tau$, $\beta = \dfrac{1}{\sqrt{1-\theta}} \geq 1$ and $\gamma_E\left(\dfrac{\Psi_E(v)}{n}\right) \geq 1$ $\left(\text{i.e., } \beta\gamma_E\left(\dfrac{\Psi_E(v)}{n}\right) \geq 1\right)$ we can prove this lemma.

For (2.42), using (2.38) we obtain :

$$\begin{aligned}
\Psi_E(v^+) &\leq n\psi_E\left(\frac{1}{\sqrt{1-\theta}}\gamma_E\left(\frac{\Psi_E(v)}{n}\right)\right)\\
&\leq n\frac{2r+1}{4}\left(\frac{1}{\sqrt{1-\theta}}\gamma_E\left(\frac{\Psi_E(v)}{n}\right)-1\right)^2\\
&\leq n\frac{2r+1}{4}\left(\frac{1+\sqrt{2\frac{\Psi_E(v)}{n}}}{\sqrt{1-\theta}}-1\right)^2\\
&\leq n\frac{2r+1}{4}\left(\frac{1+\sqrt{2\frac{\tau}{n}}-\sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2\\
&\leq n\frac{2r+1}{4}\left(\frac{\theta+\sqrt{2\frac{\tau}{n}}}{\sqrt{1-\theta}}\right)^2\\
&= \frac{2r+1}{4(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2
\end{aligned}$$

where the third inequality is derived from the second part of (2.39), while the last inequality is based on fact $1-\sqrt{1-\theta} = \dfrac{\theta}{1+\sqrt{1-\theta}} \leq \theta$.

As a result of **COROLLARY 2.2.13**, we get (2.43).

$$\begin{aligned}
\Psi_E(v^+) &\leq n\psi_E\left(\frac{1}{\sqrt{1-\theta}}\gamma_E\left(\frac{\Psi_E(v)}{n}\right)\right)\\
&\leq n\left(\frac{\left(\frac{1}{\sqrt{1-\theta}}\gamma_E\left(\frac{\Psi_E(v)}{n}\right)\right)^2-1}{2}\right)\\
&= \frac{n}{2}\left(\frac{1}{1-\theta}\gamma_E\left(\frac{\Psi_E(v)}{n}\right)^2-1\right)\\
&\leq \frac{n}{2(1-\theta)}\left(\gamma_E\left(\frac{\Psi_E(v)}{n}\right)^2+\theta-1\right)\\
&\leq \frac{n}{2(1-\theta)}\left(\left(1+\sqrt{2\frac{\Psi_E(v)}{n}}\right)^2+\theta-1\right)
\end{aligned}$$

$$= \frac{2\tau + n\theta + 2\sqrt{2n\tau}}{2(1-\theta)}$$

this concludes the proof. □

NOTATION 1. *For all $r > 1$, we denote :*

$$\overline{\Psi}_{E_0} = \frac{2r+1}{4(1-\theta)}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2 \quad (2.44)$$

$$\widehat{\Psi}_{E_0} = \frac{2\tau + n\theta + 2\sqrt{2n\tau}}{2(1-\theta)} \quad (2.45)$$

where $\overline{\Psi}_{E_0}$ and $\widehat{\Psi}_{E_0}$ are the upper bounds of $\Psi_E(v)$ for small-update and large-update, respectively.

REMARK 2.2.18. *By NOTATION 1 and the assumption that $\Psi_E(v) \leq \tau$ just before the update of μ, $\Psi_E(v^+) \leq \min\left(\overline{\Psi}_{E_0}, \widehat{\Psi}_{E_0}\right)$. For large-update methods with $\tau = O(n)$ and $\theta = \Theta(1)$, we have $\widehat{\Psi}_{E_0} = O(n)$ and for small-update methods with $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, we have $\overline{\Psi}_{E_0} = O(r)$.*

2.2.2 SECOND KERNEL FUNCTION

With a new hyperbolic-logarithmic barrier term (2.2), which we employ in the complexity analysis of our algorithm, we explore some fundamental features of the new kernel function in this section. To survey the behavior of our new kernel function (2.2), we present the following figure, which plots three different successive values of the parameter $r = \{4, 5, 6\}$.

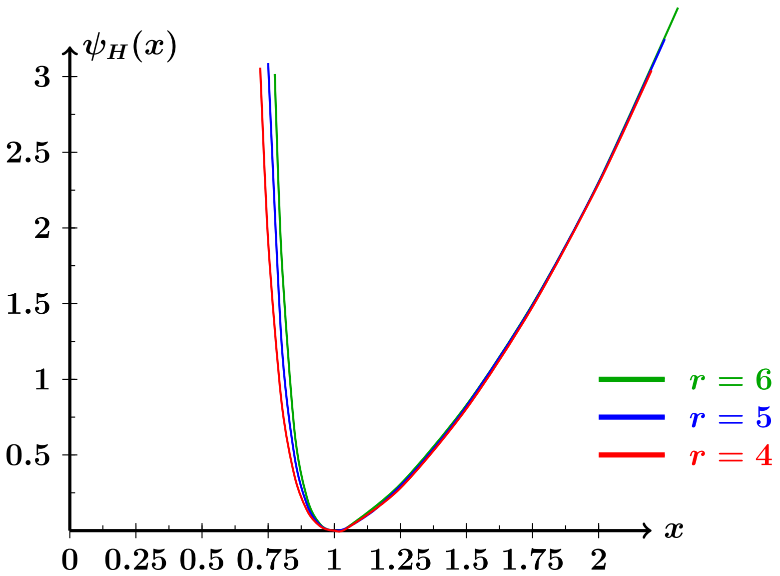

FIGURE 2.3: THE BEHAVIOR OF THE NEW KERNEL FUNCTION $\psi_H(x)$ FOR $r = \{4, 5, 6\}$.

REMARK 2.2.19. *The three conditions COND 1, COND 2 and COND 3 are verified by our novel kernel function with a hyperbolic-logarithmic barrier term (2.2), as shown in FIGURE 2.3.*

The following are the first three derivatives of (2.2) with respect to x :

$$\psi_H'(x) = x - \frac{1}{2x} - \frac{\cosh^r(x^{-1}) - \cosh^r(1)}{2\tanh(1)\cosh^r(1)\, x^{r+1}} - \frac{\tanh(x^{-1})\cosh^r(x^{-1})}{2\tanh(1)\cosh^r(1)\, x^{r+2}}. \tag{2.46}$$

$$\begin{aligned}\psi_H''(x) &= 1 + \frac{1}{2x^2} + \frac{r+1}{2}\left(\frac{\cosh^r(x^{-1}) - \cosh^r(1)}{\tanh(1)\cosh^r(1)\, x^{r+2}} + \frac{2\tanh(x^{-1})\cosh^r(x^{-1})}{\tanh(1)\cosh^r(1)\, x^{r+3}}\right)\\ &\quad + \frac{r\tanh^2(x^{-1})\cosh^r(x^{-1}) + \cosh^{r-2}(x^{-1})}{2\tanh(1)\cosh^r(1)\, x^{r+4}}.\end{aligned} \tag{2.47}$$

$$\begin{aligned}\psi_H'''(x) &= -\left(\frac{1}{x^3} + \frac{r+2}{2}\left(\frac{r+1}{\tanh(1)\cosh^r(1)}\left(\frac{\cosh^r(x^{-1}) - \cosh^r(1)}{x^{r+3}} + \right.\right.\right.\\ &\quad \left.\frac{3\tanh(x^{-1})\cosh^r(x^{-1})}{x^{r+4}}\right) + 3\frac{r\tanh^2(x^{-1})\cosh^r(x^{-1}) + \cosh^{r-2}(x^{-1})}{x^{r+5}}\Bigg)\\ &\quad \left. + \frac{r^2\tanh^3(x^{-1})\cosh^r(x^{-1}) + (3r-2)\tanh(x^{-1})\cosh^{r-2}(x^{-1})}{2\tanh(1)\cosh^r(1)\, x^{r+6}}\right).\end{aligned} \tag{2.48}$$

NOTATION 2. *The notations shown below are intended to help with the research.*

$$\begin{aligned} a &= \tanh(1)\cosh^r(1).\\ f(x) &= \cosh^r(x^{-1}) - \cosh^r(1).\\ g(x) &= \tanh(x^{-1})\cosh^r(x^{-1}).\\ h(x) &= r\tanh^2(x^{-1})\cosh^r(x^{-1}) + \cosh^{r-2}(x^{-1}).\\ q(x) &= r^2\tanh^3(x^{-1})\cosh^r(x^{-1}) + (3r-2)\tanh(x^{-1})\cosh^{r-2}(x^{-1}).\end{aligned}$$

We offer several features in the following lemma before beginning the analysis.

LEMMA 2.2.20. *For the function* (2.1), *we have the following properties for all* $r \geq 4$ *:*

$$\cosh(x^{-1}) > 0 \quad \textit{and} \quad 0 < \tanh(x^{-1}) < 1 \quad \textit{for} \quad x > 0. \tag{2.49}$$

$$ax^r + f(x) > 0, \quad g(x) > 0, \quad h(x) > 0 \quad \textit{and} \quad q(x) > 0 \quad \textit{for} \quad x > 0. \tag{2.50}$$

$$\cosh^r(x^{-1})\left(\frac{\tanh(x^{-1})}{ax^{r+2}} - \frac{1}{(ax)^r}\right) > 0 \quad \textit{for} \quad 0 < x < 1. \tag{2.51}$$

PROOF. (2.49) is valid since for all $x > 0$ the two functions $\tanh(x^{-1})$ and $\cosh(x^{-1})$ decrease in $]0, +\infty[$ with :

$$\lim_{x\to 0^+} \tanh\left(x^{-1}\right) = 1, \lim_{x\to+\infty} \tanh\left(x^{-1}\right) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \cosh\left(x^{-1}\right) = 1 > 0.$$

We can see from NOTATION 2 and (2.49) that the four functions f, g, h and q decrease in in $]0, +\infty[$ because for every $x > 0$ we have :

$$
\begin{aligned}
&(ax^r + f(x))' = -\frac{r}{x^2}\left(ax^{r+1} + \tanh\left(x^{-1}\right)\cosh^r\left(x^{-1}\right)\right) < 0,\\
&g'(x) = -\frac{1}{x^2}\left(\cosh^{r-2}\left(x^{-1}\right) + r\tanh^2\left(x^{-1}\right)\cosh^r\left(x^{-1}\right)\right) < 0,\\
&h'(x) = -\frac{1}{x^2}\left((3r-2)\tanh\left(x^{-1}\right)\cosh^{r-2}\left(x^{-1}\right) + r^2\tanh^3\left(x^{-1}\right)\cosh^r\left(x^{-1}\right)\right) < 0,\\
&q'(x) = -\frac{1}{x^2}\left((3r-2)\cosh^{r-4}\left(x^{-1}\right) + (6r^2-8r+4)\tanh^2\left(x^{-1}\right)\cosh^{r-2}\left(x^{-1}\right)\right.\\
&\qquad\left. + r^3\tanh^4\left(x^{-1}\right)\cosh^r\left(x^{-1}\right)\right) < 0,
\end{aligned}
$$

with :

$$
\lim_{x\to+\infty} ax^r + f(x) > 0,\ \lim_{x\to+\infty} g(x) = 0,\ \lim_{x\to+\infty} h(x) = 1 > 0 \quad \text{and} \quad \lim_{x\to+\infty} q(x) = 0
$$

the expected result (**2.50**) is subsequently obtained.

To demonstrate (**2.51**), we put :

$$
\Upsilon(x) = \cosh^r\left(x^{-1}\right)\left(\frac{\tanh\left(x^{-1}\right)}{ax^{r+2}} - \frac{1}{(ax)^r}\right)
$$

and we compute :

$$
\begin{aligned}
\Upsilon'(x) &= r\cosh^r\left(x^{-1}\right)\left(\frac{1}{a^r x^{p+1}} + \frac{\tanh\left(x^{-1}\right)}{a^r x^{r+2}}\right) - \cosh^r\left(x^{-1}\right)\left(\frac{(r+2)\tanh\left(x^{-1}\right)}{ax^{r+3}}\right.\\
&\quad\left. + \frac{r\tanh^2\left(x^{-1}\right) + \cosh^{-2}\left(x^{-1}\right)}{ax^{r+4}}\right)
\end{aligned}
$$

according to $0 < x < 1$, $a > 0$, $r \geq 4$, (**2.50**) and $\tanh^2\left(x^{-1}\right) + \tanh\left(x^{-1}\right) > 1$ we get :

$$
\begin{aligned}
\Upsilon'(x) &\leq -\cosh^r\left(x^{-1}\right)\left(\frac{2\left(\tanh^2\left(x^{-1}\right) + \tanh\left(x^{-1}\right) - 1\right) + \cosh^{-2}\left(x^{-1}\right)}{a^r x^{r+2}}\right)\\
&\leq 0.
\end{aligned}
$$

As a result, we infer that the function $\Upsilon(x)$ decreases in $]0,1[$ and we obtain :

$$
\lim_{x\to 1}\Upsilon(x) = 1 - a^{-r} > 0,
$$

so for all $x \in]0,1[$, we have $\Upsilon(x) > 0$. □

By employing (**2.2**), (**2.46**), (**2.47**) and (**2.50**), it is straightforward to validate the three conditions **COND 1**, **COND 2** and **COND 3** of **DEFINITION 2.2.1** and verify that the function defined by (**2.2**) is a barrier kernel function.

COND 1.

$$
\begin{aligned}
\psi_H(1) &= \frac{1^2-1}{2} + \frac{1}{2r}\left(\frac{\cosh^r(1) - \cosh^r(1)}{a1^r} - \log(1^r)\right) = 0,\\
\psi_H'(1) &= 1 - \frac{1}{2}\left(\frac{\cosh^r(1) - \cosh^r(1)}{a1^{r+1}} + \frac{\tanh(1)\cosh^r(1)}{a1^{r+2}}\right) = 0.
\end{aligned}
$$

COND 2.

$$
\psi_H''(x) = 1 + \frac{1}{2x^2} + \frac{r+1}{2a}\left(\frac{f(x)}{x^{r+2}} + \frac{2g(x)}{x^{r+3}}\right) + \frac{h(x)}{2ax^{r+4}} > 0,
$$

this disparity results from the fact that $r \geq 4$, $x > 0$ and NOTATION 2.

COND 3.

$$\lim_{x\to 0^+} \psi_H(x) = \lim_{x\to 0^+} \frac{x^2-1}{2} + \frac{1}{2r}\left(\frac{\cosh^r(x^{-1}) - \cosh^r(1)}{ax^r} - \log(x^r)\right) = +\infty,$$

$$\lim_{x\to +\infty} \psi_H(x) = \lim_{x\to +\infty} \frac{x^2-1}{2} + \frac{1}{2r}\left(\frac{\cosh^r(x^{-1}) - \cosh^r(1)}{ax^r} - \log(x^r)\right) = +\infty.$$

REMARK 2.2.21. *Furthermore, (2.47) and (2.50) suggest that we have for each $x > 0$:*

$$\psi_H''(x) \geq 1 > 0, \quad \textit{for all } x > 0. \tag{2.52}$$

We may now explore the essential characteristics of our function $\psi_H(x)$.

LEMMA 2.2.22. *Let $\psi_H(x)$ be as defined in (2.2). Then $\psi_H(x)$ is an eligible kernel function.*

PROOF. In order to satisfy LEMMA 2.2.3, the function $\psi_H(x)$ must simply meet the following requirements (2.19), (2.20), (2.21) and (2.22). To do this, we use NOTATION 2 and the functions (2.2), (2.46), (2.47), (2.48).

(a) $$x\psi_H''(x) + \psi_H'(x) = 2x + \frac{1}{2a}\left(\frac{rf(x)}{x^{r+1}} + \frac{(2r+1)g(x)}{x^{r+2}} + \frac{h(x)}{x^{r+3}}\right) > 0,$$
for all $x \in]0,1[$.

(b) $$x\psi_H''(x) - \psi_H'(x) = \frac{1}{x} + \frac{1}{2a}\left(\frac{(r+2)f(x)}{x^{r+1}} + \frac{(2r+3)g(x)}{x^{r+2}} + \frac{h(x)}{x^{r+3}}\right) > 0,$$
for all $x > 1$.

(c) We may deduce from (2.50), the positivity of x and r that $\psi_H'''(x) < 0$ for all $x > 0$.

(d) For all $0 < x < 1$, we have :

$$\begin{aligned}
2\psi_H''(x)^2 - \psi_H'(x)\psi_H'''(x) = \; & 2 + \frac{3}{x^2} + \frac{1}{4a}\Bigg(2(r+1)\left(\frac{(r+6)f(x)}{x^{r+2}} + \frac{(3r+14)g(x)}{x^{r+3}}\right) \\
& + \frac{((r+1)(2-r)-2)f(x) + 2(3r+10)h(x)}{x^{r+4}} + \frac{2q(x)}{x^{r+5}} \\
& + \frac{((r+1)(2-3r)-2)g(x)}{x^{r+5}} - \frac{(3r-2)h(x)}{x^{r+6}} - \frac{q(x)}{x^{r+7}}\Bigg) \\
& + \frac{1}{4a^2}\Bigg(r(r+1)\left(\frac{f^2(x)}{x^{2r+4}} + \frac{4f(x)g(x)}{x^{2r+5}}\right) - \frac{rf(x)g(x)}{x^{2r+6}} \\
& + \frac{(r+1)(5r+2)g^2(x)}{x^{2r+6}} + \frac{(5r+2)g(x)h(x) - f(x)q(x)}{x^{2r+7}} \\
& + \frac{2h^2(x) - g(x)q(x)}{x^{2r+8}} > 0,
\end{aligned}$$

the evidence is now complete. □

REMARK 2.2.23. *Since the kernel function $\psi_H(x)$ defined by (2.1) is efficient. Then, $\psi_H(x)$ satisfies the following conditions. The sign $(+)$ indicate whether a condition is satisfied.*

(2.23)	(2.24)	(2.25)	(2.26)	(2.27)	(2.28)	(2.29)	(2.30)	(2.31)
+	+	+	+	+	+	+	+	+

TABLE 2.1: SOME CONDITIONS OF THE FUNCTION $\psi_H(x)$.

COROLLARY 2.2.24. *For $r \geq 4$, we have :*

$$\psi_H(x) \leq \frac{x^2-1}{2}.$$

PROOF. For all $x \geq 1$ and $r \geq 4$, we obtain :

$$\begin{aligned} \Upsilon(x) &= \psi_H(x) - \frac{x^2-1}{2} \\ &= \frac{1}{2r}\left(\frac{\cosh^r(x^{-1}) - \cosh^r(1)}{ax^r} - \log(x^r)\right) \\ &\leq 0 \end{aligned}$$

this last inequity is the result of the following :

$$\Upsilon'(x) = -\frac{1}{2a}\left(\frac{a}{x} + \frac{f(x)}{x^{r+1}} + \frac{g(x)}{x^{r+2}}\right).$$

The function $\Upsilon(x)$ then decreases monotonically with respect to x, giving us :

$$\Upsilon(x) \leq \Upsilon(1) = 0. \qquad \square$$

LEMMA 2.2.25. *For all $r \geq 4$, the kernel function $\psi_H(x)$ hold the following results :*

$$\frac{1}{2}(x-1)^2 \leq \psi_H(x) \leq \frac{1}{2}\left(\psi_H'(x)\right)^2, \quad x > 0 \tag{2.53}$$

$$\psi_H(x) \leq \frac{\vartheta r + \iota}{2}(x-1)^2, \quad x \geq 1 \tag{2.54}$$

with $\vartheta = 1 + \dfrac{\tanh(1)}{2}, \quad \iota = \dfrac{5}{2} + \dfrac{1}{2\tanh(1)\cosh^2(1)}.$

PROOF. For (2.53). From (2.17), we get :

$$\psi_H(x) = \int_1^x \int_1^\xi \psi_2''(z)\,dz\,d\xi \geq \int_1^x \int_1^\xi dz\,d\xi = \frac{1}{2}(x-1)^2, \quad \forall x > 0.$$

and

$$\begin{aligned} \psi_H(x) = \int_1^x \int_1^\xi \psi_H''(z)\,dz\,d\xi &\leq \int_1^x \int_1^\xi \psi_H''(\xi)\psi_H''(z)\,dz\,d\xi \\ &= \int_1^x \psi_H''(\xi)\psi_H'(\xi)\,d\xi = \frac{1}{2}\left(\psi_H'(x)\right)^2. \end{aligned}$$

We just need to compute (2.54) from the right portion of the inequality (2.30) and (2.47) :

$$\begin{aligned} \psi_H''(1) &= r + \frac{5}{2} + \frac{r\tanh^2(1)\cosh^r(1) + \cosh^{r-2}(1)}{2a} \\ &= \left(1 + \frac{\tanh^2(1)\cosh^r(1)}{2a}\right)r + \frac{5}{2} + \frac{\cosh^{r-2}(1)}{2a} \\ &= \vartheta r + \iota \end{aligned}$$

consequently,

$$\begin{aligned}\psi_H(x) &\leq \frac{1}{2}\psi_2''(1)(x-1)^2 \\ &= \frac{\vartheta r+\iota}{2}(x-1)^2, \quad x \geq 1\end{aligned}$$

this concludes the proof. □

LEMMA 2.2.26. *Let $\gamma_H(s) : [0,+\infty[\rightarrow [1,+\infty[$ be the inverse function of $\psi_H(x)$ for all $x > 1$. Then, we have :*

$$1+\sqrt{\frac{2s}{\vartheta r+\iota}} \leq \gamma_H(s) \leq 1+\sqrt{2s}, \quad r \geq 4. \tag{2.55}$$

PROOF. For (**2.55**), let $\psi_H(x) = s$, $x \geq 1$ i.e. $x = \gamma_H(s)$, $r \geq 0$. Using the left portion of the inequality (**2.53**), we get :

$$s = \psi_H(x) \geq \frac{1}{2}(x-1)^2 \text{ this suggests that } x = \gamma_H(s) \leq 1+\sqrt{2s}.$$

and from (**2.38**), we get :

$$s = \psi_H(x) \leq \frac{\vartheta r+\iota}{2}(x-1)^2 \text{ this suggests that } x = \gamma_H(s) \geq 1+\sqrt{\frac{2s}{\vartheta r+\iota}}.$$ □

LEMMA 2.2.27. *Let $\rho_H(s) : [0,+\infty[\rightarrow]0,1]$ be the inverse function of $-\frac{1}{2}\psi_H'(x)$ for all $x \in]0,1]$. Then, we have :*

$$\frac{\cosh\left(x^{-1}\right)}{ax} \leq (4s+2)^{\frac{1}{r}}, \quad r \geq 4. \tag{2.56}$$

PROOF. Let $s = -\frac{1}{2}\psi_H'(x)$ for $0 < x \leq 1$. Because of the definition of ρ_H and (**2.46**), we get :

$$\begin{aligned} s = -\frac{1}{2}\psi_H'(x) \;\Leftrightarrow\; & s = -\frac{1}{2}\left(x - \frac{1}{2x} - \frac{1}{2a}\left(\frac{f(x)}{x^{r+1}} + \frac{g(x)}{x^{r+2}}\right)\right) \\ \Leftrightarrow\; & \frac{g(x)}{2ax^{r+2}} = 2s + x - \frac{1}{2x} - \frac{f(x)}{2ax^{r+1}} \leq 2s+1 \end{aligned} \tag{2.57}$$

this inequality holds since $x \leq 1$, (**2.50**) and may be deduced from (**2.51**).

$$s = -\frac{1}{2}\psi_H'(x) \Leftrightarrow \left(\frac{\cosh\left(x^{-1}\right)}{ax}\right)^r \leq 4s+2$$

hence

$$s = -\frac{1}{2}\psi_H'(x) \Leftrightarrow \frac{\cosh\left(x^{-1}\right)}{ax} \leq (4s+2)^{\frac{1}{r}}.$$ □

The lemma that follows demonstrates a relationship between(**2.16**) and $\Psi_H(v)$.

LEMMA 2.2.28. *Let $\delta(v)$ be defined as in (**2.16**). Then, we have :*

$$\sqrt{2}\delta(v) \geq \sqrt{\Psi_H(v)}.$$

PROOF. (**2.14**), (**2.16**) and (**2.53**), respectively, provide the following result :

$$\begin{aligned}
\Psi_H(v) &= \sum_{i=1}^{n} \psi_H(v_i) \\
&\leq \sum_{i=1}^{n} \frac{1}{2}\left(\psi_H^{'}(v_i)\right)^2 \\
&= \frac{1}{2}\|\nabla\Psi_H(v)\|^2 \\
&= 2\delta^2(v)
\end{aligned}$$

this is the end of the proof. $\square$

In the lemma below, we construct two upper bounds for the effect of a μ-update on the value of $\Psi_H(v)$.

LEMMA 2.2.29. *Let $0 < \theta < 1$ and $v^+ = \dfrac{v}{\sqrt{1-\theta}}$. If $\Psi_H(v) \leq \tau$, then :*

$$\Psi_H(v^+) \leq \frac{\vartheta r + \iota}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2 \tag{2.58}$$

$$\Psi_H(v^+) \leq \frac{2\tau + n\theta + 2\sqrt{2n\tau}}{2(1-\theta)}. \tag{2.59}$$

PROOF. For (2.58), since $\dfrac{1}{\sqrt{1-\theta}} \geq 1$ and $\gamma_H\left(\dfrac{\Psi_H(v)}{n}\right) \geq 1$, we have $\dfrac{\gamma_H\left(\frac{\Psi_H(v)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. Using THEOREM 2.2.8 with $\beta = \dfrac{1}{\sqrt{1-\theta}}$, $\Psi_H(v) \leq \tau$, (2.54) and the second inequality in (2.55), we obtain :

$$\begin{aligned}
\Psi_H(v^+) &\leq n\psi_H\left(\frac{1}{\sqrt{1-\theta}}\gamma_H\left(\frac{\Psi_H(v)}{n}\right)\right) \\
&\leq n\frac{\vartheta r + \iota}{2}\left(\frac{1}{\sqrt{1-\theta}}\gamma_H\left(\frac{\Psi_H(v)}{n}\right) - 1\right)^2 \\
&\leq n\frac{\vartheta r + \iota}{2}\left(\frac{1 + \sqrt{2\frac{\Psi_H(v)}{n}}}{\sqrt{1-\theta}} - 1\right)^2 \\
&\leq n\frac{\vartheta r + \iota}{2}\left(\frac{1 + \sqrt{2\frac{\tau}{n}} - \sqrt{1-\theta}}{\sqrt{1-\theta}}\right)^2 \\
&\leq n\frac{\vartheta r + \iota}{2}\left(\frac{\theta + \sqrt{2\frac{\tau}{n}}}{\sqrt{1-\theta}}\right)^2 \\
&= \frac{\vartheta r + \iota}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2
\end{aligned}$$

where the last inequality holds because $1 - \sqrt{1-\theta} = \dfrac{\theta}{1+\sqrt{1-\theta}} \leq \theta$. (2.59) is given by THEOREM 2.2.8 with $\beta = \dfrac{1}{\sqrt{1-\theta}}$, (2.55) and COROLLARY 2.2.24, we get :

$$\begin{aligned}
\Psi_H(v) &\leq n\psi_H\left(\frac{1}{\sqrt{1-\theta}}\gamma_H\left(\frac{\Psi_H(v)}{n}\right)\right) \\
&\leq \frac{n}{2}\left(\left(\frac{1}{\sqrt{1-\theta}}\gamma_H\left(\frac{\Psi_H(v)}{n}\right)\right)^2 - 1\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n}{2(1-\theta)}\left(\gamma_H\left(\frac{\Psi_H(v)}{n}\right)+\theta-1\right)\\
&\leq \frac{n}{2(1-\theta)}\left(\left(1+\sqrt{2\frac{\Psi_H(v)}{n}}\right)^2+\theta-1\right)\\
&= \frac{2\tau+n\theta+2\sqrt{2n\tau}}{2(1-\theta)}.
\end{aligned}
$$

This completes the evidence. □

NOTATION 3. *We will use* $\overline{\Psi}_{H_0}$ *and* $\widehat{\Psi}_{H_0}$ *for the upper bounds of* $\Psi_H(v)$ *for small-update and large-update, respectively, as follows :*

$$
\overline{\Psi}_{H_0} = \frac{\vartheta r+\iota}{2(1-\theta)}\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2 \tag{2.60}
$$

$$
\widehat{\Psi}_{H_0} = \frac{2\tau+n\theta+2\sqrt{2n\tau}}{2(1-\theta)}. \tag{2.61}
$$

REMARK 2.2.30. *By* **NOTATION 3** *and the assumption that* $\Psi_H(v)\leq\tau$ *just before the update of* μ, $\Psi_H(v^+)\leq\min\left(\overline{\Psi}_{H_0},\widehat{\Psi}_{H_0}\right)$. *For large-update methods with* $\tau=O(n)$ *and* $\theta=\Theta(1)$, *we have* $\widehat{\Psi}_{H_0}=O(n)$, *and for small-update methods with* $\tau=O(1)$ *and* $\theta=\Theta(\frac{1}{\sqrt{n}})$, *we have* $\overline{\Psi}_{H_0}=O(r)$.

2.3 ANALYSE OF THE ALGORITHM

This section's objective is to calculate the iteration bounds for large- and small-update methods based on the two kernel functions (2.1) and (2.2). However, it must first determine the value of the step size α and express the decrease in proximity function during an inner iteration. it is widely known that the value of μ is fixed during an inner iteration.

2.3.1 VALUE FOR α

We determine a default step size α in this subsection. Every step of algorithm shown in **FIGURE 2.1** results in a new iteration (z^+,w^+). Using (2.8) and (2.13), we therefore obtain :

$$
z^+ = z+\alpha\Delta z = z\left(e+\alpha\frac{\Delta z}{z}\right) = z\left(e+\alpha\frac{d_z}{v}\right) = \frac{z}{v}\left(v+\alpha d_z\right)
$$
$$
w^+ = w+\alpha\Delta w = w\left(e+\alpha\frac{\Delta w}{w}\right) = w\left(e+\alpha\frac{d_w}{v}\right) = \frac{w}{v}\left(v+\alpha d_w\right)
$$

thus we have :

$$
v^+ = \sqrt{\frac{z^+w^+}{\mu}} = \sqrt{(v+\alpha d_z)(v+\alpha d_w)}.
$$

from the definition of the proximity after a feasible step and (2.24), it is clear that :

$$\begin{aligned}\Psi(v^+) &= \Psi\left(\sqrt{(v+\alpha d_z)(v+\alpha d_w)}\right)\\ &\leq \frac{1}{2}\left(\Psi(v+\alpha d_z)+\Psi(v+\alpha d_w)\right)\end{aligned}$$

For $\alpha > 0$, we consider the following functions :

$$f(\alpha) = \Psi(v^+) - \Psi(v) \tag{2.62}$$

$$f_1(\alpha) = \frac{1}{2}\left(\Psi(v+\alpha d_z)+\Psi(v+\alpha d_w)\right) - \Psi(v) \tag{2.63}$$

we can easily see that these two functions check :

$$f(\alpha) \leq f_1(\alpha) \quad \text{and} \quad f(0) = f_1(0) = 0.$$

The following are the two successive derivatives of $f_1(\alpha)$ with respect to α :

$$f_1^{'}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi^{'}\left(v_i+\alpha(d_z)_i\right)(d_z)_i + \psi^{'}\left(v_i+\alpha(d_w)_i\right)(d_w)_i\right) \tag{2.64}$$

using the second equation in (2.15) and (2.16), we have :

$$f_1^{'}(0) = \frac{1}{2}\left\langle \nabla\Psi(v), (d_z+d_w)\right\rangle = -\frac{1}{2}\|\nabla\Psi(v)\|^2 = -2\delta(v)^2. \tag{2.65}$$

And

$$f_1^{''}(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi^{''}\left(v_i+\alpha(d_z)_i\right)(d_z)_i^2 + \psi^{''}\left(v_i+\alpha(d_w)_i\right)(d_w)_i^2\right) \tag{2.66}$$

where $(d_z)_i$ and $(d_w)_i$ denote the i^{th} components of the vectors d_z and d_w, respectively. Since $f_1^{''}(\alpha) > 0$, $f_1(\alpha)$ is strictly convex in α unless $d_z = d_w = 0$.

NOTATION 4. *For the remainder of the paper, we will use :*

$$v_{min} = \min_{i\in\{1,\ldots,n\}} v_i, \quad \delta = \delta(v), \quad \Psi_E = \Psi_E(v) \quad \textit{and} \quad \Psi_H = \Psi_H(v).$$

We are now ready to determine the value of the step size, for this purpose we show the following lemmas which are valid for all kernel functions that satisfy (2.21). (The reader can refer to [6] for the proof of lemmas)

LEMMA 2.3.1. *(Lemma 4.1, [6]) Let $f_1(\alpha)$ be defined as in (2.63) and δ as in (2.16). Then :*

$$f_1^{''}(\alpha) \leq 2\delta^2\psi^{''}\left(v_{min} - 2\alpha\delta\right). \tag{2.67}$$

PROOF. Due to (2.16) and the first equation in (2.12), we have :

$$\begin{aligned}4\delta^2 &= \|d_z+d_w\|^2\\ &= \|d_z\|^2+\|d_w\|^2+2\langle d_z,d_w\rangle\\ &= \|d_z\|^2+\|d_w\|^2+2\langle d_z,\Lambda d_w\rangle\\ &\geq \|d_z\|^2+\|d_w\|^2.\end{aligned}$$

the last inequality follows from the monotonicity of the *LCP* (see ASSUMPTION 3 and REMARK 2.1.1), this implies that the matrix $\Lambda = DMD$ is positive semidefinite. Hence we have $\|d_z\| \leq 2\delta$ and $\|d_w\| \leq 2\delta$. Therefore :

$$v_i + \alpha(d_z)_i \geq v_{min} - 2\alpha\delta \quad \text{and} \quad v_i + \alpha(d_w)_i \geq v_{min} - 2\alpha\delta \quad \text{for } 1 \leq i \leq n$$

according to (2.21) $\psi''(t)$ is monotonically decreasing and from (2.66), we obtain :

$$\begin{aligned} f_1''(\alpha) &\leq \frac{1}{2}\delta^2\psi''(v_{min} - 2\alpha\delta)\sum_{i=1}^{n}\left((d_z)_i^2 + (d_w)_i^2\right) \\ &= 2\delta^2\psi''(v_{min} - 2\alpha\delta) \end{aligned}$$

which complete the proof. □

LEMMA 2.3.2. *(Lemma 4.2, [6]) $f_1'(\alpha) \leq 0$ certainly holds if α satisfies the inequality :*

$$-\psi'(v_{min} - 2\alpha\delta) + \psi'(v_{min}) \leq 2\delta. \tag{2.68}$$

Let $\rho(s) : [0, +\infty[\rightarrow]0, 1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(x)$ to the interval $]0, 1]$.

LEMMA 2.3.3. *(Lemma 4.3, [6]) The largest step size α that satisfies (2.68) is given by :*

$$\overline{\alpha} := \frac{\rho(\delta) - \rho(2\delta)}{2\delta}.$$

LEMMA 2.3.4. *(Lemma 4.4, [6]) Let $\overline{\alpha}$ be as defined in LEMMA 2.3.3. Then :*

$$\overline{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}. \tag{2.69}$$

In what follows we define the default step size $\alpha^\star$ as follows :

$$\alpha^\star = \frac{1}{\psi''(\rho(2\delta))} \tag{2.70}$$

according to LEMMA 2.3.4, we have : $\overline{\alpha} \geq \alpha^\star$.

2.3.2 DECREASE OF Ψ

The proximity function Ψ is now expressed as decreasing during an inner iteration with the default step size $\alpha^\star$, as provided in (2.70). To this purpose, we established at the findings listed below.

LEMMA 2.3.5. *(Lemma 12, [65]) Let $h(x)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(x)$ attain its global minimum at its stationary point $x^* > 0$. If $h''(x)$ is increasing with respect to x, then one has for any $x \in [0, x^*]$:*

$$h(x) \leq \frac{xh'(0)}{2}.$$

LEMMA 2.3.6. *(Lemma 4.5, [6]) If the step size α satisfies $\alpha \leq \overline{\alpha}$, then :*

$$f(\alpha) \leq -\alpha\delta^2. \tag{2.71}$$

Using LEMMA 2.3.6 and (2.70), we have the following theorem.

THEOREM 2.3.7. *(Theorem 4.6, [6]) With $\alpha^\star$ being the default step size, as given by (2.70). Then, we have :*

$$f(\alpha^\star) \leq -\frac{\delta^2}{\psi''\left(\rho(2\delta)\right)}. \tag{2.72}$$

2.3.3 ITERATION BOUND

We conclude this section with a theorem that estimates the total number of iterations of our algorithm given in FIGURE 2.1. This means counting how many inner iterations are required to obtain the situation $\Psi \leq \tau$. Whereas Ψ_k signifies subsequent values in the same outer iteration and $k = 1, \ldots, K$, K denotes the total number of inner iterations in the outer iteration, where Ψ_0 represents the value Ψ following the μ-update and we use the following lemma to do so.

LEMMA 2.3.8. *(Proposition 1.3.2, [64]) Let a sequence $x_k > 0$, $k = 0, \ldots, K$ that verifies :*

$$x_{k+1} \leq x_k - \kappa x_k^{1-\nu} \quad \textit{with} \quad \kappa > 0, \quad 0 < \nu \leq 1 \quad \textit{and} \quad k = 0, \ldots, K,$$

then : $K \leq \dfrac{x_0^\nu}{\kappa\nu}$.

Using the LEMMA 2.3.8 for $x_k = \Psi_k$, we can get the following lemma :

LEMMA 2.3.9. *Let K be the total number of inner iterations in the outer iteration. Then we have :*

$$K \leq \frac{\Psi_0^\nu}{\kappa\nu},$$

where : $\kappa > 0, \quad 0 < \nu \leq 1$.

PROOF. Using the definition of $f(\alpha)$ (2.62) and LEMMA 2.3.6 with $\kappa \leq \overline{\alpha}$, we have :

$$f(\kappa) = \Psi_{k+1} - \Psi_k \leq -\kappa\delta^2$$

suppose that they exist $\kappa > 0$ and $0 < \nu \leq 1$, such that :

$$\Psi_{k+1} - \Psi_k \leq -\kappa\Psi^{1-\nu} \tag{2.73}$$

according to LEMMA 2.3.8 with $x_k = \Psi_k$, we obtain the desired result. □

The following theorem, which provides an upper bound for the total number of iterations, is implied by the previous reasoning.

THEOREM 2.3.10. *If $\tau \geq 1$, the total number of iteration is bounded above by :*

$$\frac{\Psi_0^\nu}{\kappa\theta\nu}\log\frac{n}{\varepsilon}.$$

***PROOF*.** The number of outer iterations is bounded above by $\frac{1}{\theta}\log\frac{n}{\varepsilon}$ (see [70]). By multiplying the number of outer iterations by the number of inner iterations, we deduce the upper bound for the total number of iterations as follows :

$$K\frac{\log\frac{n}{\varepsilon}}{\theta} \leq \frac{\Psi_0^\nu}{\kappa\nu}\frac{\log\frac{n}{\varepsilon}}{\theta}.$$ □

2.3.4 APPLICATION TO THE TWO KERNEL FUNCTIONS

Throughout the subsection, we assume that $\min(\Psi_E, \Psi_H) \geq \tau \geq 1$. Using LEMMA 2.2.16 and LEMMA 2.2.28, we get $\max(\Psi_E, \Psi_H) \leq 2\delta^2$ then $\sqrt{2}\delta \geq \sqrt{\tau} \geq 1$. Thus, We apply the results of the previous subsections to obtain iteration bounds for large- and small-update methods based on the two kernel functions introduced before.

LEMMA 2.3.11. *Let ρ_E, ρ_H and $\overline{\alpha}$ be as defined in* LEMMA 2.2.15, LEMMA2.2.27 *and* LEMMA 2.3.4*, respectively. Then, we have :*

$$\overline{\alpha}_E \geq \frac{1}{1+r(8\delta+2)^{\frac{r+1}{r}}}, \quad r>1. \tag{2.74}$$

$$\overline{\alpha}_H \geq \frac{1}{1+2a^{r+3}(r+1)(8\delta+2)^{\frac{r+4}{r}}}, \quad r\geq 4. \tag{2.75}$$

where $\overline{\alpha}_E$ and $\overline{\alpha}_H$ are the largest step size of the worst case associated with ψ_E and ψ_H, respectively.

***PROOF*.** For (2.74), to facilitate the calculation for all $x \in]0,1]$, we put $x = \rho_E(2\delta)$ in (2.34). Then, we obtain :

$$\begin{aligned}\psi_E''(\rho_E(2\delta)) &= 1+\frac{1}{2}\left((r-1)e^{(r-1)(1-\rho_E(2\delta))}+\frac{r}{(\rho_E(2\delta))^{r+1}}\right)\\ &\leq 1+\frac{r}{2}\left(e^{(r-1)(1-\rho_E(2\delta))}+\frac{1}{(\rho_E(2\delta))^{r+1}}\right).\end{aligned} \tag{2.76}$$

Due to (2.40) and (2.41), the terms can be reduced as follows :

$$(\rho_E(2\delta))^{-(r+1)} = x^{-(r+1)} \leq (8\delta+2)^{\frac{r+1}{r}}$$

and

$$e^{(r-1)(1-\rho_E(2\delta))} = e^{(r-1)(1-x)} \leq 8\delta+2.$$

substitution in (2.76) gives :

$$\begin{aligned}\psi''_E(\rho_E(2\delta)) &\leq 1+\frac{r}{2}\left((8\delta+2)+(8\delta+2)^{\frac{r+1}{r}}\right)\\ &\leq 1+r\,(8\delta+2)^{\frac{r+1}{r}}\end{aligned}$$

using **LEMMA 2.3.4**, we get :

$$\overline{\alpha}_E \geq \frac{1}{\psi''_E(\rho_E(2\delta))} \geq \frac{1}{1+r\,(8\delta+2)^{\frac{r+1}{r}}}.$$

For (2.75), according to (2.47) with $x=\rho_H(2\delta)\in]0,1]$, we have :

$$\begin{aligned}\psi''_H(\rho_H(2\delta)) &= 1+\frac{1}{2\rho_H(2\delta)^2}+\frac{r+1}{2a}\left(\frac{f(\rho_H(2\delta))}{\rho_H(2\delta)^{r+2}}+\frac{2g(\rho_H(2\delta))}{\rho_H(2\delta)^{r+3}}\right)+\frac{h(\rho_H(2\delta))}{2a\rho_H(2\delta)^{r+4}}\\ &\leq 1+\frac{(r+1)\left(a+f(\rho_H(2\delta))+2g(\rho_H(2\delta))\right)+h(\rho_H(2\delta))}{2a\rho_H(2\delta)^{r+4}}\end{aligned} \tag{2.77}$$

Because of **NOTATION 2** and (2.49), we get :

$$\begin{aligned}a+f(\rho_H(2\delta)) &= a+\cosh^r\left(\rho_H^{-1}(2\delta)\right)-\cosh^r(1)\leq\cosh^r\left(\rho_H^{-1}(2\delta)\right)\\ g(\rho_H(2\delta)) &= \tanh\left(\rho_H^{-1}(2\delta)\right)\cosh^r\left(\rho_H^{-1}(2\delta)\right)\leq\cosh^r\left(\rho_H^{-1}(2\delta)\right)\\ h(\rho_H(2\delta)) &= r\tanh^2\left(\rho_H^{-1}(2\delta)\right)\cosh^r\left(\rho_H^{-1}(2\delta)\right)+\cosh^{r-2}\left(\rho_H^{-1}(2\delta)\right)\\ &\leq (r+1)\cosh^r\left(\rho_H^{-1}(2\delta)\right)\end{aligned}$$

by combining (2.77) and (2.56), we obtain :

$$\begin{aligned}\psi''_H(\rho_H(2\delta)) &\leq 1+\frac{2(r+1)}{a}\left(\frac{\cosh\left(\rho_H^{-1}(2\delta)\right)}{\rho_H(2\delta)}\right)^{r+4}\\ &\leq 1+2a^{r+3}(r+1)(8\delta+2)^{\frac{r+4}{r}}\end{aligned}$$

using **LEMMA 2.3.4**, we get :

$$\overline{\alpha}_H \geq \frac{1}{\psi''_H(\rho_H(2\delta))} \geq \frac{1}{1+2a^{r+3}(r+1)(8\delta+2)^{\frac{r+4}{r}}}$$

this demonstrates the lemma. □

For our algorithm, we define the default step size $\alpha^\star_E$ $\left(\alpha^\star_E\leq\overline{\alpha}_E\right)$ and $\alpha^\star_H$ $\left(\alpha^\star_H\leq\overline{\alpha}_H\right)$ which associated with (2.1) and (2.2), respectively, as follows :

$$\alpha^\star_E = \frac{1}{1+r\,(8\delta+2)^{\frac{r+1}{r}}}. \tag{2.78}$$

$$\alpha^\star_H = \frac{1}{1+2a^{r+3}(r+1)(8\delta+2)^{\frac{r+4}{r}}}. \tag{2.79}$$

At this stage, we use the default step sizes $\alpha^\star_E$ and $\alpha^\star_H$, which is given by (2.78) and (2.79), respectively, to indicate the reduction in the proximity function Ψ_E and Ψ_H during an inner iteration.

LEMMA 2.3.12. *Let $\alpha^\star_E$ and $\alpha^\star_H$ be defined as in (2.78) and (2.79), respectively. Then, we have :*

$$f(\alpha^\star_E) \leq -\frac{1}{120r}\Psi_E^{\frac{r-1}{2r}}. \tag{2.80}$$

$$f(\alpha^\star_H) \leq -\frac{1}{118(1+2a^{r+3})(r+1)}\Psi_H^{\frac{r-4}{2r}}. \tag{2.81}$$

PROOF. By combining (**2.78**) and $\sqrt{2}\delta \geq \sqrt{\Psi_E}$ then (**2.79**) and $\sqrt{2}\delta \geq \sqrt{\Psi_H}$ with **THEOREM 2.3.7**, we may demonstrate this lemma. We discover :

$$
\begin{aligned}
f(\alpha_E^\star) &\leq \frac{-\delta^2}{\psi_E''\left(\rho_E(2\delta)\right)} \\
&\leq \frac{-\delta^2}{1+r\left(8\delta+2\right)^{\frac{r+1}{r}}} \\
&\leq \frac{-\delta^2}{\sqrt{2}\delta^{\frac{r+1}{r}}+r\left(8\delta+2\sqrt{2}\delta\right)^{\frac{r+1}{r}}} \\
&\leq \frac{-\delta^{2-\frac{r+1}{r}}}{\sqrt{2}+r\left(8+2\sqrt{2}\right)^{\frac{r+1}{r}}} \\
&\leq \frac{-\delta^{2-\frac{r+1}{r}}}{\sqrt{2}+118r} \\
&\leq \frac{-\Psi_E^{\frac{r-1}{2r}}}{120r}
\end{aligned}
$$

and

$$
\begin{aligned}
f(\alpha_H^\star) &\leq \frac{-\delta^2}{\psi_H''\left(\rho_H(2\delta)\right)} \\
&\leq \frac{-\delta^2}{1+2a^{r+3}(r+1)(8\delta+2)^{\frac{r+4}{r}}} \\
&\leq \frac{-\delta^2}{(1+2a^{r+3})(r+1)(8\delta+2\sqrt{2}\delta)^{\frac{r+4}{r}}} \\
&\leq \frac{-\delta^{2-\frac{r+4}{r}}}{118(1+2a^{r+3})(r+1)} \\
&\leq \frac{-\Psi_H^{\frac{r-4}{2r}}}{118(1+2a^{r+3})(r+1)}
\end{aligned}
$$

which completes the proof. □

Our aim in this last part is to compute iteration bounds for large- and small-update methods based on our new kernel functions (**2.1**) and (**2.2**). Using the definition of $f(\alpha)$ (i.e., (**2.62**)), (**2.73**), (**2.80**) and (**2.81**), we have :

$$
\nu_E = \frac{r+1}{2r}, \qquad \kappa_E = \frac{1}{120r}.
$$

and

$$
\nu_H = \frac{r+4}{2r}, \qquad \kappa_H = \frac{1}{118(1+2a^{r+3})(r+1)}.
$$

From **LEMMA 2.3.9** one can easily deduce that :

$$
\begin{aligned}
K_E \leq \frac{1}{\kappa_E \nu_E}\Psi_{E_0}^{\nu_E} &\leq \frac{240r^2}{r+1}\Psi_{E_0}^{\frac{r+1}{2r}} \\
&\leq \frac{240r^2}{r}\Psi_{E_0}^{\frac{r+1}{2r}} \\
&\leq 240r\Psi_{E_0}^{\frac{r+1}{2r}} \qquad (2.82)
\end{aligned}
$$

and

$$K_H \leq \frac{1}{\kappa_H \nu_H} \Psi_{H_0}^{\nu_H} \leq \frac{236(1+2a^{r+3})r(r+1)}{r+4} \Psi_{H_0}^{\frac{r+4}{2r}} \leq \frac{236(1+2a^{r+3})r(r+1)}{r+1} \Psi_{H_0}^{\frac{r+4}{2r}} \leq 236(1+2a^{r+3})r \Psi_{H_0}^{\frac{r+4}{2r}} \tag{2.83}$$

where K_E and K_H are the total number of inner iterations in the outer iteration for the algorithm based on (**2.1**) and (**2.2**), respectively.

The above explanation implies the following theorem, which gives the two upper bounds for the total number of iterations of the algorithm based on (**2.1**) and (**2.2**), respectively.

THEOREM 2.3.13. *The total number of iterations required to obtain the optimal solution for (**LCP**) is bounded by :*

$$240 r \Psi_{E_0}^{\frac{r+1}{2r}} \frac{\log \frac{n}{\varepsilon}}{\theta}, \quad \textit{for the algorithm based on } (\mathbf{2.1}). \tag{2.84}$$

$$236(1+2a^{r+3}) r \Psi_{H_0}^{\frac{r+4}{2r}} \frac{\log \frac{n}{\varepsilon}}{\theta}, \quad \textit{for the algorithm based on } (\mathbf{2.2}). \tag{2.85}$$

PROOF. According to **THEOREM 2.3.10** with (**2.82**) then with (**2.83**), we obtain an upper bounds for the total number of iterations required by our algorithm based on (**2.1**) and (**2.2**) by the two equations (**2.84**) and (**2.85**), respectively. □

Using **REMARK 2.2.18** and **REMARK 2.2.30**, we obtain the complexity results of small- and large-update methods, which we summarize in the following table.

THE KERNEL FUNCTION :	LARGE-UPDATE METHODS :	SMALL-UPDATE METHODS :
$\psi_E(t)$ with $r > 1$	$O\left(rn^{\frac{r+1}{2r}} \log \frac{n}{\varepsilon}\right)$	$O\left(r^{\frac{3r+1}{2r}} \sqrt{n} \log \frac{n}{\varepsilon}\right)$
$\psi_H(t)$ with $r \geq 4$	$O\left(rn^{\frac{r+4}{2r}} \log \frac{n}{\varepsilon}\right)$	$O\left(r^{\frac{3r+4}{2r}} \sqrt{n} \log \frac{n}{\varepsilon}\right)$

TABLE 2.2: COMPLEXITY RESULTS OF LARGE- AND SMALL-UPDATE METHODS.

REMARK 2.3.14.

► *If we substitute any constant value for r, the iteration complexity of the small-update method becomes $O(\sqrt{n}\log \frac{n}{\varepsilon})$.*

► *We obtain the best-known complexity bound for large-update methods namely $O\left(\sqrt{n}\log n \log \frac{n}{\varepsilon}\right)$ if we take $r = \dfrac{\log n}{2}$ for the algorithm based on the first kernel function (**2.1**) and $r = 2\log n$ for the algorithm based on the second kernel function (**2.2**).*

The summary of our study is given in the following table.

THE KERNEL FUNCTIONS $\psi_i(x)$:	$i = E$ WITH $r > 1$ (2.1) :	$i = H$ WITH $r \geq 4$ (2.2) :
$\gamma_i(s) \geq$	$1 + \sqrt{\frac{4s}{2r+1}}$	$1 + \sqrt{\frac{2s}{\vartheta r + \iota}}$
$\gamma_i(s) \leq$	$1 + \sqrt{2s}$	$1 + \sqrt{2s}$
$\rho_i(s) \geq$	$\frac{1}{(4s+2)^{\frac{1}{r}}}$	$\frac{\cosh\left(x^{-1}\right)}{ax} \leq (4s+2)^{\frac{1}{r}}$
$\overline{\Psi}_{i_0} =$	$\frac{2r+1}{4(1-\theta)}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2$	$\frac{\vartheta r + \iota}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2$
$\widehat{\Psi}_{i_0} =$	$\frac{2\tau + n\theta + 2\sqrt{2n\tau}}{2(1-\theta)}$	$\frac{2\tau + n\theta + 2\sqrt{2n\tau}}{2(1-\theta)}$
$\alpha_i^* =$	$\frac{1}{1 + r\left(8\delta + 2\right)^{\frac{r+1}{r}}}$	$\frac{1}{1 + 2a^{r+3}(r+1)(8\delta+2)^{\frac{r+4}{r}}}$
$f(\alpha_i^*) \leq$	$-\frac{\Psi_E^{\frac{r-1}{2r}}}{120r}$	$-\frac{\Psi_H^{\frac{r-4}{2r}}}{118(1+2a^{r+3})(r+1)}$
$K_i \leq$	$240r\Psi_{E_0}^{\frac{r+1}{2r}}$	$236(1+2a^{r+3})r\Psi_{H_0}^{\frac{r+4}{2r}}$
The total number of iterations	$240r\Psi_{E_0}^{\frac{r+1}{2r}}\frac{\log\frac{n}{\varepsilon}}{\theta}$	$236(1+2a^{r+3})r\Psi_{H_0}^{\frac{r+4}{2r}}\frac{\log\frac{n}{\varepsilon}}{\theta}$
Complexity of small- and large-update	$O\left(r^{\frac{3r+1}{2r}}\sqrt{n}\log\frac{n}{\varepsilon}\right)$ $O\left(rn^{\frac{r+1}{2r}}\log\frac{n}{\varepsilon}\right)$	$O\left(r^{\frac{3r+4}{2r}}\sqrt{n}\log\frac{n}{\varepsilon}\right)$ $O\left(rn^{\frac{r+4}{2r}}\log\frac{n}{\varepsilon}\right)$
The best complexity for small-update	obtained for any constant $r > 1$	obtained for any constant $r \geq 4$
The best complexity for large-update	obtained for $r = \frac{1}{2}\log n$	obtained for $r = 2\log n$

TABLE 2.3: PROPERTIES OF OUR TWO FUNCTIONS $\psi_E(x)$ AND $\psi_H(x)$.

2.4 NUMERICAL RESULTS

In this section, we present some numerical results on some linear complementarity problems to confirm the effectiveness of our two proposed functions, where the experiment manipulation in the **Dev-Cpp 5.11 TDM-GCC 4.9.2 Setup** which executes on any simple computer using the algorithm given in FIGURE 2.1 with an initial strictly feasible point (z_0, w_0) and a positive parameter μ_0 such that $\Psi_i(z_0, w_0; \mu_0) \leq \tau$ for $i = E, H$.

Throughout the algorithm, we assume that the accuracy parameter ε is 10^{-6}, the threshold parameter τ is $\sqrt{n}$, the barrier update θ is $0.15, 0.3, 0.6$ and 0.95, the practical step size α is given by $\alpha_{pra} = \rho \min\left(\alpha_z, \alpha_w\right)$ where

$$\alpha_z = \min_{i=1,\ldots,n} \begin{cases} -\dfrac{z_i}{\Delta z_i} & \text{if } \Delta z_i < 0 \\ 1 & \text{else} \end{cases} \quad \text{and} \quad \alpha_w = \min_{i=1,\ldots,n} \begin{cases} -\dfrac{w_i}{\Delta w_i} & \text{if } \Delta w_i < 0 \\ 1 & \text{else} \end{cases}$$

with $\rho \in (0, 1)$.

We continue in this manner until μ^* is small enough (i.e., $n\mu^* < \varepsilon$) and v^* agree on $\Psi_i(v^*) \leq \tau$, at which time we claim that we have discovered the optimal solution (z^*, w^*) to (2.3) and for the system (2.4) by using (1.3) we find the optimal solution to corresponding (AVE) denoted u^*. Finally, we use **Iter** and **CPU** to represent the number of iterations and the time required to discover the solution, respectively.

On the following test problems of varying sizes, we performed numerical comparisons between the kernel functions shown in the table below.

i	KERNEL FUNCTIONS $\psi_i(x)$:	LARGE-UPDATE :	SMALL-UPDATE :	REF :
R	$\frac{x^2-1}{2} - \log(x)$	$O\left(n \log \frac{n}{\varepsilon}\right)$	$O\left(\sqrt{n} \log \frac{n}{\varepsilon}\right)$	[70]
B	$x^2 - 1 - \log(x) + \frac{x^{-r}-1}{r}$	$O\left(rn^{\frac{r+1}{2r}} \log \frac{n}{\varepsilon}\right)$	$O\left(r\sqrt{n} \log \frac{n}{\varepsilon}\right)$	[16]
L	$(r+1)x^2 - x^{-r} - (r+2)x$	$O\left(r^{\frac{2r+1}{2r}} n^{\frac{r+1}{2r}} \log \frac{n}{\varepsilon}\right)$	$O\left(\sqrt{n} \log \frac{n}{\varepsilon}\right)$	[54]
A	$x^2 - 1 - \frac{x^{-2r+1}-1}{-2r+1} - \frac{x^{-r+1}-1}{-r+1}$	$O\left(rn^{\frac{2r+1}{4r}} \log \frac{n}{\varepsilon}\right)$	$O\left(r^2\sqrt{n} \log \frac{n}{\varepsilon}\right)$	[11]
M	$r\frac{x^2-1}{2} + \frac{4}{\pi}\left(e^{r\left(\tan\left(\frac{\pi}{2+2x}\right)\right)-1} - 1\right)$	$O\left(\sqrt{n} \log^2 n \log \frac{n}{\varepsilon}\right)$	$O\left(\sqrt{n} \log \frac{n}{\varepsilon}\right)$	[60]

TABLE 2.4: SOME KERNEL FUNCTIONS AND ITS COMPLEXITY RESULTS.

REMARK 2.4.1. *The time taken by our algorithm,* **CPU**, *is calculated in seconds and includes both converting the problems (AVE and CQP) to LCP, calculating the solution to LCP and deducing the solution to original problems.*

EXPERIMENT 1. *Consider the linear complementarity problem* (2.3), *in which the matrix M and vector q are provided by :*

$$M = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \quad \textit{and} \quad q = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{P1}$$

where the LCP corresponding to (**P1**) *has a unique solution since the matrix M is positive definite.*

The strictly feasible beginning point is chosen as follows :

$$z_0 = \begin{pmatrix} 0.009 & 0.009 & 0.009 \end{pmatrix}^t$$
$$\textit{and} \quad w_0 = \begin{pmatrix} 1.036 & 1.027 & 1.018 \end{pmatrix}^t$$

we conclude our algorithm with the following solution :

$$z^* = \begin{pmatrix} 0.000001 & 0.000001 & 0.000001 \end{pmatrix}^t$$
$$\textit{and} \quad w^* = \begin{pmatrix} 1.000005 & 1.000005 & 1.000002 \end{pmatrix}^t,$$

where the findings are given in the table below.

$\psi_i(x) \setminus \theta$	$\theta = 0.15$		$\theta = 0.3$		$\theta = 0.6$		$\theta = 0.95$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_{E,1.1}(x)$	68	0.63	39	0.44	22	0.34	10	0.29
$\psi_{E,2}(x)$	73	0.88	41	0.52	24	0.36	13	0.33
$\psi_{E,4}(x)$	79	0.95	43	0.67	26	0.58	16	0.47
$\psi_{H,4}(x)$	73	0.61	39	0.40	23	0.32	12	0.23
$\psi_{H,5}(x)$	76	0.83	46	0.49	27	0.35	16	0.30
$\psi_{H,6}(x)$	83	0.94	48	0.66	29	0.53	18	0.45
$\psi_R(x)$	77	0.98	51	0.71	35	0.45	25	0.39
$\psi_{B,1.1}(x)$	70	0.69	42	0.48	23	0.36	14	0.32
$\psi_{B,4}(x)$	84	0.98	47	0.68	28	0.61	19	0.50
$\psi_{A,2}(x)$	80	0.88	44	0.68	25	0.46	17	0.41
$\psi_{A,4}(x)$	88	0.96	49	0.73	27	0.57	18	0.46
$\psi_{M,1.1}(x)$	84	0.68	47	0.55	27	0.49	19	0.37

TABLE 2.5: NUMERICAL EXPERIMENTS OF (**P1**) USING VARIOUS KERNEL FUNCTIONS.

EXPERIMENT 2. *In this experiment, we select the value of barrier degree* $\mathbf{r}$ *that provides the optimal complexity for large-updates for each parametrized function. When we present two convex quadratic programming that may be expressed in the form of a linear complementarity problem as follows :*

$$M = \begin{pmatrix} 2 & 1 & 0 & 3 & -3 \\ 1 & 4 & 0 & 4 & 2 \\ 0 & 0 & 6 & -2 & 1 \\ -3 & -4 & 2 & 0 & 0 \\ 3 & -2 & -1 & 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{P2}$$

The problem (**P2**) *has a unique solution since :*

$$\lambda_1(M) = \lambda_2(M) = 1.0177, \quad \lambda_3(M) = \lambda_4(M) = 2.1350, \quad \lambda_5(M) = 5.6946,$$

The initial iteration of the corresponding LCP to (**P2**) *is established by :*

$$z_0 = \begin{pmatrix} 1 & 0.25 & 2 & 1 & 0.5 \end{pmatrix}^t$$
$$\text{and} \quad w_0 = \begin{pmatrix} 4.75 & 8 & 11.5 & 1 & 1.5 \end{pmatrix}^t$$

the unique solution of the corresponding LCP is :

$$z^* = \begin{pmatrix} 0.000002 & 0.000002 & 0.000004 & 0.000002 & 0.000002 \end{pmatrix}^t$$
$$\text{and} \quad w^* = \begin{pmatrix} 1.000007 & 1.00026 & 1.000023 & 0.999991 & 0.999998 \end{pmatrix}^t.$$

for our two functions, the numerical results are displayed as follows :

$\psi_i(x) \setminus \theta$	$\theta = 0.15$		$\theta = 0.3$		$\theta = 0.6$		$\theta = 0.95$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_E(x)$	114	0.79	57	0.71	28	0.65	16	0.64
$\psi_H(x)$	118	0.76	59	0.65	30	0.63	17	0.54
$\psi_R(x)$	127	0.94	69	0.91	37	0.83	21	0.75
$\psi_B(x)$	119	0.81	61	0.79	31	0.69	19	0.67
$\psi_L(x)$	122	0.76	62	0.72	31	0.65	19	0.59
$\psi_A(x)$	126	0.88	67	0.81	37	0.81	20	0.80
$\psi_M(x)$	122	0.80	64	0.75	35	0.73	19	0.70

TABLE 2.6: NUMERICAL EXPERIMENTS OF (**P2**) USING VARIOUS KERNEL FUNCTIONS.

$$M = \begin{pmatrix} 4 & -2 & 0 & 0 & 1 & 1 \\ -2 & 4 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -5 & 0 & -1 & 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{P3}$$

*For the problem (**P3**)So that we can easily prove that M is positive definite because :*

$$\lambda_1(M) = \lambda_2(M) = 0.0584, \quad \lambda_3(M) = 0.0862, \quad \lambda_4(M) = \lambda_5(M) = 1.2823,$$
$$\lambda_6(M) = 5.2324.$$

Our algorithm's starting iteration is determined by :

$$z_0 = \begin{pmatrix} 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 \end{pmatrix}^t$$
$$\text{and} \quad w_0 = \begin{pmatrix} 1.4 & 1.6 & 1.1 & 1.1 & 0.7 & 0.3 \end{pmatrix}^t$$

the numerical results for our functions are shown as follows :

$\psi_i(x) \setminus \theta$	$\theta = 0.15$		$\theta = 0.3$		$\theta = 0.6$		$\theta = 0.95$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_E(x)$	120	0.97	62	0.92	31	0.82	19	0.61
$\psi_H(x)$	124	0.91	65	0.84	35	0.78	20	0.57
$\psi_R(x)$	134	1.00	74	0.94	40	0.85	23	0.78
$\psi_B(x)$	125	1.01	66	0.99	35	0.84	21	0.65
$\psi_L(x)$	127	0.98	66	0.92	36	0.84	21	0.74
$\psi_A(x)$	134	0.96	72	0.86	40	0.88	24	0.71
$\psi_M(x)$	129	0.95	69	0.92	41	0.81	22	0.78

TABLE 2.7: NUMERICAL EXPERIMENTS OF (**P3**) USING VARIOUS KERNEL FUNCTIONS.

the unique solution to the corresponding LCP is :

$$z^* = \begin{pmatrix} 0.000002 & 0.000002 & 0.000002 & 0.000002 & 0.000001 & 0.000001 \end{pmatrix}^t$$
$$\text{and} \quad w^* = \begin{pmatrix} 1.000006 & 1.00008 & 1.000001 & 1.000002 & 0.999995 & 0.999988 \end{pmatrix}^t.$$

EXPERIMENT 3. *(Problem 1, [4]) Consider the following generic version of the absolute value equation GAVE :*

$$A=\begin{pmatrix} 8 & 0 & -1 & 1 & -20\\ 1 & 1 & 1 & 4 & 25\\ 1 & -5 & 0 & 8 & -10\\ 0 & 8 & 1 & -6 & 1\\ 3 & 5 & -3 & 0 & 10 \end{pmatrix},\quad B=\begin{pmatrix} -1.5 & 0 & 1.5 & 0.5 & 0.1\\ 0 & 0.25 & 1 & 0 & 0.5\\ 1 & 0.6 & 1 & 0.4 & 0.5\\ 0 & 0.3 & 1 & 1 & 0\\ 1 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{P4}$$

$$\text{and} \qquad b=\begin{pmatrix} -11.4 & 33.75 & -2.5 & 6.3 & 17 \end{pmatrix}^t$$

The matrix M and vector q of the accompanying LCP (2.4) are provided by (P4) :

$$M=\begin{pmatrix} 1.790264 & 0.150318 & -0.314473 & 0.08150 & -0.006921\\ -0.770851 & 0.793484 & -0.323572 & -0.285865 & -0.041592\\ 0.985098 & -0.071539 & 0.657381 & -0.262439 & -0.144348\\ -0.872306 & -0.245658 & -0.270367 & 0.832010 & -0.127792\\ 0.066357 & 0.028791 & 0.019048 & 0.057723 & 0.994563 \end{pmatrix},$$

$$q=\begin{pmatrix} 0.999793 & 1.000065 & 1.000469 & 1.000402 & 1.000009 \end{pmatrix}^t.$$

with $\sigma_{min}(A)=2.8215>\sigma_{max}(B)=2.7434$ then the LCP has a unique solution (see **REMARK 1.8.12***). The strictly feasible starting point is given by :*

$$z_0=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$$

$$\text{and} \quad w_0=\begin{pmatrix} 2.70083 & 0.37167 & 2.164621 & 0.316289 & 2.166491 \end{pmatrix}^t$$

$\psi_i(x)\setminus\theta$	$\theta=0.15$		$\theta=0.3$		$\theta=0.6$		$\theta=0.95$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_{E,1.1}(x)$	110	0.87	57	0.75	32	0.66	16	0.63
$\psi_{E,2}(x)$	111	0.93	59	0.83	33	0.79	17	0.77
$\psi_{E,4}(x)$	113	1.00	59	0.94	35	0.86	22	0.83
$\psi_{H,4}(x)$	112	0.73	59	0.67	34	0.59	20	0.55
$\psi_{H,5}(x)$	117	1.38	61	1.23	36	1.05	23	0.96
$\psi_{H,6}(x)$	119	1.40	63	1.31	47	1.19	28	1.05
$\psi_{R}(x)$	112	0.90	57	0.75	34	0.69	17	0.60
$\psi_{L,5}(x)$	127	1.57	69	1.25	39	1.17	24	1.09
$\psi_{L,6}(x)$	133	2.11	80	2.05	54	1.95	30	1.62
$\psi_{A,2}(x)$	114	1.04	66	0.92	36	0.89	21	0.77
$\psi_{A,4}(x)$	118	1.37	71	1.35	39	1.28	24	1.18
$\psi_{M,1.1}(x)$	115	1.67	72	1.10	35	0.84	21	0.79

TABLE 2.8: NUMERICAL EXPERIMENTS OF (**P4**) USING VARIOUS KERNEL FUNCTIONS.

the above table presents some numerical results of our algorithm based on several kernel functions, demonstrating the usefulness of our two functions (2.1) and (2.2) with varied values of parameters θ and r. The unique solutions of the corresponding LCP and GAVE are :

$$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t,$$

$$w^* = \begin{pmatrix} 0.999793 & 1.000065 & 1.000469 & 1.000403 & 1.000009 \end{pmatrix}^t.$$

$$\text{and} \quad u^* = |u^*| = \begin{pmatrix} 0.999793 & 1.000065 & 1.000469 & 1.000403 & 1.000009 \end{pmatrix}^t.$$

EXPERIMENT 4. *Consider the specific case of the absolute value equation (3.1), in which the matrix A and vector b are provided by :*

$$A = \begin{pmatrix} 2n & n & 1 & \dots & \dots & 1 \\ n & 2n & n & 1 & \dots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & n & 2n & n \\ 1 & \dots & \dots & 1 & n & 2n \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 4n-3 \\ 5n-4 \\ \vdots \\ 5n-4 \\ 4n-3 \end{pmatrix} \tag{P5}$$

If we use (P5) and $n = 4$ we get :

$$\begin{pmatrix} 8 & 4 & 1 & 1 \\ 4 & 8 & 4 & 1 \\ 1 & 4 & 8 & 4 \\ 1 & 1 & 4 & 8 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} - \begin{pmatrix} |u_1| \\ |u_2| \\ |u_3| \\ |u_4| \end{pmatrix} = \begin{pmatrix} 13 \\ 16 \\ 16 \\ 13 \end{pmatrix}$$

where $\sigma_{min}(A) = 1.171572 > 1$. As a result, the matrix M and vector q of the corresponding LCP are given as follows :

$$M = \begin{pmatrix} 1.507936 & -0.412698 & 0.253968 & -0.158730 \\ -0.412698 & 1.793651 & -0.539682 & 0.253968 \\ 0.253968 & -0.539682 & 1.793651 & -0.412699 \\ -0.158730 & 0.253968 & -0.412698 & 1.507936 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

the above problem is readily verified as strictly monotone since all dominant principal minors of the related matrix are strictly positive and the strictly feasible beginning point is given by :

$$z_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^t$$

$$\text{and} \quad w_0 = \begin{pmatrix} 2.190476 & 2.095239 & 2.095238 & 2.190476 \end{pmatrix}^t$$

*Now, we solve the problem (**P5**) with $n = 4$ using our method, which is based on two functions and some prior functions with different values of θ. Furthermore, we suppose that the value of the barrier degree r varies for the parametrized functions. The numerical outcomes are as follows :*

$\psi_i(x) \setminus \theta$	$\theta = 0.15$		$\theta = 0.3$		$\theta = 0.6$		$\theta = 0.95$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_{E,1.1}(x)$	107	0.63	52	0.53	24	0.48	11	0.38
$\psi_{E,2}(x)$	110	0.73	52	0.67	25	0.66	12	0.55
$\psi_{E,4}(x)$	112	0.91	58	0.82	27	0.73	18	0.68
$\psi_{H,4}(x)$	106	0.52	52	0.49	25	0.46	14	0.35
$\psi_{H,5}(x)$	110	0.63	55	0.58	26	0.52	15	0.48
$\psi_{H,6}(x)$	113	0.82	56	0.62	29	0.58	18	0.50
$\psi_R(x)$	108	0.71	54	0.68	27	0.66	14	0.63
$\psi_{B,1.1}(x)$	107	0.70	52	0.67	25	0.65	12	0.60
$\psi_{B,4}(x)$	113	1.05	58	0.95	27	0.83	19	0.74
$\psi_{L,5}(x)$	152	1.53	83	1.27	56	1.17	38	1.01
$\psi_{L,6}(x)$	167	1.99	98	1.37	57	1.24	48	1.13
$\psi_{A,2}(x)$	120	0.97	60	0.95	38	0.90	33	0.87
$\psi_{A,4}(x)$	123	1.16	67	1.06	40	1.02	36	0.95

TABLE 2.9: NUMERICAL EXPERIMENTS OF (**P5**) USING VARIOUS KERNEL FUNCTIONS.

The following are the unique solution of the corresponding LCP and AVE :

$$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^t, \quad w^* = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^t$$
$$\text{and} \quad u^* = |u^*| = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^t,$$

respectively.

EXPERIMENT 5. *According to* **EXAMPLE 1.7.2**, *the function f, the beginning point, the final point and the vector b_h of the related ordinary differential equation are given as follows :*

$$\begin{aligned} & f(x) = -1, \quad u(0) = u(1) = -\frac{1}{h^2}, \quad x \in [0,1] \\ & \text{and} \quad b_h = \begin{pmatrix} \frac{1}{h^2} - 1 & -1 & \dots & -1 & \frac{1}{h^2} - 1 \end{pmatrix}^t \end{aligned} \tag{P6}$$

*the equivalent LCP (**2.3**) for $n = 8$ is provided by :*

$$M = \begin{pmatrix} 1.0228 & 0.0206 & 0.0181 & 0.0155 & 0.0126 & 0.0096 & 0.0065 & 0.0033 \\ 0.0206 & 1.0409 & 0.0361 & 0.0308 & 0.0251 & 0.0191 & 0.0129 & 0.0065 \\ 0.0181 & 0.0361 & 1.0535 & 0.0457 & 0.0372 & 0.0283 & 0.0191 & 0.0096 \\ 0.0155 & 0.0308 & 0.0457 & 1.0600 & 0.0489 & 0.0372 & 0.0251 & 0.0126 \\ 0.0126 & 0.0251 & 0.0372 & 0.0489 & 1.0600 & 0.0457 & 0.0308 & 0.0155 \\ 0.0096 & 0.0191 & 0.0283 & 0.0372 & 0.0457 & 1.0535 & 0.0361 & 0.0181 \\ 0.0065 & 0.0129 & 0.0191 & 0.0251 & 0.0308 & 0.0361 & 1.0409 & 0.0206 \\ 0.0033 & 0.0065 & 0.0096 & 0.0126 & 0.0155 & 0.0181 & 0.0206 & 1.0228 \end{pmatrix}$$

and

$$q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$$

one can easily check that the problem is strictly monotone since all the eigenvalues of M are strictly positive, as shown below :

$$\lambda_1 = 1.2281, \quad \lambda_2 = 1.0542, \quad \lambda_3 = 1.0250, \quad \lambda_4 = 1.0151, \quad \lambda_5 = 1.0106,$$
$$\lambda_6 = 1.0083, \quad \lambda_7 = 1.0070 \quad \textit{and} \quad \lambda_8 = 1.0064.$$

The strictly feasible beginning point is provided by :

$$z_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$$
$$\textit{and} \quad w_0 =$$
$$\begin{pmatrix} 2.108942 & 2.191847 & 2.247692 & 2.275788 & 2.275788 & 2.247692 & 2.191846 & 2.108942 \end{pmatrix}^t$$

the numerical results for our two functions with various values of θ, r and $n = 8$ are as follows :

$\psi_i(x) \setminus \theta$	$\theta = 0.15$		$\theta = 0.3$		$\theta = 0.6$		$\theta = 0.95$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_{E,1.1}(x)$	118	0.96	62	0.90	34	0.89	22	0.85
$\psi_{E,2}(x)$	122	1.20	66	0.99	39	0.94	26	0.86
$\psi_{E,4}(x)$	128	1.22	72	1.13	45	1.01	32	0.95
$\psi_{H,4}(x)$	120	0.92	64	0.88	36	0.73	22	0.69
$\psi_{H,5}(x)$	126	1.06	69	0.95	40	0.76	27	0.73
$\psi_{H,6}(x)$	131	1.07	75	1.01	45	0.99	34	0.93
$\psi_R(x)$	128	0.98	72	0.96	45	0.93	26	0.89
$\psi_{B,2}(x)$	124	1.24	70	1.04	40	0.97	30	0.94
$\psi_{B,4}(x)$	128	1.30	74	1.16	46	1.01	34	0.98
$\psi_{L,6}(x)$	137	1.15	84	1.12	47	1.07	36	0.98
$\psi_{M,2}(x)$	125	1.24	69	1.09	41	0.98	29	0.90
$\psi_{M,4}(x)$	132	1.33	76	1.14	49	1.05	37	0.99

TABLE 2.10: NUMERICAL EXPERIMENTS OF (P6) USING VARIOUS KERNEL FUNCTIONS.

the unique solutions of the corresponding LCP and of the AVE, respectively, are :

$$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t, \quad w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$$
$$\textit{and} \quad u^* = |u^*| = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t.$$

EXPERIMENT 6. *We consider example 2 in* **[5]** *with just slight alterations in vector h. As a result of* **EXAMPLE 1.7.3**, *the matrix H and vector h of the related hydrodynamic equations are provided by :*

$$H = \frac{-1}{2}\begin{pmatrix} 50 & 5 & 0 & \dots & \dots & \dots & 0 \\ 5 & 50 & 5 & 0 & \dots & \dots & 0 \\ 0 & 5 & 50 & 5 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 5 & 50 & 5 & 0 \\ 0 & \dots & \dots & 0 & 5 & 50 & 5 \\ 0 & \dots & \dots & \dots & 0 & 5 & 50 \end{pmatrix} \quad \text{and} \quad h = \frac{-1}{2}\begin{pmatrix} 53 \\ 58 \\ 58 \\ \vdots \\ 58 \\ 58 \\ 53 \end{pmatrix}. \tag{P7}$$

The corresponding LCP (2.4) for $n = 16$ is provided by :

$$M = \begin{pmatrix} 1.0421 & -0.0044 & 0.0005 & 0 & \dots & \dots & \dots & \dots & 0 \\ -0.0044 & 1.0426 & -0.0045 & 0.0005 & 0 & \dots & \dots & \dots & 0 \\ 0.0005 & -0.0045 & 1.0426 & -0.0045 & 0.0005 & 0 & \dots & \dots & 0 \\ 0 & 0.0005 & -0.0045 & 1.0426 & -0.0045 & 0.0005 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0.0005 & -0.0045 & 1.0426 & -0.0045 & 0.0005 & 0 \\ 0 & \dots & \dots & 0 & 0.0005 & -0.0045 & 1.0426 & -0.0045 & 0.0005 \\ 0 & \dots & \dots & \dots & 0 & 0.0005 & -0.0045 & 1.0426 & -0.0044 \\ 0 & \dots & \dots & \dots & \dots & 0 & 0.0005 & -0.0044 & 1.0421 \end{pmatrix}$$

and

$$q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$$

The problem is clearly monotone since all dominant principal minors of the associated matrix are strictly positive, and the strictly feasible beginning point is given by :

$$z_0 = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix}^t$$

and

$$w_0 = \begin{pmatrix} 1.5191 & 1.5171 & 1.5173 & 1.5172 & 1.5172 & 1.5172 & 1.5172 & 1.5172 \\ 1.5172 & 1.5172 & 1.5172 & 1.5172 & 1.5172 & 1.5173 & 1.5171 & 1.5191 \end{pmatrix}^t$$

the following are the unique solutions of the corresponding LCP and of the AVE :

$$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t,$$

$$w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$$

$$\text{and} \quad u^* = |u^*| = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$$

$\psi_i(x) \setminus \theta$	$\theta = 0.15$		$\theta = 0.3$		$\theta = 0.6$		$\theta = 0.95$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\psi_{E,1.1}(x)$	112	0.73	57	0.65	29	0.58	18	0.52
$\psi_{E,2}(x)$	114	0.85	61	0.72	32	0.63	23	0.58
$\psi_{E,4}(x)$	122	0.94	67	0.85	40	0.74	30	0.65
$\psi_{H,4}(x)$	119	0.71	64	0.69	39	0.62	26	0.45
$\psi_{H,5}(x)$	115	0.64	58	0.50	30	0.48	23	0.42
$\psi_{H,6}(x)$	125	0.82	70	0.76	42	0.64	32	0.60
$\psi_{R}(x)$	129	0.86	74	0.75	46	0.67	37	0.59
$\psi_{B,1.1}(x)$	113	0.86	59	0.80	31	0.74	20	0.60
$\psi_{B,4}(x)$	124	0.99	70	0.82	43	0.77	33	0.64
$\psi_{B,6}(x)$	134	1.03	79	0.96	51	0.82	42	0.72
$\psi_{L,5}(x)$	123	0.96	68	0.82	40	0.78	27	0.73
$\psi_{L,6}(x)$	128	1.01	73	0.91	45	0.85	35	0.79
$\psi_{A,1.1}(x)$	114	0.80	57	0.76	30	0.62	20	0.55
$\psi_{A,4}(x)$	125	1.05	70	0.87	42	0.76	32	0.69
$\psi_{A,6}(x)$	131	1.08	76	0.98	49	0.79	39	0.75
$\psi_{M,5}(x)$	122	0.80	67	0.76	41	0.69	29	0.56
$\psi_{M,6}(x)$	130	0.87	75	0.81	47	0.75	38	0.69

TABLE 2.11: NUMERICAL EXPERIMENTS OF (P7) USING VARIOUS KERNEL FUNCTIONS.

respectively. The numerical results for our two functions and other functions with varied values of θ and r for $n = 16$ are provided below :

COMMENTS 1. *Based on the findings of this section, we conclude that our two kernel functions (2.1) and (2.2) are more effective than previous functions. We note that, for each θ that is taken into account, the barrier degree r delivers superior iteration numbers in the smallest period of time. When the two new kernel functions are compared, taking into account the results obtained for the value of barrier degree r that provides the best-known complexity, in general, the first function $\psi_E(x)$ solves the problems with the fewest number of repetitions and the second function $\psi_H(x)$ in the shortest time. These numerical results confirm and agree with our theoretical findings.*

CONCLUSION

In this chapter, we used two new kernel functions to make theoretical, algorithmic and numerical adjustments in the primal–dual *IPM* for the linear complementarity problem (2.3). Regarding the theoretical and algorithmic implementation our objective is to study the complexity analysis of the primal-dual interior-point algorithm described in **FIGURE 2.1**. We demonstrated that the complexity bound for large-update methods based on the considered kernel function nor a logarithmic barrier term (2.1) and the new kernel function with a hyperbolic-logarithmic barrier term (2.2) are $O\left(rn^{\frac{r+1}{2r}}\log\frac{n}{\varepsilon}\right)$ and $O\left(rn^{\frac{r+4}{2r}}\log\frac{n}{\varepsilon}\right)$, respectively. We proved that with the special choice of its parameter r, i.e., $r = \dfrac{\log n}{2}$ for (2.1) and $r = 2\log n$ for (2.2), the algorithm has the best-known complexity bound for large-update methods, namely, $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$. Finally, we touch on the numerical implementation that confirms the efficacy of our two kernel functions by presenting some numerical experiments of the some problems (**P1**), (**P4**), (**P5**), (**P2**), (**P3**), (**P6**) and (**P7**) using to solve *LCP* and other problems where we notice that the closer the value of parameter r to the special choice and whenever the value of barrier update parameter θ is close to 1, the number of iterations obtained and the time taken should be as few as possible.

3

Solving absolute value equation using two new smoothing functions

Introduction

In this chapter, we are interested in a smoothing-type algorithm for solving the absolute value equation which is extracted from the system (**LCP**). The extremely important point is that we reformulate the absolute value equation as a set of smooth equations, as illustrated in **[71]**. Then, we propose two additional smoothing functions to solve the system of equations that have the same role as the four functions used by *Saheya et al.* **[71]** by utilizing a smooth approximation of the absolute function since it is regarded as a nonsmooth function.

The main contribution of this work is that it demonstrates that the solution of the absolute value equation is entirely connected to the solution of $H(\nu, u) = 0_{\mathbb{R}^{n+1}}$ based on ϕ_{trig} and ϕ_{exp} using some Newton-type approach. Then, under **Assumption 1**, we show that our method is well-defined ; moreover, we show that this last assumption assures that every sequence created by our algorithm converges to an *AVE* solution. The suggested approach is identical to that described in **[71]**, **[43]** and has also been proposed for tackling other types of issues (see **[22]** and **[42]**). Such as the results of this chapter are taken from **Paper 2**.

i	Smoothing functions $\phi_i(\nu, s)$:	The max norm :	Ref :
1	$\nu\left(\ln\left(1+e^{-\frac{s}{\nu}}\right)+\ln\left(1+e^{\frac{s}{\nu}}\right)\right)$	1.4ν	[71]
2	$\begin{cases} s & \text{if} \quad s \geq \frac{\nu}{2}, \\ \frac{s^2}{\nu}+\frac{\nu}{4} & \text{if} \quad -\frac{\nu}{2} < s < \frac{\nu}{2}, \\ -s & \text{if} \quad s \leq -\frac{\nu}{2}. \end{cases}$	$\frac{\nu}{4}$	[71]
3	$\sqrt{4\nu^2+s^2}$	2ν	[71]
4	$\begin{cases} \frac{s^2}{2\nu} & \text{if} \quad \lvert s\rvert \leq \nu, \\ \lvert s\rvert-\frac{\nu}{2} & \text{if} \quad \lvert s\rvert > \nu. \end{cases}$	$\frac{\nu}{2}$	[71]

Table 3.1: The max norm of some smoothing functions.

TABLE 3.1 displays the maximum norm based on different smoothing functions researched in the literature. The results are distributed as follows : SECTION 3.1 presents the two smoothing functions, investigates their characteristics, and depicts their graphs after transforming the absolute value equation into a system of smooth equations. SECTION 3.2 describes the algorithm and introduces some essential features. SECTION 3.3 discusses the algorithm's convergence findings. In SECTION 3.4, we present some numerical results from our investigations.

We recall that the specific version of the absolute value equation (*AVE*) with given $A \in \mathcal{M}_n$ and $b \in \mathbb{R}^n$ consists in locating a vector $u \in \mathbb{R}^n$ such that :

$$Au - |u| = b, \tag{3.1}$$

where $|u|$ signifies the component-wise absolute value of u and if (3.1) obtained via (LCP), then $A = (M - I)^{-1}(M + I), b = 2(M - I)^{-1}q, u = w - z$ and $|u| = w + z$.

3.1 SMOOTHING REFORMULATION

In this part, we will first explain the key point of the used method, define it and explore the two new smoothing functions. The graphs of ϕ_{trig} and ϕ_{exp} are then displayed. In addition, we discuss the particular solvability of *AVE* (3.1). Finally, we calculate the distance between our functions and the function $|s|$ and compare them with some functions proposed before.

3.1.1 PRINCIPLE OF THE METHOD

Furthermore, because the absolute function is nonsmooth, we cannot simply employ classical Newton methods to solve the *AVE*, instead, as previously stated, the problem (3.1) will be rewritten as a collection of smoothing equations. To achieve our goal, we define the function $H : \mathbb{R}_{++} \times \mathbb{R}^n \to \mathbb{R}^{n+1}$ as follows :

$$H(\nu, u) = \begin{pmatrix} \nu \\ Au - \Phi(\nu, u) - b \end{pmatrix} \tag{3.2}$$

where $\nu > 0$, $u \in \mathbb{R}^n$ and $\Phi : \mathbb{R}_{++} \times \mathbb{R}^n \to \mathbb{R}^n$ is provided by :

$$\Phi(\nu, u) = \begin{pmatrix} \phi(\nu, u_1) \\ \vdots \\ \phi(\nu, u_n) \end{pmatrix} \tag{3.3}$$

with $\phi : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$ is a smoothing function.

REMARK 3.1.1. *We will apply the approach employed by Saheya et al.* [71] *and Jiang et al.* [43]

throughout this chapter. More specifically, the function ϕ_p presented in **[43]** *is strongly semi-smooth on $\mathbb{R}^2$, whereas the four functions utilized in* **[71]** *and our two functions ϕ_{trig}, ϕ_{exp} are continuously differentiable on $\mathbb{R}^2$.*

3.1.2 SMOOTHING FUNCTIONS

In this subsection, we present two families of novel smoothing functions that are essentially required in the smoothing Newton technique for solving (**3.1**).

DEFINITION 3.1.2. *$\phi : \mathbb{R}_{++} \times \mathbb{R} \to \mathbb{R}$ is a smoothing function of the absolute function if it satisfies the following conditions :*

COND 4. *ϕ is continuously differentiable at any $(\nu, s) \in \mathbb{R}_{++} \times \mathbb{R}$.*

COND 5. $\lim_{\nu \to 0} \phi(\nu, s) = |s|$ *for any s.*

Now, for the smoothing Newton method $\forall \nu > 0$ and $\forall s \in \mathbb{R}$ we introduce the two novel functions $\phi_{trig}(\nu, s)$ and $\phi_{exp}(\nu, s)$ as follows :

$$\phi_{trig}(\nu, s) = \frac{2s}{\pi} \tan^{-1}\left(\frac{2s}{\nu}\right) \tag{3.4}$$

and

$$\phi_{exp}(\nu, s) = \begin{cases} \dfrac{\nu}{2\pi}\left(e^{\frac{\pi^2 s^2 - \nu^2}{\nu^2}} + 1\right) & \text{if } \quad |s| \leq \dfrac{\nu}{\pi} \\ \\ |s| & \text{if } \quad |s| > \dfrac{\nu}{\pi} \end{cases} \tag{3.5}$$

PROPERTIES OF THE NEW SMOOTHING FUNCTIONS

In fact, to establish that functions (**3.4**) and (**3.5**) satisfy **DEFINITION 3.1.2** we use a theoretical and graphical way.

▶ **THEORETICALLY.** We propose the following proposition for the theoretical investigation.

PROPOSITION 3.1.3. *Let ϕ_i for $i = trig, exp$ be defined as in* (**3.4**) *and* (**3.5**)*, respectively, as smoothing functions.*

PROOF. To demonstrate this statement, we must establish the following :

COND 4.

We just need to compute $\dfrac{\partial \phi_i(\nu, s)}{\partial s}$ and $\dfrac{\partial \phi_i(\nu, s)}{\partial \nu}$, then explain the continuity of these last two functions.

▶ For $i = trig$

$$\frac{\partial\phi_{trig}(\nu,s)}{\partial s}=\frac{2}{\pi}\tan^{-1}\left(\frac{2s}{\nu}\right)+\frac{4\nu s}{\pi(4s^2+\nu^2)} \tag{3.6}$$

and

$$\frac{\partial\phi_{trig}(\nu,s)}{\partial \nu}=-\frac{4s^2}{\pi(4s^2+\nu^2)}$$

then, it is obvious that $\frac{\partial\phi_{trig}(\nu,s)}{\partial s}$ and $\frac{\partial\phi_{trig}(\nu,s)}{\partial \nu}$ are continuous. As a result, ϕ_{trig} is continuously differentiable at any $(\nu,s)\in\mathbb{R}_{++}\times\mathbb{R}$.

► For $i=exp$

$$\frac{\partial\phi_{exp}(\nu,s)}{\partial s}=\begin{cases}1 & \text{if} \quad s>\frac{\nu}{\pi}\\ \frac{\pi s}{\nu}e^{\frac{\pi^2s^2-\nu^2}{\nu^2}} & \text{if} \quad -\frac{\nu}{\pi}\leq s\leq\frac{\nu}{\pi}\\ -1 & \text{if} \quad s<-\frac{\nu}{\pi}\end{cases} \tag{3.7}$$

and

$$\frac{\partial\phi_{exp}(\nu,s)}{\partial \nu}=\begin{cases}0 & \text{if} \quad s>\frac{\nu}{\pi}\\ \frac{1}{2\pi}\left(\left(1-\frac{2\pi^2s^2}{\nu^2}\right)e^{\frac{\pi^2s^2-\nu^2}{\nu^2}}+1\right) & \text{if} \quad -\frac{\nu}{\pi}\leq s\leq\frac{\nu}{\pi}\\ 0 & \text{if} \quad s<-\frac{\nu}{\pi}\end{cases}$$

it is obvious that $\frac{\partial\phi_{exp}(\nu,s)}{\partial s}\in C^1$ and $\frac{\partial\phi_{exp}(\nu,s)}{\partial \nu}\in C^1$ since :

$$\lim_{s\to\frac{\nu}{\pi}}\frac{\partial\phi_{exp}(\nu,s)}{\partial s}=\lim_{s\to\frac{\nu}{\pi}}\frac{\pi s}{\nu}e^{\frac{\pi^2s^2-\nu^2}{\nu^2}}=1,$$

$$\lim_{s\to-\frac{\nu}{\pi}}\frac{\partial\phi_{exp}(\nu,s)}{\partial s}=\lim_{s\to-\frac{\nu}{\pi}}\frac{\pi s}{\nu}e^{\frac{\pi^2s^2-\nu^2}{\nu^2}}=-1,$$

$$\lim_{s\to\frac{\nu}{\pi}}\frac{\partial\phi_{exp}(\nu,s)}{\partial \nu}=\lim_{s\to\frac{\nu}{\pi}}\frac{1}{2\pi}\left(\left(1-\frac{2\pi^2s^2}{\nu^2}\right)e^{\frac{\pi^2s^2-\nu^2}{\nu^2}}+1\right)=0$$

and

$$\lim_{s\to-\frac{\nu}{\pi}}\frac{\partial\phi_{exp}(\nu,s)}{\partial \nu}=\lim_{s\to-\frac{\nu}{\pi}}\frac{1}{2\pi}\left(\left(1-\frac{2\pi^2s^2}{\nu^2}\right)e^{\frac{\pi^2s^2-\nu^2}{\nu^2}}+1\right)=0$$

the previous arguments indicate that ϕ_{exp} is continuously differentiable at any $(\nu,s)\in\mathbb{R}_{++}\times\mathbb{R}$.

COND 5.

According to the definition of ϕ_i for $i=trig, exp$ in (3.4) and (3.5) respectively, it is obvious that :

$$\lim_{\mu\to 0}\phi_i(\nu,s)=|s|=\begin{cases}s & \text{if} \quad s\geq 0\\ -s & \text{if} \quad s\leq 0\end{cases}$$

Since we know for every $s\in\mathbb{R}$ for the first function :

$$\lim_{\nu\to 0^+}\tan^{-1}\left(\frac{2s}{\nu}\right)=\frac{\pi}{2} \quad \text{and} \quad \lim_{\nu\to 0^-}\tan^{-1}\left(\frac{2s}{\nu}\right)=-\frac{\pi}{2}$$

we have the following for the second function :

$$\lim_{\nu \to 0} \phi_{exp}(\nu, s) = \begin{cases} \lim\limits_{\nu \to 0} \dfrac{\nu}{2\pi} \left(e^{\frac{\pi^2 s^2 - \nu^2}{\nu^2}} + 1 \right) & \text{if} \quad |s| \leq 0 \\ |s| & \text{if} \quad |s| > 0 \end{cases}$$

This completes the evidence. □

► **GEOMETRICALLY** The characteristics can first be proven geometrically (through graphs). Using **FIGURES 3.1-3.2**, we can observe that when ν goes to zero, our two smoothing functions ϕ_{trig} and ϕ_{exp} converge to an absolute function, so clarifying the second characteristic, **COND 5**.

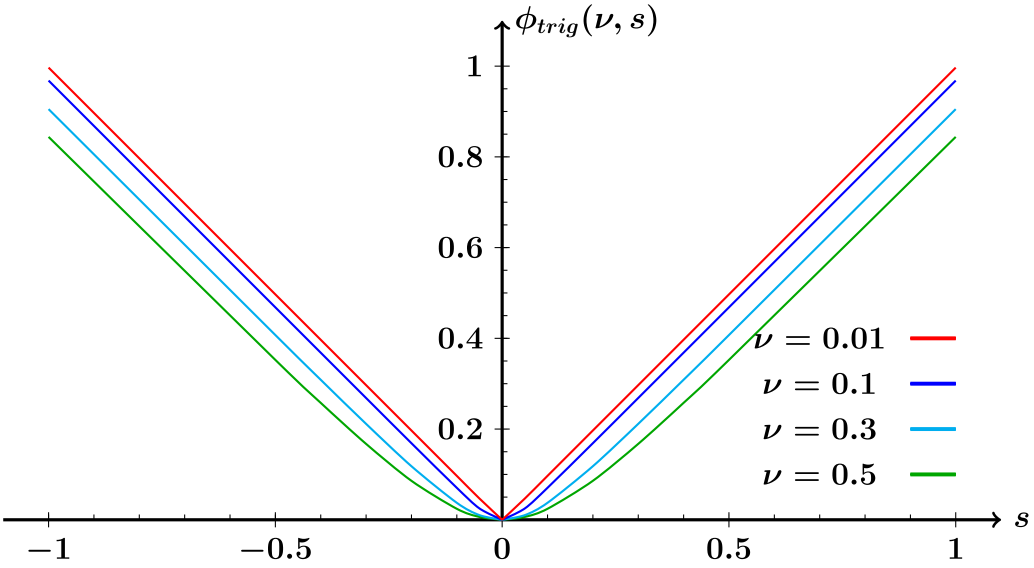

FIGURE 3.1: GRAPHS OF $\phi_{trig}(\nu, s)$ WITH DIFFERENT VALUES OF ν.

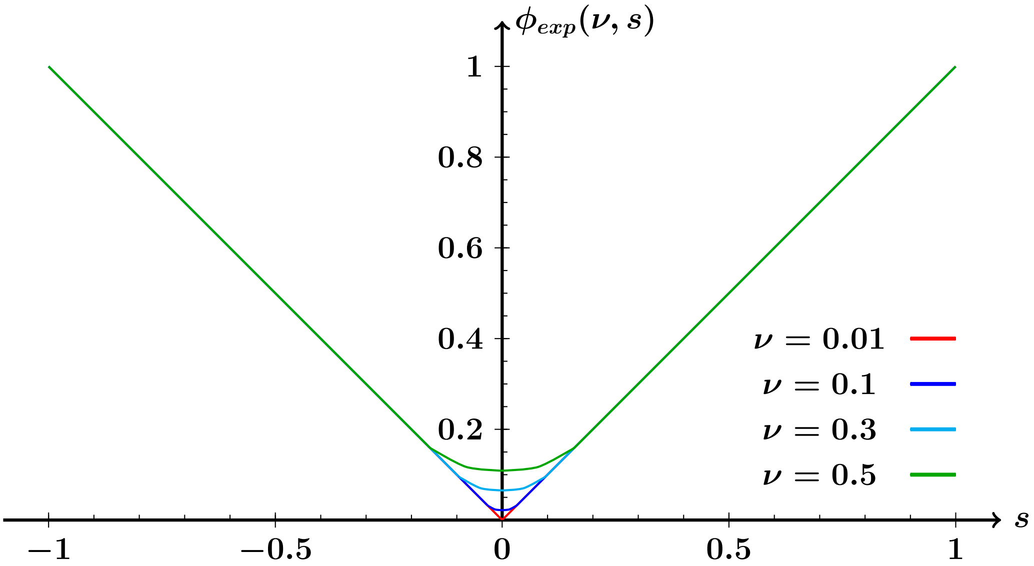

FIGURE 3.2: GRAPHS OF $\phi_{exp}(\nu, s)$ WITH DIFFERENT VALUES OF ν.

To facilitate further study on our approach, we demonstrate that the two functions ϕ_{trig} and ϕ_{exp} achieve the following property.

LEMMA 3.1.4. *For any* $(\nu, s) \in \mathbb{R}_{++} \times \mathbb{R}$, *we have the following result :*

$$-1 \leq \frac{\partial \phi_i(\nu, s)}{\partial s} \leq 1, \qquad \textit{for } i = trig, exp.$$

PROOF. We obtain the following when we compute the second derivatives of (3.6) and (3.7) with regard to s :

$$\frac{\partial^2 \phi_{trig}(\nu, s)}{\partial s^2} = \frac{8\pi\nu^3}{\pi^2(4s^2 + \nu^2)^2}$$

and

$$\frac{\partial^2 \phi_{exp}(\nu, s)}{\partial s^2} = \begin{cases} 0 & \text{if} \quad |s| > \dfrac{\nu}{\pi} \\ \left(\dfrac{\pi}{\nu} + \dfrac{2\pi^3 s^2}{\nu^3}\right) e^{\frac{\pi^2 s^2 - \nu^2}{\nu^2}} & \text{if} \quad |s| \leq \dfrac{\nu}{\pi} \end{cases}$$

we can readily demonstrate that the two functions $\dfrac{\partial^2 \phi_{trig}(\nu, s)}{\partial s^2}$ and $\dfrac{\partial^2 \phi_{exp}(\nu, s)}{\partial s^2}$ are non-negative for any $(\nu, s) \in \mathbb{R}_{++} \times \mathbb{R}$. As a result, the two functions (3.6) and (3.7) are increasing, yielding :

$$\lim_{s \to -\infty} \frac{\partial \phi_i(\nu, s)}{\partial s} \leq \frac{\partial \phi_i(\nu, s)}{\partial s} \leq \lim_{s \to +\infty} \frac{\partial \phi_i(\nu, s)}{\partial s} \Rightarrow \left| \frac{\partial \phi_i(\nu, s)}{\partial s} \right| \leq 1.$$

this concludes the proof. □

It is now necessary to give the gradients of (3.4) and (3.5), which were used in the convergence analysis and numerical implementations.

$$\nabla \phi_{trig}(\nu, s) = \begin{pmatrix} \dfrac{\partial \phi_{trig}(\nu, s)}{\partial \nu} \\ \dfrac{\partial \phi_{trig}(\nu, s)}{\partial s} \end{pmatrix} = \begin{pmatrix} -\dfrac{4s^2}{\pi(4s^2 + \nu^2)} \\ \dfrac{2}{\pi} \tan^{-1}\left(\dfrac{2s}{\nu}\right) + \dfrac{4\nu s}{\pi(4s^2 + \nu^2)} \end{pmatrix}$$

and

$$\nabla \phi_{exp}(\nu, s) = \begin{pmatrix} \dfrac{\partial \phi_{exp}(\nu, s)}{\partial \nu} \\ \\ \dfrac{\partial \phi_{exp}(\nu, s)}{\partial s} \end{pmatrix} = \begin{pmatrix} \begin{cases} 0 & \text{if} \quad |s| > \dfrac{\nu}{\pi} \\ \dfrac{1}{2\pi}\left(\left(1 - \dfrac{2\pi^2 s^2}{\nu^2}\right) e^{\frac{\pi^2 s^2 - \nu^2}{\nu^2}} + 1\right) & \text{if} \quad |s| \leq \dfrac{\nu}{\pi} \end{cases} \\ \begin{cases} 1 & \text{if} \quad s > \dfrac{\nu}{\pi} \\ \dfrac{\pi s}{\nu} e^{\frac{\pi^2 s^2 - \nu^2}{\nu^2}} & \text{if} \quad |s| \leq \dfrac{\nu}{\pi} \\ -1 & \text{if} \quad s < -\dfrac{\nu}{\pi} \end{cases} \end{pmatrix}.$$

3.1.3 THE UNIQUE SOLVABILITY OF THE AVE (3.1)

We note that the *AVE* (3.1) is uniquely solved for every $b \in \mathbb{R}^n$ under **ASSUMPTION 1** (see **THEOREM 1.8.13** and its proof). Now, using **DEFINITION 3.1.2** and **PROPOSITION 3.1.3**, we can derive the corresponding reformulation for the *AVE* (3.1) based on each ϕ_i for $i = trig, exp$.

PROPOSITION 3.1.5. *(Proposition* 2.2, [71]*) Let* $\Phi^i(\mu, u)$ *be similarly defined as in* (3.3) *based on each* ϕ_i *for* $i = trig, exp$*. Then, we have :*

(*a*) $H_i(\nu, u) = 0_{\mathbb{R}^{n+1}}$ *if and only if* u *solves the AVE* (3.1).

(*b*) H_i *is continuously differentiable on* $\mathbb{R}^{n+1} \setminus \{0\}$ *with* J_{H_i} *the Jacobian matrix of* H_i *is given by :*

$$J_{H_i}(\nu, u) = \begin{pmatrix} 1 & 0^t_{\mathbb{R}^n} \\ -\nabla_\nu \Phi_i(\nu, u) & A - J_{\Phi_{iu}}(\nu, u) \end{pmatrix} \tag{3.8}$$

where

$$\nabla_\nu \Phi_i(\nu, u) = \begin{pmatrix} \dfrac{\partial \phi_i(\nu, u_1)}{\partial \nu} \\ \vdots \\ \dfrac{\partial \phi_i(\nu, u_n)}{\partial \nu} \end{pmatrix} \quad \text{and} \quad J_{\Phi_{iu}}(\nu, u) = \begin{pmatrix} \dfrac{\partial \phi_i(\nu, u_1)}{\partial u_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \dfrac{\partial \phi_i(\nu, u_n)}{\partial u_n} \end{pmatrix}.$$

PROOF. We assume that $H_i(\nu, u) = 0_{\mathbb{R}^{n+1}}$ for the first property. Using (3.2), we obtain :

$$\begin{pmatrix} \nu \\ Au - \Phi_i(\nu, u) - b \end{pmatrix} = \begin{pmatrix} 0 \\ 0_{\mathbb{R}^n} \end{pmatrix} \tag{3.9}$$

(3.3), **COND 5** and the fact that $\nu = 0$ (as a result of the first inequality in (3.9)) lead to the following conclusion :

$$Au - \Phi_i(\nu, u) - b = 0_{\mathbb{R}^n} \quad \Leftrightarrow \quad Au - |u| - b = 0_{\mathbb{R}^n}.$$

then u is a solution to *AVE* (3.1). We repeat the process for the inverse implication. For the second feature, it is sufficient to show the continuous behavior of $-\nabla_\nu \Phi_i(\nu, u)$ and $A - J_{\Phi_{iu}}(\nu, u)$. **COND 4** is utilized to do this. □

REMARK 3.1.6. *The preceding proposition's first property demonstrates that the AVE* (3.1) *has a solution if and only if* $H_i(\nu, u) = 0_{\mathbb{R}^{n+1}}$.

LEMMA 3.1.7. *(Lemma* 2.4, [43]*) Under* **ASSUMPTION 1**, *the function* $h(u) = \|Au - |u| - b\|$ *is level-bounded.*

PROOF. For each $\kappa \in \mathbb{R}$ we designate $L(\kappa) := \{u \in \mathbb{R}^n \backslash h(u) \leq \kappa\}$. To prove that the function $h(u)$ is level-bounded, it suffices to demonstrate that the sequence $\{\|u\|\}$ which proves the condition $u \in L(\kappa)$ is bounded. We assume that the sequence $\{\|u\|\}$ is unbounded, then :

$$\begin{aligned} h(u) = \|Au - |u| - b\| &\geq \|Au\| - \|u\| - \|b\| \\ &\geq (\sigma_{min}(A) - 1)\|u\| - \|b\| \\ &\geq +\infty \end{aligned}$$

where the second inequality comes from $\sigma_{min}(A)\|u\| \leq \|Au\|$ and the last inequality derives from **ASSUMPTION 1**, which contradicts the fact that $h(u) \leq \kappa$. □

3.1.4 SOME COMPARISONS

As previously explained, $\phi_i(\nu, s)$ for $i = trig, exp$ approximates the function $|s|$, we evaluate two considerations in determining which one most correctly approximates the absolute function. We also consider if our two functions are better than the four functions presented in **TABLE 3.1** or not.

► **GEOMETRICALLY.** Using the graphs (see **FIGURE 3.3-3.4**).

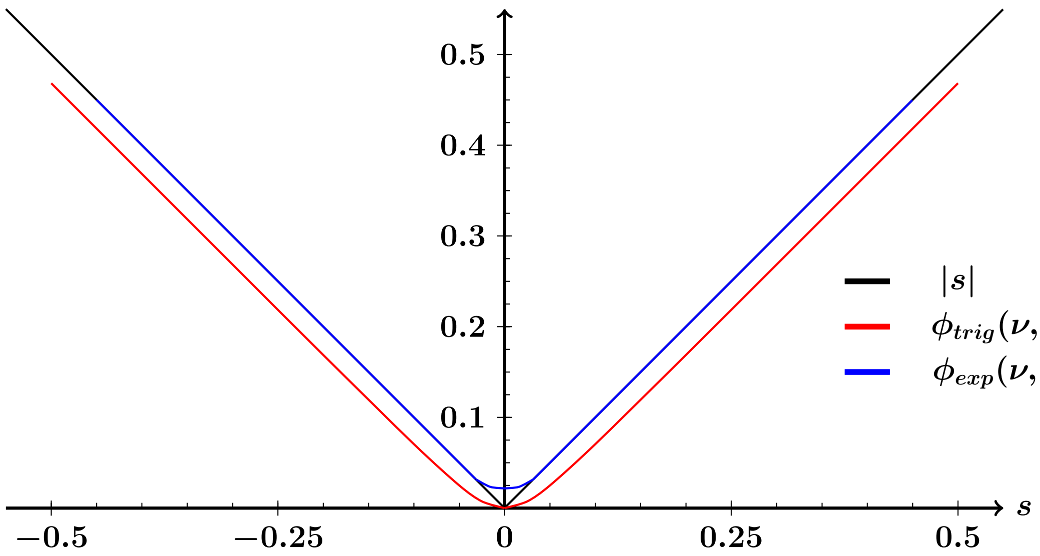

FIGURE 3.3: GRAPHS OF $|s|$, $\phi_{trig}(\nu, s)$ AND $\phi_{exp}(\nu, s)$ WITH $\nu = 0.1$.

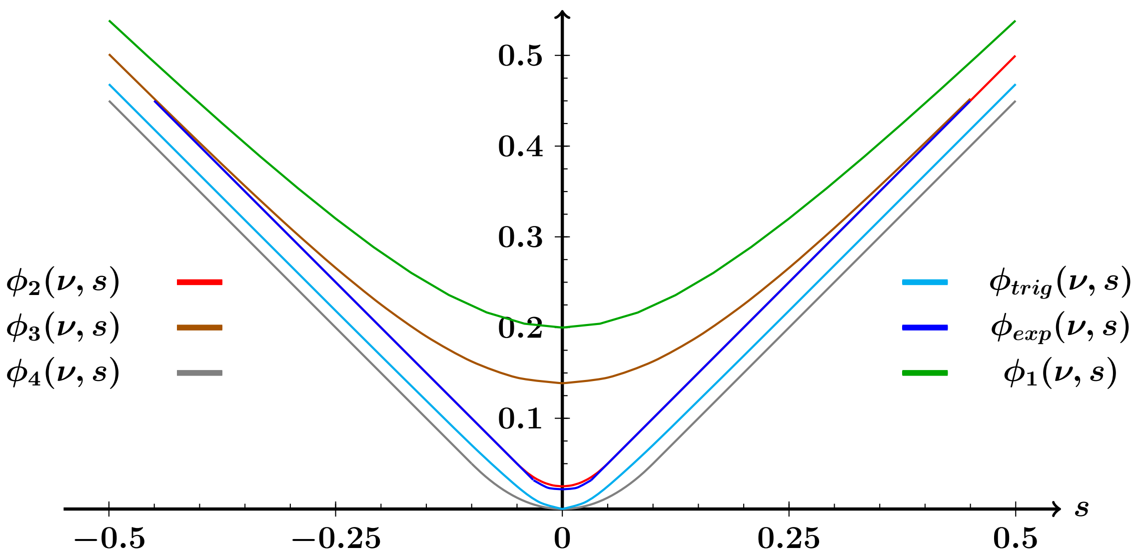

FIGURE 3.4: GRAPHS OF $\phi_{trig}(\nu, s)$, $\phi_{exp}(\nu, s)$ AND $\phi_i(\nu, s)$ FOR $i = 1, 2, 3, 4$ WITH $\nu = 0.1$.

REMARK 3.1.8. *According to* **FIGURES 3.3-3.4**, ϕ_{exp} *is the one that best approximates the absolute function in the sense that it is closest to* $|s|$ *among all* ϕ_{trig} *and* ϕ_{exp}.

► **THEORETICALLY.** To theoretically corroborate **REMARK 3.1.8** in this section, we must compute the distance between ϕ_{trig}, ϕ_{exp} and $|s|$, using the max norm. Given this, we propose the following :

PROPOSITION 3.1.9. *For each given $\nu > 0$, ϕ_{trig} and ϕ_{exp} defined by* (3.4) *and* (3.5)*, respectively, satisfy :*

$$\|\phi_{trig}(\nu, s) - |s|\|_\infty \approx 0.32\nu, \tag{3.10}$$

$$\|\phi_{exp}(\nu, s) - |s|\|_\infty \approx 0.22\nu, \tag{3.11}$$

PROOF. The distance between two real-valued functions $\phi_i(\nu, s)$ and $|s|$ for $i = trig, exp$ is defined as follows :

$$\|\phi_i(\nu, s) - |s|\|_\infty = \max_{s\in\mathbb{R}} |\phi_i(\nu, s) - |s||.$$

For (3.10), we have :

$$\lim_{|s|\to\infty} |\phi_{trig}(\nu, s) - |s|| = \frac{\nu}{\pi} \quad \text{and} \quad |\phi_{trig}(\nu, 0)| = 0,$$

hence :

$$\|\phi_{trig}(\nu, s) - |s|\|_\infty = \frac{\nu}{\pi} \approx 0.32\nu.$$

For (3.11), since $\lim_{|s|\to\infty} |\phi_{exp}(\mu, s) - |s|| = 0$, we obtain :

$$\begin{aligned} \|\phi_{exp}(\nu, s) - |s|\|_\infty &= \max_{s\in\mathbb{R}} |\phi_{exp}(\nu, s) - |s|| \\ &= |\phi_{exp}(\nu, 0)| \\ &= \left|\frac{\nu}{2\pi}(e^{-1} + 1)\right| \\ &\approx 0.22\nu. \end{aligned}$$

the evidence is now complete. □

REMARK 3.1.10. *As illustrated in* [71]*, we may compare* (3.4) *and* (3.5) *to a few well-known functions. As a result of* **PROPOSITION 3.1.9** *and* **TABLE 3.1***, and for a constant ν greater than zero, we obtain :*

$$\begin{aligned} &\|\phi_3(\nu, s) - |s|\|_\infty > \|\phi_1(\nu, s) - |s|\|_\infty > \|\phi_{trig}(\nu, s) - |s|\|_\infty, \\ &\|\phi_4(\nu, s) - |s|\|_\infty > \|\phi_2(\nu, s) - |s|\|_\infty > \|\phi_{exp}(\nu, s) - |s|\|_\infty. \end{aligned} \tag{3.12}$$

Furthermore, we may compare our two functions (3.4) and (3.5) side by side, as seen in the **FIGURES 3.3-3.4** :

$$\phi_{exp}(\nu, s) > |s| > \phi_{trig}(\nu, s) \tag{3.13}$$

where " $>$ " means better performance. **PROPOSITION 3.1.9** leads us to the following conclusion :

$$\|\phi_{exp}(\nu, s) - |s|\|_\infty < \|\phi_{trig}(\nu, s) - |s|\|_\infty. \tag{3.14}$$

REMARK 3.1.11. (3.12), (3.13) *and* (3.14) *indicate that* ϕ_{exp} *is the best approximation of the absolute function among* ϕ_i, $i = 1, 2, 3, 4$ *and* ϕ_{trig}.

The results of **REMARK 3.1.10** and **REMARK 3.1.11** do not provide a conclusive answer as to whether our function ϕ_{exp} is the best approximation of the function $|s|$ among ϕ_i, $i = trig, exp$, and the four functions listed in **TABLE 3.1**. Until verified numerically, all of this remains simply a conjecture.

3.2 A SMOOTHING-TYPE ALGORITHM

In this part, we will describe a smoothing-type approach for solving the system of smooth equations $H_i(\nu, u) = 0$, $i = trig, exp$ by using various Newton-type procedures at each iteration by letting $\nu > 0$ and $H_i \to 0$, $i = trig, exp$ such that a solution of (3.1) may be discovered.

In the smoothing-type approach, we first select the following parameters : $0 \leq \delta \leq 1$, $0 \leq \sigma \leq 1$, $\nu_0 > 0$, an arbitrary vector $u_0 \in \mathbb{R}^n$ and $\beta > 1$ fulfilling $\min\{1, \|H_i(v_0)\|^2\} \leq \beta\mu_0$, $i = trig, exp$. To obtain the search direction Δv_k, we pick $\tau_k := \min\{1, \|H_i(v_k)\|\}$, $i = trig, exp$ and solve the system (NE). Then we take the step size $\alpha_k = \max\{1, \delta, \delta^2, \ldots\}$ such that it checks the system (LS). This technique is performed until a fresh iteration w_k is found that fulfills $\|H_i(v_k)\| = 0$, $i = trig, exp$.

The generic form of a smoothing-type method based on ϕ_{trig} and ϕ_{exp} is provided by :

INPUT :

The parameters $\delta, \sigma \in (0, 1)$, $\nu_0 > 0$, $u_0 \in \mathbb{R}^n$ and $\beta > 1$.
$\mathrm{e}_0 := (1, 0) \in \mathbb{R} \times \mathbb{R}^n$ with $\min\{1, \|H_i(\nu_0, u_0)\|^2\} \leq \beta\nu_0$.

ITERATION :

BEGIN :

$v_0 := (\nu_0, u_0)$;
WHILE ($\|H_i(v_k)\| \neq 0$) **DO**
BEGIN :
set $\tau_k := \min\{1, \|H_i(v_k)\|\}$;
compute $\Delta v_k = (\Delta\nu_k, \Delta u_k) \in \mathbb{R} \times \mathbb{R}^n$ by using :

$$H_i(v_k) + J_{H_i}(v_k)\Delta v_k = \frac{\tau_k^2 \mathrm{e}_0}{\beta}; \qquad \text{(NE)}$$

let α_k be the maximum of the values $1, \delta, \delta^2, \ldots$ such that :

$$\|H_i(v_{k+1})\| \leq \left(1 - \sigma\left(1 - \frac{1}{\beta}\right)\alpha_k\right)\|H_i(v_k)\|; \qquad \text{(LS)}$$

$v_{k+1} := v_k + \alpha_k \Delta v_k$.
END.

END.

FIGURE 3.5: A SMOOTHING-TYPE ALGORITHM.

3.2.1 ALGORITHM PROPERTIES

The following proposition states some basic features of the aforementioned algorithm.

PROPOSITION 3.2.1. *Let the sequence $\{v_k\}$ be generated by our algorithm. Then, we have :*

(a) *The sequences $\{\|H_i(v_k)\|\}$, $i = trig, exp$ and $\{\tau_k\}$ are monotonically decreasing.*

(b) *$\tau_k^2 \leq \beta\nu_k$ holds for all k.*

(c) *The sequence $\{\nu_k\}$ is monotonically decreasing and $\nu_k > 0$ for all k.*

PROOF. For (a). First, we use the line search (**LS**) to discover :

$$\begin{aligned} \|H_i(v_{k+1})\| &= \|H_i(v_k + \alpha_k \Delta v_k)\| \\ &\leq \left(1 - \sigma\left(1 - \frac{1}{\beta}\right)\alpha_k\right)\|H_i(v_k)\| \\ &\leq \gamma\|H_i(v_k)\| \\ &< \|H_i(v_k)\| \end{aligned} \tag{3.15}$$

where the final inequality is followed by $0 < \gamma < 1$, the sequence $\{\|H_i(v_k)\|\}$ is monotonically decreasing. Second, it is clear from the definition of τ_k and (**3.15**) that $\tau_{k+1} < \tau_k$, i.e., the sequence $\{\tau_k\}$, is monotonically decreasing.

We utilize recurrence proof for (b). For $k = 0$, we have :

$$\tau_0^2 := (\min\{1, \|H_i(v_0)\|\})^2 \leq \beta\nu_0$$

then we presume that the expression is correct for k, i.e., $\tau_k^2 \leq \beta\nu_k$ and we investigate its applicability for $k+1$.

$$\begin{aligned} \tau_{k+1}^2 - \beta\nu_{k+1} &= \tau_{k+1}^2 - \beta(\nu_k + \alpha_k\Delta\nu_k) \\ &= \tau_{k+1}^2 - \beta\nu_k - \beta\alpha_k\left(-\nu_k + \frac{\tau_k^2}{\beta}\right) \\ &= \tau_{k+1}^2 - (1-\alpha_k)\beta\nu_k - \alpha_k\tau_k^2 \\ &\leq \tau_{k+1}^2 - (1-\alpha_k)\tau_k^2 - \alpha_k\tau_k^2 \\ &\leq 0 \end{aligned}$$

where the second equality arises from the first equation in (**NE**), the first inequality follows from our assumption $\tau_k^2 \leq \beta\nu_k$, and the third inequality follows from the second half of (a) ($\{\tau_k\}$ is monotonically decreasing). As a consequence, the intended result.

Using the first equation in (**NE**) and (b), we get (c) :

$$\begin{aligned} \nu_{k+1} &= \nu_k + \alpha_k\Delta\nu_k \\ &= (1-\alpha_k)\nu_k + \alpha_k\frac{\tau_k^2}{\beta} \\ &\leq (1-\alpha_k)\nu_k + \alpha_k\nu_k \\ &= \nu_k \end{aligned}$$

and

$$\begin{aligned}\nu_k &= \nu_{k-1} + \alpha_{k-1}\Delta\nu_{k-1}\\ &= (1-\alpha_{k-1})\nu_{k-1} + \alpha_{k-1}\frac{\tau_{k-1}^2}{\beta}\\ &\geq (1-\alpha_{k-1})\frac{\tau_{k-1}^2}{\beta} + \alpha_{k-1}\frac{\tau_{k-1}^2}{\beta}\\ &> 0\end{aligned}$$

this completes the evidence. □

The solvability of Newton's equations (**NE**) is demonstrated by the following theorem.

THEOREM 3.2.2. *Under* **ASSUMPTION 1** *the matrix* $J_{H_i}(\nu, u)$ *given in* (**3.8**) *based on* ϕ_{trig} *and* ϕ_{exp} *is invertible for all* $(\nu, u) \in \mathbb{R}_{++} \times \mathbb{R}^n$.

PROOF. For $i = trig, exp$ and for any $(\nu, u) \in \mathbb{R}_{++} \times \mathbb{R}^n$, it is straightforward to establish from (**3.8**) that $J_{H_i}(\nu, u)$ is invertible if and only if $A - J_{\Phi_{iu}}(\nu, u)$ is nonsingular since otherwise for some $x \in \mathbb{R}^n \setminus \{0\}$ we have that $(A - J_{\Phi_{iu}}(\nu, u))\, x = 0_{\mathbb{R}^n}$. Then :

$$\begin{aligned}\sigma_{min}^2(A) &= \lambda_{min}(A^tA)\\ &\leq \|Ax\|^2\\ &= \|J_{\Phi_{iu}}(\nu, u)x\|^2\\ &\leq \lambda_{max}^2\left(J_{\Phi_{iu}}(\nu, u)\right)\\ &\leq 1\end{aligned}$$

where the first and second inequalities result from :

$$\begin{aligned}\|Ax\|^2 &\geq \min_{\|x\|=1} \langle Ax, Ax\rangle\\ &= \lambda_{min}(A^tA)\end{aligned}$$

and

$$\begin{aligned}\|J_{\Phi_{iu}}(\nu, u)x\|^2 &\leq \max_{\|x\|=1} \langle J_{\Phi_{iu}}(\nu, u)x, J_{\Phi_{iu}}(\nu, u)x\rangle\\ &= \lambda_{max}^2\left(J_{\Phi_{iu}}(\nu, u)\right),\end{aligned}$$

respectively, and **LEMMA 3.1.4** implies the final inequality, which contradicts **ASSUMPTION 1**. In view of this, the matrix $A - J_{\Phi_{iu}}(\nu, u)$ is invertible. □

REMARK 3.2.3. *The system of equations* (**NE**) *is solved using* **THEOREM 3.2.2**.

Now, using the same reasoning as in **[42]**, we obtain the following theorem.

THEOREM 3.2.4. *The line search* (**LS**) *is well-defined.*

PROOF. The proof principle for this theorem is the same as in **[42]** (for additional information, see Remark 2.1 (iv)). We used the following formula for $i = trig, exp$:

$$\Upsilon_i(\alpha_k) = \kappa\left(\|H_i(v_{k+1})\| - \|H_i(v_k)\|\right) - \alpha_k \frac{(H_i(v_k))^t J_{H_i}(v_k)\Delta v_k}{\|H_i(v_k)\|}. \tag{3.16}$$

where $\kappa > 1$. It is clear from the system (**NE**), the specification of the parameter τ_k and the two facts $(H_i(v_k))^t H_i(v_k) = \|H_i(v_k)\|^2$, $(H_i(v_k))^t e_0 = \|H_i(v_k)\|$ that :

$$\begin{aligned}
(H_i(v_k))^t \nabla H_i(v_k)\Delta v_k &= (H_i(v_k))^t \left(-H_i(v_k) + \frac{\tau_k^2 e_0}{\beta}\right) \\
&= \|H_i(v_k)\| \left(-\|H_i(v_k)\| + \frac{\tau_k^2}{\beta}\right) &(3.17)\\
&\leq \|H_i(v_k)\|^2 \left(-1 + \frac{\tau_k}{\beta}\right) &(3.18)
\end{aligned}$$

we get the following from (**3.18**) and **PROPOSITION 3.2.1** :

$$\Upsilon_i(\alpha_k) \leq \alpha_k \|H_i(v_k)\| \left(1 - \frac{\tau_k}{\beta}\right) \quad \Rightarrow \quad \Upsilon_i(\alpha_k) \leq \|H_i(v_k)\|. \tag{3.19}$$

Then, as a result of (**3.16**), (**3.17**) and (**3.19**) we get :

$$\begin{aligned}
\kappa\|H_i(v_{k+1})\| &= \Upsilon_i(\alpha_k) + \kappa\|H_i(v_k)\| + \alpha_k \frac{(H_i(v_k))^t J_{H_i}(v_k)\Delta v_k}{\|H_i(v_k)\|} \\
&\leq \Upsilon_i(\alpha_k) + \kappa\|H_i(v_k)\| + \frac{\alpha_k}{\|H_i(v_k)\|}\|H_i(v_k)\|^2 \left(-1 + \frac{\tau_k}{\beta}\right) \\
&\leq \left(\kappa + 1 - \left(1 - \frac{\tau_k}{\beta}\right)\alpha_k\right)\|H_i(v_k)\|
\end{aligned}$$

furthermore, we may obtain :

$$\|H_i(v_{k+1})\| \leq \left(\frac{\kappa+1}{\kappa} - \frac{1}{\kappa}\left(1 - \frac{\tau_k}{\beta}\right)\alpha_k\right)\|H_i(v_k)\|$$

in order to finish the proof and reach the desired result, we choose the value of $\sigma = \frac{\tau_0}{\kappa}$ and proceed to the limit when κ goes to $+\infty$. □

The following conclusion may now be applied based on **REMARK 3.2.3** and **THEOREM 3.2.4**.

COROLLARY 3.2.5. *Let us suppose* **ASSUMPTION 1** *is correct. The algorithm illustrated in* **FIGURE 3.5** *is well-defined.*

3.3 CONVERGENCE ANALYSIS

In this section, we show that every sequence $\{v_k\}$ created using the procedure indicated in **FIGURE 3.5** has a certain set of features, given the assumptions we make. The most important feature, the boundness of the sequence $\{v_k\}$, is described by the following theorem.

THEOREM 3.3.1. *Assume* **ASSUMPTION 1** *is correct. The algorithm then generates the sequence* $\{v_k\}$ *depending on each* ϕ_i *where* $i = trig, exp$ *is bounded.*

PROOF. It is sufficient to show that the two sequences $\{\nu_k\}$ and $\{u_k\}$ are bounded in order to show that $\{v_k\}$ is bounded. Based on (3.2) and the first property in **PROPOSITION 3.2.1**, it follows that the sequence $\{\|H_i(v_k)\|\}$ is bounded for $i = trig, exp$. As a result, it is straightforward to see that the sequences $\{\nu_k\}$ and $\{\|Au_k - \Phi_i(\nu_k, u_k) - b\|\}$ are bound. Furthermore, we are aware that for any k, we have :

$$\|Au_k\| \geq \sigma_{min}(A)\|u_k\|$$

as a result, we believet :

$$\|\Phi_{trig}(\nu_k, u_k)\| \leq \|u_k\| \quad \text{and} \quad \|\Phi_{exp}(\nu_k, u_k)\| \leq \|u_k\| + \frac{\|\nu_k\|}{\pi}$$

when $\{u_k\}$ is supposed to be is unbounded in the following, we obtain :

▶ For $i = trig$, we get :

$$\begin{aligned}
\|Au_k - \Phi_{trig}(\nu_k, u_k) - b\| &\geq \|Au_k\| - \|\Phi_{trig}(\nu_k, u_k)\| - \|b\| \\
&\geq (\sigma_{min}(A) - 1)\|u_k\| - \|b\| \\
&\geq +\infty
\end{aligned}$$

▶ For $i = exp$, we get :

$$\begin{aligned}
\|Au_k - \Phi_{exp}(\nu_k, u_k) - b\| &\geq \|Au_k\| - \|\Phi_{exp}(\nu_k, u_k)\| - \|b\| \\
&\geq (\sigma_{min}(A) - 1)\|u_k\| - \frac{\|\nu_k\|}{\pi} - \|b\| \\
&\geq +\infty
\end{aligned}$$

when **ASSUMPTION 1** is followed by the two last inequalities, which in both cases show that the sequence $\{\|Au_k - \Phi_i(\nu_k, u_k) - b\|\}$ is unbounded, resulting in a contradiction. As a result, the sequence $\{u_k\}$ has been bounded. □

As previously stated, the sequence $\{v_k\}$ leads to a solution for *AVE* (3.1). To that goal, the following two theorems provide the convergence result of the method proposed in **FIGURE 3.5**.

THEOREM 3.3.2. *(Theorem* 3.1, **[42]***) Suppose that* **ASSUMPTION 1** *is satisfied and let* $\{v_k\}$ *the sequence generated by our algorithm. Then, any accumulation point of the sequence* $\{u_k\}$ *is a solution of the AVE* (3.1).

PROOF. By using the first property in **PROPOSITION 3.2.1** and **THEOREM 3.3.1**, it is clear that there exists a point $v^* = (\nu^*, u^*)$ such that

$$\lim_{k\to+\infty} v_k = v^* \Rightarrow \lim_{k\to+\infty} \|H_i(v_k)\| = \|H_i(v^*)\|, \text{ with } i = trig, exp.$$

In the following, for $i = trig, exp$ we distingue two cases :

► If $\|H_i(v^*)\| > 0$. Then, we have that $\nu^* > 0$ and $\lim\limits_{k\to+\infty} \alpha_k = 0$ consequently for all sufficiently large k, $\widehat{\alpha}_k := \dfrac{\alpha_k}{\delta}$ does not satisfy the line search (**LS**), i.e.,

$$\|H_i\left(v_k + \widehat{\alpha}_k \Delta v_k\right)\| > \left(1 - \sigma\left(1 - \frac{1}{\beta}\right)\widehat{\alpha}_k\right)\|H_i\left(v_k\right)\|$$

therefore :

$$\frac{\|H_i\left(v_{k+1}\right)\| - \|H_i\left(v_k\right)\|}{\widehat{\alpha}_k} > -\sigma\left(1 - \frac{1}{\beta}\right)\|H_i\left(v_k\right)\| \tag{3.20}$$

since $\nu^* > 0$, it follows that H_i is continuously differentiable at v^*. Let $k \to +\infty$ and due to (**3.19**) then we obtain :

$$\begin{aligned} -\sigma\left(1 - \frac{1}{\beta}\right)\|H_i\left(v^*\right)\| &< \frac{(H_i(v^*))^t J_{H_i}(v^*)\Delta v^*}{\|H_i(v^*)\|} \\ &\leq \left(-1 + \frac{\tau^*}{\beta}\right)\|H_i\left(v^*\right)\| \\ &\leq \left(-1 + \frac{1}{\beta}\right)\|H_i\left(v^*\right)\| \end{aligned} \tag{3.21}$$

where the last inequality follows from $\tau^* := \min\{1, \|H_i(v^*)\|\}$ so that the latter requires that $\tau^* \leq 1$. (**3.21**) indicates that :

$$-\sigma\left(1 - \frac{1}{\beta}\right) \leq -1 + \frac{1}{\beta} \Rightarrow (1 - \sigma)\left(1 - \frac{1}{\beta}\right) \leq 0 \Rightarrow 1 \leq \frac{1}{\beta}$$

which contradicts the fact that $\beta > 1$. Hence $H_i(v^*) = 0_{\mathbb{R}^{n+1}}$.

► If $\|H_i(v^*)\| = 0$, it is evident and we conclude that u^* is a solution of (**3.1**) this implies the desired result. □

We find that the local quadratic convergence rate of the algorithm given in **FIGURE 3.5**, which is equivalent to the results of Theorems 3.2 in **[42]** and 4.4 in **[43]**.

THEOREM 3.3.3. *Assume* **ASSUMPTION 1** *is true and let $v^* = (\nu^*, u^*)$ be an accumulation point of the sequence $\{v_k\}$ created by the algorithm shown in* **FIGURE 3.5**. *If all matrices V are supplied by :*

$$V = \begin{pmatrix} 1 & 0 \\ 0_{\mathbb{R}^n} & A + diag(d_j) \end{pmatrix} \in \partial H_i(v^*)$$

with $i = trig, exp$, $d_j \in [-1, 1]$, $j = 1, \ldots, n$ and $\partial H_i(v^*) = \lim\limits_{v_k \to v^*} J_{H_i}(v_k)$ is invertible. Then, we have :

(a) The whole sequence $\{v_k\}$ converges to v^*.

(b) $\|v_{k+1} - v^*\| = o\left(\|v_k - v^*\|\right)$.

(c) $\nu_{k+1} = \nu_k^2$.

3.4 NUMERICAL RESULTS

In this section, we report the numerical results of the algorithm illustrated in FIGURE 3.5 for resolving the special and general forms of absolute value equations. Each experiment was modified using the **Dev-Cpp 5.11 TDM-GCC 4.9.2 Setup** and running on a PC with no calculator station. Throughout the process, our algorithm will terminate when $\|H_i\,(v_k)\,\| \leq 10^{-6}$, $i = trig, exp$ or there have been more than 100 iterations. We examine the following factors :

$$\delta = 0.5, \quad \sigma = 0.0001, \quad \nu_0 = 0.1 \quad \text{and} \quad \beta = \max\left\{1, \frac{1.01 * \tau_0^2}{\nu}\right\}.$$

Iter and **CPU** denote the number of iterations and the time taken by our algorithm in seconds to achieve the solution u^* and (z^*, w^*), respectively. Now, we consider two types of problems.

PROBLEMS WITH FIXED SIZE

EXPERIMENT 1. *Due to* (**P1**) *(see page* **60***), the matrix* A *and the vector* b *according to the corresponding absolute value equation* (**3.1**) *are given as follows :*

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 3 \end{pmatrix} \quad \textit{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

where the absolute value equation corresponding to (**P1**) *has a unique solution because*

$$\sigma_{min}(A) = 3.1670 > 1.$$

The beginning point for this problem is given by :

$$u_0 = \begin{pmatrix} 0.027 & 0.018 & 0.009 \end{pmatrix}^t$$

the numerical results are shown in the table below.

$\phi_i(\nu, s)$	$\phi_{trig}(\nu, s)$	$\phi_{exp}(\nu, s)$	$\phi_1(\nu, s)$	$\phi_2(\nu, s)$	$\phi_3(\nu, s)$	$\phi_4(\nu, s)$
Iter	21	18	25	19	24	22
CPU	0.63	0.54	0.77	0.65	0.69	0.75

TABLE 3.2: NUMERICAL EXPERIMENTS OF (**P1**) USING VARIOUS SMOOTHING FUNCTIONS.

the unique solutions to AVE and LCP corresponding to (**P1**)*, respectively, are :*

$$u^* = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^t,$$
$$z^* = \frac{1}{2}\left(|u^*| - u^*\right) = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^t$$
$$\textit{and} \quad w^* = \frac{1}{2}\left(|u^*| + u^*\right) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^t.$$

EXPERIMENT 2. *Because of* **REMARK 1.8.5**, *the corresponding AVEs for the LCPs presented in* (**P2**) *and* (**P3**) *(see page* **61-62**), *respectively, are :*

$$A = \begin{pmatrix} 0.9503 & 0.0745 & 0.1698 & -0.1905 & 0.4679 \\ 0.0745 & 0.8882 & 0.0787 & -0.3810 & -0.3685 \\ 0.1698 & 0.0787 & 1.3644 & 0.0952 & 0.0124 \\ 0.1905 & 0.3810 & -0.0952 & 1.2857 & 0.0952 \\ -0.4679 & 0.3685 & -0.0124 & 0.0952 & 1.1284 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 20 \\ 8 \\ -12 \\ -2 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1.4444 & -0.2222 & -0.1111 & 0.3333 & 0.1111 & -0.3333 \\ -0.2222 & 0.9111 & 0.1556 & 0.3333 & -0.1556 & -0.3333 \\ -0.1111 & 0.1556 & -0.0222 & -0.3333 & -0.9778 & 0.3333 \\ 0.3333 & 0.3333 & -0.3333 & -1 & 0.3333 & 0 \\ -0.1111 & 0.1556 & 0.9778 & -0.3333 & 0.0222 & 0.3333 \\ 0.3333 & 0.3333 & -0.3333 & 0 & 0.3333 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} 6 & 14 & 0 & 0 & -8 & -16 \end{pmatrix}^t$$

The two above AVEs have unique solutions since the corresponding LCPs (**P2**) *and* (**P3**) *are an* $\mathscr{P}$*-LCPs (see* **COROLLARY 1.8.9** *and* **EXPERIMENT 2** *in* **CHAPTER 2**).

For the first AVE, the beginning point is chosen as follows :

$$u_0 = \begin{pmatrix} 3.75 & 7.75 & 9.5 & 0 & 1 \end{pmatrix}^t$$

the results are summarized in the following table.

$\phi_i(\nu, s)$	$\phi_{trig}(\nu, s)$	$\phi_{exp}(\nu, s)$	$\phi_1(\nu, s)$	$\phi_2(\nu, s)$	$\phi_3(\nu, s)$	$\phi_4(\nu, s)$
Iter	25	20	26	22	21	29
CPU	0.76	0.56	0.85	0.67	0.79	0.81

TABLE 3.3: NUMERICAL EXPERIMENTS OF (**P2**) USING VARIOUS SMOOTHING FUNCTIONS.

we end our algorithm with the following solutions for AVE and LCP :

$$u^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t,$$
$$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$$
$$\text{and} \quad w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t.$$

For the second AVE, the first viable iteration is given by :

$$u_0 = \begin{pmatrix} 1.3 & 1.5 & 1 & 1 & 0.6 & 0.2 \end{pmatrix}^t$$

the findings are presented in the table below :

$\phi_i(\nu, s)$	$\phi_{trig}(\nu, s)$	$\phi_{exp}(\nu, s)$	$\phi_1(\nu, s)$	$\phi_2(\nu, s)$	$\phi_3(\nu, s)$	$\phi_4(\nu, s)$
Iter	40	34	46	38	45	41
CPU	1.06	0.87	1.34	1.09	1.22	1.31

TABLE 3.4: NUMERICAL EXPERIMENTS OF (**P3**) USING VARIOUS SMOOTHING FUNCTIONS.

our algorithm concludes with the following solutions :

$$u^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t,$$
$$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$$
$$\text{and} \quad w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t.$$

EXPERIMENT 3. *The matrix* A *and vector* b *of the* (**AVE**) *are provided by* (**P4**) *(see page* **63***) :*

$$A = \begin{pmatrix} 5.951812 & -13.357438 & -1.441769 & 8.096392 & -56.164700 \\ -41.445786 & 112.610451 & 20.080326 & -77.108437 & 478.393616 \\ -8.951807 & 8.357422 & 4.441766 & -8.096379 & 46.164635 \\ 21.385540 & -50.140556 & -11.465863 & 37.228912 & -190.682709 \\ 36.626507 & -75.020065 & -20.923693 & 46.746983 & -381.526062 \end{pmatrix},$$
$$b = \begin{pmatrix} -57.9157 & 491.5301 & 40.9157 & -194.6747 & -395.0964 \end{pmatrix}^t$$

where the AVE has a unique solution since $\sigma_{min}(A) = 1.028468 > 1$. *The following provides a starting point :*

$$u_0 = \begin{pmatrix} 1.70083 & -0.62833 & 1.164621 & -0.683711 & 1.166491 \end{pmatrix}^t,$$

the following are the numerical results :

$\phi_i(\nu, s)$	$\phi_{trig}(\nu, s)$	$\phi_{exp}(\nu, s)$	$\phi_1(\nu, s)$	$\phi_2(\nu, s)$	$\phi_3(\nu, s)$	$\phi_4(\nu, s)$
Iter	66	59	74	61	73	73
CPU	1.59	1.14	1.64	1.33	1.61	1.62

TABLE 3.5: NUMERICAL EXPERIMENTS OF (**P4**) USING VARIOUS SMOOTHING FUNCTIONS.

the AVE and LCP, respectively, have a unique solutions :

$$u^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t,$$
$$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$$
$$\text{and} \quad w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t.$$

PROBLEMS WITH VARIABLE SIZE

EXPERIMENT 4. *The absolute value equation corresponding to the problem* (**P5**) *(see page* **64***) has a unique solution because, as shown in the table below,* $\sigma_{min}(A) > 1$ *for all* $n = 2, 4, 8, 16, 32, 64,$

n	2	4	8	16	32	64
$\sigma_{min}(A)$	2	2.1459	1.8443	1.5108	1.2807	1.1471

TABLE 3.6: SOME MINIMAL SINGULAR VALUES OF MATRIX A.

As a result, the beginning point for this problem is given by :

$$u_0 = \begin{pmatrix} 1.190476 & 1.095239 & \dots & 1.095239 & 1.190476 \end{pmatrix}^t$$

the unique solutions of corresponding AVE and LCP are :

$$u^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t,$$
$$z^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \end{pmatrix}^t$$
$$\text{and} \quad w^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t.$$

the numerical results for various n values are shown in the table below.

$\phi_i(\nu, s) \setminus n$		2	4	8	16	32	64
$\phi_{trig}(\nu, s)$	Iter	15	20	27	43	55	77
	CPU	0.72	1.03	1.82	2.09	2.88	4.75
$\phi_{exp}(\nu, s)$	Iter	11	19	21	32	48	71
	CPU	0.42	0.98	1.36	1.91	2.28	3.43
$\phi_1(\nu, s)$	Iter	15	25	30	49	57	79
	CPU	0.81	1.19	2.00	2.14	2.94	4.91
$\phi_2(\nu, s)$	Iter	12	20	23	35	49	73
	CPU	0.53	1.02	1.40	1.96	2.33	3.73
$\phi_3(\nu, s)$	Iter	15	22	29	45	56	78
	CPU	0.82	1.13	1.62	1.98	2.44	3.91
$\phi_4(\nu, s)$	Iter	12	20	24	36	50	74
	CPU	0.83	1.06	1.96	2.11	2.91	4.86

TABLE 3.7: NUMERICAL EXPERIMENTS OF (**P5**) USING VARIOUS SMOOTHING FUNCTIONS.

EXPERIMENT 5. *By applying the* **DEFINITION 1.7.6** *condition, the related AVE to problem* (**P6**) *(see page* **65***) has a unique solution because we know that the corresponding matrix satisfies :*

$$\|A_h^{-1}\| \leq \frac{1}{8} < 1$$

by calculating the norm, we find that $\|A_h^{-1}\| = 0.1024$. The initial iteration of the associated AVE is given by :

$$u_0 = \begin{pmatrix} 1.108942 & 1.191847 & 1.247692 & 1.275788 & & \dots \\ \dots & & 1.275788 & 1.247692 & 1.191847 & 1.108942 \end{pmatrix}^t$$

the unique solutions of the associated AVE and LCP are given by :

$$u^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t,$$
$$z^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \end{pmatrix}^t$$
$$\text{and} \quad w^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t,$$

the numerical findings for various smoothing functions are as follows :

$\phi_i(\nu,s) \setminus n$		2	4	8	16	32	64
$\phi_{trig}(\nu,s)$	**Iter**	12	14	30	45	55	69
	CPU	0.60	0.69	0.77	0.90	0.98	1.09
$\phi_{exp}(\nu,s)$	**Iter**	10	11	26	42	50	67
	CPU	0.54	0.63	0.74	0.86	0.92	1.03
$\phi_1(\nu,s)$	**Iter**	18	25	35	53	60	75
	CPU	0.68	0.80	0.87	1.02	0.14	1.26
$\phi_2(\nu,s)$	**Iter**	11	15	28	43	56	69
	CPU	0.58	0.65	0.79	0.88	0.96	1.07
$\phi_3(\nu,s)$	**Iter**	15	21	34	49	57	71
	CPU	0.62	0.75	0.81	0.96	1.00	1.12
$\phi_4(\nu,s)$	**Iter**	12	19	31	47	58	70
	CPU	0.65	0.76	0.85	0.99	1.04	1.18

TABLE 3.8: NUMERICAL EXPERIMENTS OF (**P6**) USING VARIOUS SMOOTHING FUNCTIONS.

EXPERIMENT 6. *For the problem* (**P7**) *(see page* **67***), the matrix* A *and vector* **b** *of the related AVE are provided by :*

$$A = \begin{pmatrix} 49 & 5 & 0 & \dots & \dots & \dots & 0 \\ 5 & 49 & 5 & 0 & \dots & \dots & 0 \\ 0 & 5 & 49 & 5 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 5 & 49 & 5 & 0 \\ 0 & \dots & \dots & 0 & 5 & 49 & 5 \\ 0 & \dots & \dots & \dots & 0 & 5 & 49 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 53 \\ 58 \\ 58 \\ \vdots \\ 58 \\ 58 \\ 53 \end{pmatrix}.$$

The numerical outcomes are as follows :

$\phi_i(\nu,s) \setminus n$		2	4	8	16	32	64
$\phi_{trig}(\nu,s)$	**Iter**	10	17	31	38	54	64
	CPU	0.49	0.63	0.71	0.76	0.96	0.98
$\phi_{exp}(\nu,s)$	**Iter**	09	12	24	34	53	60
	CPU	0.45	0.59	0.69	0.74	0.92	0.97
$\phi_1(\nu,s)$	**Iter**	15	21	38	47	67	71
	CPU	0.59	0.70	0.82	0.85	1.05	1.11
$\phi_2(\nu,s)$	**Iter**	10	14	29	41	56	72
	CPU	0.46	0.61	0.70	0.77	0.95	1.01
$\phi_3(\nu,s)$	**Iter**	13	19	35	44	64	68
	CPU	0.52	0.64	0.75	0.80	0.99	1.03
$\phi_4(\nu,s)$	**Iter**	12	16	34	43	54	63
	CPU	0.55	0.67	0.76	0.82	1.02	1.07

TABLE 3.9: NUMERICAL EXPERIMENTS OF (**P7**) USING VARIOUS SMOOTHING FUNCTIONS.

where the initial iteration is provided by :

$$u_0 = \begin{pmatrix} 1.0191 & 1.0171 & 1.0173 & 1.0172 & \dots & 1.0172 & 1.0173 & 1.0171 & 1.0191 \end{pmatrix}^t$$

and the unique solutions are provided by :

$$u^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t,$$
$$z^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \end{pmatrix}^t$$
$$\text{and} \quad w^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t.$$

*This is due to the fact that the related AVE of (**P7**) has a unique solution for* $n = 2, 4, 8, 16, 32, 64$, *because :*

n	2	4	8	16	32	64
$\sigma_{min}(-2H - I)$	48	40.9098	39.6031	39.1703	39.0453	39.0117

TABLE 3.10: SOME MINIMAL SINGULAR VALUES OF MATRIX $-2H - I$.

COMMENTS 2. *We infer from the results of this section that our two functions (**3.4**) and (**3.5**) are more effective than the prior functions.The difference between our new smoothing functions and those of Saheya et al. **[71]** becomes large in terms of the number of iterations and computation time. Furthermore,* ϕ_{exp} *is the best of all functions* ϕ_{trig}, ϕ_1, ϕ_2, ϕ_3 *and* ϕ_4, *confirming the geometrical and theoretical conclusions reported in **REMARK 3.1.10** and **REMARK 3.1.11**. The order of numerical performance from good to bad is :*

$$\phi_{exp} > \phi_2 > \phi_{trig} > \phi_4 > \phi_3 > \phi_1, \quad \textit{for iterations.}$$
$$\phi_{exp} > \phi_2 > \phi_{trig} > \phi_3 > \phi_4 > \phi_1, \quad \textit{for time taken.}$$

CONCLUSION

In this chapter, we presented theoretical and numerical contributions. Indeed, the introduction of two new smoothing functions (**3.4**) and (**3.5**) was used to resolve the special form of the absolute value equation, especially, which is extracted from (**LCP**). We have validated the theoretical claims by observing that the absolute value equation is rewritten as a set of smooth equations, as shown in **[71]**, since the absolute function is believed to be non-smooth. We also examine some of the characteristics of these two functions. We show that the smoothing-type algorithm based on our two functions has a local quadratic convergence rate that is well-defined and effective under suitable assumption. With some numerical implementations, it was finally found that ϕ_{exp} is the best choice of smoothing function to work with the proposed smoothing-type algorithm, while it also best approximates the function $|s|$. In other words, ϕ_{exp} is more effective than the smoothing function ϕ_{trig}, although they are the two most effective compared to the previous smoothing functions in terms of number of iterations and time taken.

GENERAL CONCLUSION

First and foremost, this thesis is highly intriguing since it allows us to continue working and researching on the topic of optimization, which has been one of the most significant areas that catches our attention since the completion of the master's thesis.

In this thesis, we investigated the topic of algorithmic complexity in two of the most known methods of optimization algorithms for solving the linear complementarity problem. In order to achieve our goal and respond to the introduction's questions, we articulated our research in two parts parts :

In the first part, we provide a primal–dual interior-point method according to two new parametric kernel functions, where the first function generalizes the algorithmic complexity achieved by *Bai et al.* **[7]** and the second function has a new hyperbolic-logarithmic barrier term. As a result, we examined the algorithm proposed in **FIGURE 2.1** using the two new kernel functions (**2.1**) and (**2.2**). Then, using some essential properties of the two proposed kernel functions, we proved that in two cases, our algorithms have the best-known complexity bounds for small- and large-update methods, namely, $O(\sqrt{n}\log\frac{n}{\varepsilon})$ and $O\left(\sqrt{n}\log n\log\frac{n}{\varepsilon}\right)$, respectively, with the special choice of their barrier degrees. Finally, we confirmed these theoretical results using some numerical experiments then we notice that our two new functions are more effective than some kernel functions and the first function solves the problems with the fewest number of iterations and the second function in the quickest time when our kernel functions are compared together.

On the other hand, we offer a smoothing-type approach as illustrated in **[71]** the absolute value equation is recast as a collection of smooth equations because the absolute function is thought to be non-smooth. Then, we suggest two novel smoothing functions (**3.4**) and (**3.5**) to solve the system of smooth equations. We also look at some of the features of these two smoothing functions, which are essential to the complexity of the algorithm. We show that the algorithms based on our two new functions are well-defined and have an effective local quadratic convergence rate under appropriate assumption. In addition, we have proven that our two smoothing functions are more effective than other smoothing functions in a theoretic and geometric manner, specifically function (**3.5**) is the best among all. In the end, we present some numerical experiments on some problems to demonstrate the effectiveness of our two new functions and confirms all our previous results.

Lastly, we draw the conclusion that our four proposed functions in this thesis achieve the best theoretical and numerical results in both methods and we confirm numerically the relation between the solutions of *AVE* and *LCP* (i.e., results of **COROLLARY 1.8.15** and **REMARK 1.8.16**) in addition to the numerical successes of the interior-point approach in comparison to the smoothing-type approach when we compare the two methods, although they are lacking in solution precision (see **APPENDIX**).

VARIOUS PROPOSALS FOR FURTHER RESEARCH

True, this time period has provided us with several experiences that have allowed us to actualize certain concepts, but research has no bounds as long as there is a desire to optimize and discover superior features. As a result, the findings of this study suggest that future research should concentrate on :

- ► The generalization of the complexity bounds based on a kernel function with a new type of barrier term for interior-point methods to solve the *SDLCP*.
- ► Studied other iterative methods used to solve the linear complementarity problem and will compare them numerically to see which one is more effective.
- ► Whether the two smoothing functions proposed in this work can be employed for solving the absolute value equation associated with the second-order cone (*SOCAVE*).
- ► Propose an infeasible interior-point algorithm with a full-Newton step for the linear complementarity problem based on our two kernel functions.

APPENDIX

To compare the smoothing-type and interior-point algorithms, we must compare the best function for each approach that achieves the fewest number of iterations and the shortest time for all tests. To that purpose, we provide the table below :

PROBLEMS :	SMOOTHING-TYPE ALGORITHM :		INTERIOR-POINT ALGORITHM :			
	$\phi_{exp}(\nu, s)$		$\psi_E(x)$		$\psi_H(x)$	
	Iter	**CPU**	**Iter**	**CPU**	**Iter**	**CPU**
(**P1**)	**18**	**0.54**	**10**	**0.29**	**12**	**0.23**
(**P2**)	**20**	**0.56**	**16**	**0.64**	**17**	**0.54**
(**P3**)	**34**	**0.87**	**19**	**0.61**	**20**	**0.57**
(**P4**)	**59**	**1.14**	**16**	**0.63**	**20**	**0.55**
(**P5**)	**19**	**0.98**	**11**	**0.38**	**14**	**0.35**
(**P6**)	**26**	**0.74**	**22**	**0.85**	**22**	**0.69**
(**P7**)	**34**	**0.69**	**18**	**0.52**	**23**	**0.42**

For the problems with variable size, we take into consideration the sizes used in **CHAPTER 2**, i.e., $n = 4$ for (**P5**), $n = 8$ for (**P6**) and $n = 16$ for (**P7**).

Regarding solutions, we provide the following table :

PROBLEMS :	ST ALGORITHM :	IP ALGORITHM :
(**P1**)	$z^* = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^t$	$z^* = \begin{pmatrix} 0.000001 & 0.000001 & 0.000001 \end{pmatrix}^t$
	$w^* = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^t$	$w^* = \begin{pmatrix} 1.000005 & 1.000005 & 1.000002 \end{pmatrix}^t$
	$u^* = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^t$	$u^* = \begin{pmatrix} 1.000003 & 1.000002 & 1.000001 \end{pmatrix}^t$
(**P2**)	$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$	$z^* = \begin{pmatrix} 0.000002 & 0.000002 & 0.000004 & 0.000002 & 0.000002 \end{pmatrix}^t$
	$w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$	$w^* = \begin{pmatrix} 1.000007 & 1.00026 & 1.000023 & 0.999991 & 0.999998 \end{pmatrix}^t$
	$u^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$	$u^* = \begin{pmatrix} 1.000005 & 1.000023 & 1.000019 & 0.999989 & 0.999996 \end{pmatrix}^t$
(**P3**)	$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$	$z^* = \begin{pmatrix} 0.000002 & 0.000002 & 0.000002 & 0.000002 & 0.000001 & 0.000001 \end{pmatrix}^t$
	$w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$	$w^* = \begin{pmatrix} 1.000006 & 1.00008 & 1.000001 & 1.000002 & 0.999995 & 0.999988 \end{pmatrix}^t$
	$u^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$	$u^* = \begin{pmatrix} 1.000005 & 1.000007 & 1 & 1 & 0.999993 & 0.999986 \end{pmatrix}^t$
(**P4**)	$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$	$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$
	$w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$	$w^* = \begin{pmatrix} 0.999793 & 1.000065 & 1.000469 & 1.000403 & 1.000009 \end{pmatrix}^t$
	$u^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$	$u^* = \begin{pmatrix} 0.999793 & 1.000065 & 1.000469 & 1.000403 & 1.000009 \end{pmatrix}^t$
(**P5**)	$z^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \end{pmatrix}^t$	$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}^t$
	$w^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t$	$w^* = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^t$
	$u^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t$	$u^* = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^t$
(**P6**)	$z^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \end{pmatrix}^t$	$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$
	$w^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t$	$w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$
	$u^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t$	$u^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$
(**P7**)	$z^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \end{pmatrix}^t$	$z^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^t$
	$w^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t$	$w^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$
	$u^* = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}^t$	$u^* = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^t$

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